SEPARATION COORDINATES, MODULI SPACES AND STASHEFF POLYTOPES

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ABSTRACT. We show that the orthogonal separation coordinates on the sphere S^n are naturally parametrised by the real version of the Deligne-Mumford-Knudsen moduli space $\bar{M}_{0,n+2}(\mathbb{R})$ of stable curves of genus zero with $n+2$ marked points. We use the combinatorics of Stasheff polytopes tessellating $\bar{M}_{0,n+2}(\mathbb{R})$ to classify the different canonical forms of separation coordinates and deduce an explicit construction of separation coordinates as well as of Stäckel systems from the mosaic operad structure on $\bar{M}_{0,n+2}(\mathbb{R})$.

1. INTRODUCTION

Separation of variables is one of the oldest techniques in mathematical physics, which still remains one of the most effective and powerful tools in the theory of integrable systems. Its quantum version initiated by Lamé [17] is actually more natural than the classical one developed approximately at the same time by Jacobi [12], when one has to consider the Hamilton-Jacobi equation rather than the equations of motion.

The general theory of separation coordinates goes back to Stäckel [27] and Levi-Civita [19] and was developed further by Eisenhart [10]. In the particular case of the sphere S^n, which we will be interested in, the first (and the most important) example, that of elliptic coordinates, was introduced already in 1859 by C. Neumann [23]. The classification problem of all separation coordinates on S^n had been studied in detail by Olevski [24] and Kalnins & Miller [13]. In particular, in the last paper to describe the answer the authors used a sophisticated graphical procedure similar to the one developed by Vilenkin to describe the polyspherical coordinates [30].

Despite all these advances almost nothing has been known about the global geometry of the space of separation coordinates. In the present paper we fill this gap by describing the topology and algebraic geometry of this space in the case of S^n. In particular, we link the graphical procedures from [13] and [30] with the rich combinatorial theory of *associahedra*, or *Stasheff polytopes*, introduced by Stasheff in 1963 in homotopy theory [28].

The reason this has never been done before is probably that the algebraic equations involved seemed far too complicated for a direct solution. Interestingly, Stäckel already commented on this in his habilitation thesis from 1891, where he notes [27, p. 6]:

> *Die Diskussion dieser Gleichungen ergab, dass es für $n = 2$ drei wesentlich verschiedene Formen der Gleichung $H = 0$ giebt, bei*

> *denen diese notwendigen Bedingungen erfüllt sind, und da diese*
> *Gleichungen auch wirklich Separation der Variabeln gestatten, ist*
> *die Frage für den Fall $n = 2$ vollständig erledigt. Aber schon für*
> *$n = 3$ werden die algebraischen Rechnungen so umständlich, dass*
> *mir eine weitere Verfolgung dieses Weges aussichtslos erschien.* [1]

In a sense, our goal in this paper is to accomplish Stäckel's computations for arbitrary n, using a substantial progress made in the theory of moduli spaces in the last few decades. More precisely, we prove the following result.

Main Theorem. *The Stäckel systems on S^n with diagonal algebraic curvature tensor form a smooth projective variety isomorphic to the real Deligne-Mumford-Knudsen moduli space $\bar{M}_{0,n+2}(\mathbb{R})$ of stable genus zero curves with $n + 2$ marked points.*

As a corollary we have that the set X_n of equivalence classes of separation coordinates on the sphere S^n modulo the orthogonal group is in one-to-one correspondence with the quotient space $Y_n = \bar{M}_{0,n+2}(\mathbb{R})/S_{n+1}$. Since the real version of the Deligne-Mumford-Knudsen moduli space $\bar{M}_{0,n+2}(\mathbb{R})$ is known to be tessellated by $(n+1)!/2$ copies of the Stasheff polytope K_n after Kapranov [15] and Devadoss [7], we can use the known results about $\bar{M}_{0,n+2}(\mathbb{R})$ and associahedra [8] to describe the combinatorial structure of X_n. In particular, we use the mosaic operad [7] to give an explicit construction for Stäckel systems and separation coordinates.

Note that in this way we establish (and exploit) a surprising correspondence between two seemingly completely unrelated objects – separation coordinates on a sphere on one hand and stable genus zero curves with marked points on the other hand – revealing yet another guise of the famous moduli space $\bar{M}_{0,n}(\mathbb{R})$.

The algebraic nature of the problem of separation of variables was explicitly revealed in [25], where the Nijenhuis integrability conditions for Killing tensors were reduced to purely algebraic equations for the associated algebraic curvature tensors. The statement of the Main Theorem stems from a thorough analysis of these equations in the first non-trivial case $n = 3$, done in [26]. Its proof is based on the recent work by Aguirre, Felder and the second author [1], where the moduli space $\bar{M}_{0,n+2}$ was identified with the set of the Gaudin subalgebras in the Kohno-Drinfeld Lie algebra $\mathfrak{t}_{n+1}$. We show that the Killing tensors on S^n with diagonal algebraic curvature tensor satisfy the defining relations of the Kohno-Drinfeld Lie algebra, which allows us to make the link with the main result of [1].

2. Separation coordinates, Killing tensors and Stäckel systems

2.1. **Separation coordinates.** Recall that the Hamilton-Jacobi equation

$$\tfrac{1}{2} g^{ij} \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j} = E$$

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[1] "For $n = 2$ the discussion of these equations yielded three essentially different forms of the equation $H = 0$ [the Hamilton-Jacobi equation], for which the necessary conditions are satisfied, and since these equations indeed allow a separation of variables, the question is completely settled in the case $n = 2$. However, already for $n = 3$ the algebraic computations become so cumbersome, that it seemed hopeless to me to pursue this approach further."

separates in local coordinates $x^1, \ldots, x^n$ on a Riemannian manifold M^n if it admits a solution of the form

$$W(x^1, \ldots, x^n; \underline{c}) = W_1(x^1; \underline{c}) + \ldots + W_n(x^n; \underline{c}), \qquad \det\left(\frac{\partial^2 W}{\partial x^i \partial c_j}\right) \neq 0,$$

depending on n parameters $\underline{c} = (c_1, \ldots, c_n)$. Note that if we reparametrise each coordinate x^i with a strictly monotonic function Φ_i, the Hamilton-Jacobi equation is still separable in the new coordinates $\Phi_i(x^i)$. The same is true for a permutation of the variables. In order to avoid this arbitrariness, we consider different coordinate systems as equivalent if they are related by such transformations. By abuse of language we will call a corresponding equivalence class simply *separation coordinates*. Equivalently, we can think of separation coordinates as the (unordered) system of coordinate hypersurfaces defined by the equations $x^i = $ constant. The separation coordinates are called *orthogonal*, if the normals of these hypersurfaces are mutually orthogonal.

The main tool in studying orthogonal separation coordinates are Killing tensors that satisfy a certain condition. The details of this relation will be explained in the rest of this section, with emphasis on spheres.

2.2. Killing tensors.

Definition 1. A *Killing tensor* on a Riemannian manifold (M, g) is an element $K \in \Gamma(S^2 T^* M)$ satisfying, in any coordinate system x^α ($\alpha = 1, \ldots, n$), the equation

$$(2.1) \qquad \nabla_\alpha K_{\beta\gamma} + \nabla_\beta K_{\gamma\alpha} + \nabla_\gamma K_{\alpha\beta} = 0,$$

where ∇ is the Levi-Civita connection of the metric g.

Note that the metric g is trivially a Killing tensor, because it is covariantly constant: $\nabla_\alpha g_{\beta\gamma} = 0$. Here we will be concerned with Killing tensors on the standard round sphere S^n, regarded as the hypersurface

$$S^n = \{x \in V \colon \|x\| = 1\} \subset V$$

of unit vectors in an $(n+1)$-dimensional Euclidean vector space V, equipped with the induced metric g.

Definition 2. An *algebraic curvature tensor* on a vector space V is an element $R \in (V^*)^{\otimes 4}$ satisfying the usual (algebraic) symmetries of a Riemannian curvature tensor, namely:

$(2.2a) \qquad R_{jikl} = -R_{ijkl} = R_{ijlk}$ \hspace{2em} (antisymmetry)

$(2.2b) \qquad R_{klij} = R_{ijkl}$ \hspace{2em} (pair symmetry)

$(2.2c) \qquad R_{ijkl} + R_{iklj} + R_{iljk} = 0$ \hspace{2em} (Bianchi identity).

The space of Killing tensors K on $S^n \subset V$ is naturally isomorphic to the space of algebraic curvature tensors R on V [21]. This isomorphism is explicitly given by the formula

$$(2.3) \quad K_x(v, w) := R(x, v, x, w) = \sum_{i,j,k,l=1}^{n+1} R_{ijkl} x^i x^k v^j w^l, \qquad x \in S^n, \quad v, w \in T_x S^n,$$

where we consider a point $x \in S^n$ as well as the tangent vectors $v, w \in T_x S^n$ as vectors in V satisfying $\|x\| = 1$ and $v, w \perp x$. The above isomorphism is equivariant under the natural actions of the isometry group $O(V)$ on Killing tensors and on algebraic curvature tensors respectively.

2.3. **Stäckel systems.** In Definition 1, a Killing tensor is a symmetric bilinear form $K_{\alpha\beta}$ on the manifold M. In what follows we will interpret it in two other ways, each of which gives rise to a Lie bracket and hence to a Lie algebra generated by Killing tensors. On one hand, we can use the metric to identify the symmetric bilinear form $K_{\alpha\beta}$ with a symmetric endomorphism $K^\alpha{}_\beta$. Interpreted in this way, the space of Killing tensors generates a Lie subalgebra of $\Gamma(\mathrm{End}(TM))$ with respect to the commutator bracket

$$(2.4) \qquad\qquad\qquad\qquad [K, L] = KL - LK.$$

On the other hand, we can interpret a Killing tensor $K_{\alpha\beta}$ as a function $K_{\alpha\beta}p^\alpha p^\beta$ on the total space of the cotangent bundle T^*M which is quadratic in the fibres. Interpreted in this way, the space of Killing tensors generates a Lie subalgebra of $C^\infty(T^*M)$ with respect to the Poisson bracket

$$(2.5) \qquad\qquad \{K, L\} = \sum_{\alpha=1}^n \left(\frac{\partial K}{\partial x^\alpha} \frac{\partial L}{\partial p_\alpha} - \frac{\partial L}{\partial x^\alpha} \frac{\partial K}{\partial p_\alpha} \right).$$

Definition 3. A *Stäckel system* on an n-dimensional Riemannian manifold is an n-dimensional space of Killing tensors which mutually commute with respect to both of the following brackets:

(1) the commutator bracket (2.4)
(2) the Poisson bracket (2.5)

Remark 1. In the initial definition given by Stäckel, condition (2) is replaced by an integrability condition on the eigen spaces of the Killing tensors [27]. The equivalence of both definitions is proven in [2].

It can be shown that every Stäckel system contains a Killing tensor with simple eigenvalues [2]. Moreover, the n distributions given by the orthogonal complements of its eigendirections are integrable. Hence they define n hypersurface foliations with orthogonal normals or, equivalently, orthogonal coordinates. On the other hand, every Killing tensor commuting with the above and having simple eigenvalues defines the same coordinates. In this manner each Stäckel system defines a unique coordinate system. It is a classical result that these are separation coordinates and that every system of orthogonal separation coordinates arises in this way from a Stäckel system:

Theorem 1 ([27, 10, 2])**.** *There is a bijective correspondence between Stäckel systems and orthogonal separation coordinates.*

Remark 2. A priori, the above result is only a local result. However, any local Killing tensor field on a sphere can be extended to a global one (see e. g. [4]). Hence the same is true for the corresponding separation coordinates. That is why we can use the above result for a global classification of orthogonal separation coordinates on S^n.

2.4. **Killing tensors with diagonal algebraic curvature tensor.** In the following we will only consider Killing tensors on S^n whose algebraic curvature tensor is diagonal in the following sense.

Definition 4. Due to the symmetries (2.2a) and (2.2b), we can interpret an algebraic curvature tensor R on V as a symmetric bilinear form on $\Lambda^2 V$. We say that R is *diagonal* in an orthonormal basis $\{e_i : 1 \leqslant i \leqslant n+1\}$ of V, if it is diagonal

as a bilinear form on $\Lambda^2 V$ in the associated basis $\{e_i \wedge e_j : 1 \leqslant i < j \leqslant n+1\}$. In components, this simply means that $R_{ijkl} = 0$ unless $\{i,j\} = \{k,l\}$.

Restricting to Killing tensors with diagonal algebraic curvature tensor does not mean any loss of generality. The reason is the following refinement of Theorem 1 for spheres.

Theorem 2 ([4])**.** *Necessary and sufficient conditions for the existence of an orthogonal separable coordinate system for the Hamilton-Jacobi equation on S^n are that there are n Killing tensors with diagonal algebraic curvature tensor, one of which is the metric, which are linearly independent (locally) and pairwise commute with respect to the Poisson bracket.*

Remark 3. By this theorem condition (2) in Definition 3 implies condition (1) on S^n. As we will show in Section 5 below, both conditions are actually equivalent for S^n.

The restriction to separation coordinates which are orthogonal does not constitute a loss of generality either, because of the following result.

Theorem 3 ([13])**.** *All separation coordinates on S^n are equivalent to orthogonal separation coordinates.*

The equivalence corresponds to a linear change of the so-called ignorable coordinates, on which the metric does not depend (see [13]). We will consider separation coordinates up to this equivalence throughout this paper.

Theorems 2 and 3 reduce the classification of separation coordinates on spheres to the purely algebraic problem of finding certain abelian subalgebras in the following two Lie algebras.

Definition 5. We denote by $\mathfrak{o}_{n+1} \subset \Gamma(\text{End}(TS^n))$ and by $\mathscr{D}_{n+1} \subset C^\infty(T^*S^n)$ the Lie subalgebras generated by Killing tensors with diagonal algebraic curvature tensor under the commutator bracket (2.4) and the Poisson bracket (2.5) respectively.

By definition, a diagonal algebraic curvature tensor R is uniquely determined by the diagonal elements R_{ijij} for $1 \leqslant i < j \leqslant n+1$. Indeed, the symmetries (2.2a) and (2.2b) determine the components $R_{ijij} = -R_{ijji} = R_{jiji} = -R_{jiij}$ for $i < j$. And if we set all other components to zero, the resulting tensor R satisfies all symmetries (2.2) of an algebraic curvature tensor. For fixed i and j, let K_{ij} be the Killing tensor with diagonal algebraic curvature tensor R given by

$$(2.6) \qquad R_{ijij} = -R_{ijji} = R_{jiji} = -R_{jiij} = 1$$

and all other components zero. Then K_{ij} with $i < j$ form a basis of the space of Killing tensors on S^n with diagonal algebraic curvature tensor and constitute a set of generators for both Lie algebras, $\mathfrak{o}_{n+1}$ and $\mathscr{D}_{n+1}$. The following two propositions show that they satisfy the same relations in $\mathfrak{o}_{n+1}$ and in $\mathscr{D}_{n+1}$.

Proposition 1. *Let K_{ij}, $1 \leqslant i < j \leqslant n+1$ be the basis of the space of Killing tensors on S^n with diagonal algebraic curvature tensor, as defined above. For convenience we set $K_{ji} := K_{ij}$. Then K_{ij} satisfy the following relations in $\mathfrak{o}_{n+1}$:*

$$(2.7a) \qquad [K_{ij}, K_{kl}] = 0 \qquad\qquad \text{if } i,j,k,l \text{ are distinct}$$
$$(2.7b) \qquad [K_{ij}, K_{ik} + K_{jk}] = 0 \qquad \text{if } i,j,k \text{ are distinct.}$$

Proof. We can extend the Killing tensor K on $T_x S^n$ to a symmetric tensor $\hat{K}$ on $V = T_x S^n \oplus \mathbb{R}x$ by omitting the restriction $v, w \perp x$ in (2.3). The antisymmetry (2.2a) implies that $K_x(v, x) = 0$ for any $v \in V$, so $\hat{K}$ is the extension of K by zero. Consequently, we have $[\hat{K}_{ij}, \hat{K}_{kl}] = [\widehat{K_{ij}, K_{kl}}]$, so that it is sufficient to check the above relations on the corresponding extensions. To do so, consider the Killing tensor K_{ij} at a point $x \in S^n$. By (2.3) and the definition (2.6) of the diagonal algebraic curvature tensor of K_{ij} we have

$$\hat{K}_{ij}(v, w) = \sum_{a,b=1}^{n+1} (R_{abab} x^a x^a v^b w^b + R_{abba} x^a x^b v^b w^a)$$
$$= x^i x^i v^j w^j + x^j x^j v^i w^i - x^i x^j v^i w^j - x^i x^j v^j w^i.$$

Let us put all indices down for convenience. Then we have

$$\hat{K}_{ij} = \begin{pmatrix} x_j^2 & -x_i x_j \\ -x_i x_j & x_i^2 \end{pmatrix},$$

where we left only non-zero (i-th and j-th) rows and columns. This already proves relation (2.7a). To check the remaining relation (2.7b), we compute

$$
\begin{aligned}
(2.8) \quad [\hat{K}_{ij}, \hat{K}_{jk}] &= \left[\begin{pmatrix} x_j^2 & -x_i x_j & 0 \\ -x_i x_j & x_i^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & x_k^2 & -x_j x_k \\ 0 & -x_j x_k & x_j^2 \end{pmatrix} \right] . \\
&= x_i x_j x_k \begin{pmatrix} 0 & -x_k & x_j \\ x_k & 0 & -x_i \\ -x_j & x_i & 0 \end{pmatrix}
\end{aligned}
$$

Here we omitted rows and columns other than i, j, k, because they are zero. In the same way we compute $[\hat{K}_{ij}, \hat{K}_{ik}]$ and verify (2.7b). $\qquad \square$

Proposition 2. *As elements of $\mathscr{D}_{n+1}$ the generators K_{ij} satisfy the following relations:*

$$(2.9a) \qquad \{K_{ij}, K_{kl}\} = 0 \qquad\qquad \text{if } i, j, k, l \text{ are distinct}$$
$$(2.9b) \qquad \{K_{ij}, K_{ik} + K_{jk}\} = 0 \qquad\qquad \text{if } i, j, k \text{ are distinct.}$$

Proof. The function on $T^* S^n$ given by K_{ij} is

$$(2.10a) \qquad K_{ij}(x, p) = x_j^2 p_i^2 + x_i^2 p_j^2 - 2 x_i x_j p_i p_j = (x_i p_j - x_j p_i)^2.$$

This already proves relation (2.9a). In order to verify relation (2.9b), we compute

$$
\begin{aligned}
(2.10b) \qquad \{K_{ij}, K_{jk}\} &= \frac{\partial K_{ij}}{\partial x_j} \frac{\partial K_{jk}}{\partial p_j} - \frac{\partial K_{ij}}{\partial p_j} \frac{\partial K_{jk}}{\partial x_j} \\
&= 4(x_i p_j - x_j p_i)(x_j p_k - x_k p_j)(x_k p_i - x_i p_k),
\end{aligned}
$$

which is clearly antisymmetric with respect to i and j. $\qquad \square$

The next proposition says that there are no more relations between the generators and their brackets, both in $\mathfrak{d}_{n+1}$ as well as in $\mathscr{D}_{n+1}$, provided $n \geqslant 2$.

Proposition 3. *The generators K_{ij} and the commutator brackets $[K_{ij}, K_{jk}]$ with $1 \leqslant i < j < k \leqslant n+1$ are linearly independent in $\mathfrak{d}_{n+1}$ for $n \geqslant 2$. The same is true for K_{ij} and the Poisson brackets $\{K_{ij}, K_{jk}\}$ as elements of $\mathscr{D}_{n+1}$.*

Proof. Since K_{ij} are symmetric and $[K_{ij}, K_{jk}]$ are antisymmetric, it suffices to check the linear independence of both sets independently. The elements K_{ij} are linearly independent by definition. To prove the linear independence of $[K_{ij}, K_{jk}]$ suppose that

$$\sum_{1 \leqslant i < j < k \leqslant n+1} \lambda_{ijk}[K_{ij}, K_{jk}] = 0.$$

For each triple (p, q, r) with $1 \leqslant p < q < r \leqslant n+1$ consider a point $x \in S^n$ with $x_m = 0$ if and only if $m \notin \{p, q, r\}$. Due to (2.8) we have that $[K_{ij}, K_{jk}] = 0$ at this point x for all $i < j < k$ unless $(i, j, k) = (p, q, r)$. Hence all $\lambda_{pqr} = 0$.

In the case of $\mathscr{D}_{n+1}$ we note that K_{ij} are quadratic in momenta while $\{K_{ij}, K_{jk}\}$ are cubic; the rest of the proof is the same. $\square$

2.5. **Stäckel systems generated from special conformal Killing tensors.**
The generic Stäckel systems on a sphere can be constructed from special conformal Killing tensors.

Definition 6. A *special conformal Killing tensor* on a Riemannian manifold is a symmetric tensor $L_{\alpha\beta}$ satisfying

$$\nabla_\gamma L_{\alpha\beta} = \lambda_\alpha g_{\beta\gamma} + \lambda_\beta g_{\alpha\gamma} \qquad\qquad \lambda = \tfrac{1}{2}\nabla \operatorname{tr} L.$$

The space of special conformal Killing tensors parametrises geodesically equivalent metrics, which are metrics having the same set of unparametrised geodesics. Their importance in our context stems from the fact that

$$K := L - (\operatorname{tr} L)g$$

defines a Killing tensor, as one immediately checks, and the fact that every Killing tensor of this form is contained in a Stäckel system. The latter follows easily from the Nijenhuis integrability conditions applied to K (see for example [26]). Thus, in the generic case where L (and hence K) has pairwise different eigenvalues, it defines a system of separation coordinates. The corresponding Stäckel system is spanned by the coefficients of the polynomial

(2.11) $$\operatorname{Adj}(L - \lambda \operatorname{Id}) = \sum_{i=0}^{n-1} K_i \lambda^i,$$

where $\operatorname{Adj} X$ denotes the adjugate matrix, i.e. the transpose of the cofactor matrix of X [3].

We can deduce the corresponding separation coordinates directly from the special conformal Killing tensor L, since its eigenvalues are constant on the corresponding coordinate hypersurfaces [5]. This means that the eigenvalues of L can be taken as coordinate functions. On $S^n \subset V$ the situation is further simplified. The reason is that under certain conditions, which are met in this case, every special conformal Killing tensor L on M is the restriction of a covariantly constant symmetric tensor $\hat{L}$ on the metric cone over M and vice versa (see for example [20]). Here the metric cone over $S^n \subset V$ is nothing but V, so the determination of separation coordinates on $S^n \subset V$ arising from a special conformal Killing tensor reduces to computing the eigenvalues of the restriction L of a constant symmetric tensor $\hat{L}$ on V.

2.6. **Two extremal cases: elliptic and polyspherical coordinates.** As a matter of example, let us consider two extremal cases. The generic case of orthogonal separation coordinates on the sphere consists of elliptic coordinates and can be obtained from a special conformal Killing tensor L with simple eigenvalues as described above. For $\hat{L}$ with (constant) eigenvalues $\Lambda_1 < \Lambda_2 < \ldots < \Lambda_{n+1}$ the eigenvalues $\lambda_1(x), \ldots, \lambda_n(x)$ of L at a point $x = (x_1, \ldots, x_{n+1}) \in S^n$ are the solutions of the equation

$$(2.12) \qquad \sum_{k=1}^{n+1} \frac{x_k^2}{\Lambda_k - \lambda} = 0 \qquad \|x\|^2 = 1$$

which can be ordered to satisfy

$$\Lambda_1 < \lambda_1(x) < \Lambda_2 < \lambda_2(x) < \cdots < \lambda_n(x) < \Lambda_{n+1}.$$

This is nothing else but the defining equation for the classical *elliptic coordinates* on the sphere S^n introduced in 1859 by C. Neumann [23]. Note that shifting or multiplying the parameters $\Lambda_1 < \Lambda_2 < \ldots < \Lambda_{n+1}$ by a constant results in a mere reparametrisation of the same coordinate system. Therefore elliptic coordinates form an $(n-1)$-parameter family of separation coordinates on S^n.

The other extreme, having no continuous parameters at all, are *polyspherical coordinates* considered by Vilenkin [30, 31]. Each of these coordinate systems is given in terms of Cartesian coordinates by starting with $\boldsymbol{x}(\varnothing) := 1$ on $S^0 \subset \mathbb{R}^1$ and then defining recursively $\boldsymbol{z} = \boldsymbol{z}(\varphi_1, \ldots, \varphi_{n-1})$ on $S^{n-1} \subset \mathbb{R}^n$ from $\boldsymbol{x} = \boldsymbol{x}(\varphi_1, \ldots, \varphi_{n_1-1})$ on $S^{n_1-1} \subset \mathbb{R}^{n_1}$ and $\boldsymbol{y} = \boldsymbol{y}(\varphi_{n_1}, \ldots, \varphi_{n_1+n_2-2})$ on $S^{n_2-1} \subset \mathbb{R}^{n_2}$ by setting

$$(2.13) \qquad \boldsymbol{z} = (\boldsymbol{x}\cos\varphi_{n-1}, \boldsymbol{y}\sin\varphi_{n-1})$$

for $n = n_1 + n_2$. Since this involves a choice of a splitting $n = n_1 + n_2$ in each step, polyspherical coordinates on S^{n-1} are parametrised by planar rooted binary trees with n leaves.[2] For example, the standard spherical coordinates correspond to the binary tree where each right child is a leaf.

2.7. **The residual action of the isometry group.** Theorem 2 implies that for a Stäckel system on a sphere we can always find an isometry which takes all Killing tensors in this Stäckel system to Killing tensors having diagonal algebraic curvature tensors. This means that the space of Killing tensors with diagonal algebraic curvature tensor defines a slice for the action of the isometry group. If we want to classify separation coordinates up to isometries, we have to take into account that the stabiliser of this slice in the isometry group is not trivial.

Due to the symmetries (2.2a) and (2.2b), the space of algebraic curvature tensors is a subspace of the space $S^2\Lambda^2 V$ of symmetric forms on $\Lambda^2 V$. The natural action of the isometry group $O(V)$ on this space is given as follows. Mapping an orthonormal basis $\{e_i : 1 \leqslant i \leqslant n+1\}$ of V to the basis $\{(e_i \wedge e_j)/\sqrt{2} : 1 \leqslant i < j \leqslant n+1\}$ of $\Lambda^2 V$ defines a map

$$(2.14) \qquad\qquad O(V) \to O(\Lambda^2 V),$$

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[2]When Vilenkin introduced polyspherical coordinates in [30], he used trees which are not binary. The description with binary trees appeared in Vilenkin and Klimyk [31, Chap. 10.5] and both are completely equivalent.

since the latter basis is orthonormal with respect to the scalar product on $\Lambda^2 V$ induced from the one on V. Via the action of $O(\Lambda^2 V)$ on $\Lambda^2 V$ this defines an action of $O(V)$ on $S^2 \Lambda^2 V$ and hence on algebraic curvature tensors.

In general, the subgroup in $O(V)$ leaving the space of diagonal bilinear forms on V invariant is the subgroup of signed permutation matrices, acting by permutations and sign changes of the chosen basis $\{e_i\}$ in V. This group is the symmetry group of the hyperoctahedron in V with vertices $\pm e_i$ and is isomorphic to the semidirect product $S_N \ltimes \mathbb{Z}_2^N$, where $N = \dim V = n + 1$.

The stabiliser in $O(V)$ of the space of diagonal algebraic curvature tensors is now the preimage under (2.14) of the stabiliser of diagonal bilinear forms on $\Lambda^2 V$. Since the latter consists of permutations and sign changes of the basis $\{e_i \wedge e_j\}$ in $\Lambda^2 V$, this is just the group of permutations and sign changes of the basis elements e_i, i.e. the group described in the preceding paragraph. Note that the normal subgroup of sign changes, which is isomorphic to $\mathbb{Z}_2^N$, acts trivially on diagonal bilinear forms. Hence the action descends to the quotient $(S_N \ltimes \mathbb{Z}_2^N)/\mathbb{Z}_2^N \cong S_N$. Summarising the above, we have:

Proposition 4. *The stabiliser in the isometry group* $O(V)$ *of the space of Killing tensors on* $S^n \subset V$ *with diagonal algebraic curvature tensor is the hyperoctahedral group and isomorphic to the semidirect product* $S_N \ltimes \mathbb{Z}_2^N$, *where* $N = n + 1$. *This action descends to an action of* S_N *given by*

$$(2.15) \qquad \sigma(K_{ij}) = K_{\sigma(i)\sigma(j)}, \qquad \sigma \in S_N.$$

3. Gaudin subalgebras of the Kohno-Drinfeld Lie algebra and the moduli space $\bar{M}_{0,n+1}$.

We describe now the result of [1], which plays a key role for us. The (real version of) the *Kohno-Drinfeld Lie algebra* $\mathfrak{t}_n$ $(n = 2, 3, \dots)$ is defined as the quotient of the free Lie algebra over $\mathbb{R}$ with generators $t_{ij} = t_{ji}$, $i \neq j \in \{1, \dots, n\}$, by the ideal generated by the relations

$$(3.1) \qquad [t_{ij}, t_{kl}] \;=\; 0, \qquad \text{if } i, j, k, l \text{ are distinct,}$$
$$(3.2) \qquad [t_{ij}, t_{ik} + t_{jk}] \;=\; 0, \qquad \text{if } i, j, k \text{ are distinct.}$$

This Lie algebra appeared in Kohno's work as the holonomy Lie algebra of the complement to the union of the diagonals $z_i = z_j$, $i < j$ in $\mathbb{C}^n$ (which is also a configuration space of n distinct points on the complex plane) and in Drinfeld's work as the value space of the universal Knizhnik-Zamolodchikov connection (see the references in [1]).

Gaudin subalgebras of Kohno-Drinfeld Lie algebras were introduced in [1] as the abelian Lie subalgebras of maximal dimension contained in the linear span $\mathfrak{t}_n^1$ of the generators t_{ij}. The main class of examples is provided by Gaudin's models of integrable spin chains

$$(3.3) \qquad \mathfrak{g}_n(z) = \left\{ \sum_{1 \leqslant i < j \leqslant n} \frac{b_i - b_j}{z_i - z_j} \, t_{ij}, \quad b \in \mathbb{R}^n \right\}.$$

Note that they are parametrised by $z \in \Sigma_n/\mathrm{Aff}$, where

$$\Sigma_n = \mathbb{R}^n \smallsetminus \bigcup_{i<j} \{z \in \mathbb{R}^n \,|\, z_i = z_j\}$$

is the configuration space of n distinct ordered points on the real line and Aff is the group of affine maps $z \mapsto az + b$, $a \neq 0$, acting diagonally on $\mathbb{R}^n$. A different type of example, which came from the representation theory of the symmetric group, is given by the *Jucys–Murphy subalgebras* spanned by

(3.4) $t_{12}, \quad t_{13} + t_{23}, \quad t_{14} + t_{24} + t_{34}, \quad \ldots$

(see [29] and references therein).

The main result of [1] is the following.

Theorem 4. [1] *Gaudin subalgebras in $\mathfrak{t}_n$ form a nonsingular algebraic subvariety of the Grassmannian $G(n-1, n(n-1)/2)$ of $(n-1)$-dimensional subspaces in $\mathfrak{t}_n^1$, isomorphic to the moduli space $\bar{M}_{0,n+1}$ of stable curves of genus zero with $n+1$ marked points.*

In fact, the result holds for any quotient of $\mathfrak{t}_n$ where both the generators t_{ij}, $1 \leqslant i < j \leqslant n$, and the brackets $[t_{ij}, t_{jk}]$, $1 \leqslant i < j < k \leqslant n$, are linearly independent (see remark 2.6 in [1]), and over any field.

The most popular version of the moduli space $\bar{M}_{0,n+1}$ – appearing, for example, in the celebrated Witten's conjecture – is defined over $\mathbb{C}$. It is a particular (Deligne-Mumford) compactification of the configuration space $M_{0,n+1}(\mathbb{C})$ of $n+1$ distinct labelled points in $\mathbb{C}P^1$ modulo $PGL_2(\mathbb{C})$ studied by Knudsen [16], who proved that it is a smooth projective variety. The compactification $\bar{M}_{0,n+1}(\mathbb{C})$ includes the singular rational curves with double point singularities and with the following properties: the graph of components is a tree (genus zero) and each irreducible component contains at least three marked or singular points (stability condition).

However, we need the real version $\bar{M}_{0,n+1}(\mathbb{R})$, which we discuss next.

4. The real version $\bar{M}_{0,n+1}(\mathbb{R})$ and Stasheff polytopes

4.1. **Topology.** The real version $\bar{M}_{0,n+1}(\mathbb{R})$ was studied in more detail by Kapranov [15] and Devadoss [7]. By Knudsen's theorem, which works over $\mathbb{R}$ as well, $\bar{M}_{0,n+1}(\mathbb{R})$ is a smooth real manifold of dimension $n-2$. It can be described as an iterated blow-up of $\mathbb{R}P^{n-2}$ [15, 7, 6]. $\bar{M}_{0,4}(\mathbb{R})$ is simply $\mathbb{R}P^1$ and $\bar{M}_{0,5}(\mathbb{R})$ is a non-orientable surface with Euler characteristic -3, which is a connected sum of five copies of $\mathbb{R}P^2$.

The topology of $\bar{M}_{0,n+1}(\mathbb{R})$ becomes increasingly complicated when n grows. It is known to be aspherical (Davis et al. [6]). The Euler characteristic can be given explicitly by

$$\chi\big(\bar{M}_{0,n+1}(\mathbb{R})\big) = (-1)^{\frac{n-2}{2}}(n-1)!!(n-3)!!$$

for even n (and zero for odd n), see [7]. A description of the cohomology is more complicated than in the complex case, as found by Etingof et al. in [11].

4.2. **Combinatorics.** Fortunately, a lot of information about $\bar{M}_{0,n+1}(\mathbb{R})$ is encapsulated in a well studied remarkable polytope known as *associahedron*, or *Stasheff polytope K_n*. Namely, $\bar{M}_{0,n+1}(\mathbb{R})$ is tessellated by $n!/2$ copies of K_n, see [15, 7].

K_n was first described by Stasheff as a combinatorial object in the homotopy theory of H-spaces [28] (see the history of this in Stasheff's contribution to [22]). Its first realisation as a convex polytope is usually ascribed to Milnor. By now we have several geometric realisations of Stasheff polytopes, see e.g. [9] and references therein. K_n is a convex polytope of dimension $n-2$: K_3 is a segment, K_4 is

a pentagon and K_5 is the polyhedron shown in Figure 1, which can be obtained combinatorially by cutting off three skew edges from a cube.
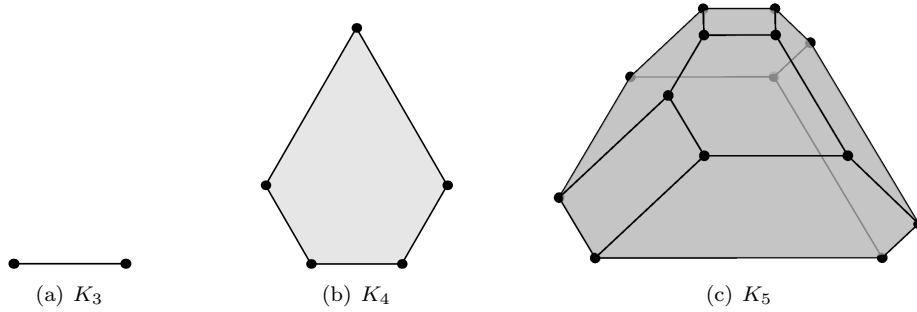

(a) K_3 (b) K_4 (c) K_5

FIGURE 1. Stasheff polytopes.

The faces of K_n of codimension d are in one-to-one correspondence with dissections of a based $(n+1)$-gon by d non-intersecting diagonals (see e.g. [8]). In particular, the vertices of K_n correspond to the triangulations of the $(n+1)$-gon by non-intersecting diagonals and their number is C_{n-1}, where

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

is the *Catalan number*.

Alternatively, the faces of K_n can be labelled by non-isomorphic planar rooted trees with n leaves (see e.g. [7]). These are simply the dual graphs of the dissected polygons, cut off at the edges of the polygon. In particular, the vertices of K_n correspond to binary rooted trees. For $n = 4$ this is depicted in Figure 2.
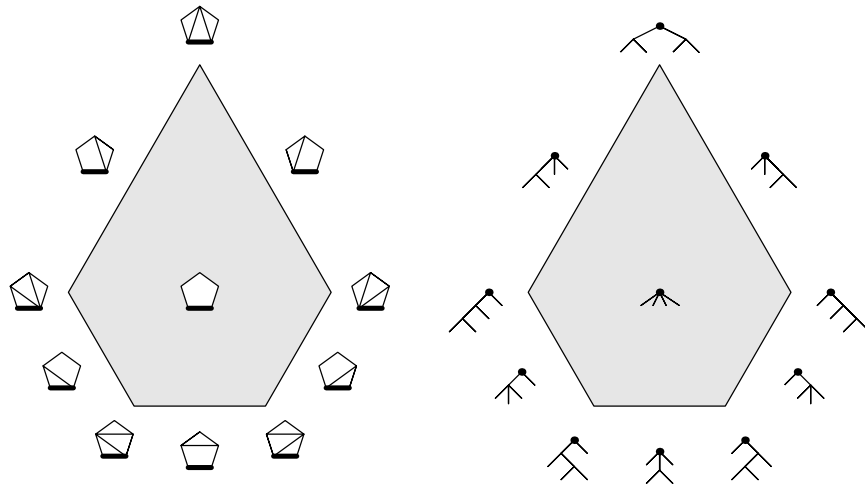

FIGURE 2. Labellings of K_4 by dissections of a based pentagon (left) and planar rooted trees with four leaves (right).

The Stasheff polytope K_n admits a realisation with the dihedral symmetry D_{n+1}, which is the symmetry group of a regular $(n+1)$-gon [18].

4.3. **Operad structure.** The sequence of moduli spaces $\bar{M}_{0,n+1}(\mathbb{R})$ for $n = 1, 2, \ldots$ carries a natural operad structure, called the "mosaic operad" [7].

Definition 7. An *operad structure* on a sequence of objects $\mathcal{O}(n)$ is a composition map

$$\circ: \quad \mathcal{O}(k) \times \mathcal{O}(n_1) \times \cdots \times \mathcal{O}(n_k) \quad \longrightarrow \quad \mathcal{O}(n_1 + \cdots + n_k)$$
$$(y, x_1, \ldots, x_k) \quad \mapsto \quad y \circ (x_1, \ldots, x_k)$$

together with a right action

$$\star: \quad \mathcal{O}(n) \times S_n \quad \longrightarrow \quad \mathcal{O}(n)$$
$$(x, \pi) \quad \mapsto \quad x \star \pi$$

of the permutation group S_n on each object $\mathcal{O}(n)$, satisfying the following axioms:

Identity: There is a distinguished element $1 \in \mathcal{O}(1)$ satisfying

$$y \circ (1, \ldots, 1) = y = 1 \circ y.$$

Associativity:

$$z \circ (y_1 \circ x_1, \ldots, y_k \circ x_k) = \big(z \circ (y_1, \ldots, y_k)\big) \circ (x_1, \ldots, x_k),$$

Equivariance:

$$(y \star \pi) \circ (x_1, \ldots, x_k) = y \circ \big((x_1, \ldots, x_k) \star \pi\big)$$
$$y \circ (x_1 \star \pi_1, \ldots, x_k \star \pi_k) = \big(y \circ (x_1, \ldots, x_k)\big) \star (\pi_1, \ldots, \pi_k),$$

where S_k acts on $(x_1, \ldots, x_k)$ by permutation and $(\pi_1, \ldots, \pi_k)$ on $\mathcal{O}(n_1 + \cdots + n_k)$ under the inclusion $S_{n_1} \times \cdots \times S_{n_k} \hookrightarrow S_{n_1 + \cdots + n_k}$.

In terms of dissected polygons, the operad structure on $\mathcal{O}(n) := \bar{M}_{0,n+1}(\mathbb{R})$ is given by gluing the k $(n_i + 1)$-gons x_i with their base to the k non-base edges of the $(k+1)$-gon y to form the $(n_1 + \ldots + n_k + 1)$-gon $y \circ (x_1, \ldots, x_k)$ with the base of y as base. If y is dissected by d_0 diagonals and x_i by d_i diagonals, then $y \circ (x_1, \ldots, x_k)$ is dissected by $d_0 + d_1 + \cdots d_k + k$ diagonals, namely the diagonals of y and $x_1, \ldots, x_k$ plus the k glued pairs of edges which become diagonals after gluing. On planar trees, the composition $y \circ (x_1, \ldots, x_k)$ is given by grafting the k trees x_i with their root to the leaves of the tree y.

The operad structure on Stasheff polytopes defines a map

$$\circ: K_k \times K_{n_1} \times \cdots \times K_{n_k} \hookrightarrow K_{n_1 + \cdots + n_k}$$

whose image is a codimension k face of $K_{n_1 + \cdots + n_k}$. This yields a decomposition of the faces of a Stasheff polytope into products of Stasheff polytopes [28].

5. The correspondence

After these preparations we are now in a position to state our main result. By Propositions 1 and 2 the defining relations of the Kohno-Drinfeld Lie algebra $\mathfrak{t}_n$ are satisfied in the Lie algebras $\mathfrak{d}_n$ and $\mathscr{D}_n$ (c.f. Definition 5). This provides surjective Lie algebra morphisms

(5.1) $$\mathfrak{t}_n \longrightarrow \mathfrak{d}_n \qquad\qquad \mathfrak{t}_n \longrightarrow \mathscr{D}_n,$$

given by mapping the generator t_{ij} to the Killing tensor K_{ij}. Under these morphisms the linear span $\mathfrak{t}_n^1$ of the t_{ij} is isomorphic to the space of Killing tensors – interpreted as endomorphisms in $\mathfrak{d}_n$ respectively as quadratic functions on the cotangent bundle in $\mathscr{D}_n$. Thus the above morphisms map Gaudin subalgebras in

the Kohno-Drinfeld Lie algebra $\mathfrak{t}_n$ to Stäckel systems on S^{n-1} with diagonal algebraic curvature tensor. Proposition 3 now shows that this defines an isomorphism between Gaudin subalgebras and Stäckel systems. Now using Theorem 4 we have the following correspondence:

Theorem 5. *The Stäckel systems on S^n with diagonal algebraic curvature tensor form a nonsingular algebraic subvariety of the Grassmannian $G(n, n(n+1)/2)$ of n-planes in the space of Killing tensors with diagonal algebraic curvature tensor, which is isomorphic to the real Deligne-Mumford-Knudsen moduli space $\bar{M}_{0,n+2}(\mathbb{R})$ of stable genus zero curves with $n+2$ marked points.*

Note that since $\bar{M}_{0,n+2}(\mathbb{R})$ can be considered as a compactification of the configuration space $\Sigma_{n+1}/\mathrm{Aff}$ of $n+1$ ordered distinct points on a real line modulo the affine group, we have a natural action of the symmetric group S_{n+1} on $\bar{M}_{0,n+2}(\mathbb{R})$.

Corollary 1. *The space X_n of equivalence classes of orthogonal separation coordinates on the sphere S^n modulo the orthogonal group $O(n+1)$ is naturally homeomorphic to the quotient space $Y_n = \bar{M}_{0,n+2}(\mathbb{R})/S_{n+1}$.*

Proof. This follows directly from Theorems 1 and 5. It suffices to note that by Proposition 4 the morphisms (5.1) are equivariant with respect to the S_{n+1}-action on $\mathfrak{t}_{n+1}$ and $\mathfrak{d}_{n+1}$ respectively $\mathscr{D}_{n+1}$. Therefore the isomorphism in Theorem 5 is S_{n+1}-equivariant, so that the corresponding quotients are homeomorphic. Note that while the space of Stäckel systems on $S^n \subset V$ with diagonal algebraic curvature tensor depends on the choice of an orthonormal basis in the ambient space V (for which the algebraic curvature tensors are diagonal), the quotient does not. Therefore the homeomorphism is natural. $\square$

Since $\bar{M}_{0,n+2}(\mathbb{R})$ is tessellated by $(n+1)!/2$ copies of the Stasheff polytope K_{n+1}, we can use it to describe the quotient. The interior of K_{n+1} corresponds to the classical elliptic coordinates (2.12) on the sphere S^n. The $n+1$ distinct real parameters $(\Lambda_1, \ldots, \Lambda_{n+1}) \in \Sigma_{n+1}$ they depend on are the eigenvalues of the symmetric tensor $\hat{L}$ on $\mathbb{R}^{n+1}$ which restricts to the corresponding special conformal Killing tensor L on S^n. Shifting or scaling them only reparametrises the corresponding coordinates. Hence the actual parameter space is the quotient $\Sigma_{n+1}/\mathrm{Aff}$, which is nothing else but the open moduli space $M_{0,n+2}(\mathbb{R})$. Thus we have the following important

Corollary 2. *All orthogonal separation coordinates on S^n belong to the closure of the Neumann family of elliptic coordinates. The possible degenerations of the Neumann family correspond to the faces of the Stasheff polytope K_{n+1}.*

The first part is probably not surprising for the experts (see the similar claim in the complex case in [14]), but we have not seen it explicitly stated and proved in the literature. In Section 6.2 we will show that rather than by actually performing the limiting process explicitly (as in [14]), the limiting cases can be better understood by composing generic separation coordinates (that is elliptic coordinates) of lower dimensions under the operad composition. The same holds true for the corresponding Stäckel systems.

Because we have $(n+1)!/2$ Stasheff polytopes K_{n+1} tiling $\bar{M}_{0,n+2}(\mathbb{R})$, the quotient $Y_n = \bar{M}_{0,n+2}(\mathbb{R})/S_{n+1}$ is actually only "a half" of K_{n+1} with some identification between the faces. In the interior of the polytope the identification is given by the action of $\mathbb{Z}_2 \subset D_{n+2}$, corresponding to a reflection in the dihedral

group, realised as an isometry of K_{n+1} (see above). If we are using the blow-up description of $\bar{M}_{0,n+2}(\mathbb{R})$ [15, 7, 6], then it corresponds to the longest element $(1, 2, \ldots, n, n+1) \mapsto (n+1, n, \ldots, 2, 1)$ in the symmetric group S_{n+1}, mapping the A_n Weyl chamber into the opposite one. For K_4 this is just a reflection symmetry of the pentagon as indicated in Figure 4.

The identification of the faces is more sophisticated. Probably the best way to describe them is using the "twisting along the diagonal" operation for the dissected $(n+1)$-gon introduced in [7, 8]. On planar trees this corresponds to reversing the ordering of a node's children. Trees which are equivalent under this operation are called "dyslexic".

As a matter of illustration, let us consider the least non-trivial example $n = 2$, depicted in Figure 3. The parameter space $\bar{M}_{0,4}(\mathbb{R}) \cong \mathbb{R}P^1$ is just a circle,
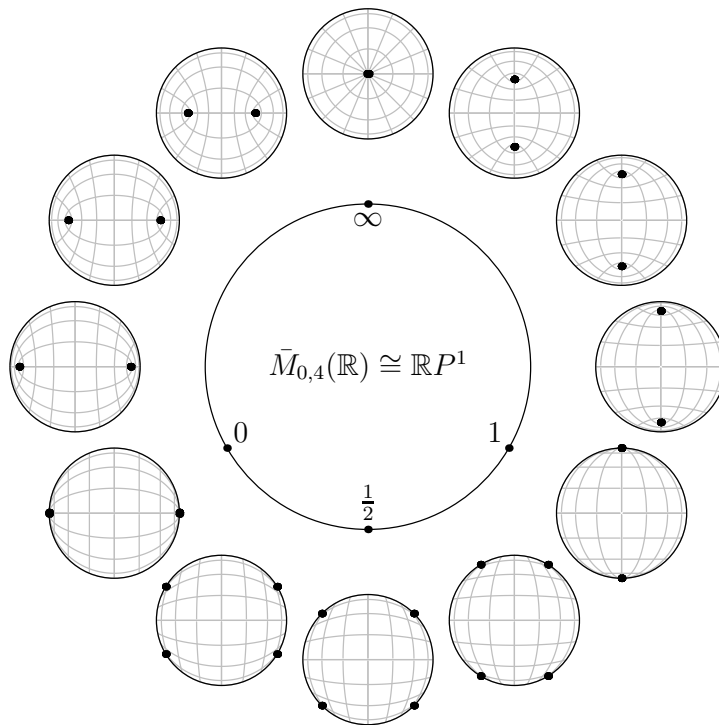

FIGURE 3. Orthogonal separation coordinates on S^2 parametrised by $\bar{M}_{0,4}(\mathbb{R}) \cong \mathbb{R}P^1$.

parametrised by the affine-invariant cross ratio $\tau = \frac{\Lambda_2 - \Lambda_1}{\Lambda_3 - \Lambda_1}$. It is tessellated by three copies of the Stasheff polytope K_3, the tiles being the arcs between the points $\tau = 0, 1, \infty$. We can identify K_3 with the arc $[0, 1]$, for which $\Lambda_1 < \Lambda_2 < \Lambda_3$. The symmetry group S_3 acts by permuting Λ_1, Λ_2 and Λ_3 and hence the points 0, 1 and ∞. The quotient $Y_2 = \bar{M}_{0,4}(\mathbb{R})/S_3$ can be identified with the arc $[0, \frac{1}{2}]$. This is "a half" of K_3, where $\tau = 0$ corresponds to spherical coordinates and $\tau = 1/2$ to the "lemniscatic" case, which is just a particular case of elliptic coordinates when $\Lambda_2 = (\Lambda_1 + \Lambda_3)/2$. In this example there is no identification in place, because the dimension is too low.

6. Applications

6.1. **Enumerating separation coordinates.** For a Stasheff polytope, the number of non-equivalent faces of a given dimension can be given by the following Devadoss-Read formula [8]. Let $A(x,y) = \sum a_{mn} x^m y^n$ be the formal series solution of the functional equation

$$(6.1) \qquad A(x,y) = y + \frac{1}{2}\left(\frac{A(x,y)^2}{1 - A(x,y)} + \frac{(1 + A(x,y))A(x^2,y^2)}{1 - A(x^2,y^2)}\right).$$

There is no closed formula for $A(x,y)$, but one uses Equation (6.1) to find the coefficients a_{mn} recursively. The claim is that the coefficient a_{mn} is the number of non-equivalent faces of K_n of codimension $m - 1$. Devadoss and Read proved this using a combinatorial technique going back to Pólya [8].

Using Table 2 from [8], we get the number of non-equivalent canonical forms of separation coordinates on S^n for $n \leqslant 10$, as listed in Table 1. The 1's on the

TABLE 1. Number of canonical forms for separation coordinates on S^n, ordered by increasing number of independent continuous parameters.

	0	1	2	3	4	5	6	7	8	9	total
S^2	1	1									2
S^3	2	3	1								6
S^4	3	8	5	1							17
S^5	6	20	22	8	1						57
S^6	11	49	73	46	11	1					191
S^7	23	119	233	206	87	15	1				684
S^8	46	288	689	807	485	147	19	1			2482
S^9	98	696	1988	2891	2320	1021	236	24	1		9275
S^{10}	207	1681	5561	9737	9800	5795	1960	356	29	1	35127

diagonal correspond to elliptic coordinates, the numbers in the first column to polyspherical coordinates. Note that the sequence $1, 2, 3, 6, 11, 23, \ldots$ is the sequence of Wedderburn-Etherington numbers, i.e. the number of non-planar binary rooted trees with $n + 1$ leaves [32]. This reflects the fact that we can parametrise polyspherical coordinates by planar binary rooted trees and that the notions *dyslexic* and *non-planar* coincide for binary trees.

In the first row we have 1 and 1, corresponding to spherical and elliptic coordinates on S^2 respectively, as discussed in Section 5. The numbers in the second row – 2, 3, 1 – are in perfect agreement with the results of Eisenhart [10], Olevski [24] and Kalnins & Miller [13]. They correspond to spherical and cylindrical coordinates (2), two types of Lamé rotational coordinates plus Lamé subgroup reduction (3) and elliptic coordinates (1) respectively. Their identification with the faces of the Stasheff polytope K_4 is indicated in Figure 4, in comparison to the different labellings shown in Figure 2. Polyspherical coordinates comprise, for example, the usual spherical coordinates plus cylindrical coordinates. Note that "the half" of K_4, obtained as the quotient under the reflection symmetry indicated in Figure 4, is a quadrilateral and that for the quotient space $Y_3 = \bar{M}_{0,5}(\mathbb{R})/S_4$ we have to identify
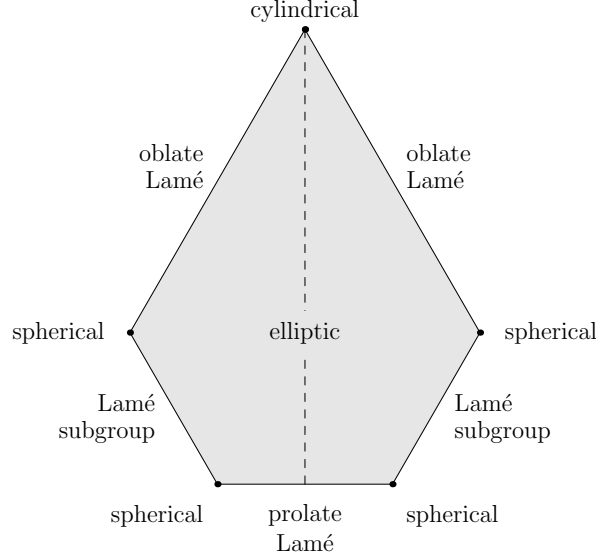
FIGURE 4. The Stasheff polytope K_4, labelled by separation coordinates on S^3.

its two adjacent vertices that are labelled by spherical coordinates and joined by Lamé subgroup reduction.

In accordance with our description, the total number of canonical forms for separation coordinates on S^n, indicated in the last column in Table 1, is the number of dyslexic planar rooted trees with $n + 1$ leaves [33].

6.2. **Constructing separation coordinates via the mosaic operad.** The correspondence in Theorem 5 transfers the natural operad structure on $\bar{M}_{0,n+2}(\mathbb{R})$ to orthogonal separation coordinates on S^n and thereby yields a uniform construction procedure for separation coordinates on spheres. Implicitly this structure is already present in [13] (see, in particular, formula (3.14) therein). Let us now make this operad structure explicit. We begin with the operad structure on $\mathcal{O}(n) := \mathbb{R}^n$ given by

$$\circ : \quad \mathbb{R}^k \times \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_k} \quad \longrightarrow \quad \mathbb{R}^{n_1 + \cdots + n_k}$$
$$(\boldsymbol{y}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) \quad \mapsto \quad \boldsymbol{y} \circ (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) := (y_1 \boldsymbol{x}_1, \ldots, y_k \boldsymbol{x}_k)$$

together with the permutation action of S_n on $\mathbb{R}^n$. This operad structure descends to an operad structure on $\mathcal{O}(n) := S^{n-1} \subset \mathbb{R}^n$, since $(y_1 \boldsymbol{x}_1)^2 + \cdots (y_k \boldsymbol{x}_k)^2 = y_1^2 + \cdots y_k^2 = 1$ for $\boldsymbol{y} \in S^{k-1}$ and $\boldsymbol{x}_\alpha \in S^{n_\alpha - 1}$. Note that the composition map

$$\circ : S^{k-1} \times S^{n_1 - 1} \times \cdots \times S^{n_k - 1} \longrightarrow S^{n_1 + \cdots + n_k - 1}$$

describes the k-fold join $S^{n_1 - 1} \star \cdots \star S^{n_k - 1} \cong S^{n_1 + \cdots + n_k - 1}$ of the spheres $S^{n_1 - 1}, \ldots,$ $S^{n_k - 1}$. This operad structure on spheres induces an operad structure on (local) coordinates on spheres, since coordinates $\boldsymbol{x}_0 = \boldsymbol{x}_0(\varphi_{01}, \ldots, \varphi_{0,k-1})$ on $S^{k-1} \subset \mathbb{R}^k$ and $\boldsymbol{x}_\alpha = \boldsymbol{x}_\alpha(\varphi_{\alpha 1}, \ldots, \varphi_{\alpha, n_\alpha - 1})$ on $S^{n_\alpha - 1} \subset \mathbb{R}^{n_\alpha}$ for $\alpha = 1, \ldots, k$ determine coordinates

$$\boldsymbol{x} = \boldsymbol{x}(\varphi_{01}, \ldots, \varphi_{0,k-1}, \varphi_{11}, \ldots, \varphi_{1,n_1-1}, \ldots, \varphi_{k1}, \ldots, \varphi_{k,n_k-1})$$

on $S^{n_1+\cdots+n_k-1}$, given by setting

$$(6.2) \qquad\qquad \boldsymbol{x} := \boldsymbol{x}_0 \circ (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k).$$

The interior of a Stasheff polytope corresponds to elliptic coordinates and its faces are products of Stasheff polytopes. Therefore we can construct all orthogonal separation coordinates on spheres (modulo isometries) from elliptic coordinates by composing them in a recursive manner via the operad composition (6.2). Just start with trivial coordinates $\boldsymbol{x}(\varnothing) = 1$ on a certain number of zero dimensional spheres S^0 and take elliptic coordinates for $\boldsymbol{x}_0$ in each step. The different choices one has when iterating this composition are given by the trees labelling the corresponding separation coordinates. That is, the rooted trees describe the hierarchy of iterated decompositions of a sphere as joins of lower dimensional spheres. This parallels the decomposition of the faces of a Stasheff polytope into products of lower dimensional Stasheff polytopes.

Note that the construction (2.13) of Vilenkin's polyspherical coordinates on S^{n-1} corresponds to the special case $k = 2$ of the above construction, starting from trivial coordinates $\boldsymbol{x}(\varnothing) = 1$ on n copies of S^0 and using the (elliptic) coordinates $\boldsymbol{x}_0(\varphi) = (\cos\varphi, \sin\varphi)$ on $S^{k-1} = S^1$ in each step.

Moreover, this operad structure on separation coordinates also explains Kalnins & Miller's graphical procedure [13]. Namely, adding in an "irreducible block" a leaf to each box which is not joined to another block and replacing each irreducible block by a node, the graphs in [13] become the trees arising from the operad structure.

6.3. **Constructing Stäckel systems via the mosaic operad.** We now explain how this operad structure manifests itself on the level of Stäckel systems. To this end, let $I_1 \cup \cdots \cup I_k = I$ be a partition of $I = \{1, \ldots, n\}$ with $|I_\alpha| =: n_\alpha$ and set $I_0 := \{1, \ldots, k\}$. We denote by $\mathfrak{d}_n^1$ the space of Killing tensors on S^{n-1} with diagonal algebraic curvature tensor and define the injections

$$\iota_0: \quad \mathfrak{d}_k^1 \quad \hookrightarrow \mathfrak{d}_{n_1+\cdots+n_k}^1 \qquad \iota_0(K_{\alpha\beta}) := \sum_{a\in I_\alpha, b\in I_\beta} K_{ab} \qquad \alpha, \beta \in I_0$$

$$\iota_\alpha: \quad \mathfrak{d}_{n_\alpha}^1 \hookrightarrow \mathfrak{d}_{n_1+\cdots+n_k}^1 \qquad \iota_\alpha(K_{ij}) := K_{ij} \qquad\qquad i, j \in I_\alpha.$$

Proposition 5. *Let Σ_0 be a Stäckel system on S^{k-1} and Σ_α be Stäckel systems on $S^{n_\alpha-1}$ for $\alpha = 1, \ldots, k$, all consisting of Killing tensors with diagonal algebraic curvature tensor. Then*

$$(6.3) \qquad \Sigma_0 \circ (\Sigma_1, \ldots, \Sigma_k) := \iota_0(\Sigma_0) \oplus \iota_1(\Sigma_1) \oplus \cdots \oplus \iota_k(\Sigma_k)$$

is a Stäckel system on $S^{n_1+\cdots+n_k-1}$. Moreover, this operation together with the S_n-action (2.15) defines an operad structure on those Stäckel systems on S^{n-1} that consist of Killing tensors with diagonal algebraic curvature tensor.

Proof. First observe that the sum on the right hand side of (6.3) is indeed a direct sum. Hence its dimension is $n_1 + \cdots + n_k - 1$. By Definition 3 and Remark 3, we have to show that all Killing tensors in this subspace commute. That is, a Killing tensor from $\iota_p(\Sigma_p)$ and another one from $\iota_q(\Sigma_q)$ commute for all $p, q = 0, 1, \ldots, k$. For $p, q \neq 0$ this is evident. For $p = q = 0$ one readily checks that the inclusion ι_0 preserves the relations (2.7) and hence maps commuting Killing tensors to commuting Killing tensors. In the remaining case $p \neq 0 = q$ the commutator

$$[\iota_p(K_{ij}), \iota_0(K_{\alpha\beta})] = [K_{ij}, \sum_{a\in I_\alpha, b\in I_\beta} K_{ab}] \qquad i, j \in I_p, \quad \alpha \neq \beta \in I_0$$

is zero unless $p = \alpha$ or $p = \beta$. But if $p = \alpha$, the sum over $a \in I_\alpha$ only contributes non-zero terms for $a = i$ and $a = j$. So the above commutator reduces to

$$\sum_{b \in I_\alpha} [K_{ij}, K_{ib} + K_{jb}] = 0$$

due to the relations (2.7), and similarly for $p = \beta$. This proves that (6.3) is a Stäckel system.

To check that this composition defines an operad is straightforward. The identity element is the empty Stäckel system on S^0 and equivariance is obvious. Associativity can be shown by taking subdivisions $I_\alpha = I_{\alpha 1} \cup \cdots \cup I_{\alpha k_\alpha}$ of I_α for all $\alpha \in I_0$ and considering the corresponding inclusions for Killing tensors. The details will be left to the reader. $\qquad \square$

To give an example, let us construct the Stäckel system for standard spherical coordinates on S^{n-1} by choosing $k = 2$ with the Stäckel system Σ_0 on $S^{k-1} = S^1$ spanned by K_{12}, starting from empty Stäckel systems on n copies of S^0 and taking $n_2 = 1$ in each step. This yields the Stäckel system spanned by

$$K_{12}, \quad K_{13} + K_{23}, \quad K_{14} + K_{24} + K_{34}, \quad \ldots$$

and shows that the Jucys-Murphy subalgebras (3.4) in the Kohno-Drinfeld Lie algebra correspond to standard spherical coordinates.

7. Outlook

We have shown that the theory of Deligne-Mumford-Knudsen moduli spaces and Stasheff polytopes provides the right framework for the classification and construction of all orthogonal separation coordinates on spheres. In particular, we elucidated the natural algebro-geometric structure of the parameter space classifying isometry classes of separation coordinates, which for a long time had only been known as a mere set, and gave a precise description of its topology.

It would be very interesting to see if the same approach will work in a more general situation. In particular, one can use the algebraic approach of [26] to study the orthogonal separation coordinates for all (pseudo-)Riemannian constant curvature manifolds, such as hyperbolic space $\mathbb{H}^n$. The question is whether the corresponding moduli spaces of separation coordinates are related to any known algebro-geometric moduli spaces or families of polyhedra. This will be subject for future research.

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