

Nonlinear Fokker-Planck equation: stability, distance and the corresponding extremal problem in the spatially inhomogeneous case

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Abstract

We start with a global Maxwellian M_k , which is a stationary solution, with the constant total density ($\rho(t) \equiv \tilde{\rho}$), of the Fokker-Planck equation. The notion of distance between the function M_k and an arbitrary solution f (with the same total density $\tilde{\rho}$ at the fixed moment t) of the Fokker-Planck equation is introduced. In this way, we essentially generalize the important Kullback-Leibler distance, which was studied before. Using this generalization, we show local stability of the global Maxwellians in the spatially inhomogeneous case. We compare also the energy and entropy in the classical and quantum cases.

1 Introduction

We consider the Fokker-Planck equation

$$\frac{\partial f}{\partial t} = \Delta_v f - v \cdot \nabla_x f + \operatorname{div}_v(vf(1 + kf)), \quad (1.1)$$

where $t \in \mathbb{R}$ stands for time, $x = (x_1, x_2, \dots, x_n) \in \Omega$ stands for space coordinates, $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ is velocity and \mathbb{R} denotes the real axis. This non-linear Fokker-Planck equation serves as a kinetic model for bosons ($k > 0$) and fermions ($k < 0$).

The important notion of Kullback-Leibler distance [7] is essentially generalized in our paper and new conditional extremal problems, which appear in this way, are solved. The solutions $f(t, x, v)$ of the Fokker-Planck equation are studied in the bounded domain Ω of the x -space. Such an approach essentially changes the usual situation, that is, the total energy depends on t and the notion of distance (between a stationary solution and an arbitrary solution of the Fokker-Planck equation) includes the x -space. Thus, the notion of distance remains well-defined also in the spatially inhomogeneous case. Recall that the Kullback-Leibler distance, which has many applications (see, e.g., [5, 18, 21] and references therein), is defined only in the spatially homogeneous case.

In our previous paper [12] we studied a model case of the one dimensional x -space. Here the case $\dim \Omega \geq 1$ is dealt with. Furthermore, using our generalization of the Kullback-Leibler distance, we show local stability of global Maxwellians in the spatially inhomogeneous case.

The comparison of the energy and entropy in the classical and quantum cases is an important subject (see [1, 13–17, 22] and references therein). Here, we compare these energy and entropy for the situation described by the Fokker-Planck equation. It is especially interesting for the applications that the fermion and boson cases are essentially different.

Our definition of the quantum entropy S_k ($k \neq 0$) is slightly different from the previous definitions (see [3, 10]). We show that the natural requirement

$$S_{k \rightarrow S_c}, \quad k \rightarrow 0 \quad (S_c = S_0 \text{ is the classical entropy}) \quad (1.2)$$

is not fulfilled in the case of the old definition, however (1.2) holds for our modified definition (see Section 2). Some necessary definitions are given in Subsection 2.1. An important functional, which attains its maximum at the

function M_k is introduced there. The distance between solutions and the corresponding extremal problem are studied in Sections 2 and 4. Our results on the Fokker-Planck equation are mostly related to the corresponding results (from [16, 17]) on the Boltzmann equation but the theorems from Section 4 have no analogs in the Boltzmann case.

We use the standard notation $|v| = \sqrt{v_1^2 + \dots + v_n^2}$ and C_0^1 denotes the class of differentiable functions $f(x, v)$, which tend to zero sufficiently rapidly when v tends to infinity.

2 The extremal problem

2.1 Preliminaries

Here, we present some well-known notions and results connected with equation (1.1). It is required that the distribution function $f(t, x, v)$ satisfies the inequalities

$$f(t, x, v) \geq 0, \quad 1 + kf(t, x, v) \geq 0 \quad (k \in \mathbb{R}), \quad (2.1)$$

and we set $f \log f = 0$ for the case that $f = 0$ and $(1 + kf) \log(1 + kf) = 0$ for $1 + kf = 0$. Then, the mapping

$$\Phi(f) := f \log f \quad \text{for } k = 0, \quad (2.2)$$

$$\Phi(f) := f \log f - \frac{1}{k}(1 + kf) \log(1 + kf) + f \quad \text{for } k \neq 0, \quad (2.3)$$

is well-defined. Now, the entropy is given by the equality

$$S(t, f) = S(t) = - \int_{\Omega} \int_{\mathbb{R}^n} \Phi(f) dv dx. \quad (2.4)$$

The notions of density $\rho(t, x)$, total density $\rho(t)$, mean velocity $u(t, x)$, energy $E(t, x)$, and total energy $E(t)$ are introduced via formulas:

$$\rho(t, x) = \int_{\mathbb{R}^n} f(t, x, v) dv, \quad \rho(t) = \int_{\Omega} \rho(t, x) dx, \quad (2.5)$$

$$u(t, x) = (1/\rho(t, x)) \int_{\mathbb{R}^n} v f(t, x, v) dv, \quad (2.6)$$

$$E(t, x) = \int_{\mathbb{R}^n} \frac{|v|^2}{2} f(t, x, v) dv, \quad E(t) = \int_{\Omega} \int_{\mathbb{R}^n} \frac{|v|^2}{2} f(t, x, v) dv dx. \quad (2.7)$$

We assume that the domain Ω is bounded, and so its volume is also bounded:

$$\text{Vol}(\Omega) = V_\Omega < \infty. \quad (2.8)$$

2.2 The free energy functional and extremal problem

We introduce the "free energy" functional

$$F(f) = F(f(t)) = S(t) - E(t), \quad (2.9)$$

where $S(t)$ and $E(t)$ are defined by formulas (2.4) and (2.7), respectively. Next, we use the calculus of variations (see [6]) and find the function f_{max} which maximizes the functional (2.9), where the parameters t and $\rho(t) = \tilde{\rho} > 0$ are fixed. The corresponding Euler's equation takes the form

$$-\frac{|v|^2}{2} - \log f + \log(1 + kf) + \mu = 0. \quad (2.10)$$

From the last relation we obtain

$$f/(1 + kf) = C \exp \{-|v|^2/(2)\}, \quad C := e^{-\mu}. \quad (2.11)$$

Formula (2.11) implies that

$$f = M_k = \frac{C \exp \{-|v|^2/2\}}{1 - kC \exp \{-|v|^2/2\}}, \quad (2.12)$$

that is, f coincides with the global Maxwellian M_k .

In view of the requirement $\rho(t) = \tilde{\rho}$, the constant C in the equality (2.12) is derived from the relation

$$V_\Omega \int_{\mathbb{R}^n} \frac{C \exp \{-|v|^2/2\}}{1 - kC \exp \{-|v|^2/2\}} dv = \tilde{\rho}. \quad (2.13)$$

The function f given by (2.11) (or, equivalently, by (2.12)) is nonsingular and satisfies conditions (2.1) and $\rho(t) = \tilde{\rho} > 0$ if and only if

$$C > 0, \quad 1 - kC > 0. \quad (2.14)$$

In Subsection 2.3 we prove that there is a unique value C satisfying relations (2.13) and (2.14). Let us show that F indeed attains its maximum on the

global Maxwellian M_k corresponding to such C . According to (2.4), (2.7) and (2.9) the "free energy" F admits the representation

$$F = \int_{\Omega} \int_{\mathbb{R}^n} \Psi(f) dv dx, \quad \Psi := -\frac{|v|^2}{2} f - \Phi. \quad (2.15)$$

Taking into account (2.2), (2.3), (2.14) and (2.15), we have the inequality

$$\frac{\delta^2}{\delta f^2} \Psi = -\frac{1}{(1+kf)f} < 0, \quad (2.16)$$

and the next proposition follows.

Proposition 2.1 *Under condition (2.14), the functional F given by (2.9) attains its maximum on the function M_k of the form (2.12) (where C is defined in (2.13)), that is,*

$$G(f) = F(M_k) - F(f) > 0 \quad (f \neq M_k). \quad (2.17)$$

Remark 2.2 *In this subsection we introduced the important functional F , which attains its maximum on the global Maxwellians and the generalization G (see (2.17)) of the Kullback-Leibler distance. As opposed to the Kullback-Leibler distance, which is defined in the x -homogeneous case, the distance G is well-defined for the functions f , which depend on x .*

Later in this section we will consider the energy E , entropy S and free energy F of the global Maxwellians. The following sections are dedicated to the study of the general solutions of the Fokker-Planck equation.

2.3 Comparison of the classical and quantum characteristics

Let us calculate the integral on the left-hand side of (2.13). Using spherical coordinates, we have

$$C \int_{\mathbb{R}^n} \frac{\exp\{-|v|^2/2\}}{1 - kC \exp\{-|v|^2/2\}} dv = \omega_{n-1} C \int_0^\infty \frac{r^{n-1} \exp\{-r^2/2\}}{1 - kC \exp\{-r^2/2\}} dr, \quad (2.18)$$

where the surface area of the $(n-1)$ -sphere of radius 1 is

$$\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \quad (2.19)$$

and $\Gamma(z)$ is the gamma function. Euler's integral representation of the gamma function easily yields

$$\int_0^\infty e^{-ar^2} r^{n-1} dr = \frac{1}{2} a^{-n/2} \Gamma(n/2) \quad (a > 0). \quad (2.20)$$

Notation 2.3 According to (2.12) and (2.13), the value C corresponding to M_k depends on k . We denote this value by C_k .

Taking into account (2.13) and (2.18)–(2.20), we obtain

$$(2\pi)^{n/2} V_\Omega C_k L_{n/2}(kC_k) = \tilde{\rho}, \quad (2.21)$$

where

$$L_{n/2}(z) = \frac{2^{1-(n/2)}}{\Gamma(n/2)} \int_0^\infty \frac{e^{-r^2/2}}{1 - ze^{-r^2/2}} r^{n-1} dr = \sum_{m=1}^\infty \frac{z^{m-1}}{m^{n/2}}. \quad (2.22)$$

We note that the series representation $L_{n/2}(z) = \sum_{m=1}^\infty \frac{z^{m-1}}{m^{n/2}}$ does not hold for $|z| > 1$, and we use only the first equality in (2.22) for the case that $z < -1$. Using the first equality in (2.22), we derive the following statement.

Proposition 2.4 The function $L_{n/2}(z)$ increases strictly monotonically in the interval $-\infty \leq z < 1$ and

$$L_{n/2}(0) = 1; \quad (2.23)$$

$$L_{1/2}(1) = L_1(1) = \infty; \quad L_{n/2}(1) < \infty, \quad n > 2. \quad (2.24)$$

Proposition 2.4 implies the next two corollaries.

Corollary 2.5 If $k > 0$ (boson case) and either $n = 1$ or $n = 2$, then equation (2.21) has one and only one solution C_k such that $C_k > 0$ and $kC_k < 1$.

Corollary 2.6 If $k > 0$ (boson case), $n > 2$ and

$$(2\pi)^{n/2} V_\Omega L_{n/2}(1) > k\tilde{\rho}, \quad (2.25)$$

then equation (2.21) has one and only one solution C_k such that $C_k > 0$ and $kC_k < 1$.

Remark 2.7 *The function $L_{n/2}(z)$ belongs to the class of the L-functions [9] and is connected with the famous (see, e.g., [19]) Riemann zeta-function*

$$\zeta(z) = \sum_{p=1}^{\infty} \frac{1}{p^z}; \quad \Re z > 1 \quad (2.26)$$

by the relation

$$L_{n/2}(1) = \zeta(n/2). \quad (2.27)$$

Hence, we have the equalities

$$L_{3/2}(1) = \zeta(3/2) = 2.612, \quad L_2(1) = \zeta(2) = 1.645, \quad (2.28)$$

$$L_{5/2}(1) = \zeta(5/2) = 1.341, \quad L_3(1) = \zeta(3) = 1.202. \quad (2.29)$$

Let us study the fermion case $k < 0$. Taking into account Notation 2.3, we may later (differently from the constants C_k) consider C as a variable. In view of (2.22), the next proposition is valid.

Proposition 2.8 *Assume that $k < 0$. Then, $CL_{n/2}(kC)$ increases strictly monotonically with respect to C ($0 \leq C < \infty$), and $CL_{n/2}(kC) \rightarrow \infty$ for $C \rightarrow \infty$.*

Corollary 2.9 *If $k < 0$ (fermion case), then equation (2.21) has one and only one solution C_k such that $C_k > 0$.*

The second inequality in (2.14) holds in the fermion case (i.e., in the case $C = C_k$ and $k < 0$) automatically.

Finally, we consider in this section the energy of the global Maxwellian:

$$\begin{aligned} E_k = E(M_k) &= \frac{1}{2} \int_{\Omega} \int_{\mathbb{R}^n} |v|^2 \frac{C_k \exp\{-|v|^2/2\}}{1 - kC_k \exp\{-|v|^2/2\}} dv dx \\ &= \frac{1}{2} \omega_{n-1} V_{\Omega} C_k \int_0^{\infty} r^{n+1} \frac{\exp\{-r^2/2\}}{1 - kC_k \exp\{-r^2/2\}} dr. \end{aligned} \quad (2.30)$$

It is immediate from (2.21) that

$$\tilde{\rho}/L_{n/2}(kC_k) = (2\pi)^{n/2} V_{\Omega} C_k, \quad (2.31)$$

and so, using (2.19) and (2.22), we rewrite (2.30) in the form

$$E_k = \left(\frac{n\tilde{\rho}}{2} \right) \frac{L_{(n/2)+1}(kC_k)}{L_{n/2}(kC_k)}. \quad (2.32)$$

We note that the corresponding classical energy E_c (i.e., the energy for the case $k = 0$) is given by the formula

$$E_c = \frac{n\tilde{\rho}}{2}. \quad (2.33)$$

The points $kC = \pm 1$ are called the critical points in boson and fermion theories. (Recall that the series representation of $L_{n/2}(z)$ in (2.22) does not hold for $|z| > 1$.)

Proposition 2.10 *Let the condition*

$$-1 \leq kC_k < 1 \quad (2.34)$$

be fulfilled. Then, we have the inequalities

$$E_{q,B} < E_c < E_{q,F} \quad \text{for } n \geq 1, \quad (2.35)$$

where E_q denotes the energy in the quantum case (i.e., the case $k \neq 0$), $E_{q,B}$ stands for the energy in the boson case $k > 0$ and $E_{q,F}$ stands for the energy in the fermion case $k < 0$.

Proof. Taking into account the second equality in (2.22), we obtain:

$$\frac{L_{(n/2)+1}(kC)}{L_{n/2}(kC)} < 1 \quad \text{for } k > 0, \quad kC < 1. \quad (2.36)$$

Moreover, we will show that

$$\frac{L_{(n/2)+1}(kC)}{L_{n/2}(kC)} > 1 \quad \text{for } k < 0, \quad kC \geq -1. \quad (2.37)$$

For this purpose, we compare sums of two consequent terms (with numbers $2p$ and $2p + 1$) in the Taylor series representations (2.22) of $L_{(n/2)+1}(z)$ and $L_{n/2}(z)$, and derive that

$$\begin{aligned} & |z|^{2p-1} \left(\left(-\frac{1}{(2p)^{l+1}} + \frac{|z|}{(2p+1)^{l+1}} \right) - \left(-\frac{1}{(2p)^l} + \frac{|z|}{(2p+1)^l} \right) \right) \\ &= |z|^{2p-1} \left(\frac{2p-1}{(2p)^{l+1}} - \frac{2p|z|}{(2p+1)^{l+1}} \right). \end{aligned} \quad (2.38)$$

Furthermore, it is immediate that

$$|z|^{2p-1} \left(\frac{2p-1}{(2p)^{l+1}} - \frac{2p|z|}{(2p+1)^{l+1}} \right) = \frac{|z|^{2p-1}}{(2p)^l} \left(\frac{2p-1}{2p} - |z| \left(\frac{2p}{2p+1} \right)^{l+1} \right). \quad (2.39)$$

Finally, it is easy to see that, for $-1 \leq z < 0$, $l = n/2$, $n \geq 2$, we have

$$\frac{2p-1}{2p} - |z| \left(\frac{2p}{2p+1} \right)^{l+1} \geq \frac{2p-1}{2p} - \left(\frac{2p}{2p+1} \right)^2 > 0, \quad (2.40)$$

and relations (2.38)–(2.40) imply (2.37). It remains to prove (2.37) for the case $n = 1$. We easily calculate directly that

$$\frac{2p-1}{2p} - \frac{8}{10} \left(\frac{2p}{2p+1} \right)^{3/2} > 0, \quad (2.41)$$

which yields

$$\frac{2p-1}{2p} - |z| \left(\frac{2p}{2p+1} \right)^{3/2} > 0 \quad \text{for} \quad -0.8 \leq z < 0. \quad (2.42)$$

Formulas (2.38), (2.39) and (2.42) show that

$$L_{3/2}(kC) > L_{1/2}(kC) \quad \text{for} \quad -0.8 \leq kC < 0. \quad (2.43)$$

We use connections between Lerch zeta functions and Riemann zeta functions and take into account the estimate [11, sequence A078434] in order to calculate that up to the first two decimal places we have

$$L_{3/2}(-1) = 0.76. \quad (2.44)$$

Taking into account that (for $-1 \leq z < 0$) the series in (2.22) is an alternating series satisfying the Leibniz criterion, we obtain the inequalities

$$0.65 < L_{1/2}(-0.8) < 0.6589. \quad (2.45)$$

In view of Proposition 2.4, the functions $L_{1/2}$ and $L_{3/2}$ increase monotonically on the interval $[-1, -0.8]$. Hence, relations (2.44) and (2.45) imply that

$$L_{3/2}(kC) > L_{1/2}(kC) \quad \text{for} \quad -1 \leq kC \leq -0.8. \quad (2.46)$$

Inequalities (2.43) and (2.46) prove that (2.37) holds also for $n = 1$. Thus, it is proved that (2.37) is valid for all $n \geq 1$. Inequalities (2.35) follow directly from (2.32), (2.33), (2.36) and (2.37). \square

Remark 2.11 *The proof of formula (2.37), for the case that $n = 1$, shows that Conjecture 6.1 from [16] is valid.*

Formulas (2.21) and (2.23) yield

$$C_0 = (2\pi)^{-n/2} \tilde{\rho} / V_\Omega. \quad (2.47)$$

Lemma 2.12 *The following inequalities are valid:*

$$C_k > C_0 \quad \text{for } k < 0; \quad C_k < 2C_0 \quad \text{for } -1 < 2kC_0 < 0; \quad (2.48)$$

$$C_k < C_0 \quad \text{for } 0 < kC_0 < 1. \quad (2.49)$$

Proof. Let $k < 0$. Then, according to Proposition 2.4, we have

$$L_{n/2}(kC_0) < L_{n/2}(0) = 1. \quad (2.50)$$

Using relations (2.21), (2.47) and (2.50), we obtain

$$C_k L_{n/2}(kC_k) = C_0 > C_0 L_{n/2}(kC_0). \quad (2.51)$$

Hence, Proposition 2.8 implies that the first inequality in (2.48) holds.

Next, let $-1 < 2kC_0 < 0$. Again using Proposition 2.4, we see that

$$2L_{n/2}(2kC_0) > 2L_{n/2}(-1). \quad (2.52)$$

Taking into account (2.22) and (2.23), from (2.52) we derive

$$2L_{n/2}(2kC_0) > L_{n/2}(0) = 1. \quad (2.53)$$

Thus, we obtain

$$2C_0 L_{n/2}(2kC_0) > C_0 = (2\pi)^{-n/2} \tilde{\rho} / V_\Omega. \quad (2.54)$$

In view of (2.21) and (2.54), we have $2C_0 L_{n/2}(2kC_0) > C_k L_{n/2}(kC_k)$. Then, as in the proof of the first inequality in (2.48), we apply Proposition 2.8 and see that the second inequality in (2.48) holds.

Finally, let $0 < kC_0 < 1$. Since $L_{n/2}(z)$ is increasing (see Proposition 2.4), we have

$$L_{n/2}(kC_0) > L_{n/2}(0) = 1, \quad (2.55)$$

and, moreover, $CL_{n/2}(kC)$ also increases strictly monotonically. According to (2.21), (2.47) and (2.55), the relations

$$C_0L_{n/2}(kC_0) > C_0 = C_kL_{n/2}(kC_k) \quad (2.56)$$

hold. Therefore, since $CL_{n/2}(kC)$ is monotonic, we see that $C_k < C_0$. \square

It follows from Lemma 2.12 that C_k is bounded in the neighborhood of $k = 0$. The behavior of E_q , $F_q = F(M_k)$ and the entropy $S_q = S(M_k)$ in the punctured neighborhood of $k = 0$ is given in the next proposition.

Proposition 2.13 *For $k \rightarrow 0$, we have the following asymptotic relations:*

$$E_q - E_c = -\frac{n\tilde{\rho}kC_0}{4(2^{n/2})} + O(k^2) \quad (E_c = E(M_0)), \quad (2.57)$$

$$S_q - S_c = -\frac{(n-2)\tilde{\rho}kC_0}{4(2^{n/2})} + O(k^2) \quad (S_c = S(M_0)), \quad (2.58)$$

$$F_q - F_c = \frac{\tilde{\rho}kC_0}{2(2^{n/2})} + O(k^2) \quad (F_c = F(M_0)). \quad (2.59)$$

Proof. Step 1. In order to calculate the entropy $S(M_k)$ we recall definitions (2.4) and (2.12) of S and M_k , respectively, and use the equalities

$$M_k = g/(1 - kg), \quad 1 + kM_k = (1 - kg)^{-1}, \quad g := C_k e^{-|v|^2/2}, \quad (2.60)$$

which simplify the expressions $\Phi(M_k)$ for Φ given by (2.2) and (2.3):

$$\Phi(M_k) = M_k(1 + \log g) + (1/k) \log(1 - kg) \quad \text{for } k \neq 0, \quad (2.61)$$

$$\Phi(M_0) = g \log g \quad \text{for } M_0 = g. \quad (2.62)$$

Recall definitions (2.5) and (2.7) of ρ and E and recall that for $f = M_k$ we have $\rho(t) \equiv \text{const} = \tilde{\rho}$. Substituting $\log g = \log C_k - (1/2)|v|^2$ into (2.61) and then substituting (2.61) into (2.4), we obtain

$$S(M_k) = E_q - (1 + \log C_k)\tilde{\rho} - \frac{1}{k}V_\Omega \int_{\mathbb{R}^n} \log(1 - kg)dv \quad \text{for } k \neq 0. \quad (2.63)$$

Substituting $\log g = \log C_k - (1/2)|v|^2$ into (2.62) and then substituting (2.62) into (2.4), we obtain

$$S(M_0) = S_c = E_c - \tilde{\rho} \log C_0, \quad (2.64)$$

where $E_c = E(M_0)$ and C_0 is the value of C_k for the case that $k = 0$ (recall Notation 2.3). Taking into account the definition (2.7) of energy and using spherical coordinates and integration by parts, we rewrite (2.63):

$$S(M_k) = E_q - (1 + \log C_k) \tilde{\rho} + \frac{2}{n} E_q = \left(1 + \frac{2}{n}\right) E_q - (1 + \log C_k) \tilde{\rho} \quad \text{for } k \neq 0. \quad (2.65)$$

According to (2.33), we have $\tilde{\rho} = 2E_c/n$. Therefore, for $k \neq 0$ formulas (2.64) and (2.65) imply that

$$S(M_k) - S(M_0) = S_q - S_c = \frac{n+2}{n} (E_q - E_c) - \tilde{\rho} \log(C_k/C_0), \quad (2.66)$$

Hence, taking into account (2.9) and (2.66) we derive

$$F_q - F_c = \frac{2}{n} (E_q - E_c) - \tilde{\rho} \log(C_k/C_0). \quad (2.67)$$

Step 2. The equalities in (2.51) and (2.56) yield

$$\frac{kC_0}{L_{n/2}(kC_k)} = kC_k. \quad (2.68)$$

In view of formula (2.47) and Lemma 2.12, we see that the values $|kC_k|$ and $\sup_{|z| \leq \varepsilon} \left| \frac{d}{dz} \left(\frac{kC_0}{L_{n/2}(z)} \right) \right|$ are small for small values of $|k|$. Thus, we can apply the iteration method to the equation $z = \frac{kC_0}{L_{n/2}(z)}$ in order to derive

$$C_k = C_0 + O(k), \quad k \rightarrow 0. \quad (2.69)$$

Taking into account the series expansion in (2.22) and formulas (2.68) and (2.69), we obtain

$$C_k/C_0 = 1/L_{n/2}(kC_k) = 1 - kC_k/2^{n/2} + O(k^2). \quad (2.70)$$

Furthermore, from (2.69) and (2.70) we have

$$\log(C_k/C_0) = -kC_0/2^{n/2} + O(k^2). \quad (2.71)$$

Using formulas (2.32), (2.33), (2.69) and the series expansion in (2.22), we see that (2.57) holds. According to (2.57) and (2.71) we may rewrite (2.66) in the form (2.58). Finally, in view of (2.57) and (2.71), we rewrite (2.67) in the form (2.59). \square

Corollary 2.14 *The following inequalities hold for small values of k :*

$$E_{q,B} < E_c < E_{q,F}; \quad S_{q,B} < S_c < S_{q,F} \quad \text{for } n > 2; \quad F_{q,F} < F_c < F_{q,B}. \quad (2.72)$$

3 General-type solutions

3.1 Dissipative and conservative solutions

1. In this section we study general solutions $f(t, x, v)$ (satisfying (2.1)) of the Fokker-Planck equation (1.1). The total energy flux through the surface $\partial\Omega$ per unit time is given by the equalities

$$\begin{aligned} A(f, \Omega) &:= \int_{\Omega} \int_{\mathbb{R}^n} (|v|^2/2) v \cdot \nabla_x f(t, x, v) dv dx \\ &= \int_{\partial\Omega} \int_{\mathbb{R}^n} (|v|^2/2) (v \cdot n(y)) f(t, y, v) dv d\sigma, \end{aligned} \quad (3.1)$$

where $\partial\Omega$ is the boundary of the Ω , and the integral $\int_{\partial\Omega} g d\sigma$ is the surface integral with $n(y)$ being the outward unit normal to that surface, $y \in \partial\Omega$. The second equality in (3.1) is immediate from Gauss-Ostrogradsky divergence formula.

The total density flux through the surface $\partial\Omega$ per unit time has the form

$$B(f, \Omega) := \int_{\Omega} \int_{\mathbb{R}^n} v \cdot \nabla_x f(t, x, v) dv dx = \int_{\partial\Omega} \int_{\mathbb{R}^n} (v \cdot n(y)) f(t, y, v) dv d\sigma. \quad (3.2)$$

Definition 3.1 *By $D(\Omega)$, we denote the class of functions $f(t, x, v)$ satisfying the Fokker-Planck equation (1.1), inequalities (2.1) and the condition $A(f, \Omega) \geq 0$ for all t (i.e., the class of the dissipative solutions f).*

Definition 3.2 *By $C(\Omega)$, we denote the class of functions $f(t, x, v)$ satisfying the Fokker-Planck equation (1.1), inequalities (2.1) and the condition $A(f, \Omega) = 0$ for all t (i.e., the class of the conservative solutions f).*

It is obvious that $C(\Omega) \subset D(\Omega)$.

Proposition 3.3 *Let $f(t, x, v)$ satisfy (1.1) and (2.1), and assume that for all $y \in \partial\Omega$ the equality*

$$f(t, y, v) = f(t, y, -v) \quad (3.3)$$

holds. Then, $f(t, x, v) \in C(\Omega)$.

Proof. Taking into account (3.3), we derive

$$\int_{\mathbb{R}^n} (|v|^2/2)(v \cdot n(y))f(t, x, v)vdv = 0. \quad (3.4)$$

It is immediate from (3.1) and (3.4) that $A(f, \Omega)=0$. \square

Corollary 3.4 *Maxwellians M_k given by (2.12) are conservative, that is, $M_k \in C(\Omega)$.*

The **bounce-back condition** (3.3) means that particles arriving with a certain velocity to the boundary $\partial\Omega$ will bounce back with an opposite velocity (see [20, p.16]).

3.2 Boundedness

Introducing the function

$$s(r) = r \log r - \frac{1}{k}(1 + kr) \log(1 + kr), \quad (3.5)$$

we see that

$$s'(r) = \log(r/(1 + kr)), \quad s''(r) = (r(1 + kr))^{-1}, \quad (3.6)$$

and obtain the proposition below.

Proposition 3.5 *The function $s(r)$ is convex on the semi-axis $r \geq 0$ for the case that $k > 0$ and on the interval $0 \leq r < 1/|k|$ for the case that $k < 0$.*

We consider functions g such that

$$g \geq 0, \quad 1 + kg > 0, \quad \frac{1}{2} \int_{\mathbb{R}^n} |v|^2 g(v)dv < \infty. \quad (3.7)$$

For $k < 1$ and Z given by

$$Z = 1/(e^{|v|^2/2} - k), \quad (3.8)$$

the convexity of s implies that

$$s(g) - s(Z) \geq s'(Z)(g - Z). \quad (3.9)$$

Using (3.6), (3.8), (3.9) and the equality

$$Z/(1+kZ) = e^{-|v|^2/2}, \quad (3.10)$$

we easily derive

$$-s(g) \leq \frac{|v|^2}{2}g - s(Z) - \frac{|v|^2}{2}Z. \quad (3.11)$$

Taking into account the equality

$$1+kZ = 1/(1-ke^{-|v|^2/2}), \quad (3.12)$$

we rewrite (3.11) in the form

$$-s(g) \leq \frac{|v|^2}{2}g - \frac{1}{k} \log(1-ke^{-|v|^2/2}). \quad (3.13)$$

The proposition below follows from (3.13) and the inequality $\log(1+a) \leq a$ for $a \geq 0$.

Proposition 3.6 *Assume that $k < 0$ and g satisfies (3.7). Then we have*

$$-|v|^2g/2 - s(g) \leq e^{-|v|^2/2}. \quad (3.14)$$

Remark 3.7 *For the case $k = -1$, the inequality (3.14) was derived in [2].*

Recall definitions (2.4), (2.7) and (2.9) of S , E and F , respectively, and note that S is expressed via Φ . Compare the expression (2.3) for Φ with the expression for s . Thus, we see that formula (3.14) yields the following corollary.

Corollary 3.8 *Let the conditions of Proposition 3.6 be fulfilled. Then there exists a positive constant C such that*

$$F(g) \leq C. \quad (3.15)$$

In a similar way, from (3.13) and series representation of $\log(1+a)$ we obtain the next corollary.

Corollary 3.9 *Let $0 < k < 1$ and assume that g satisfies (3.7). Then there exists a positive constant C such that*

$$F(g) \leq C. \quad (3.16)$$

If either conditions of Corollary 3.8 or Corollary 3.9 hold, we put

$$\tilde{C} = \inf F(g). \quad (3.17)$$

Passing to the limit and using Corollaries 2.5, 2.6 and 2.9, we could show (under rather general conditions) the existence of the global Maxwellian $M_{\tilde{C}}$ such that

$$F(M_{\tilde{C}}) = \tilde{C}. \quad (3.18)$$

Later we assume that $M_{\tilde{C}}$ exists.

4 Lyapunov functional

The Lyapunov functional for equation (1.1) has the form

$$\tilde{G}(f) = \tilde{C} - F(f), \quad (4.1)$$

where $F(f)$ is defined by (2.9). Recall that, for the generalization $G(f)$ (given by (2.17)) of the Kullback-Leibler distance, we always assumed that $\rho(f)$ is fixed at the moment t (i.e., $\rho(f, t) \equiv \tilde{\rho}$). We do not assume this in the present section. In other words, we consider \tilde{G} on a wider set of solutions. Clearly, under conditions of Corollaries 3.8 or 3.9, the inequality

$$\tilde{G}(f) \geq 0 \quad (4.2)$$

holds. Thus, if a stationary solution $M_{\tilde{C}}$ of (1.1) satisfies the equality (3.18), the value $\tilde{G}(f) = F(M_{\tilde{C}}) - F(f)$ at t may be considered as a distance between $M_{\tilde{C}}$ and f at the time t .

In this section we substitute condition (2.1) by a stronger condition with the strict inequalities

$$f(t, x, v) > 0, \quad 1 + kf(t, x, v) > 0 \quad (k \in \mathbb{R}). \quad (4.3)$$

Hence, we can consider equation (1.1) in the form

$$\frac{\partial f}{\partial t} = -v \cdot \nabla_x f + \operatorname{div}_v \left(f(1 + kf) \nabla_v \left(\log \frac{f}{1 + kf} + 1 + |v|^2/2 \right) \right). \quad (4.4)$$

We multiply (4.4) by $\left(\log \frac{f}{1 + kf} + 1 + |v|^2/2 \right)$ and integrate over \mathbb{R}^n (applying also integration by parts with respect to the variables v_i) and Ω . Under

natural assumptions on the decay of f and $\partial f/\partial v_i$ at infinity (so that the values at $\pm\infty$ disappear in the formulas for the integration by parts), we obtain

$$\begin{aligned} & \int_{\Omega} \int_{\mathbb{R}^n} \left(\log \frac{f}{1+kf} + 1 + |v|^2/2 \right) \frac{\partial f}{\partial t} dv dx \\ &= - \int_{\Omega} \int_{\mathbb{R}^n} \left(\log \frac{f}{1+kf} + 1 + |v|^2/2 \right) v \cdot \nabla_x f dv dx \\ & \quad - \int_{\Omega} \int_{\mathbb{R}^n} f(1+kf) \left| \nabla_v \left(\log \frac{f}{1+kf} + 1 + |v|^2/2 \right) \right|^2 dv dx. \end{aligned} \quad (4.5)$$

Let us introduce the total flow of the entropy across the boundary Ω :

$$U(f, \Omega) = - \int_{\Omega} \int_{\mathbb{R}^n} \left(\log \frac{f}{1+kf} + 1 \right) v \cdot \nabla_x f dv dx = - \int_{\Omega} \int_{\mathbb{R}^n} v \cdot \nabla_x \Phi(f) dv dx, \quad (4.6)$$

where Φ is defined by (2.2) and (2.3). Using Gauss-Ostrogradsky formula, we also have

$$U(f, \Omega) = - \int_{\partial\Omega} \int_{\mathbb{R}^n} (v \cdot n(y)) \Phi(f) dv d\sigma. \quad (4.7)$$

It follows from (4.5) and (4.6) that

$$\frac{d\tilde{G}}{dt} \leq U(f, \Omega) - A(f, \Omega), \quad (4.8)$$

where the function $A(f, \Omega)$ is defined by relation (3.1). Indeed, relations (3.1) and (4.6) yield the equality

$$U(f, \Omega) - A(f, \Omega) = - \int_{\Omega} \int_{\mathbb{R}^n} \left(\log \frac{f}{1+kf} + 1 + |v|^2/2 \right) v \cdot \nabla_x f dv dx, \quad (4.9)$$

whereas formulas (2.1), (4.5) and (4.9) imply that

$$U(f, \Omega) - A(f, \Omega) \geq \int_{\Omega} \int_{\mathbb{R}^n} \left(\log \frac{f}{1+kf} + 1 + |v|^2/2 \right) \frac{\partial f}{\partial t} dv dx. \quad (4.10)$$

Furthermore, according to (2.4), (2.7), (2.9) and (4.1), we have

$$\frac{\partial \tilde{G}}{\partial t} = \frac{\partial E}{\partial t} - \frac{\partial S}{\partial t} = \int_{\Omega} \int_{\mathbb{R}^n} \left(\log \frac{f}{1+kf} + 1 + |v|^2/2 \right) \frac{\partial f}{\partial t} dv dx,$$

and so (4.8) is immediate from (4.10).

Using inequality (4.8), we derive the following assertion.

Theorem 4.1 *Assume that $f \in C_0^1$ is a dissipative solution of (1.1), which satisfies (2.1), and that the inequality*

$$U(f, \Omega) \leq 0 \tag{4.11}$$

holds. Then the inequality $(\partial \tilde{G} / \partial t) \leq 0$ is valid.

Corollary 4.2 *Assume that the conditions of Theorem 4.1 are fulfilled and $\tilde{G}(f, t_0) < \delta$. Then, the inequality $\tilde{G}(f, t) < \delta$ holds for all $t > t_0$.*

Thus, the distance \tilde{G} between a Maxwellian $M_{\tilde{c}}$ satisfying (3.18) and f satisfying (4.11) decreases. Taking into account the definition of the Lyapunov stability, we proved the following important result.

Theorem 4.3 *The stationary solution $M_{\tilde{c}}$ is locally stable (i.e., Lyapunov stable) in the class of the dissipative solutions f satisfying inequalities (2.1) and (4.11).*

We note that $M_{\tilde{c}}$ does not depend on x , and so definitions (3.1) and (4.6) imply that $A(M_{\tilde{c}}, \Omega) = U(M_{\tilde{c}}, \Omega) = 0$.

Remark 4.4 *The earlier results on the local stability for the Fokker-Planck equation (see [2]) were obtained only for the spatially homogeneous case. Theorems 4.1 and 4.3 have no analogs also in the case of the Boltzmann equation.*

Concluding remark. Following [13–17], we compare classical and quantum results, that is, determinate and probabilistic cases. As in the case of the Boltzmann equation [16, 17], we use a special extremal principle, which is based on the ideas of game theory. (We note that extremal principles remain central in modern physics.) The "players" in the game described by the Fokker-Planck equation are the total energy E and entropy S , the "gain" in the game is the functional F and the strategy in the game is determinate in the classical case and probabilistic in the quantum case. It is of interest that the inequalities $E_{q,B} < E_c < E_{q,F}$, $S_{q,B} < S_c < S_{q,F}$ (for $n > 2$), and $F_{q,F} < F_c < F_{q,B}$ (see (2.72)) hold for small k in our game.

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