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CONJECTURE ON THE VALUE OF $\pi\left(10^{26}\right)$, THE NUMBER OF PRIMES $< 10^{26}$

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ABSTRACT. Based on the first 25 known values of $\pi\left(10^n\right)$, the number of primes less than 10^n with $1 \leq n \leq 25$, we propose a conjectured value range of $\pi\left(10^{26}\right)$ calculated by using polynomial interpolations with two corrective functions obtained by Thiele interpolations on relative differences of exact and interpolated values of $\pi\left(10^n\right)$. The conjectured range value is in agreement with values obtained by the Eulerian logarithmic integral and with the Riemann functions.

1. Introduction

For $x \in \mathbb{Z}^+$, it is known (see e.g. [1]) that if $\pi(x)$ is the number of primes$\leq x$, then

$$lim_{x\to\infty}\left(\frac{\pi(x)}{\frac{x}{log(x)}}\right)=1 \tag{1.1}$$

meaning in the general sense that the relative error of approximating $\pi(x)$ by $\left(\frac{x}{log(x)}\right)$ approaches 0 as x approaches infinity. Several better approximations to $\pi(x)$ for $x \lesssim 10^{25}$ are given for example by the offset logarithmic integral or Eulerian logarithmic integral (see e.g. [2])

$$Li(x)=li\left(x\right)-li\left(2\right)=\int_2^x\frac{dt}{log\left(t\right)} \tag{1.2}$$

in function of the logarithmic integral $li(x)=\int_0^x\frac{dt}{log(t)}$, or by the Riemann function (see e.g. [3])

$$R\left(x\right)=\sum_{j=1}^{\infty}\frac{\mu\left(j\right)}{j}li\left(x^{\frac{1}{j}}\right) \tag{1.3}$$

where $\mu\left(j\right)$ is the Moebius function and $j \in \mathbb{Z}^+$, or by an even better function

$$R\left(x\right)-\frac{1}{log\left(x\right)}+\frac{1}{\pi}arctan\left(\frac{\pi}{log\left(x\right)}\right) \tag{1.4}$$

Other approximation formulas to $\pi(x)$ include Legendre's [4], Lehmer's [5], and Meissel's [6].

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Values of $\pi(10^n)$ are known for $1 \le n \le 25$ (see [7]). The values of $\pi(10^{24})$ and $\pi(10^{25})$ were calculated relatively recently, respectively in 2010 [8] and in June 2013 [9].

In this paper, we propose a simple method based on polynomial interpolations of the known 25 first values of $\pi(10^n)$ introducing two corrective functions to calculate a conjectured value of $\pi\left(10^{26}\right)$.

2. Method

The method we use includes six steps, some involving interpolations[1].

Step 1: First, polynomial functions

$$P_n(x) = \sum_{i=0}^{n-1} \left(a_{i,n} x^i\right) \tag{2.1}$$

with coefficients $a_{i,n} \in \mathbb{R}$ and $i, n, x \in \mathbb{Z}^+$, are calculated by polynomial interpolations over the values of $\pi(10^n)$ up to n for $2 \le n \le 25$, yielding 23 polynomials, given in Appendix 1, such that the equality

$$\pi(10^x) = P_n(x) \tag{2.2}$$

holds exactly or $1 \le x \le n$.

Step 2: For each value of n until $n = 24$, the next value for $x = n+1$ is calculated by (2.1) yielding $P_n(n+1)$ and compared to the value of $\pi(10^{n+1})$. The relative differences

$$\delta_{n+1} = \frac{\pi(10^{n+1}) - P_n(n+1)}{\pi(10^{n+1})} \tag{2.3}$$

are then calculated. As $P_n(n+1) < \pi(10^{n+1})$ for $2 \le n \le 24$, δ_{n+1} is such that $0 < \delta_{n+1} < 1$ and decreases with increasing values of n (see Fig. 2.1).

Step 3: For each set value of n, corrective functions Φ_{n+1} in the form of continued fractions

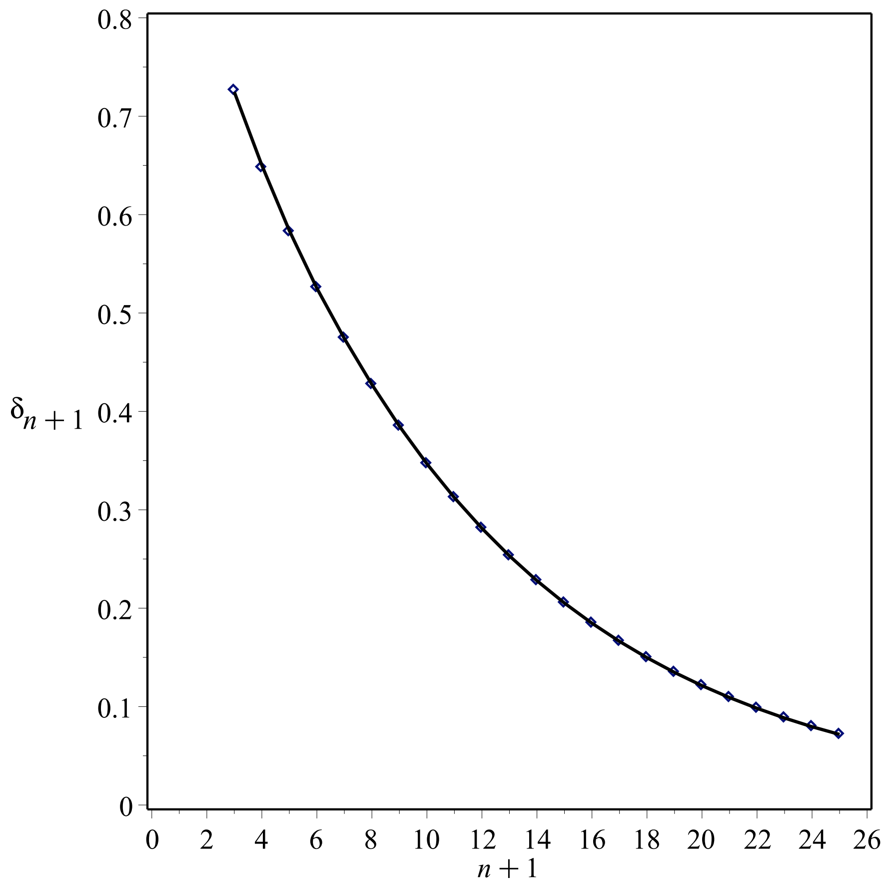
$$\Phi_{n+1}(x) = c_1 + \cfrac{x-3}{c_2 + \cfrac{x-4}{c_3 + \dots \ddots \dots + \cfrac{x-(n-3)}{c_{n-4} + \cfrac{x-(n-2)}{c_{n-3} + \cfrac{x-(n-1)}{c_{n-2} + Kx}}}}} \tag{2.4}$$

with coefficients $c_i, K \in \mathbb{R}$ and $i, n, x \in \mathbb{Z}^+$, $1 \le i \le n-2$, are calculated by Thiele interpolations over the $(n-1)$ values of δ_{n+1} for $3 \le n \le 24$, yielding 22 corrective functions Φ_{n+1}. The coefficients c_i and K of Φ_{n+1} for $n = 24$ are given in Appendix 2.

Step 4: The predictive power of using the 22 corrective functions Φ_{n+1} to calculate a next interpolated value of $\pi(10^{x+1})$ knowing the first x values of $\pi(10^x)$ with $1 \le x \le n$ for all values of n, $1 \le n \le 24$, is verified by taking the nearest integer

[1] Curve Fitting package of Maple 16.00 with 50 digits precision

FIGURE 2.1. δ_{n+1}in function of $(n+1)$



to $(P_x(x+1)/(1-\Phi_{n+1}(x+1)))$ and compared to $\pi(10^{x+1})$. It yields that the equality

$$\pi(10^{x+1}) = Round\left(\frac{P_x(x+1)}{1-\Phi_{n+1}(x+1)}\right) \tag{2.5}$$

holds exactly for all values of $1 \le x \le n$ and of n, $1 \le n \le 24$, where Φ_4 should be taken for $1 \le n \le 3$ and $Round(X)$ is the nearest integer to X ($X \in \mathbb{R}$).

Step 5: To interpolate the next unknown value for $n = 26$ of $\pi(10^{26})$, as Φ_{26} is obviously unknown, we use a relation similar to (2.5) with $x = 25$ and Φ_{25} instead of Φ_{26}, to obtain a very approximate interpolated value of $\pi(10^{26})$

$$Round\left(\frac{P_{25}(26)}{1-\Phi_{25}(26)}\right) = 1699246738822618041025224 \tag{2.6}$$

Step 6: To improve this interpolation, we have to find an additional correction to the corrective function Φ_{25}. We performed a similar interpolation in May 2013 (unpublished) on the first 24 values of $\pi(10^n)$ before the announcement of the calculated value of $\pi(10^{25}) = 176846309399143769411680$ by J. Buethe, J. Franke,

TABLE 1. Relative differences δ'_{n+1}

$n+1$	δ'_{n+1}	$n+1$	δ'_{n+1}	$n+1$	δ'_{n+1}
5	3.10676×10^{-2}	12	1.23708×10^{-4}	19	5.44808×10^{-7}
6	-4.15297×10^{-3}	13	-3.31760×10^{-5}	20	1.72692×10^{-7}
7	2.85895×10^{-5}	14	4.84531×10^{-5}	21	-3.64154×10^{-7}
8	7.32975×10^{-4}	15	-1.62333×10^{-5}	22	-7.56044×10^{-8}
9	5.14623×10^{-3}	16	1.11746×10^{-6}	23	-9.45864×10^{-9}
10	-8.04909×10^{-3}	17	-4.32482×10^{-6}	24	2.62139×10^{-8}
11	5.18791×10^{-3}	18	-4.83262×10^{-6}	25	1.34117×10^{-8}

A. Jost and T. Kleinjung on 1st June 2013 [9]. This previous interpolation yielded an interpolated value of

$$Round\left(\frac{P_{24}(25)}{1-\Phi_{24}(25)}\right) = 1768463070273346927638889 < \pi(10^{25}) \tag{2.7}$$

giving a relative difference

$$\delta'_{n+1} = \frac{\pi(10^{n+1}) - Round\left(\frac{P_n(n+1)}{1-\Phi_n(n+1)}\right)}{\pi(10^{n+1})} \tag{2.8}$$

of $\delta'_{25} \approx 1.34117 \times 10^{-8}$.

To estimate the error made by using Φ_n instead of Φ_{n+1} in (2.5) when interpolating to find a value of $\pi(10^{n+1})$, the relative differences δ'_{n+1} are calculated for $4 \leq n \leq 24$, yielding 21 relative differences δ'_{n+1} (2.8), whose approximate values are reported in Table 1. Their absolute value decreases with increasing n as shown in Fig. 2.2.

Assuming that $\left|\delta'_{n+1}\right|$ continues to decrease with n, an interpolated value of $\left|\delta'_{26}\right| \lesssim 10^{-8}$ can be expected. A new Thiele interpolation over the last five values of $\left|\delta'_{n+1}\right|$ for $20 \leq n \leq 24$ yields a second corrective function $|\psi_{n+1}(n)|$

$$|\psi_{n+1}(n)| = d_1 + \cfrac{n-20}{d_2 + \cfrac{n-21}{d_3 + \cfrac{n-22}{d_4 + Mn}}} \tag{2.9}$$

with $d_i, M \in \mathbb{R}$ and $i, n \in \mathbb{Z}^+$, $1 \leq i \leq 4$. The coefficients d_i and M are given in Appendix 3. For $n = 25$, (2.9) yields $|\psi_{26}| \approx 7.07767 \times 10^{-9}$. Assuming $\psi_{26} \approx \pm 7.1 \times 10^{-9}$ yields a conservative assumption for the value of $\pi(10^{26})$

$$\pi(10^{26}) \approx 1699246738822618041025224 \pm 12064651845640588 \tag{2.10}$$

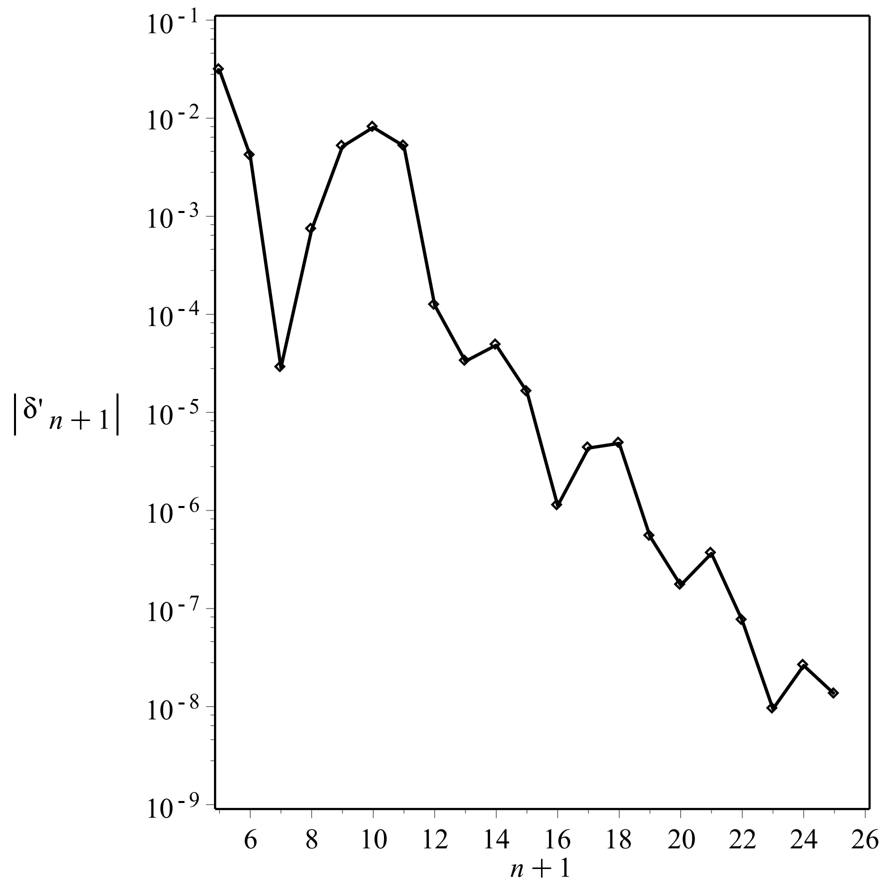
or

$$1699246726757966195384636 \lesssim \pi(10^{26}) \lesssim 1699246750887269886665812 \tag{2.11}$$

Assuming further that ψ_{26} would be positive like δ'_{25} and δ'_{24} and that it would be included in the range $7 \times 10^{-9} \lesssim \psi_{26} \lesssim 7.1 \times 10^{-9}$, the value range for $\pi(10^{26})$ reduces to

$$\pi(10^{26}) \approx 1699246738822618041025224^{+12064651845640588}_{+11894727171758326} \tag{2.12}$$

FIGURE 2.2. $\left|\delta'_{n+1}\right|$ in function of $n+1$



or

$$1699246750717345212783550 \lesssim \pi(10^{26}) \lesssim 1699246750887269886665812 \tag{2.13}$$

3. Discussion

It is interesting to compare this value range to those values obtained with the approximating functions of Section 1. Table 2 shows for $24 \le n \le 26$ the values of $Round\left(\frac{10^n}{log(10^n)}\right)$, $Round\left(Li(10^n)\right)$ (1.2), $Round\left(R\left(10^n\right)\right)$ (1.3) and the associated relative differences

$$\delta''_n = \frac{\pi(10^n) - Round\left(f(n)\right)}{\pi(10^n)} \tag{3.1}$$

where $f(n)$ is either $\left(\frac{10^n}{log(10^n)}\right)$, or $Li(10^n)$, or $R\left(10^n\right)$, and where $\pi(10^{24}) = 18435599767349200867866$ and $\pi(10^{25}) = 176846309399143769411680$.

Note that in calculating $R\left(10^n\right)$, the summation in (1.3) is made for $1 \le j \le 1000$, as the value of $R\left(10^n\right)$ does not change for higher values of j, due mainly

Table 2. Values of $Round\left(\frac{10^n}{log(10^n)}\right)$, $Round\left(Li(10^n)\right)$ (1.2), $Round\left(R\left(10^n\right)\right)$ (1.3) for $24 \leq n \leq 26$ and associated $\delta_n^{''}$ (2.12)

n	$f(n)$	$Round\left(f(n)\right)$	$\delta_n^{''}$
24	$\frac{10^n}{log(10^n)}$	18095603412635492818797	1.84424×10^{-2}
24	$Li(10^n)$	18435599767366347775143	-9.30098×10^{-13}
24	$R\left(10^n\right)$	18435599767347541878147	8.99884×10^{-14}
25	$\frac{10^n}{log(10^n)}$	173717792761300731060452	1.76906×10^{-2}
25	$Li(10^n)$	176846309399198930392618	-3.11915×10^{-13}
25	$R\left(10^n\right)$	176846309399141934626966	1.03750×10^{-14}
26	$\frac{10^n}{log(10^n)}$	1670363391935583952504342	-
26	$Li(10^n)$	1699246750872593033005723	-
26	$R\left(10^n\right)$	1699246750872419991992147	-

to the operation of rounding to the nearest integer. Similarly, the values obtained by rounding to the nearest integer of the function (1.4) does not differ from those obtained by rounding $R\left(10^n\right)$ for the same reason.

It is seen that, for $n = 24$ and 25, the values obtained with $\frac{10^n}{log(10^n)}$ are relatively far from the real values of $\pi(10^n)$ with $\delta_n^{''}$ of the order 10^{-2} , while the values of $Li(10^n)$ and $R\left(10^n\right)$ gives much better results with $\delta_n^{''}$ of the order 10^{-13}... 10^{-14}. For $n = 26$, the values of $Li(10^{26})$ and $R\left(10^{26}\right)$ are within the range given by (2.13).

4. Conclusion

We proposed a simple method to interpolate a conjectured value of $\pi\left(10^{26}\right)$, based on polynomial interpolations of the known 25 first values of $\pi\left(10^n\right)$ with two corrective functions, the first based on a Thiele interpolation of the relative differences of the exact and interpolated values of $\pi\left(10^n\right)$ and the second obtained by a new Thiele interpolation of the last five values of the relative differences of the exact and interpolated values of $\pi\left(10^n\right)$ with a previous value of the first corrective function. The range obtained for the conjectured value of $\pi\left(10^{26}\right)$ contains those values calculated by the Eulerian logarithmic integral and with the Riemann functions.

Appendix 1: Polynomials $P_n\left(x\right)$ for $2 \leq n \leq 25$

$P_2\left(x\right) = 21x - 17$

$P_3\left(x\right) = 61x^2 - 162x + 105$

$P_4\left(x\right) = \left(398/3\right)x^3 - 735x^2 + \left(3892/3\right)x - 691$

$P_5\left(x\right) = \left(1397/6\right)x^4 - \left(6587/3\right)x^3 + \left(44485/6\right)x^2 - \left(31033/3\right)x + 4897$

$P_6\left(x\right) = \left(41269/120\right)x^5 - \left(118219/24\right)x^4 + \left(648877/24\right)x^3 - \left(1679165/24\right)x^2 + \left(1677731/20\right)x - 36372$

$P_7\left(x\right) = \left(63053/144\right)x^6 - \left(2124317/240\right)x^5 + \left(10324961/144\right)x^4 - \left(14150231/48\right)x^3 + \left(46161541/72\right)x^2 - \left(6885127/10\right)x + 278893$

$P_8(x) = (1230899/2520)\,x^7 - (1059103/80)\,x^6 + (106869757/720)\,x^5 - (42511909/48)\,x^4$
$+ (2168305201/720)\,x^3 - (692786273/120)\,x^2 + (199310224/35)\,x - 2182905$

$P_9(x) = (1087519/2240)\,x^8 - (12232463/720)\,x^7 + (8059007/32)\,x^6$
$- (295746589/144)\,x^5 + (9612782119/960)\,x^4 - (5337867053/180)\,x^3$
$+ (17329512589/336)\,x^2 - (190033621/4)\,x + 17392437$

$P_{10}(x) = (8766223/20160)\,x^9 - (1526557/80)\,x^8 + (3642052523/10080)\,x^7$
$- (61717169/16)\,x^6 + (73322957917/2880)\,x^5 - (6425876741/60)\,x^4$
$+ (119710398973/420)\,x^3 - (11000451139/24)\,x^2 + (13960806758/35)\,x - 140399577$

$P_{11}(x) = (1285948381/3628800)\,x^{10} - (13829848163/725760)\,x^9 + (2713678729/6048)\,x^8$
$- (734294140229/120960)\,x^7 + (8994784761253/172800)\,x^6 - (10167707046167/34560)\,x^5$
$+ (80107171523893/72576)\,x^4 - (488994253144639/181440)\,x^3 + (204682947179773/50400)\,x^2$
$- (1697281382717/504)\,x + 1145548804$

$P_{12}(x) = (1762482209/6652800)\,x^{11} - (62163411143/3628800)\,x^{10} + (356291415727/725760)\,x^9$
$- (496320428783/60480)\,x^8 + (1914173704139/21600)\,x^7 - (111749346412919/172800)\,x^6$
$+ (261253448575099/80640)\,x^5 - (4021281732287857/362880)\,x^4 + (22852595048399971/907200)\,x^3$
$- (1810394687049689/50400)\,x^2 + (113126344733737/3960)\,x - 9429344450$

$P_{13}(x) = (87613503863/479001600)\,x^{12} - (1117825763711/79833600)\,x^{11}$
$+ (4178914904089/8709120)\,x^{10} - (671147586733/69120)\,x^9$
$+ (1870673383324673/14515200)\,x^8 - (2850595750775761/2419200)\,x^7$
$+ (66036047077439107/8709120)\,x^6 - (50008128480417883/1451520)\,x^5$
$+ (1188002313137637187/10886400)\,x^4 - (2241074007266561/9600)\,x^3$
$+ (105572714697948823/332640)\,x^2 - (562059733555247/2310)\,x + 78184159413$

$P_{14}(x) = (730505748461/6227020800)\,x^{13} - (418827227947/39916800)\,x^{12}$
$+ (202948195226041/479001600)\,x^{11} - (37036465604101/3628800)\,x^{10}$
$+ (338543894892361/2073600)\,x^9 - (2209549040385367/1209600)\,x^8$
$+ (638513890320991673/43545600)\,x^7 - (309191577104894683/3628800)\,x^6$
$+ (3885522866329964983/10886400)\,x^5 - (1921510055977553161/1814400)\,x^4$
$+ (1532239610437270573/712800)\,x^3 - (517946743840607861/184800)\,x^2$
$+ (749473155761448241/360360)\,x - 652321589048$

$P_{15}(x) = (68044412311/968647680)\,x^{14} - (434610313091/59875200)\,x^{13}$
$+ (54461331244787/159667200)\,x^{12} - (1154924881829021/119750400)\,x^{11}$
$+ (2655624146858351/14515200)\,x^{10} - (1274125884612191/518400)\,x^9$
$+ (814648425163663999/33868800)\,x^8 - (1890972821539241113/10886400)\,x^7$
$+ (336391248296695009 7/3628800)\,x^6 - (39446931450255536567/10886400)\,x^5$
$+ (20279531504080349887 9/19958400)\,x^4 - (28079184621772453729/1425600)\,x^3$
$+ (416132103978867941789/16816800)\,x^2 - (70617762061263719/3960)\,x$
$+5471675518942$

$P_{16}(x) = (51581436427121/1307674368000)\,x^{15} - (29037678164927/6227020800)\,x^{14}$
$+ (2356528303232491/9340531200)\,x^{13} - (3963130920435319/479001600)\,x^{12}$
$+ (188475479592111883/1026432000)\,x^{11} - (42323388417591161/14515200)\,x^{10}$
$+ (1561486132330065612 7/457228800)\,x^9 - (13050538335127845887/43545600)\,x^8$
$+ (258822026039619096035 3/1306368000)\,x^7 - (53527147712466477239/5443200)\,x^6$
$+ (232250648536987989971/6415200)\,x^5 - (214241913350759156389/2217600)\,x^4$
$+ (5441912683776903632418 37/3027024000)\,x^3 - (524844387326399456539/2402400)\,x^2$
$+ (6906580916276240896/45045)\,x - 46109760908179$

$$
\begin{aligned}
P_{17}(x) = {} & (17450310541093/836911595520)\,x^{16} - (7313219107110283/2615348736000)\,x^{15} \\
& + (180531219567405389/1046139494400)\,x^{14} - (121365437052931943/18681062400)\,x^{13} \\
& + (3833898578147855953/22992076800)\,x^{12} - (44388621420665486663/14370048000)\,x^{11} \\
& + (311968634799119772881/7315660800)\,x^{10} - (408405984743715765221/914457600)\,x^{9} \\
& + (10468950307805472153895 1/29262643200)\,x^{8} - (5730415250350129280461 9/2612736000)\,x^{7} \\
& + (58761752776286891555629 1/5748019200)\,x^{6} - (25656021657153776870717 3/718502400)\,x^{5} \\
& + (98141462322431555247406 7/1076275200)\,x^{4} - (29710845425841425154523003/18162144000)\,x^{3} \\
& + (3892769448567569833753 99/201801600)\,x^{2} - (10582916504243951337 1/80080)\,x \\
& + 390148002619146
\end{aligned}
$$

$$
\begin{aligned}
P_{18}(x) = {} & (1851587512783157/177843714048000)\,x^{17} - (32892317466569501/20922789888000)\,x^{16} \\
& + (256329638772689/2335132800)\,x^{15} - (4918740790637408989/1046139494400)\,x^{14} \\
& + (103257725801778476353/747242496000)\,x^{13} - (3394793051603326648 21/114960384000)\,x^{12} \\
& + (6362604138669081813 91/13412044800)\,x^{11} - (1709364029573132658 01/292626432)\,x^{10} \\
& + (58215415330924617167453/10450944000)\,x^{9} - (603914728576698573791677 1/146313216000)\,x^{8} \\
& + (1362372951548947987220519/5748019200)\,x^{7} - (600894522401170143356521 9/5748019200)\,x^{6} \\
& + (227851260505746171411272429 9/653837184000)\,x^{5} \\
& - (1243038724423425067320005221/145297152000)\,x^{4} \\
& + (25683899275037271281761 39/172972800)\,x^{3} - (34379474783938054453879 67/201801600)\,x^{2} \\
& + (13986816548395306033025 9/12252240)\,x - 3313027022947168
\end{aligned}
$$

$$
\begin{aligned}
P_{19}(x) = {} & (2628028871421403/533531142144000)\,x^{18} - (105675755827312 01/12703122432000)\,x^{17} \\
& + (21003369878917183/321889075200)\,x^{16} - (8251376607822726667/2615348736000)\,x^{15} \\
& + (705870926762785432 1/67060224000)\,x^{14} - (137168476397716450849/53374464000)\,x^{13} \\
& + (2294454618933119106045 1/482833612800)\,x^{12} - (271196454331659216727/399168000)\,x^{11} \\
& + (556013030981016791428933/73156608000)\,x^{10} - (2446752331753149321520481/36578304000)\,x^{9} \\
& + (1840753991498633195715463/3973939200)\,x^{8} - (72170805307845647867807051/28740096000)\,x^{7} \\
& + (8274015071966546316157978352 9/7846046208000)\,x^{6} \\
& - (98402301347766134600009057/2918916000)\,x^{5} \\
& + (107115515185413881741729737/1341204480)\,x^{4} \\
& - (81430400765252671443569075 9/6054048000)\,x^{3} \\
& + (1191473629707981909464035 39/791683200)\,x^{2} - (12106097398209854931846 49/12252240)\,x \\
& + 28223319434109668
\end{aligned}
$$

$$
\begin{aligned}
P_{20}(x) = {} & (26934382640709253/1216451004088320 0)\,x^{19} \\
& - (166368869850866779/400148356608000)\,x^{18} \\
& + (2987733695853594407/82081714176000)\,x^{17} \\
& - (35383617261994314 59/1793381990400)\,x^{16} \\
& + (2115788491722200305 91/2853107712000)\,x^{15} \\
& - (321879470215961698267 61/15692092416000)\,x^{14} \\
& + (31322308257655919826305 3/7242504192000)\,x^{13} \\
& - (102885153829060335904329 7/1448500838400)\,x^{12} \\
& + (44456420445740629307608553/4828336128000)\,x^{11} \\
& - (2124675056281416789418 87/2239488000)\,x^{10} \\
& + (3760078141213234941907434283/4828336128000)\,x^{9} \\
& - (490634366409004846568085244 1/965667225600)\,x^{8} \\
& + (481253715711529530059356254 7/183891708000)\,x^{7} \\
& - (24754931088221306129141635793 63/23538138624000)\,x^{6} \\
& + (98676721740339474188796234 1/3048192000)\,x^{5} \\
& - (554787666780034170195370470 1/7472424960)\,x^{4}
\end{aligned}
$$

$+\,(125190395324618857278613349737/102918816000)\,x^{3}$
$-\,(41062191060690947895024739489/30875644800)\,x^{2}$
$+\,(15342006664059599043795083/17907120)\,x - 241120506972982862$

$P_{21}\left(x\right) = (768801874165960931/810967336058880000)\,x^{20}$
$-\,(47895830419641353593/243290200817664000)\,x^{19}$
$+\,(19593696496463403689/1024379792916480)\,x^{18}$
$-\,(448205658674858604491/388022648832000)\,x^{17}$
$+\,(8712644329419234113383/179338199040000)\,x^{16}$
$-\,(189707965396142126979173/125536739328000)\,x^{15}$
$+\,(1809328805921664964890847/50214695731200)\,x^{14}$
$-\,(253665818771705300538215719/376610217984000)\,x^{13}$
$+\,(527355591384145604585438497/52672757760000)\,x^{12}$
$-\,(460925183825799784732199789/3862668902400)\,x^{11}$
$+\,(252657778262904116233611461/220723937280)\,x^{10}$
$-\,(170656737803291291159227967873/19313344512000)\,x^{9}$
$+\,(1906512687609116773290392565521/34871316480000)\,x^{8}$
$-\,(2302249554842995226554178587199/8559323136000)\,x^{7}$
$+\,(19560002409333744035028716869169/18830510899200)\,x^{6}$
$-\,(199482007364920203714291611479/64576512000)\,x^{5}$
$+\,(54603156665489883803994128245399/7939451520000)\,x^{4}$
$-\,(753730761427013331958837118357/68612544000)\,x^{3}$
$+\,(156740472492042038497972499701/13332664800)\,x^{2}$
$-\,(866115476410575028425322373/116396280)\,x + 2065285115524899931$

$P_{22}\left(x\right) = (54386159471042293/140359731240960000)\,x^{21}$
$-\,(11339777731555550441/128047474114560000)\,x^{20}$
$+\,(6933583440707809527311/729870602452992000)\,x^{19}$
$-\,(16277845641786522178019/25609494822912000)\,x^{18}$
$+\,(173069291634903343162099/5820339732480000)\,x^{17}$
$-\,(185313460297730292870869/179338199040000)\,x^{16}$
$+\,(31253190030180758167460353/1129830653952000)\,x^{15}$
$-\,(439746760244290352119569907/753220435968000)\,x^{14}$
$+\,(27854308762263416011926122669/2824576634880000)\,x^{13}$
$-\,(7087441471779667149333596851/52672757760000)\,x^{12}$
$+\,(17272738385198816463545470907/11588006707200)\,x^{11}$
$-\,(74078609744831895304816687447/5518098432000)\,x^{10}$
$+\,(85271649858214871482393447550287/869100503040000)\,x^{9}$
$-\,(544996898468027472985563167444741/941525544960000)\,x^{8}$
$+\,(70149627027741059244452076836923/25677969408000)\,x^{7}$
$-\,(957972246454320768062382326886223/94152554496000)\,x^{6}$
$+\,(39105801276089863543963319222962243/1333827855360000)\,x^{5}$
$-\,(504262702012069575930635459206387/7939451520000)\,x^{4}$
$+\,(71182532364843711432355416393311/718331328000)\,x^{3}$
$-\,(36474259397560749064143053993/350859600)\,x^{2}$
$+\,(7533688390536925014719660137/116396280)\,x - 17731276931934494721$

$P_{23}\left(x\right) = (626047699915867799/4132355616829440000)\,x^{22}$
$-\,(32308010721489584677/851515702861824000)\,x^{21}$
$+\,(2511103473848270746499/561438924963840000)\,x^{20}$

$- (990899107137455767879/3003582726144000)\, x^{19}$
$+ (21862957755711217114748881/128047474114560000)\, x^{18}$
$- (255701609845973121847979/388022648832000)\, x^{17}$
$+ (119722559289831616224557863/6083703521280000)\, x^{16}$
$- (1227534164901745056978902207/2636271525888000)\, x^{15}$
$+ (100182659337689089936042566167/11298306539520000)\, x^{14}$
$- (2154242664974814138465530011/15692092416000)\, x^{13}$
$+ (3693244759563863089340159781247/2124467896320000)\, x^{12}$
$- (348679419716275796227603114639/19313344512000)\, x^{11}$
$+ (24313863183439316679673602226004513/158176291553280000)\, x^{10}$
$- (225569651273130061058657092854493/210901722071040)\, x^{9}$
$+ (187595874338792344136694981328643/31039303680000)\, x^{8}$
$- (3134444555753150961218198128157/114124308480)\, x^{7}$
$+ (46582581480572090585766685040140853/470762772480000)\, x^{6}$
$- (24628252584588755381459107318208291/88921857024000)\, x^{5}$
$+ (332629942234296028174359960597644779/568586874240000)\, x^{4}$
$- (73343429897691545576959449692313389/82129215168000)\, x^{3}$
$+ (741767456331546132245359568800763/806626220400)\, x^{2}$
$- (65620214252713224536332366223/116396280)\, x + 152553697445181546607$

$P_{24}(x) = (97841530496558649293/1723467782592331776000)\, x^{23}$
$- (335407509903912323681/21615398611107840000)\, x^{22}$
$+ (454501108157980384073/227070854096486400)\, x^{21}$
$- (107504314873384627683143/663518729502720000)\, x^{20}$
$+ (8983776081040688437620797/973160803270656000)\, x^{19}$
$- (50404505281634117870005549/128047474114560000)\, x^{18}$
$+ (1168666259954948513855852177/89633231880192000)\, x^{17}$
$- (2471024795530257064602656791/7189831434240000)\, x^{16}$
$+ (51511683265529304575508645967/7030057402368000)\, x^{15}$
$- (721535570132495990554385229809/5649153269760000)\, x^{14}$
$+ (30370138820999258718781973500777/16570849591296000)\, x^{13}$
$- (46160207442983253249607104925813/2124467896320000)\, x^{12}$
$+ (408425601328540276858577344312109/1917288382464000)\, x^{11}$
$- (1907340749270059838043121409492039/1106127912960000)\, x^{10}$
$+ (10084417668013132593085446666361763/878757175296000)\, x^{9}$
$- (898532173217897912413387346270287/14411105280000)\, x^{8}$
$+ (876114367621403488143688018877125577/3201186852864000)\, x^{7}$
$- (695775343904264963827651396107198739/727542466560000)\, x^{6}$
$+ (26751505111131816927397223600866889/10266151896000)\, x^{5}$
$- (79477104547597150222616625089859530191/14783258730240000)\, x^{4}$
$+ (403589896543329394729142860419921947/50190075936000)\, x^{3}$
$- (52511426579628409245398898318755171/6453009763200)\, x^{2}$
$+ (26325492596728861363119710723467/5354228880)\, x - 1315069260003198192788$

$P_{25}(x) = (1810237126686092753419/886354859618913484800000)\, x^{24}$
$- (62771250251033894473907/10340806695553990656000)\, x^{23}$
$+ (11511991109112409288411081/13488008733331292160000)\, x^{22}$
$- (308610477223222584935011/4087275373736755200)\, x^{21}$
$+ (1651947699182756157926795881/350337889177436160000)\, x^{20}$
$- (430292416333562341190138281/1946321606541312000)\, x^{19}$

$+ (1693062186931273696473576313/209532230369280000)\,x^{18}$
$- (2440242314982415470323718869/10342295986176000)\,x^{17}$
$+ (42496596088997036408436172662149/7592461994557440000)\,x^{16}$
$- (13805367401638213159287611947619/126541033242624000)\,x^{15}$
$+ (10501406884609102136813341938733721/5965505852866560000)\,x^{14}$
$- (260923334002982532857433489649357/11047233060864000)\,x^{13}$
$+ (3149961803456717214034882917658324207/11931011705733120000)\,x^{12}$
$- (311007895030981777959409995030791881/126541033242624000)\,x^{11}$
$+ (36072334629973139334722245407290958851/1898115498639360000)\,x^{10}$
$- (3842017793050978223673061999182697157/31635258310656000)\,x^{9}$
$+ (9526409805672264569123818314621843047/14966587883520000)\,x^{8}$
$- (17325638483889432743363938986730301171/6402373705728000)\,x^{7}$
$+ (5590963217656815416105678129476360121221/608225502044160000)\,x^{6}$
$- (4938656200253742430432451463654529907/202164221952000)\,x^{5}$
$+ (16036374727879322218852223186722596976589/325231692065280000)\,x^{4}$
$- (43579059908227428615914106531732753781/602280911232000)\,x^{3}$
$+ (3563764005018304979715944223158462749/49473074851200)\,x^{2}$
$- (114930909252025411165097498628379/2677114440)\,x + 11356590626799451081145$

Appendix 2: Coefficients c_i and K of the corrective function Φ_{25}

$c_1 = 0.72619047619047619047619047619047619047619048$
$c_2 = -12.737322640345465761875385564466378778531770512029$
$c_3 = -0.84164269999731510129175214910348073427241614034768$
$c_4 = -17.908330903689032603895485760334710528008413393816$
$c_5 = -21.793051968869308781494503792492773700351128057503$
$c_6 = 0.052225391819545299875787414478370692540986477002$
$c_7 = 22.867256875351930467366237953412010183478716371242$
$c_8 = -1.707611319924344890707228312842658677594869606318$
$c_9 = -1.2751632549654008209020770141453274057116453048774$
$c_{10} = 92.538545141732012060260360179516606702270761634364$
$c_{11} = 0.15088559448367433870497668805212911805725048971437$
$c_{12} = 12.296035579537995708862927240169729240327162730035$
$c_{13} = 0.47119594517891782795780492658781237198369447024440$
$c_{14} = 103.88603190326434306252340497028079649950656584049$
$c_{15} = 0.00206293819355223802469982625483902305678068096840$
$c_{16} = -187.13045591233740512137368315625509082987760574938$
$c_{17} = -0.27349669744110325871269828279318826062620906588370$
$c_{18} = 60.84802398138390071856628905328286526305777316700$
$c_{19} = 0.038078160763329325175635232207793882248458501401699$
$c_{20} = -375.10551758837650646142692040134675795455506670138$
$c_{21} = 0.12511512946150416165180535229285204189299499111780$
$c_{22} = 71.229908224807213192533908002847675276237637077608$
$K = -2.4774348845851811138845060795239470850970682780195$

Appendix 3: Coefficients d_i and M of the corrective function $|\psi_{n+1}|$

$d_1 = -3.6415421544257471159022604244391523071564117009437 \times 10^{-7}$
$d_2 = -3485608.6856471123162187714001336338709185526509134$
$d_3 = -4.6018631805033582153977197516014111986555627816497 \times 10^{-7}$

$d_4 = -294118428.933536077747125674202428669415681997544406$
$M = 13267560.4249039279284148206562948759455904314398 45$

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