

SKEW-MONOIDAL CATEGORIES AND THE CATALAN SIMPLICIAL SET

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1. INTRODUCTION

The n th Catalan number C_n , given explicitly by $\frac{1}{n+1}\binom{2n}{n}$, is well-known to be the answer to many different counting problems. For example, it is the number of bracketings of an $(n+1)$ -fold product. Thus there are many families of sets, \mathbb{C}_n , indexed by the natural numbers, whose cardinalities are the Catalan numbers; such families might then be called ‘‘Catalan sets’’. Stanley [18, 19] describes at least 205 such families. We shall show how to define functions between these sets, in such a way as to produce a simplicial set \mathbb{C} , which is the ‘‘Catalan simplicial set’’ of the title. This is done using what seems to be a new description of the Catalan sets, which relies heavily on the Boolean algebra **2**.

Simplicial sets are abstract, combinatorial models of spaces, most often used in homotopy theory. They also arise, however, as models of higher-dimensional categories, and that is their main role in this paper: our primary goal is to show that the simplicial set \mathbb{C} encodes, in a precise sense, a particular categorical structure called a *skew-monoidal category*. We shall show that a skew-monoidal category is the same thing as a simplicial map from \mathbb{C} to another simplicial set $NCat$, the nerve of the monoidal bicategory Cat of categories and functors.

The structure of skew-monoidal category was introduced recently by Szlachányi [23] in his study of bialgebroids, which are themselves an extension of the notion of quantum group.

Thus the work presented here lies at the interface of several mathematical disciplines:

- (a) algebraic topology, because it involves simplicial sets and nerves;
- (b) combinatorics, in the form of the Catalan numbers;

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- (c) quantum groups, through recent work on bialgebroids;
- (d) logic, because of the distinguished role of the Boolean algebra $\mathbf{2}$; and
- (e) category theory.

The natural generalisation of the notion of monoid from the level of sets to categories is that of *monoidal category* [14]. Like a monoid, a monoidal category \mathcal{A} comes equipped with a unit element $I \in \mathcal{A}$ and a multiplication $\otimes: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ (now a functor, rather than a function); but unlike a monoid, the basic operations of a monoidal category do not satisfy associativity and unitality on the nose, but only up to coherent natural families of isomorphisms:

$$(1.1) \quad \begin{aligned} \lambda_A: I \otimes A &\rightarrow A & \text{and} & \quad \rho_A: A \rightarrow A \otimes I & \text{(for } A \in \mathcal{A}\text{)} \\ \alpha_{ABC}: (A \otimes B) \otimes C &\rightarrow A \otimes (B \otimes C) & \text{(for } A, B, C, \in \mathcal{A}\text{)}. \end{aligned}$$

In calling these families coherent, we mean to say that certain diagrams of derived natural transformations, built from composites of tensors of α 's, λ 's and ρ 's, must commute. The commutativity of these particular diagrams in fact implies the commutativity of *all* such diagrams: this is one form of the coherence theorem for monoidal categories [14], and justifies the choice of axioms made. Mac Lane's original formulation [13] posited five generating axioms, recalled in Section 2 below; Kelly later reduced these to two [10].

Recently, Szlachányi [23] has introduced *skew-monoidal categories*. The basic data for a skew-monoidal category are the same as for a monoidal category, except that the constraint morphisms in (1.1) are no longer required to be invertible. The axioms which are to be satisfied are Mac Lane's original five, rather than Kelly's two; the choice is substantive, as without invertibility in (1.1), the two axiomatisations are no longer equivalent. Szlachányi's motivation in [23] for defining skew-monoidal structure comes from representation theory: he identifies left bialgebroids based on a ring R with closed skew-monoidal structures on the category of left R -modules. Lack and Street have placed this result in a more general context, showing in [11] that the quantum categories of [3] can be captured as *skew monoidales*—internal skew-monoidal objects—in a suitable monoidal bicategory.

Whilst these applications justify the use of skew-monoidal structure, they do not give an intrinsic justification for the form the structure takes. There are in fact two places in the definition where a non-obvious choice has been made. The first concerns the orientation of the maps in (1.1). For example, had we taken λ to have components $A \rightarrow I \otimes A$, whilst leaving ρ and α unchanged, we would have obtained (the one-object case of) Burroni's notion of *pseudocategory* [2]; on the other hand, if we had reversed the sense of α whilst leaving λ and ρ unchanged, we would have obtained something very close to Grandis' notion [7] of *d-lax 2-category*.

The second choice concerns the axioms the maps in (1.1) must satisfy. We have already said that Mac Lane’s five axioms are no longer equivalent to Kelly’s two in the skew setting; so why, then, should we prefer the former to the latter? For monoidal categories, we justified the axioms in terms of a theorem stating that all diagrams of coherence morphisms commute. In the skew-monoidal case, we have no such justification, since a general diagram of skew-monoidal coherence morphisms need *not* commute; describing the structure these coherence morphisms determine is in fact quite subtle [12].

The objective of the paper is to provide a perspective on skew-monoidal structure which, amongst other things, makes it quite apparent why the choices made above are natural ones. To do this, we use the Catalan simplicial set \mathbb{C} mentioned above. It turns out to be quite easy to describe: it is itself a nerve, the nerve of the monoidal poset $(\mathbf{2}, \vee, 0)$. In particular, this makes it 2-coskeletal—meaning that for $n > 2$, each n -simplex boundary has a unique filler—and thus completely determined by its 0-, 1- and 2-simplices, of which it has one, two, and five respectively; more generally, the number of n -simplices is the n -th Catalan number.

Our perspective, then, is that \mathbb{C} classifies skew-monoidal structures in the sense that simplicial maps from \mathbb{C} into a suitably-defined nerve of \mathbf{Cat} are precisely skew-monoidal categories. More generally, skew monoidales in a monoidal bicategory \mathcal{K} are classified by maps from \mathbb{C} into the simplicial nerve of \mathcal{K} .

The two non-degenerate 2-simplices in \mathbb{C} encode the tensor and unit operations borne by any skew-monoidal category; the non-degenerate 3-simplices encode α , λ and ρ , with the orientations specified above; whilst the non-degenerate 4-simplices encode the skew-monoidal axioms. The coskeletality means that, in particular, the 3- and 4-simplices are completely determined by the 2-simplices, and it is in this sense that our perspective justifies the choices of coherence data and axioms for a skew-monoidal structure.

There is another well-known connection between the Catalan numbers and (skew) monoidal structure, arising from the fact that the n th Catalan number C_n is the number of ways to bracket an $(n + 1)$ -fold product. The connection described in this paper seems to be quite different; indeed, in the context of this paper C_n involves n -fold products.

This work is inspired by an old idea of Michael Johnson on how to capture not only associativity but also unitality constraints simplicially. He reminded us of this in a recent talk [8] to the Australian Category Seminar.

2. SKEW-MONOIDAL CATEGORIES AND SKEW MONOIDALES

As in the introduction, a *skew-monoidal category* is a category \mathcal{A} equipped with a unit element $I \in \mathcal{A}$, a tensor product $\otimes: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$,

and natural families of (non-invertible) constraint maps α , λ and ρ as in (1.1), all subject to the commutativity of the following diagrams—wherein tensor is denoted by juxtaposition—for all $a, b, c, d \in \mathcal{A}$:

$$(2.1) \quad \begin{array}{ccc} & (ab)(cd) & \\ \alpha \nearrow & & \searrow \alpha \\ ((ab)c)d & & a(b(c(d))) \\ \alpha 1 \searrow & & \nearrow 1\alpha \\ & (a(bc))d \xrightarrow{\alpha} a((bc)d) & \end{array}$$

$$(2.2) \quad \begin{array}{ccc} & (aI)b \xrightarrow{\alpha} a(Ib) & \\ \rho 1 \nearrow & & \searrow 1\lambda \\ ab & \xrightarrow{\text{id}} & ab \end{array}$$

$$(2.3) \quad \begin{array}{ccc} & I(ab) & \\ \alpha \nearrow & & \searrow \lambda \\ (Ia)b & \xrightarrow{\lambda 1} & ab \end{array}$$

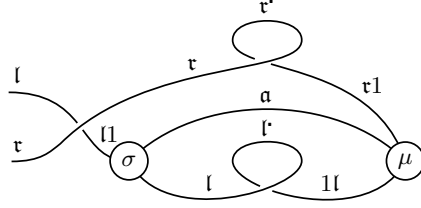
$$(2.4) \quad \begin{array}{ccc} & (ab)I & \\ \rho \nearrow & & \searrow \alpha \\ ab & \xrightarrow{1\rho} & a(bI) \end{array}$$

$$(2.5) \quad \begin{array}{ccc} & II & \\ \rho \nearrow & & \searrow \lambda \\ I & \xrightarrow{\text{id}} & I \end{array} .$$

We may also say that $(\otimes, I, \alpha, \lambda, \rho)$ is a *skew-monoidal structure* on \mathcal{A} . What we call skew-monoidal is what Szlachányi [23] calls left-monoidal, whilst what he calls right-monoidal we would call *opskew-monoidal*; an opskew-monoidal structure on \mathcal{A} is the same as a skew-monoidal structure on \mathcal{A}^{op} .

More generally, we can consider skew monoidales in a monoidal bicategory. A monoidal bicategory is a one-object tricategory in the sense of [6]; it thus comprises a bicategory \mathcal{B} equipped with a unit object I and tensor product homomorphism $\otimes: \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$ which is associative and unital only up to pseudonatural equivalences \mathbf{a} , \mathbf{l} and \mathbf{r} . The coherence of these equivalences is witnessed by invertible modifications π, μ, σ and τ , whose components are 2-cells with boundaries

those of the axioms (2.1)–(2.4) above, and an invertible 2-cell θ whose boundary is that of (2.5). The modifications π, μ, σ and τ are as in [6], though we write σ and τ for what there are called λ and ρ ; whilst $\theta: \tau_I \circ \iota_I \Rightarrow 1_I: I \rightarrow I$ can be defined from the remaining coherence data as the composite:

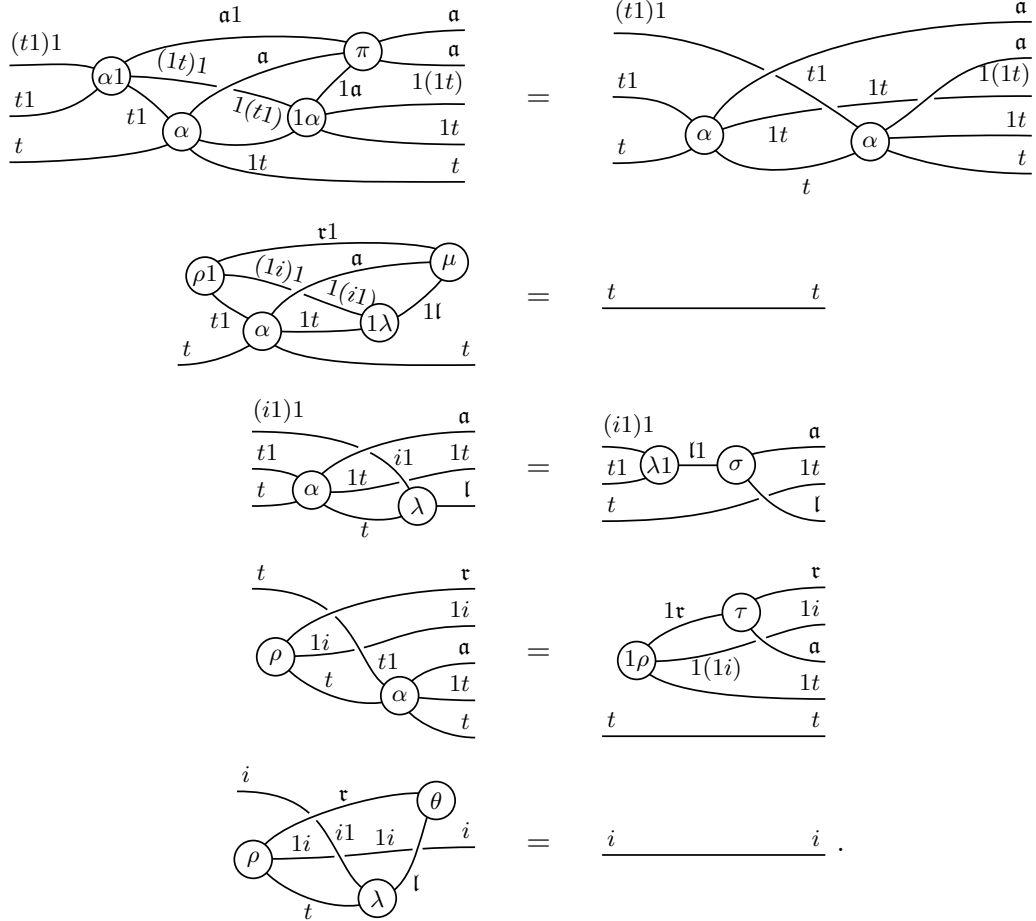


Here, and elsewhere in this paper, we use string notation to display composite 2-cells in a bicategory, with objects represented by regions, 1-cells by strings, and generating 2-cells by vertices. We orient our string diagrams with 1-cells proceeding down the page and 2-cells proceeding from left to right. If a 1-cell ξ belongs to a specified adjoint equivalence, then we will denote its specified adjoint pseudoinverse by ξ^* , and as usual with adjunctions, will draw the unit and counit of the adjoint equivalence in string diagrams as simple caps and cups. In representing the monoidal structure of a bicategory, we denote the tensor product \otimes by juxtaposition, notate the structural 1-cells $\mathbf{a}, \mathbf{l}, \mathbf{r}$ and 2-cells π, ν, λ, ρ explicitly, and use string crossings to notate pseudonaturality constraint 2-cells, and also instances of the pseudofunctoriality of \otimes of the form $(f \otimes 1) \circ (1 \otimes g) \cong (1 \otimes g) \circ (f \otimes 1)$ (the interchange isomorphisms).

With our notational conventions now established, we now define a *skew monoidal* in a monoidal bicategory \mathcal{B} to be given by an object $A \in \mathcal{B}$, unit and multiplication morphisms $i: I \rightarrow A$ and $t: A \otimes A \rightarrow A$, and (non-invertible) coherence 2-cells

$$\begin{array}{ccc}
 (A \otimes A) \otimes A & \xrightarrow{\mathbf{a}} & A \otimes (A \otimes A) \\
 t \otimes A \downarrow & \cong & \downarrow A \otimes t \\
 A \otimes A & \xrightarrow{t} & A \xleftarrow{t} A \otimes A
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccccc}
 I \otimes A & \xrightarrow{\mathbf{l}} & A & \xrightarrow{\mathbf{r}} & A \otimes I \\
 i \otimes A \downarrow & \cong & \parallel & \cong & \downarrow A \otimes i \\
 A \otimes A & \xrightarrow{t} & A & \xleftarrow{t} & A \otimes A
 \end{array}$$

subject to the following five axioms, the appropriate analogues of (2.1)–(2.5).



Note that a skew monoidal in the monoidal bicategory $(\mathbf{Cat}, \times, 1)$ is precisely a skew-monoidal category, whilst a skew monoidal in the 2-cell dual $(\mathbf{Cat}^{\text{co}}, \times, 1)$ is an opskew-monoidal category.

3. NERVES OF MONOIDAL BICATEGORIES

Before describing the simplicial set \mathbb{C} that classifies skew-monoidal categories, and more generally, skew monoidales in a monoidal bicategory, we will first describe the nerve construction by which we will assign a simplicial set $N\mathcal{B}$ to a given monoidal bicategory \mathcal{B} ; the classification of skew monoidales in \mathcal{B} will then be in terms of simplicial maps $\mathbb{C} \rightarrow N\mathcal{B}$.

First let us recall some basic definitions. We write Δ for the simplicial category; the objects are $[n] = \{0, \dots, n\}$ for $n \geq 0$ and the morphisms are order-preserving functions. Objects X of $\mathbf{SSet} = [\Delta^{\text{op}}, \mathbf{Set}]$ are called *simplicial sets*; we write X_n for $X([n])$ and call its elements *n-simplices* of X . We use the notation $d_i: X_n \rightarrow X_{n-1}$ and $s_i: X_n \rightarrow X_{n+1}$ for the face and degeneracy maps, induced by acting on X by

the maps $\delta_i: [n-1] \rightarrow [n]$ and $\sigma_i: [n+1] \rightarrow [n]$ of Δ , the respective injections and surjections for which $\delta_i^{-1}(i) = \emptyset$ and $\sigma_i^{-1}(i) = \{i, i+1\}$. An $(n+1)$ -simplex x is called *degenerate* when it is in the image of some s_i , and *non-degenerate* otherwise.

A simplicial set is called *r-coskeletal* when it lies in the image of the right Kan extension functor $[(\Delta^{(r)})^{\text{op}}, \text{Set}] \rightarrow [\Delta^{\text{op}}, \text{Set}]$, where $\Delta^{(r)} \subset \Delta$ is the full subcategory on those $[n]$ with $n \leq r$. In elementary terms, a simplicial set is *r-coskeletal* when every n -boundary with $n > r$ has a unique filler; here, an *n-boundary* in a simplicial set is a collection of $(n-1)$ -simplices (x_0, \dots, x_n) satisfying $d_j(x_i) = d_i(x_{j+1})$ for all $0 \leq i \leq j < n$; a *filler* for such a boundary is an n -simplex x with $d_i(x) = x_i$ for $i = 0, \dots, n$.

As we noted above, a monoidal bicategory is a one-object tricategory in the sense of [6]. There are several known constructions of nerves for tricategories; the one of interest to us is essentially Street's ω -categorical nerve [20], restricted from dimension ω to dimension 3, and generalised from strict to weak 3-categories. An explicit description of this nerve is given in [5]; we now reproduce the details for the case of a monoidal bicategory \mathcal{B} . For such a \mathcal{B} , the *nerve* $N\mathcal{B}$ is the simplicial set defined as follows:

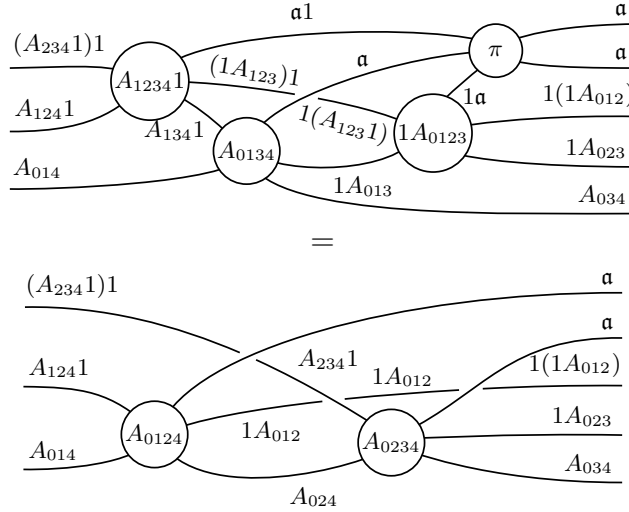
- There is a unique 0-simplex, denoted \star .
- A 1-simplex is an object A_{01} of \mathcal{B} ; its two faces are necessarily \star .
- A 2-simplex is given by objects A_{12}, A_{02}, A_{01} of \mathcal{B} together with a 1-cell $A_{012}: A_{12} \otimes A_{01} \rightarrow A_{02}$; its three faces are A_{12}, A_{02} , and A_{01} .
- A 3-simplex is given by:
 - Objects A_{ij} for each $0 \leq i < j \leq 3$;
 - 1-cells $A_{ijk}: A_{jk} \otimes A_{ij} \rightarrow A_{ik}$ for each $0 \leq i < j < k \leq 3$;
 - A 2-cell

$$\begin{array}{ccc}
 (A_{23} \otimes A_{12}) \otimes A_{01} & \xrightarrow{\mathbf{a}} & A_{23} \otimes (A_{12} \otimes A_{01}) \\
 \begin{array}{c} A_{123} \otimes 1 \\ \downarrow \end{array} & \begin{array}{c} \xRightarrow{A_{0123}} \\ \end{array} & \begin{array}{c} \downarrow 1 \otimes A_{012} \\ \end{array} \\
 A_{13} \otimes A_{01} & \xrightarrow{A_{013}} A_{03} \xleftarrow{A_{023}} & A_{23} \otimes A_{02} ;
 \end{array}$$

its four faces are $A_{123}, A_{023}, A_{013}$ and A_{012} .

- A 4-simplex is given by:
 - Objects A_{ij} for each $0 \leq i < j \leq 4$;
 - 1-cells $A_{ijk}: A_{jk} \otimes A_{ij} \rightarrow A_{ik}$ for each $0 \leq i < j < k \leq 4$;
 - 2-cells $A_{ijk\ell}: A_{ij\ell} \circ (A_{jkl} \otimes 1) \Rightarrow A_{ik\ell} \circ (1 \otimes A_{ijk}) \circ \mathbf{a}$ for each $0 \leq i < j < k < \ell \leq 4$

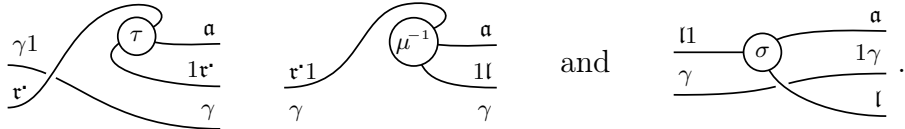
such that the 2-cell equality



holds. The five faces of this simplex are A_{1234} , A_{0234} , A_{0134} , A_{0124} and A_{0123} .

- Higher-dimensional simplices are determined by the requirement that $N\mathcal{B}$ be 4-coskeletal.

It remains to describe the degeneracy operators. The degeneracy of the unique 0-simplex is the unit object $I \in \mathcal{B}$; the two degeneracies $s_0(A), s_1(A)$ of a 1-simplex $A \in \mathcal{B}$ are the unit constraints $\mathbf{r}: A \otimes I \rightarrow A$ and $\mathbf{l}: I \otimes A \rightarrow A$; the three degeneracies $s_0(\gamma), s_1(\gamma)$ and $s_2(\gamma)$ of a 2-simplex $\gamma: B \otimes C \rightarrow A$ are the respective 2-cells



The four degeneracies of a 3-simplex are simply the assertions of certain 2-cell equalities; that these hold is a consequence of the axioms for a monoidal bicategory. Higher degeneracies are determined by coskeletality.

4. THE CATALAN SIMPLICIAL SET

As mentioned in the introduction, the *Catalan simplicial set* is the nerve of the monoidal category $(\mathbf{2}, \vee, 0)$. The category $\mathbf{2}$ has two objects 0, 1 and a single morphism $0 \rightarrow 1$. The tensor for the monoidal structure is disjunction and the unit is 0. This is actually a strict monoidal category and so can be regarded as a one-object 2-category $\Sigma\mathbf{2}$. The Catalan simplicial set \mathbb{C} is the nerve of this 2-category; its n -simplices are normal lax functors $[n] \rightarrow \Sigma\mathbf{2}$ as in [20]. In section 5 we will show that \mathbb{C} classifies skew monoidales in a monoidal bicategory.

Now consider item (fff) in Stanley's list [18] of Catalan sets. These are relations $R \subseteq [n] \times [n]$ that are reflexive, symmetric and have the

interpolation property:

$$(i, k) \text{ and } i \leq j \leq k \text{ implies } (i, j) \text{ and } (j, k) .$$

Let \mathbb{K}_n be the set of such relations. Each \mathbb{K}_n has C_n elements.

Proposition 4.1. *The assignment sending each normal lax functor $F: [n] \rightarrow \Sigma \mathbf{2}$ to*

$$\{(i, j) : F(i \leq j) = 0 \vee F(j \leq i) = 0\}$$

is a bijection $N\Sigma \mathbf{2}_n \cong \mathbb{K}_n$. Thus $|\mathbb{C}_n| = C_n$.

Proof. It is easy to see that the relation above is symmetric and reflexive. Suppose that $i \leq j \leq k$ and (i, k) is in the set above. Then we find that $F(j \leq k) \cdot F(i \leq j) \Rightarrow F(i \leq k)$ has codomain 0 and so must be the identity on 0. Then the domain is 0 and thus $F(j \leq k) = F(i \leq j) = 0$. Now notice that since $\Sigma \mathbf{2}$ has a single object and is locally posetal, normal lax functors $[n] \rightarrow \Sigma \mathbf{2}$ are completely determined by their action on 1-cells. The assignment is a bijection because each F is completely determined by which maps $i \leq j$ go to 0. \square

Corollary 4.2. *The number of non-degenerate simplices of each dimension in \mathbb{C} is given by the Motzkin sequence*

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, \dots$$

Proof. The Catalan numbers can be obtained from the Motzkin numbers [16] by taking cardinalities in

$$\mathbb{C}_n \cong \sum_{m=0}^n \binom{n}{m} \times \text{nd}_m \mathbb{C}$$

where $\text{nd}_m \mathbb{C}$ is the set of non-degenerate m -simplices in \mathbb{C} . \square

The Catalan simplicial set can also be described using ideals. The category of ordered sets and order-preserving functions is denoted by Ord . For ordered sets M and N , an *ideal* $A: M \rightarrow N$ is a subset $A \subseteq N \times M$ (that is, a relation from M to N) such that

$$q \leq j, (j, i) \in A, i \leq p \text{ implies } (q, p) \in A .$$

Composition of ideals is composition of relations. We have a 2-category Idl of ordered sets, ideals and inclusions (for example, see [4]). The identity ideal 1_M of M is $\{(j, i) : j \leq i\}$. Each order-preserving function $\xi: M \rightarrow N$ gives rise to ideals $\xi_*: M \rightarrow N$ and $\xi^*: N \rightarrow M$ defined by

$$\xi_* = \{(j, i) : j \leq \xi(i)\} \quad \text{and} \quad \xi^* = \{(i, j) : \xi(i) \leq j\} .$$

Then $1_M \leq \xi^* \xi_*$ and $\xi_* \xi^* \leq 1_N$. This means $\xi_* \dashv \xi^*$ in Idl . This defines functors $(-)_*: \text{Ord} \rightarrow \text{Idl}$ and $(-)^*: \text{Ord}^{\text{op}} \rightarrow \text{Idl}$, both the identity on objects. This is all familiar \mathcal{V} -category theory with $\mathcal{V} = \mathbf{2}$.

Let us put

$$\mathbb{S}_n = \{B \in \text{Idl}([n], [n]) : 1_{[n]} \leq B\} .$$

Then each $\xi : [m] \rightarrow [n]$ gives a function $\mathbb{S}_\xi : \mathbb{S}_n \rightarrow \mathbb{S}_m$ defined by

$$\mathbb{S}_\xi(B) = \xi^* B \xi_* = \xi^{-1}(B) = \{(p, q) : (\xi(p), \xi(q)) \in B\} .$$

Notice that $1_{[n]} \leq B$ implies $1_{[n]} \leq \xi^* \xi_* \leq \xi^* B \xi_*$. Thus we have defined a simplicial set $\mathbb{S} : \Delta^{\text{op}} \rightarrow \text{Set}$, indeed, a simplicial ordered set $\mathbb{S} : \Delta^{\text{op}} \rightarrow \text{Ord}$.

There is an isomorphism $\mathbb{S} \cong \mathbb{C}$. These ideals can be enumerated using Young diagrams (of which there are a Catalan number); this provides an alternative proof that there are a Catalan number of simplices at each dimension of \mathbb{C} .

Remark 4.3. Each Tamari lattice [15] has a Catalan number of elements. It is not too hard to define the Tamari order on each \mathbb{C}_n . It is natural to ask whether the simplicial structure on \mathbb{C} preserves the Tamari order. This is not the case: we have $\rho \leq s_1(i)$ in \mathbb{C}_3 but $d_1(\rho) = s_1(c) \not\leq i = d_1(s_1(i))$.

5. THE CATALAN SIMPLICIAL SET CLASSIFIES SKEW MONOIDALES

Once we have enumerated the simplices of \mathbb{C} in low dimensions, it will become clear that the image of every map $F : \mathbb{C} \rightarrow N\mathcal{B}$ picks out essentially the data and axioms for a skew monoidale in \mathcal{B} . More precisely, such maps are in bijection with skew monoidales in \mathcal{B}' (defined below). First, it is important to recognise that, since $N\mathcal{B}$ is 4-coskeletal, every simplicial map $F : \mathbb{C} \rightarrow N\mathcal{B}$ is completely determined by its image on the 4-truncation of its domain. In fact, if two such simplicial maps are equal on their 3-truncation, then they are equal.

We now investigate the non-degenerate n -simplices in \mathbb{C} for $n \leq 4$. It is convenient to note that all simplices above dimension 2 are uniquely determined by their faces. As such, every n -simplex a can be identified with the $(n + 1)$ -tuple $(d_0(a), d_2(a), \dots, d_n(a))$ for $n \geq 2$.

- There is a single 0-simplex, call it \star .
- There is a single non-degenerate 1-simplex c whose two faces are necessarily \star .
- There are two non-degenerate 2-simplices:

$$\begin{aligned} t &= (c, c, c) \\ i &= (s_0(\star), c, s_0(\star)) \end{aligned}$$

- There are four non-degenerate 3-simplices:

$$\begin{aligned} a &= (t, t, t, t) \\ \ell &= (i, s_1(c), t, s_1(c)) \\ r &= (s_0(c), t, s_0(c), i) \\ k &= (i, s_1(c), s_0(c), i) \end{aligned}$$

- There are nine non-degenerate 4-simplices:

$$\begin{array}{ll}
 A1 = (a, a, a, a, a) & A6 = (s_0(i), r, k, \ell, s_2(i)) \\
 A2 = (r, s_1(t), a, s_1(t), \ell) & A7 = (k, r, s_0 s_1(c), \ell, k) \\
 A3 = (r, r, s_2(t), a, s_2(t)) & A8 = (\ell, s_1(t), s_0(t), \ell, k) \\
 A4 = (s_0(t), a, s_0(t), \ell, \ell) & A9 = (k, r, s_2(t), s_1(t), r) \\
 A5 = (s_1(i), s_2(i), k, s_0(i), s_1(i)) &
 \end{array}$$

The image of F on the 4-truncation of \mathbb{C} consists of the following.

- A single object $F(c) = A$.
- Two 1-cells

$$A \otimes A \xrightarrow{F(t)} A \quad \text{and} \quad I \otimes I \xrightarrow{F(i)} A .$$

- Four 2-cells

$$\begin{array}{ccc}
 (A \otimes A) \otimes A & \xrightarrow{\alpha} & A \otimes (A \otimes A) \\
 F(t) \otimes A \downarrow & \xRightarrow{F(a)} & \downarrow A \otimes F(t) \\
 A \otimes A & \xrightarrow{F(t)} & A \xleftarrow{F(t)} A \otimes A
 \end{array}$$

$$\begin{array}{ccc}
 (A \otimes I) \otimes I & \xrightarrow{\alpha} & A \otimes (I \otimes I) \\
 r \otimes 1 \downarrow & \xRightarrow{F(r)} & \downarrow A \otimes F(i) \\
 A \otimes I & \xrightarrow{r} & A \xleftarrow{F(t)} A \otimes A
 \end{array}$$

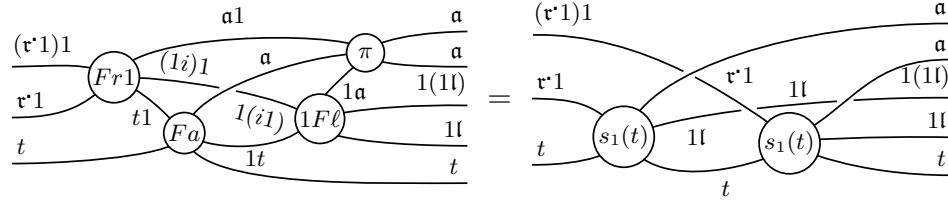
$$\begin{array}{ccc}
 (I \otimes I) \otimes A & \xrightarrow{\alpha} & I \otimes (I \otimes A) \\
 F(i) \otimes 1 \downarrow & \xRightarrow{F(\ell)} & \downarrow 1 \otimes \iota \\
 A \otimes A & \xrightarrow{F(t)} & A \xleftarrow{\iota} I \otimes A
 \end{array}$$

$$\begin{array}{ccc}
 (I \otimes I) \otimes I & \xrightarrow{\alpha} & I \otimes (I \otimes I) \\
 F(i) \otimes 1 \downarrow & \xRightarrow{F(k)} & \downarrow 1 \otimes F(i) \\
 A \otimes I & \xrightarrow{r} & A \xleftarrow{\iota} I \otimes A
 \end{array}$$

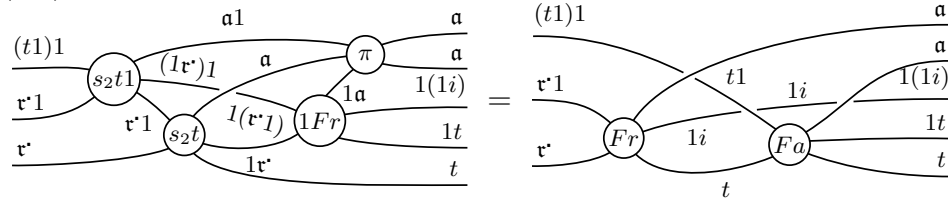
- And nine equalities:

(5.1)

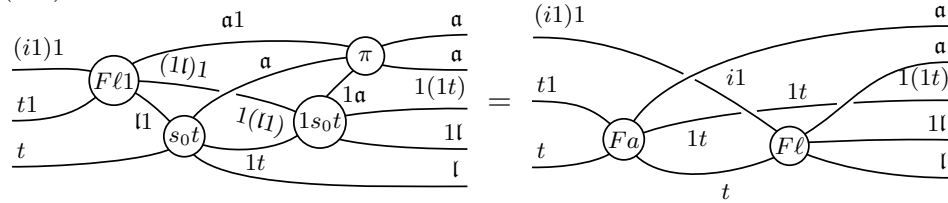
(5.2)



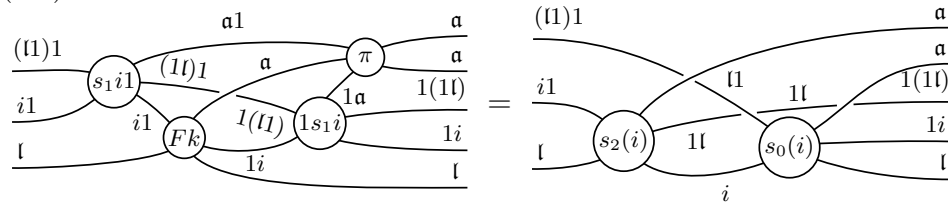
(5.3)



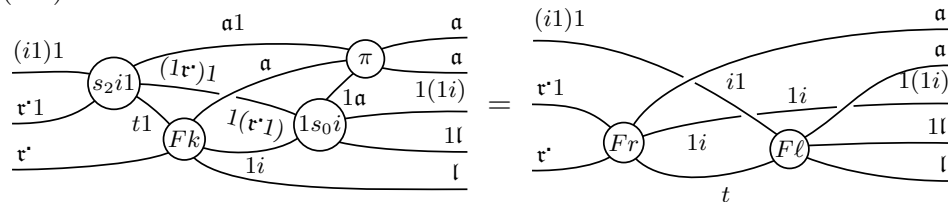
(5.4)



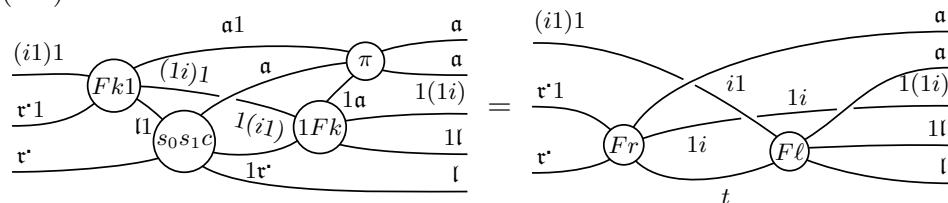
(5.5)



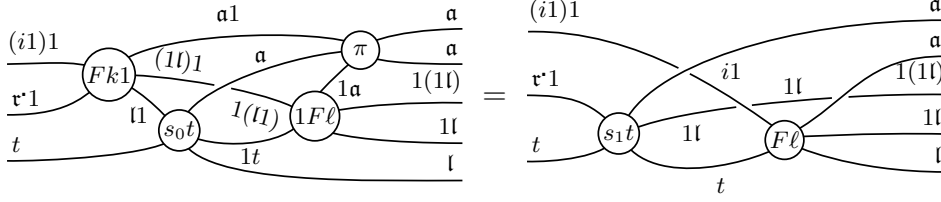
(5.6)



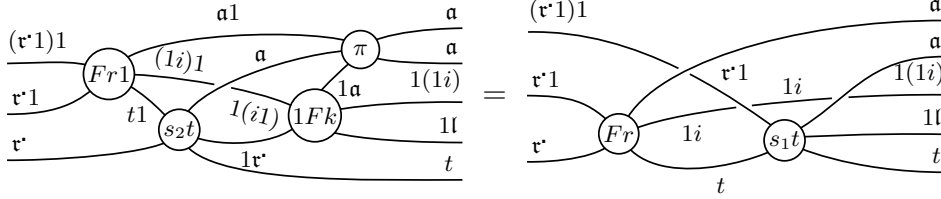
(5.7)



(5.8)



(5.9)



The similarity with skew monoidales in \mathcal{B} is immediately clear. There is however one problem: the unit map for a skew monoidale is of the form $I \rightarrow A$ but $F(i)$ is a map $I \otimes I \rightarrow F(c)$. Similarly, the left and right unit constraints for a skew monoidale have different domains and codomains than $F(r)$ and $F(\ell)$. This disparity means that the structure given by F is not strictly that of a skew monoidale. However, the difference amounts to the fact that $I \otimes I$ does not equal I .

We address this problem by considering the monoidal bicategory \mathcal{B}' with the same tensor as \mathcal{B} but whose unit object is $I \otimes I$. The unit maps for this monoidal structure are

$$(I \otimes I) \otimes A \xrightarrow{l_{I \otimes A}} I \otimes A \xrightarrow{l_A} A \quad \text{and} \quad A \otimes (I \otimes I) \xrightarrow{A \otimes r_I} A \otimes I \xrightarrow{r_A} A .$$

The invertible modifications π, μ, σ and τ are also altered accordingly. In this case the identity functor on \mathcal{B} becomes a strong monoidal functor $\mathcal{B} \rightarrow \mathcal{B}'$. Having modified the unit object in \mathcal{B} we can construct a bijection between simplicial maps $F: \mathbb{C} \rightarrow N\mathcal{B}$ and skew monoidales in \mathcal{B}' .

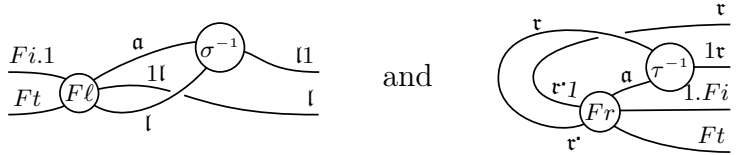
Remark 5.1. In the fundamental example $\mathcal{B} = \text{Cat}$, there is a further bijection between skew monoidales in Cat' and skew monoidales in Cat ; that is, skew-monoidal categories.

The first thing to notice is that the equality in (5.5) together with the monoidal bicategory axioms force $F(k)$ to be equal to the 2-cell

$$(5.10) \quad \begin{array}{c} \begin{array}{c} \text{r'1} \\ \mu^{-1} \\ \theta^{-1} \\ \tau(\theta) \end{array} \\ \begin{array}{c} \text{Fi.1} \\ \text{r'} \end{array} \end{array} \quad \begin{array}{c} \text{a} \\ \text{1} \\ \text{1.Fi} \\ \text{l} \end{array}$$

and thus completely specified by the coherence data of \mathcal{B} .

We now compare the data comprising the image of F with the data for a skew monoidale in \mathcal{B}' . At dimensions 0 and 1 these data are exactly equal: a single object $F(c) = A$ together with two 1-cells $F(t): A \otimes A \rightarrow A$ and $F(i): I \otimes I \rightarrow A$. At dimension two, the 2-cell $F(a)$ has the same form as the associativity constraint α for a skew monoidale; whilst, as observed above, $F(k)$ is necessarily of the form (5.10). On the other hand, the data $F(\ell)$ and $F(r)$ give rise to left and right unit constraints λ and ρ for a skew monoidale in \mathcal{B}' upon forming the composites



The assignments $F(\ell) \mapsto \lambda$ and $F(r) \mapsto \rho$ are in fact bijective, the former since it is given by composing with an invertible 2-cell, and the latter since it is given by composition with an invertible 2-cell followed by transposition under adjunction. Thus the two-dimensional data of F and of a skew monoidale in \mathcal{B}' are in bijective correspondence.

Finally, we find after some calculation that, with respect to the α , λ and ρ defined above, equations (5.1), (5.2), (5.3), (5.4), (5.6) and (5.7) express precisely the five axioms for a skew monoidale in \mathcal{B}' ; equation (5.5) specifies $F(k)$ and nothing more; whilst equations (5.8) and (5.9) are both equalities which follow using only the axioms for a monoidal bicategory. We have thus shown:

Theorem 5.2. *There is a bijection between simplicial maps $\mathbb{C} \rightarrow N\mathcal{B}$ and skew monoidales in \mathcal{B}' .*

Remark 5.3. If we consider a bicategory \mathcal{K} (not necessarily monoidal), and let $N\mathcal{K}$ be its nerve, then simplicial maps $\mathbb{C} \rightarrow N\mathcal{K}$ are monads in \mathcal{K} .

Remark 5.4. The assignation $\mathcal{B} \mapsto N(\mathcal{B})$ sending a monoidal bicategory to its nerve can be extended to a functor $N: \text{MonBicat}_s \rightarrow \text{SSet}$, where MonBicat_s is the category of monoidal bicategories and morphisms which strictly preserve all the structure. When seen in this way, the nerve functor has a left adjoint Φ ; it follows that $\Phi(\mathbb{C})$ is the free monoidal bicategory containing a skew monoidale.

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