# On the average number of subgroups of the group $\mathbb{Z}_m \times \mathbb{Z}_n$

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**Abstract**

Let $\mathbb{Z}_m$ be the group of residue classes modulo $m$. Let $s(m,n)$ and $c(m,n)$ denote the total number of subgroups of the group $\mathbb{Z}_m \times \mathbb{Z}_n$ and the number of its cyclic subgroups, respectively, where $m$ and $n$ are arbitrary positive integers. We derive asymptotic formulas for the sums $\sum_{m,n\le x} s(m,n)$, $\sum_{m,n\le x} c(m,n)$ and for the corresponding sums restricted to $\gcd(m,n)>1$, i.e., concerning the groups $\mathbb{Z}_m \times \mathbb{Z}_n$ having rank two.

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## 1 Introduction

Throughout the paper we use the notations: $\mathbb{N} := \{1,2,\ldots\}$, $\mathbb{N}_0 := \{0,1,2,\ldots\}$, $\mathbb{Z}_m$ is the additive group of residue classes modulo $m$, $\phi$ is Euler's totient function, $\tau(n)$ is the number of divisors of $n$, $\mu$ denotes the Möbius function, $\psi$ is the Dedekind function given by $\psi(n) = n\prod_{p\,|\,n}(1+1/p)$, $*$ stands for the Dirichlet convolution of arithmetic functions, $\zeta$ is the Riemann zeta-function. Let $n=\prod_p p^{\nu_p(n)}$ be the prime power factorization of $n\in\mathbb{N}$, where the product is over the primes $p$ and all but a finite number of the exponents $\nu_p(n)$ are zero. Furthermore, let $\gamma_k$ ($k\in\mathbb{N}_0$) denote the Stieltjes constants defined by

$$\gamma_k := \lim_{x\to\infty}\left(\sum_{n\le x}\frac{(\log n)^k}{n} - \frac{(\log x)^{k+1}}{k+1}\right),$$

where $\gamma_0=\gamma$ is the Euler-Mascheroni constant. We note that the constants $\gamma_k$ are connected to the coefficients of the Laurent series expansion of the function $\zeta(s)$ about its pole $s=1$, namely,

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty}\frac{(-1)^k\gamma_k}{k!}(s-1)^k,$$

see, e.g., A. Ivić [7, Th. 1.3].

Consider the group $G := \mathbb{Z}_m \times \mathbb{Z}_n$, where $m,n\in\mathbb{N}$ are arbitrary. Note that $G$ is isomorphic to $\mathbb{Z}_{\gcd(m,n)} \times \mathbb{Z}_{\mathrm{lcm}(m,n)}$. If $\gcd(m,n)=1$, then $G$ is cyclic, isomorphic to $\mathbb{Z}_{mn}$. If $\gcd(m,n)>1$, then $G$ has rank two. We recall that a finite Abelian group has rank $r$ if it is isomorphic to $\mathbb{Z}_{n_1}\times\cdots\times\mathbb{Z}_{n_r}$, where $n_1,\ldots,n_r\in\mathbb{N}\setminus\{1\}$ and $n_j\,|\,n_{j+1}$ ($1\le j\le r-1$). Let $s(m,n)$ and $c(m,n)$ denote the total number of subgroups of the group $G$ and the number of its cyclic subgroups, respectively.

Concerning general properties of the subgroup lattice of finite Abelian groups, see R. Schmidt [11], M. Suzuki [13]. For every $m, n \in \mathbb{N}$ one has

$$s(m,n) = \prod_p s(p^{\nu_p(m)}, p^{\nu_p(n)}), \tag{1}$$

$$c(m,n) = \prod_p c(p^{\nu_p(m)}, p^{\nu_p(n)}). \tag{2}$$

For the $p$-group $\mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}$ of rank two, with $1 \le a \le b$, the following explicit formulas hold:

$$s(p^a, p^b) = \frac{(b-a+1)p^{a+2} - (b-a-1)p^{a+1} - (a+b+3)p + (a+b+1)}{(p-1)^2}, \tag{3}$$

$$c(p^a, p^b) = 2(1 + p + p^2 + \ldots + p^{a-1}) + (b-a+1)p^a. \tag{4}$$

The formula (3) was deduced, applying Goursat's lemma for groups, by G. Călugăreanu [3, Sect. 4] and J. Petrillo [10, Prop. 2], and using the concept of the fundamental group lattice by M. Tărnăuceanu [14, Prop. 2.9], [15, Th. 3.3]. Formula (4) was given in [15, Th. 4.2]. Therefore, $s(m,n)$ and $c(m,n)$ can be computed using (1), (3) and (2), (4), respectively. The following more compact formulas were derived in [4] by a simple elementary method: For every $m, n \in \mathbb{N}$,

$$\begin{aligned} s(m,n) &= \sum_{d \mid m, e \mid n} \gcd(d,e) \\ &= \sum_{d \mid \gcd(m,n)} \phi(d)\tau(m/d)\tau(n/d) \end{aligned} \tag{5}$$

and

$$\begin{aligned} c(m,n) &= \sum_{d \mid m, e \mid n} \phi(\gcd(d,e)) \\ &= \sum_{d \mid \gcd(m,n)} (\mu * \phi)(d)\tau(m/d)\tau(n/d). \end{aligned} \tag{6}$$

See also [17, 18] for a general identity concerning the number of cyclic subgroups of an arbitrary finite Abelian group.

The identities (1) and (2) tell us that the functions $(m,n) \mapsto s(m,n)$ and $(m,n) \mapsto c(m,n)$ are multiplicative, viewed as arithmetic functions of two variables. This property follows also from the first formulas of (5) and (6), respectively. See [4, Sect. 2]. Therefore, $n \mapsto s(n) := s(n,n)$ (sequence [12, A060724]) and $n \mapsto c(n) := c(n,n)$ (sequence [12, A060648]) are multiplicative functions of a single variable and for every $n \in \mathbb{N}$ one has

$$s(n) = \sum_{d \mid n} \tau(d)\psi(n/d), \tag{7}$$

$$c(n) = \sum_{d \mid n} \psi(d), \tag{8}$$

see [4, Sect. 3]. Note that $s(n) = \sum_{d \mid n} c(d)$ $(n \in \mathbb{N})$.

In this paper we are concerned with the asymptotic properties of the Dirichlet summatory functions of $s(m,n)$, $c(m,n)$, $s(n)$ and $c(n)$. As far as we know, no such results are given in the literature. The only existing asymptotic results for the number of subgroups of finite Abelian groups having rank two concern another function, namely $t_2(n)$, see Section 4. We establish asymptotic formulas for the sums $\sum_{m,n\le x} s(m,n)$, $\sum_{m,n\le x} c(m,n)$, $S^{(2)}(x) := \sum'_{m,n\le x} s(m,n)$, $C^{(2)}(x) := \sum'_{m,n\le x} c(m,n)$, $\sum_{n\le x} s(n)$ and $\sum_{n\le x} c(n)$, where $\sum'$ means that summation is restricted to $\gcd(m,n) > 1$. Here $S^{(2)}(x)$ and $C^{(2)}(x)$ represent the number of subgroups, respectively cyclic subgroups of the groups $\mathbb{Z}_m \times \mathbb{Z}_n$ having rank two, with $m, n \le x$. Our main results are given in Section 2, while their proofs are contained in Section 3.

We remark that a compact formula for the number $s_3(n)$ of subgroups of the group $(\mathbb{Z}_n)^3$ and an asymptotic formula for the sum $\sum_{n\le x} s_3(n)$ were given in [5].

## 2 Results

The double Dirichlet series of the functions $s(m,n)$ and $c(m,n)$ can be represented by the Riemann zeta function, as shown in the next result.

**Theorem 1.** *For every $z, w \in \mathbb{C}$ with $\Re z > 1, \Re w > 1$,*

$$\sum_{m,n=1}^{\infty} \frac{s(m,n)}{m^z n^w} = \frac{\zeta^2(z)\zeta^2(w)\zeta(z+w-1)}{\zeta(z+w)}\,, \tag{9}$$

$$\sum_{m,n=1}^{\infty} \frac{c(m,n)}{m^z n^w} = \frac{\zeta^2(z)\zeta^2(w)\zeta(z+w-1)}{\zeta^2(z+w)}\,, \tag{10}$$

**Remark 1.** According to (9) and (10),

$$\sum_{m,n=1}^{\infty} \frac{s(m,n)}{m^z n^w} = \sum_{m,n=1}^{\infty} \frac{c(m,n)}{m^z n^w} \sum_{m,n=1}^{\infty} \frac{F(m,n)}{m^z n^w}\,,$$

where the function $F$ is defined by $F(m,n) = 1$ for $m = n$ and $F(m,n) = 0$ for $m \ne n$ ($m, n \in \mathbb{N}$). Therefore (see, e.g., [17] and [19] for related properties of the Dirichlet convolution of arithmetic functions of several variables and of multiple Dirichlet series),

$$s(m,n) = \sum_{d \mid \gcd(m,n)} c(m/d, n/d) \quad (m, n \in \mathbb{N})\,. \tag{11}$$

**Theorem 2.** *For large real $x$ and every fixed $\varepsilon > 0$,*

$$\sum_{m,n\le x} s(m,n) = x^2 \left( \sum_{r=0}^{3} A_r (\log x)^r \right) + O\left( x^{\frac{1117}{701}+\varepsilon} \right)\,, \tag{12}$$

$$\sum_{m,n\le x} c(m,n) = x^2 \left( \sum_{r=0}^{3} B_r (\log x)^r \right) + O\left( x^{\frac{1117}{701}+\varepsilon} \right)\,, \tag{13}$$

*where $1117/701 \approx 1.5934$, $A_r, B_r$ $(0 \le r \le 3)$ are constants,*

$$A_3 = \frac{1}{3\zeta(2)} = \frac{2}{\pi^2}, \quad A_2 = \frac{1}{\zeta(2)} \left( 3\gamma - 1 - \frac{\zeta'(2)}{\zeta(2)} \right)\,,$$

$$A_1 = \frac{1}{\zeta(2)} \left( 8\gamma^2 - 6\gamma - 2\gamma_1 + 1 - 2(3\gamma - 1)\frac{\zeta'(2)}{\zeta(2)} + 2\left( \frac{\zeta'(2)}{\zeta(2)} \right)^2 - \frac{\zeta''(2)}{\zeta(2)} \right)\,,$$

$$B_3 = \frac{1}{3\zeta^2(2)} = \frac{12}{\pi^4}, \quad B_2 = \frac{1}{\zeta^2(2)}\left(3\gamma - 1 - 2\frac{\zeta'(2)}{\zeta(2)}\right),$$

$$B_1 = \frac{1}{\zeta^2(2)}\left(8\gamma^2 - 6\gamma - 2\gamma_1 + 1 - 4(3\gamma-1)\frac{\zeta'(2)}{\zeta(2)} + 6\left(\frac{\zeta'(2)}{\zeta(2)}\right)^2 - 2\frac{\zeta''(2)}{\zeta(2)}\right).$$

**Remark 2.** Let $\Delta(x)$ denote the error term in the Dirichlet divisor problem, i.e.,

$$\Delta(x) := \sum_{n\le x} \tau(n) - x\log x - (2\gamma-1)x\,,$$

and

$$\theta_0 := \inf\{\theta :\ \Delta(x) = O(x^\theta)\}\,,$$

for $x$ large. Then $O\left(x^{\frac{1117}{701}+\epsilon}\right)$ can be readily replaced by $O\left(x^{\frac{3-\theta_0}{2-\theta_0}+\varepsilon}\right)$. Using the classic bound $\theta_0 \ge \frac{1}{3}$, one obtains $O\left(x^{8/5+\varepsilon}\right)$. The hitherto sharpest result $\theta_0 \ge \frac{131}{416}$, which is due to M. Huxley [6], gives the $O$-term stated in Theorem 2.

**Remark 3.** The constants $A_0$ and $B_0$ can be constructed from the proof below. They are quite complicated and hardly accessible to numerical evaluation, since they involve *inter alia* the infinite series $\sum_{k=1}^\infty \tau(k)\Delta(k)k^{-2}$.

In order to formulate our result concerning the sums $S^{(2)}(x)$ and $C^{(2)}(x)$ some further notations are needed. For $K \in \mathbb{N}$ and $s \in \mathbb{C}$ let

$$F_K(s) := \prod_{p^{\nu_p(K)}\,||\,K} \left(1 - \eta_p(K)\,p^{-s}\right), \quad \text{where} \quad \eta_p(K) := \frac{\nu_p(K)}{\nu_p(K)+1}\,, \tag{14}$$

$$\begin{aligned}\alpha_0(K) &:= F_K(1) = \prod_{p^{\nu_p(K)}\,||\,K} \left(1 - \eta_p(K)\,p^{-1}\right),\\ \alpha_1(K) &:= F_K'(1) = \sum_{p^*\,|\,K} \frac{\eta_{p^*}(K)}{p^*}\log p^* \prod_{p^{\nu_p(K)}\,||\,K,\ p\ne p^*} \left(1 - \frac{\eta_p(K)}{p}\right),\end{aligned} \tag{15}$$

and let

$$\beta_0(K) := \tau(K)\alpha_0(K)\,, \quad \beta_1(K) := \tau(K)\left(\alpha_0(K)(2\gamma-1) + \alpha_1(K)\right). \tag{16}$$

**Theorem 3.** *For large real $x$ and every fixed $\varepsilon > 0$,*

$$S^{(2)}(x) := \sum_{\substack{m,n\le x\\ \gcd(m,n)>1}} s(m,n) = x^2\left(\sum_{r=0}^{3} C_r(\log x)^r\right) + O\left(x^{\frac{1117}{701}+\varepsilon}\right), \tag{17}$$

$$C^{(2)}(x) := \sum_{\substack{m,n\le x\\ \gcd(m,n)>1}} c(m,n) = x^2\left(\sum_{r=0}^{3} D_r(\log x)^r\right) + O\left(x^{\frac{1117}{701}+\varepsilon}\right), \tag{18}$$

*where* $C_3 = A_3$*,* $D_3 = B_3$*,* $C_r = A_r - b_r$*,* $D_r = B_r - b_r$ $(0 \le r \le 2)$ *with* $A_r$ *and* $B_r$ $(0 \le r \le 3)$ *defined in Theorem 2 and* $b_r$ $(0 \le r \le 2)$ *given by*

$$\begin{aligned} b_2 &= \sum_{K=1}^{\infty} \mu(K) \left( \frac{\beta_0(K)}{K} \right)^2 = \prod_p \left( 1 - \frac{4}{p^2} + \frac{4}{p^3} - \frac{1}{p^4} \right) , \\ b_1 &= \sum_{K=1}^{\infty} \frac{2\mu(K)}{K^2} \beta_0(K) \left( \beta_1(K) - \beta_0(K) \log K \right) , \\ b_0 &= \sum_{K=1}^{\infty} \frac{\mu(K)}{K^2} \left( \beta_1(K) - \beta_0(K) \log K \right)^2 , \end{aligned} \tag{19}$$

*using the notation of* (15) *and* (16)*.*

**Theorem 4.** *We have*

$$\sum_{n \le x} s(n) = \frac{5\pi^2}{24} x^2 + O\left( x \log^{8/3} x \right) . \tag{20}$$

$$\sum_{n \le x} c(n) = \frac{5}{4} x^2 + O\left( x \log^{5/3} x \right) . \tag{21}$$

## 3 Proofs

**Proof of Theorem 1.** Applying the second formula of (5) we deduce for $\Re z, \Re w > 1$,

$$\sum_{m,n=1}^{\infty} \frac{s(m,n)}{m^z n^w} = \sum_{d,a,b=1}^{\infty} \frac{\phi(d)\tau(a)\tau(b)}{(da)^z (db)^w} = \sum_{d=1}^{\infty} \frac{\phi(d)}{d^{z+w}} \sum_{a=1}^{\infty} \frac{\tau(a)}{a^z} \sum_{b=1}^{\infty} \frac{\tau(b)}{b^w} ,$$

and using the familiar formulas for the latter Dirichlet series we obtain (9). The proof of (10), based on the second formula of (6) is similar. □

**Proof of Theorem 2.** We need the following result.

**Lemma 1.** *For* $m, n \in \mathbb{N}$ *let*

$$T(m,n) := \sum_{\ell \mid \gcd(m,n)} \ell \, \tau\left( \frac{m}{\ell} \right) \tau\left( \frac{n}{\ell} \right) ,$$

*i.e.,*

$$\sum_{m,n=1}^{\infty} \frac{T(m,n)}{m^z n^w} = \zeta^2(z)\zeta^2(w)\zeta(z+w-1)$$

*for* $\Re z, \Re w > 1$*. Then for an arbitrary fixed* $\varepsilon > 0$*,*

$$S(x) := \sum_{m,n \le x} T(m,n) = x^2 \left( \sum_{r=0}^{3} c_r (\log x)^r \right) + O\left( x^{\frac{1117}{701} + \varepsilon} \right) .$$

*Here* $c_3 = \frac{1}{3}$*,* $c_2 = 3\gamma - 1$*,* $c_1 = 8\gamma^2 - 6\gamma - 2\gamma_1 + 1$*. The constant* $c_0$ *can be constructed from the proof below, but is not accessible to numerical evaluation for the reason described in Remark 3.*

**Proof of Lemma 1.** For $x$ large, let $1 < y < x$ be a positive real parameter at our disposal, and put $z := \frac{x}{y}$. Further, write $M := \max(j,k)$ for short. Then,

$$S(x) = \sum_{\substack{\ell M \le x \\ \ell, j, k \in \mathbb{N}}} \ell\, \tau(j)\tau(k) = \left\{ \sum_{\substack{\ell M \le x \\ \ell \le y}} + \sum_{\substack{\ell M \le x \\ M \le z}} - \sum_{\substack{\ell M \le x \\ \ell \le y, M \le z}} \right\} \ell\, \tau(j)\tau(k) \\ =: S_1(x) + S_2(x) - S_3(x)\,, \tag{22}$$

say. In what follows[1], let $\theta$ be an arbitrary fixed real greater than $\theta_0$. Then, firstly,

$$\begin{aligned} S_3(x) &= \sum_{\ell \le y} \ell \sum_{j,k \le z} \tau(j)\tau(k) \\ &= \left(\tfrac{1}{2} y^2 + O(y)\right) \left(z \log z + (2\gamma - 1) z + O(z^\theta)\right)^2 \\ &= \tfrac{1}{2} x^2 \log^2 z + (2\gamma-1) x^2 \log z + \tfrac{1}{2}(2\gamma-1)^2 x^2 \\ &+ O\left(\frac{x^2}{y} \log^2 x\right) + O\left(x^{1+\theta} y^{1-\theta}\right) . \end{aligned} \tag{23}$$

Secondly,

$$\begin{aligned} S_1(x) &= \sum_{\ell \le y} \ell \left( \left(\frac{x}{\ell} \log \frac{x}{\ell} + (2\gamma - 1)\frac{x}{\ell}\right)^2 + O\left(\left(\frac{x}{\ell}\right)^{1+\theta}\right)\right) \\ &= \sum_{\ell \le y} \ell \left(\frac{x}{\ell} \log \frac{x}{\ell} + (2\gamma-1)\frac{x}{\ell}\right)^2 + O\left(x^{1+\theta} y^{1-\theta}\right) . \end{aligned} \tag{24}$$

By a straightforward computation,

$$\begin{aligned} &\sum_{\ell \le y} \ell \left(\frac{x}{\ell} \log \frac{x}{\ell} + (2\gamma-1)\frac{x}{\ell}\right)^2 \\ &= x^2 \sum_{\ell \le y} \frac{\log^2 \ell}{\ell} - 2x^2 (\log x + (2\gamma - 1)) \sum_{\ell \le y} \frac{\log \ell}{\ell} \\ &+ x^2 (\log x + (2\gamma-1))^2 \sum_{\ell \le y} \frac{1}{\ell} . \end{aligned} \tag{25}$$

By Euler's summation formula, for $r = 0, 1, 2$,

$$\sum_{\ell \le y} \frac{\log^r \ell}{\ell} = \frac{\log^{r+1} y}{r+1} + \gamma_r + O\left(\frac{\log^r y}{y}\right) .$$

Combining this with (25) and (24), we get

$$\begin{aligned} S_1(x) &= x^2 \left(\log^2 x (\log y + \gamma)\right. \\ &- \log x \left(\log^2 y - 2(2\gamma-1)\log y - 2\gamma(2\gamma-1) + 2\gamma_1\right) \\ &+ \frac{1}{3} \log^3 y - (2\gamma - 1)\log^2 y \\ &+ \left.(2\gamma-1)^2 \log y + \gamma(2\gamma-1)^2 - 4\gamma\gamma_1 + 2\gamma_1 + \gamma_2\right) \\ &+ O\left(\frac{x^2}{y}\log^2 x\right) + O\left(x^{1+\theta} y^{1-\theta}\right) . \end{aligned} \tag{26}$$

[1]This arrangement implies that $O(x^\theta \log x)$ can be replaced throughout by $O(x^\theta)$, etc.

Finally, with $M := \max(j,k)$,

$$S_2(x) = \sum_{j,k\le z} \tau(j)\tau(k) \sum_{\ell \le \frac{x}{M}} \ell = \sum_{j,k\le z} \tau(j)\tau(k) \left( \frac{x^2}{2M^2} + O\left(\frac{x}{M}\right)\right) . \tag{27}$$

The $O$-term here contributes overall

$$\ll x \sum_{j,k\le z} \frac{\tau(j)\tau(k)}{\sqrt{jk}} \ll x \left( \sum_{j\le z} \frac{\tau(j)}{\sqrt{j}} \right)^2 \ll xz \log^2 x = \frac{x^2}{y} \log^2 x\,. \tag{28}$$

Writing $S_2^*(x)$ for the main term in (27), we get

$$\begin{aligned} S_2^*(x) &= \frac{x^2}{2} \sum_{j,k\le z} \frac{\tau(j)\tau(k)}{\max(j^2,k^2)} \\ &= x^2 \sum_{j\le k\le z} \frac{\tau(k)}{k^2} \tau(j) - \frac{x^2}{2} \sum_{k\le z} \frac{\tau^2(k)}{k^2} =: x^2 \left( R_1(z) - \tfrac{1}{2} R_2(z) \right) . \end{aligned} \tag{29}$$

Now

$$R_2(z) = \frac{\zeta^4(2)}{\zeta(4)} + O\left(\frac{\log^3 x}{z}\right) = \frac{5\pi^4}{72} + O\left(\frac{\log^3 x}{z}\right) . \tag{30}$$

Further,

$$\begin{aligned} R_1(z) &= \sum_{k\le z} \frac{\tau(k)}{k^2} \left( k\log k + (2\gamma - 1)k + \Delta(k) \right) \\ &= \sum_{k\le z} \frac{\tau(k)}{k} (\log k + 2\gamma - 1) + \sum_{k\le z} \frac{\tau(k)\Delta(k)}{k^2} . \end{aligned} \tag{31}$$

Here the last sum equals

$$\sum_{k=1}^{\infty} \frac{\tau(k)\Delta(k)}{k^2} + O\left(z^{\theta-1}\right) =: C_1 + O\left(z^{\theta-1}\right) .$$

Moreover, using Stieltjes integral notation,

$$\begin{aligned} &\sum_{k\le z} \frac{\tau(k)}{k} (\log k + 2\gamma - 1) \\ &= \int_{1-}^{z+} \frac{\log u + 2\gamma - 1}{u} \,\mathrm{d}\left( u\log u + (2\gamma-1)u + \Delta(u) \right) \\ &= \int_1^z \frac{\log u + 2\gamma - 1}{u} \left( \log u + 2\gamma \right) \mathrm{d}u + (2\gamma-1)^2 + O(z^{\theta-1}) \\ &- \int_1^z \frac{\mathrm{d}}{\mathrm{d}u} \left( \frac{\log u + 2\gamma - 1}{u} \right) \Delta(u) \,\mathrm{d}u \\ &= \frac{1}{3} \log^3 z + \left( 2\gamma - \tfrac{1}{2} \right) \log^2 z + 2\gamma(2\gamma-1) \log z + C_2 + O\left(z^{\theta-1}\right) , \end{aligned} \tag{32}$$

where

$$C_2 := (2\gamma-1)^2 + \int_1^\infty \frac{\log u + 2(\gamma-1)}{u^2}\Delta(u)\,\mathrm{d}u\,.$$

Putting together (27) - (32), and recalling that $z = \frac{x}{y}$, we arrive at

$$\begin{aligned} S_2(x) = x^2 \bigg( &\frac{1}{3}\log^3 z + \big(2\gamma - \tfrac{1}{2}\big)\log^2 z + 2\gamma(2\gamma-1)\log z \\ &+ C_1 + C_2 - \frac{5\pi^4}{144}\bigg) + O\left(\frac{x^2}{y}\log^2 x\right) + O\left(x^{1+\theta}y^{1-\theta}\right)\,. \end{aligned} \tag{33}$$

Finally, using (23), (26), and (33) in (22), an involved but straightforward calculation yields

$$S(x) = x^2\left(\sum_{r=0}^{3} c_r (\log x)^r\right) + O\left(\frac{x^2}{y}\log^2 x\right) + O\left(x^{1+\theta}y^{1-\theta}\right)\,,$$

with $c_1, c_2, c_3$ as stated in Lemma 1. Balancing the two $O$-terms here, the optimal choice is $y = x^{\frac{1-\theta}{2-\theta}}$. This completes the proof of Lemma 1. □

Now use that (cf. Theorem 1 and Remark 1),

$$s(m,n) = \sum_{d \mid \gcd(m,n)} \mu(d) T(m/d, n/d) \quad (m,n \in \mathbb{N})\,.$$

We deduce

$$\begin{aligned} \sum_{m,n\le x} s(m,n) &= \sum_{d\le x}\mu(d)\sum_{a,b\le x/d} T(a,b) \\ &= \sum_{d\le x}\mu(d)\left(\left(\frac{x}{d}\right)^2\sum_{r=0}^{3} c_r\left(\log\frac{x}{d}\right)^r + O\left(\left(\frac{x}{d}\right)^{\frac{1117}{701}+\varepsilon}\right)\right) \\ &= x^2 V(x) + O\left(\sum_{d\le x}\left(\frac{x}{d}\right)^{\frac{1117}{701}+\varepsilon}\right)\,, \end{aligned}$$

where the error term is $O\left(x^{\frac{1117}{701}+\varepsilon}\right)$ and

$$\begin{aligned} V(x) &= \left(c_3\log^3 x + c_2\log^2 x + c_1\log x + c_0\right)\sum_{d\le x}\frac{\mu(d)}{d^2} \\ &\quad - \left(3c_3\log^2 x + 2c_2\log x + c_1\right)\sum_{d\le x}\frac{\mu(d)\log d}{d^2} \\ &\quad + \left(3c_3\log x + c_2\right)\sum_{d\le x}\frac{\mu(d)\log^2 d}{d^2} - c_3\sum_{d\le x}\frac{\mu(d)\log^3 d}{d^2} \\ &= \left(c_3\log^3 x + c_2\log^2 x + c_1\log x + c_0\right)\left(\frac{1}{\zeta(2)} + O\left(\frac{1}{x}\right)\right) \\ &\quad - \left(3c_3\log^2 x + 2c_2\log x + c_1\right)\left(\frac{\zeta'(2)}{\zeta^2(2)} + O\left(\frac{\log x}{x}\right)\right) \end{aligned}$$

$$+ (3c_3 \log x + c_2) \left( \frac{2(\zeta'(2))^2 - \zeta''(2)\zeta(2)}{\zeta^3(2)} + O\left(\frac{\log^3 x}{x}\right)\right)$$
$$-c_3 \left( c^* + O\left(\frac{\log^3 x}{x}\right)\right) ,$$

with a certain constant $c^*$, leading to the asymptotic formula (12).

From (11) we deduce by Möbius inversion that

$$c(m,n) = \sum_{d \mid \gcd(m,n)} \mu(d) s(m/d, n/d) ,$$

and obtain

$$\sum_{m,n \le x} c(m,n) = \sum_{d \le x} \mu(d) \sum_{a,b \le x/d} s(a,b) .$$

Applying now the formula (12), similar computations show the validity of (13). □

**Proof of Theorem 3.** Obviously, by (5),

$$\sum_{\substack{m,n \le x \\ \gcd(m,n) > 1}} s(m,n) = \sum_{m,n \le x} s(m,n) - \sum_{\substack{m,n \le x \\ \gcd(m,n) = 1}} \tau(m)\tau(n) . \tag{34}$$

In order to find an asymptotics for the last sum we need an auxiliary result.

**Lemma 2.** [2] *For $K$ a positive integer and $Y$ a large real variable satisfying $K \le Y^9$, it holds true that*

$$\sum_{n \le Y} \tau(Kn) = \beta_0(K) Y \log Y + \beta_1(K) Y + O\left(Y^{1/3+\varepsilon}\right) ,$$

*for every fixed $\varepsilon > 0$, uniformly in $K$, with the notations given by (15) and (16). Note that $\beta_0(K)\,,\ \beta_1(K) \ll K^{\varepsilon}$.*

**Proof of Lemma 2.** We start from the formula[3] (cf. E. C. Titchmarsh [16, (1.4.2)])

$$\frac{1}{\tau(K)} \sum_{n=1}^{\infty} \frac{\tau(Kn)}{n^s} = \zeta^2(s)\, F_K(s) \qquad (\Re s > 1) ,$$

where $F_K(s)$ is defined in (14). We can follow the classic example of the deduction of [16, Theorem 12.2], sketching only the necessary changes. With $a_n := \frac{\tau(Kn)}{\tau(K)} \le \tau(n)$ and $T := Y^{2/3}$, Perron's formula gives

$$\sum_{n \le Y} a_n = \frac{1}{2\pi i} \int_{1+\delta-iT}^{1+\delta+iT} \zeta^2(s)\, F_K(s) \frac{Y^s}{s} \,\mathrm{d}s + O\left(Y^{1/3+\varepsilon}\right) ,$$

with arbitrarily small fixed $\delta > 0$. The line of integration is now shifted to $s = -\delta + it$, $-T \le t \le T$. On the horizontal line segments $-\delta \le \sigma \le 1 + \delta$, $t = \pm T$,

$$F_K(\sigma \pm iT) \ll \prod_{p \mid K} p^{\delta} \le K^{\delta} \le Y^{9\delta} ,$$

[2] It is possible or even likely, that this result or even a sharper assertion is contained in the literature. However, the authors' attempts to find it were not successful.

[3] The authors are grateful to Professor A. Ivić for directing their attention to this identity.

hence this brings in only a harmless factor. The residue of the integrand at $s = 1$ equals

$$\alpha_0(K) Y \log Y + (\alpha_0(K)(2\gamma - 1) + \alpha_1(K))\, Y,$$

where $\alpha_0(K)$ and $\alpha_1(K)$ are defined by (15).

Furthermore, the residue of the integrand at $s = 0$ is

$$\zeta^2(0) F_K(0) = \frac{1}{4} \prod_{p^{\nu_p(K)} \,||\, K} (1 - \eta_{p,K}) \ll 1$$

uniformly in $K$. It remains to estimate the integral

$$\int_{-\delta - iT}^{-\delta + iT} \zeta^2(s)\, F_K(s) \frac{Y^s}{s} \, \mathrm{d}s \,.$$

Expanding the product which defined $F_K(s)$ we obtain $2^{\omega(K)} \ll K^{\delta'}$ terms (with $\omega(K)$ denoting the number of prime divisors of $K$) of the form $B^{-s}$ where $B$ is the product of some or all of the primes which divide $K$.

Now the proof of [16, Theorem 12.2], which involves $\zeta^2(s)$ alone, ultimately leads to estimates of the type

$$\int_1^T G(t) e^{i\Phi(t)} \, \mathrm{d}t \ll \max_{[1,T]} |G(t)| \, \max_{[1,T]} \left|\Phi''(t)\right|^{-1/2} ,$$

with $G(t), \Phi(t)$ real functions. Writing $B^{\delta - it} = B^{\delta} e^{-it \log B}$, we see that $B^\delta \le K^\delta$ contributes only a harmless factor, while $-t \log B$ does not contribute at all to $\Phi''(t)$. This completes the proof of Lemma 2. □

**Lemma 3.** *For large real $x$, let*

$$U(x) := \sum_{\substack{m,n \le x \\ \gcd(m,n) = 1}} \tau(m) \tau(n) \,.$$

*Then it follows that*

$$U(x) = x^2 \left( b_2 \log^2 x + b_1 \log x + b_0 \right) + O\left( x^{4/3 + \varepsilon} \right)$$

*for every $\varepsilon > 0$. Here the constants $b_0, b_1$ and $b_2$ are given by (19).*

**Proof of Lemma 3.** By a familiar device usually attributed to Vinogradov,

$$\begin{aligned} U(x) &= \sum_{K \le x} \mu(K) \sum_{\substack{m,n \le x \\ K \,|\, \gcd(m,n)}} \tau(m) \tau(n) \\ &= \sum_{K \le x} \mu(K) \left( \sum_{n \le \frac{x}{K}} \tau(Kn) \right)^2 . \end{aligned} \tag{35}$$

The contribution of the $K$ with $x^{9/10} < K \le x$ is small: Using that $\tau(Kn) \le \tau(K)\tau(n)$, we get

$$\sum_{x^{9/10}<K\le x} \mu(K) \left( \sum_{n\le \frac{x}{K}} \tau(Kn) \right)^2 \ll \sum_{x^{9/10}<K\le x} \tau^2(K) \left(\frac{x}{K}\right)^{2+\varepsilon} \ll x^{11/10+\varepsilon} .$$

But if $K \le x^{9/10}$, then $K \le \left(\frac{x}{K}\right)^9$, thus we may apply Lemma 2 to the inner sum in (35). In this way,

$$U(x) = \sum_{K\le x^{9/10}} \mu(K) \left( \beta_0(K) \frac{x}{K} \log \frac{x}{K} + \beta_1(K) \frac{x}{K} + O\left( \left(\frac{x}{K}\right)^{1/3+\varepsilon/2} \right) \right)^2 + O\left(x^{11/10+\varepsilon}\right) .$$

The $O$-term here contributes overall at most

$$\ll x^{4/3+\varepsilon} \sum_{K\le x^{9/10}} K^{-4/3} \ll x^{4/3+\varepsilon} .$$

We claim that this implies that

$$U(x) = x^2 \left( \sum_{K=1}^{\infty} \frac{\mu(K)}{K^2} \left(\beta_0(K)\log x + \beta_1(K) - \beta_0(K)\log K\right)^2 \right) + O\left(x^{4/3+\varepsilon}\right) . \tag{36}$$

But this is easy to see, since

$$x^2 \left( \sum_{K>x^{9/10}} \frac{\mu(K)}{K^2} \left(\beta_0(K)\log x + \beta_1(K) - \beta_0(K)\log K\right)^2 \right) \ll x^{2+\varepsilon} \sum_{K>x^{9/10}} K^{-2+\varepsilon} \ll x^{11/10+2\varepsilon} .$$

By an obvious calculation, (36) completes the proof of Lemma 3. □

Now Lemma 3 and (34), together with (12) and (13), give the asymptotics (17) and (18), respectively. □

**Proof of Theorem 4.** The asymptotic formulas (20) and (21) are direct consequences of (7) and (8), respectively, and of the known estimate

$$\sum_{n\le x} \psi(n) = \frac{15}{2\pi^2} x^2 + \mathcal{O}(x \log^{2/3} x)$$

of A. Walfisz [20, p. 100]. □

# 4 Appendix

Let $t_2(n)$ denote the sum of the numbers of subgroups of Abelian groups of order $n$ having rank $\le 2$ (up to isomorphisms). Then one has the following Dirichlet series representation, due to G. Bhowmik: For $\Re z > 1$,

$$\sum_{n=1}^{\infty} \frac{t_2(n)}{n^z} = \zeta^2(z)\zeta^2(2z)\zeta(2z-1) \prod_p \left(1 + \frac{1}{p^{2z}} - \frac{2}{p^{3z}}\right) . \tag{37}$$

The universality of this Dirichlet series was proved in [9]. For asymptotic properties of $t_2(n)$, based on formula (37) see the papers [1, 8]. More generally, for finite Abelian groups $A$ let $t_r(n) = \sum_{\#A=n,\mathrm{rank}(A)=r} s(A)$, where $\#A$ is the order of $A$, rank$(A)$ is its rank and $s(A)$ stands for the number of subgroups of $G$. See [2] for properties of the function $t_r(n)$.

Here we give a short direct proof for (37). By the Busche-Ramanujan identity for the divisor function $\tau$ (cf. [19]), the second formula of (5) can be written as

$$s(m,n) = \sum_{d \mid \gcd(m,n)} d\, \tau(mn/d^2)\,, \tag{38}$$

see [4]. Now, according to the definition of $t_2(n)$ and using (38),

$$t_2(n) = \sum_{\substack{k\ell=n \\ k \mid \ell}} s(k,\ell) = \sum_{\substack{k\ell=n \\ k \mid \ell}} \sum_{d \mid k} d\tau(k\ell/d^2) = \sum_{d^2a^2j=n} d\tau(a^2 j)\,,$$

that is

$$t_2(n) = \sum_{d^2k=n} d\tau(k)\tau(1,2;k) \quad (n \in \mathbb{N})\,, \tag{39}$$

where, as usual, $\tau(1,2;k) = \sum_{a^2b=k} 1$ (sequence [12, item A046951]). Note that $\tau(1,2;k) = \prod_p (\lfloor \nu_p(k)/2 \rfloor + 1)$. It turns out that the function $t_2(n)$ is multiplicative and (39) quickly leads to the formula (37).

Let $u_2(n)$ denote the total number of cyclic subgroups of Abelian groups of order $n$ having rank $\le 2$ (up to isomorphisms), not investigated in the literature. It follows at once from (11) and the above results for $t_2(n)$ that

$$\sum_{n=1}^{\infty} \frac{u_2(n)}{n^z} = \zeta^2(z)\zeta(2z)\zeta(2z-1) \prod_p \left(1 + \frac{1}{p^{2z}} - \frac{2}{p^{3z}}\right),$$

and

$$u_2(n) = \sum_{a^2b=n} \mu(a) t_2(b) \quad (n \in \mathbb{N})\,.$$

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