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ON SOME DETERMINANTS WITH LEGENDRE SYMBOL ENTRIES

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Abstract. In this paper we mainly focus on some determinants with Legendre symbol entries. For an odd prime p and an integer d, let $S(d,p)$ denote the determinant of the $(p-1)/2 \times (p-1)/2$ matrix whose (i,j)-entry ($1 \leqslant i,j \leqslant (p-1)/2$) is the Legendre symbol $(\frac{i^2+dj^2}{p})$. We investigate properties of $S(d,p)$ as well as some other determinants involving Legendre symbols. In Section 3 we pose 17 open conjectures on determinants one of which states that $(\frac{-S(d,p)}{p})=1$ if $(\frac{d}{p})=1$. This material might interest some readers and stimulate further research.

1. Introduction

For an $n \times n$ matrix $A=(a_{ij})_{1\leqslant i,j\leqslant n}$ over the field of complex numbers, we often write $\det A$ in the form $|a_{ij}|_{1\leqslant i,j\leqslant n}$. In this paper we study determinants with Legendre symbol entries.

Let p be an odd prime and let $(\frac{\cdot}{p})$ be the Legendre symbol. The circulant determinant

$$\left|\left(\frac{j-i}{p}\right)\right|_{0\leqslant i,j\leqslant p-1}=\begin{vmatrix} (\frac{0}{p}) & (\frac{1}{p}) & (\frac{2}{p}) & \ldots & (\frac{p-1}{p}) \\ (\frac{p-1}{p}) & (\frac{0}{p}) & (\frac{1}{p}) & \ldots & (\frac{p-2}{p}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\frac{1}{p}) & (\frac{2}{p}) & (\frac{3}{p}) & \ldots & (\frac{0}{p}) \end{vmatrix}$$

takes the value

$$\prod_{r=0}^{p-1}\sum_{k=0}^{p-1}\left(\frac{k}{p}\right)\left(e^{2\pi ir/p}\right)^k=0$$

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since $\sum_{k=0}^{p-1}(\frac{k}{p})=0$. (See [K99, (2.41)] for the evaluation of a general circulant determinant.) For the matrix $A=(a_{ij})_{1\leqslant i,j\leqslant p-1}$ with $a_{ij}=(\frac{i-j}{p})$, L. Carlitz [C59, Theorem 4] proved that its characteristic polynomial is

$$|xI_{p-1}-A|=\left(x^2-\left(\frac{-1}{p}\right)p\right)^{(p-3)/2}\left(x^2-\left(\frac{-1}{p}\right)\right),$$

where I_{p-1} is the $(p-1)\times(p-1)$ identity matrix. Putting $x=0$ in Carlitz's formula, we obtain that

$$(-1)^{p-1}|A|=\left(-\left(\frac{-1}{p}\right)\right)^{(p-1)/2}p^{(p-3)/2}=p^{(p-3)/2}.$$

For $m\in\mathbb{Z}$ let $\{m\}_p$ denote the least nonnegative residue of an integer m modulo p. For any integer $a\not\equiv 0\pmod p$, $\{aj\}_p$ $(j=1,\ldots,p-1)$ is a permutation of $1,\ldots,p-1$, and its sign is the Legendre symbol $(\frac{a}{p})$ by Zolotarev's theorem (cf. [DH] and [Z]). Therefore, for any integer $d\not\equiv 0\pmod p$ we have

$$\left|\left(\frac{i+dj}{p}\right)\right|_{0\leqslant i,j\leqslant p-1}=\left(\frac{-d}{p}\right)\left|\left(\frac{i-j}{p}\right)\right|_{0\leqslant i,j\leqslant p-1}=0 \qquad (1.1)$$

and

$$\left|\left(\frac{i+dj}{p}\right)\right|_{1\leqslant i,j\leqslant p-1}=\left(\frac{-d}{p}\right)\left|\left(\frac{i-j}{p}\right)\right|_{1\leqslant i,j\leqslant p-1}=\left(\frac{-d}{p}\right)p^{(p-3)/2}. \qquad (1.2)$$

Let p be an odd prime. In 2004, R. Chapman [Ch04] used quadratic Gauss sums to determine the values of

$$\left|\left(\frac{i+j-1}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}\quad\text{and}\quad\left|\left(\frac{i+j-1}{p}\right)\right|_{1\leqslant i,j\leqslant (p+1)/2}.$$

Since $(p+1)/2-i+(p+1)/2-j-1\equiv-(i+j)\pmod p$, we see that

$$\left|\left(\frac{i+j-1}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}=\left(\frac{-1}{p}\right)\left|\left(\frac{i+j}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}$$

and

$$\left|\left(\frac{i+j-1}{p}\right)\right|_{1\leqslant i,j\leqslant (p+1)/2}=\left|\left(\frac{i+j}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}.$$

Chapman [Ch12] also conjectured that

$$\left|\left(\frac{j-i}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}=\begin{cases}-r_p & \text{if } p\equiv 1\pmod 4,\\ 1 & \text{if } p\equiv 3\pmod 4,\end{cases} \qquad (1.3)$$

where $\varepsilon_p^{(2-(\frac{2}{p}))h(p)} = r_p + s_p\sqrt{p}$ with $r_p, s_p \in \mathbb{Z}$. (Throughout this paper, ε_p and $h(p)$ stand for the fundamental unit and the class number of the real quadratic field $\mathbb{Q}(\sqrt{p})$ respectively.) As Chapman could not solve this problem for several years, he called the determinant *evil* (cf. [Ch12]). Chapman's conjecture on his "evil" determinant was recently confirmed by M. Vsemirnov [V12, V13] via matrix decomposition.

Let $p \equiv 1 \pmod 4$ be a prime. In an unpublic manuscript written in 2003 Chapman [Ch03] conjectured that

$$s_p = \left(\frac{2}{p}\right)\left|\left(\frac{j-i}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}. \tag{1.4}$$

Note that (1.3) and (1.4) together yield an interesting identity

$$\varepsilon_p^{(2-(\frac{2}{p}))h(p)} = \left(\frac{2}{p}\right)\left|\left(\frac{j-i}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}\sqrt{p} - \left|\left(\frac{j-i}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}.$$

Taking the norm with respect to the field extension $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$, we are led to the identity

$$\left|\left(\frac{j-i}{p}\right)\right|^2_{0\leqslant i,j\leqslant (p-1)/2} - p\left|\left(\frac{j-i}{p}\right)\right|^2_{1\leqslant i,j\leqslant (p-1)/2} = (-1)^{h(p)}$$

since $N(\varepsilon_p) = -1$ (cf. Theorem 3 of [Co62, p. 185]). This provides an *explicit* solution to the diophantine equation $x^2 - py^2 = (-1)^{h(p)}$.

Now we state our first theorem.

Theorem 1.1. *Let p be an odd prime. For $d \in \mathbb{Z}$ define*

$$R(d,p) := \left|\left(\frac{i+dj}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}. \tag{1.5}$$

If $p \equiv 1 \pmod 4$, then

$$R(d,p) \equiv \left(\left(\frac{d}{p}\right)d\right)^{(p-1)/4}\ \frac{p-1}{2}! \pmod p. \tag{1.6}$$

When $p \equiv 3 \pmod 4$, we have

$$R(d,p) \equiv \begin{cases} (\frac{2}{p}) \pmod p & \text{if } (\frac{d}{p}) = 1, \\ 1 \pmod p & \text{if } (\frac{d}{p}) = -1. \end{cases} \tag{1.7}$$

Also,

$$R(-d,p) \equiv \left(\frac{2}{p}\right) R(d,p) \pmod p, \tag{1.8}$$

and

$$\left|\left(\frac{i+dj+c}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}\equiv R(d,p)\pmod p\quad\text{for all } c\in\mathbb{Z}.\tag{1.9}$$

Remark 1.1. Let p be any odd prime. By Wilson's theorem,

$$\left(\frac{p-1}{2}!\right)^2\equiv\begin{cases}-1\pmod p & \text{if } p\equiv 1\pmod 4,\\ 1\pmod p & \text{if } p\equiv 3\pmod 4.\end{cases}\tag{1.10}$$

Corollary 1.1. *Let $p\equiv 1\pmod 4$ be a prime, and write $\varepsilon_p^{h(p)}=a_p+b_p\sqrt{p}$ with $a_p,b_p\in\mathbb{Q}$, where ε_p and $h(p)$ are the fundamental unit and the class number of the real quadratic field $\mathbb{Q}(\sqrt{p})$. Then we have*

$$a_p\equiv-\frac{p-1}{2}!\pmod p\quad\text{and } h(p)\equiv 1\pmod 2.\tag{1.11}$$

Proof. By (1.6) we have

$$R(1,p)\equiv\frac{p-1}{2}!\pmod p.$$

On the other hand, Chapman [Ch04, Corollary 3] proved that

$$\left|\left(\frac{i+j}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}=\left|\left(\frac{i+j-1}{p}\right)\right|_{1\leqslant i,j\leqslant (p+1)/2}=-\left(\frac{2}{p}\right)2^{(p-1)/2}a_p.$$

So we have the first congruence in (1.11). Taking norms (with respect to the field extension $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$) of both sides of the identity $\varepsilon_p^{h(p)}=a_p+b_p\sqrt{p}$, we obtain

$$N(\varepsilon)^{h(p)}=a_p^2-pb_p^2.$$

Since

$$a_p^2\equiv\left(\frac{p-1}{2}!\right)^2\equiv-1\pmod p,$$

we must have $N(\varepsilon)=-1$ and $2\nmid h(p)$. This proves the second congruence in (1.11). $\square$

It is well known that for any odd prime p the $(p-1)/2$ squares

$$1^2,2^2,\ldots,\left(\frac{p-1}{2}\right)^2$$

give all the $(p-1)/2$ quadratic residues modulo p. So we think that it's natural to consider some Legendre symbol determinants involving binary quadratic forms.

Theorem 1.2. *Let p be any odd prime. For $d\in\mathbb{Z}$ define*

$$S(d,p):=\left|\left(\frac{i^2+dj^2}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2} \tag{1.12}$$

and

$$T(d,p):=\left|\left(\frac{i^2+dj^2}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}. \tag{1.13}$$

(i) *For any $c\in\mathbb{Z}$ with $p\nmid c$, we have*

$$S(c^2d,p)=\left(\frac{c}{p}\right)^{(p+1)/2}S(d,p)\quad\text{and}\quad T(c^2d,p)=\left(\frac{c}{p}\right)^{(p+1)/2}T(d,p). \tag{1.14}$$

If $p\equiv 1\pmod 4$, then

$$S(-d,p)=\left(\frac{2}{p}\right)S(d,p)\quad\text{and}\quad T(-d,p)=\left(\frac{2}{p}\right)T(d,p). \tag{1.15}$$

When $p\equiv 3\pmod 4$, we have

$$\left(\frac{d}{p}\right)=-1\Longrightarrow S(d,p)=0. \tag{1.16}$$

(ii) *We have*

$$\left(\frac{T(d,p)}{p}\right)=\begin{cases}(\frac{2}{p}) & \text{if } (\frac{d}{p})=1,\\ 1 & \text{if } (\frac{d}{p})=-1.\end{cases} \tag{1.17}$$

Also,

$$T(-d,p)\equiv\left(\frac{2}{p}\right)T(d,p)\pmod p \tag{1.18}$$

and

$$\left|\left(\frac{i^2+dj^2+c}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}\equiv T(d,p)\pmod p\quad\text{for all } c\in\mathbb{Z}. \tag{1.19}$$

Remark 1.2. The author conjectured that (1.16) also holds for any prime $p\equiv 1\pmod 4$. This was later confirmed by his student Xiangzi Meng in the following way: The transpose of $S(d,p)$ equals

$$\begin{aligned}\left|\left(\frac{di^2+j^2}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}&=\left(\frac{d}{p}\right)^{(p-1)/2}\left|\left(\frac{(di)^2+dj^2}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}\\ &=\left|\left(\frac{(di)^2+dj^2}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}=\left(\frac{d}{p}\right)S(d,p)\end{aligned}$$

with the help of Zolotarev's theorem.

Example 1.1. Note that

$$S(1,11)=\begin{vmatrix} -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 \end{vmatrix} = -16$$

and

$$S(2,13)=\begin{vmatrix} 1 & 1 & -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \end{vmatrix} = 0.$$

Now we present our third theorem.

Theorem 1.3. (i) *For any odd prime p, we have*

$$\left|\frac{(\frac{i+j}{p})}{i+j}\right|_{1\leqslant i,j\leqslant (p-1)/2} \equiv \begin{cases} (\frac{2}{p}) \pmod p & \text{if } p\equiv 1 \pmod 4, \\ ((p-1)/2)! \pmod p & \text{if } p\equiv 3 \pmod 4. \end{cases} \tag{1.20}$$

(ii) *Let $p\equiv 3 \pmod 4$ be a prime. Then*

$$\left|\frac{1}{i^2+j^2}\right|_{1\leqslant i,j\leqslant (p-1)/2} \equiv \left(\frac{2}{p}\right) \pmod p. \tag{1.21}$$

We are going to prove Theorems 1.1-1.3 in the next section, and pose over ten new conjectures on determinants in Section 3.

2. Proof of Theorems 1.1-1.3

Lemma 2.1 ([K05, Lemma 9])**.** *Let $P(z)=\sum_{k=0}^{n-1}a_kz^k$ be a polynomial with complex number coefficients. Then we have*

$$|P(x_i+y_j)|_{1\leqslant i,j\leqslant n} = a_{n-1}^n\prod_{k=0}^{n-1}\binom{n-1}{k}\times\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(y_j-y_i). \tag{2.1}$$

Proof of Theorem 1.1. Set $n=(p-1)/2$. For any $c\in\mathbb{Z}$, we have

$$\left|\left(\frac{i+dj+c}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2} \equiv |(i+dj+c)^n|_{0\leqslant i,j\leqslant n} \pmod p.$$

In light of Lemma 2.2,

$$\begin{aligned}&|(i+dj+c)^n|_{0\leqslant i,j\leqslant n}=|(i+dj+c-d-1)^n|_{1\leqslant i,j\leqslant n+1}\\&=\prod_{k=0}^{n}\binom{n}{k}\times\prod_{1\leqslant i<j\leqslant n+1}(i-j)(dj+c-d-1-(di+c-d-1))\\&=\frac{(n!)^{n+1}}{\prod_{k=0}^{n}k!(n-k)!}(-d)^{n(n+1)/2}\prod_{1\leqslant i<j\leqslant n+1}(j-i)^2=(-d)^{n(n+1)/2}(n!)^{n+1}.\end{aligned}$$

Therefore (1.9) holds, and also

$$R(d,p)\equiv(-d)^{(p^2-1)/8}\left(\frac{p-1}{2}!\right)^{(p+1)/2}\pmod p. \tag{2.2}$$

In the case $p\equiv 1\pmod 4$, from (2.2) we obtain

$$\begin{aligned}R(d,p)\equiv&(-d)^{(p-1)/4\ (p+1)/2}\ \frac{p-1}{2}!\left(\frac{p-1}{2}!\right)^{2(p-1)/4}\\&\equiv\left(d^{(p+1)/2}\right)^{(p-1)/4}\ \frac{p-1}{2}!\equiv\left(\left(\frac{d}{p}\right)d\right)^{(p-1)/4}\ \frac{p-1}{2}!\pmod p.\end{aligned}$$

In the case $p\equiv 3\pmod 4$, (2.2) yields

$$R(d,p)\equiv(-d)^{(p-1)/2\times(p+1)/4}\left(\frac{p-1}{2}!\right)^{2(p+1)/4}\equiv\left(\frac{-d}{p}\right)^{(p+1)/4}\pmod p$$

and hence (1.7) follows.

Now it remains to show (1.8). If $p\equiv 1\pmod 4$, then by (1.6)

$$R(-d,p)\equiv\left(\left(\frac{-d}{p}\right)(-d)\right)^{(p-1)/4}\ \frac{p-1}{2}!\equiv\left(\frac{2}{p}\right)R(d,p)\pmod p.$$

If $p\equiv 3\pmod 4$, then $(\frac{-d}{p})=-(\frac{d}{p})$ and hence we get (1.8) from (1.7).

The proof of Theorem 1.1 is now complete. □

Lemma 2.2. *Let* $p\equiv 1\pmod 4$ *be a prime. Then*

$$\left(\frac{((p-1)/2)!}{p}\right)=\left(\frac{2}{p}\right). \tag{2.3}$$

Proof. Since

$$(-4)^{(p-1)/4}=(-1)^{(p-1)/4}2^{(p-1)/2}=\left(\frac{2}{p}\right)2^{(p-1)/2}\equiv 1\pmod p,$$

for some $x\in\mathbb{Z}$ we have

$$x^4\equiv-4\equiv4\left(\frac{p-1}{2}!\right)^2\pmod p,\quad\text{i.e., } x^2\equiv\pm2\times\frac{p-1}{2}!\pmod p.$$

Therefore (2.3) holds. $\square$

Proof of Theorem 1.2(i). Let $c\in\mathbb{Z}$ with $p\nmid c$. For each $j=1,\ldots,(p-1)/2$ let $\sigma_c(j)$ be the unique $r\in\{1,\ldots,(p-1)/2\}$ such that $cj\equiv r$ or $-r\pmod p$. By a result of H. Pan [P06], the sign of the permutation σ_c equals $(\frac{c}{p})^{(p+1)/2}$. Thus

$$S(c^2d,p)=\left|\left(\frac{i^2+d\sigma_c(j)^2}{p}\right)\right|_{1\leqslant i,j\leqslant(p-1)/2}=\left(\frac{c}{p}\right)^{(p+1)/2}S(d,p).$$

Similarly the second equality in (1.14) also holds.

Now we handle the case $p\equiv1\pmod4$. As $((p-1)/2)!^2\equiv-1\pmod p$, by applying (1.14) with $c=((p-1)/2)!$ and using (2.3) we immediately get (1.15).

Assume that $p\equiv3\pmod4$. As the transpose of $S(-1,p)$ coincides with $(\frac{-1}{p})^{(p-1)/2}S(-1,p)=-S(-1,p)$, we have $S(-1,p)=0$. If $(\frac{d}{p})=-1$, then $d\equiv-c^2\pmod p$ for some integer $c\not\equiv0\pmod p$, and hence

$$S(d,p)=S(-c^2,p)=\left(\frac{c}{p}\right)^{(p+1)/2}S(-1,p)=0.$$

This proves (1.16).

So far we have proved the first part of Theorem 1.2. $\square$

Proof of Theorem 1.2(ii). Set $n=(p-1)/2$. For any $c\in\mathbb{Z}$, we have

$$\left|\left(\frac{i^2+dj^2+c}{p}\right)\right|_{0\leqslant i,j\leqslant(p-1)/2}\equiv|(i^2+dj^2+c)^n|_{0\leqslant i,j\leqslant n}\pmod p.$$

In light of Lemma 2.2,

$$\begin{aligned}&|(i^2+dj^2+c)^n|_{0\leqslant i,j\leqslant n}=|((i-1)^2+d(j-1)^2+c)^n|_{1\leqslant i,j\leqslant n+1}\\&=\prod_{k=0}^n\binom{n}{k}\times\prod_{1\leqslant i<j\leqslant n+1}((i-1)^2-(j-1)^2)(d(j-1)^2+c-d(i-1)^2-c)\\&=\frac{(n!)^{n+1}}{\prod_{k=0}^n k!(n-k)!}(-d)^{n(n+1)/2}\prod_{0\leqslant i<j\leqslant n}(j-i)^2(j+i)^2\\&=(-d)^{n(n+1)/2}(n!)^{n+1}\prod_{0\leqslant i<j\leqslant n}(i+j)^2.\end{aligned}$$

Therefore (1.19) holds, and also

$$\begin{aligned}T(d,p)\equiv&(-d)^{(p^2-1)/8}\left(\frac{p-1}{2}!\right)^{(p+1)/2}\prod_{0\leqslant i<j\leqslant (p-1)/2}(i+j)^2\\ \equiv&R(d,p)\prod_{0\leqslant i<j\leqslant (p-1)/2}(i+j)^2 \pmod p\end{aligned}$$

with the help of (2.2). Combining this with (1.8) we obtain (1.18). Note that

$$\left(\frac{T(d,p)}{p}\right)=\left(\frac{R(d,p)}{p}\right).$$

If $(\frac{d}{p})=1$, then by Theorem 1.1 we have

$$R(d,p)\equiv\begin{cases}d^{(p-1)/4}((p-1)/2)! \pmod p & \text{if } p\equiv 1 \pmod 4,\\ (\frac{2}{p}) \pmod p & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

and hence $(\frac{R(d,p)}{p})=(\frac{2}{p})$ with the help if Lemma 2.2. In the case $(\frac{d}{p})=-1$, by Theorem 1.1 we have

$$R(d,p)\equiv\begin{cases}(-d)^{(p-1)/4}((p-1)/2)! \pmod p & \text{if } p\equiv 1 \pmod 4,\\ 1 \pmod p & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

and hence $(\frac{R(d,p)}{p})=1$ with the help if Lemma 2.2. Therefore (1.17) also holds. We are done. □

Lemma 2.3. *Let p be any odd prime. For any $d\in\mathbb{Z}$ with $(\frac{d}{p})=-1$, we have the new congruence*

$$\prod_{x=1}^{(p-1)/2}(x^2-d)\equiv(-1)^{(p+1)/2}\,2 \pmod p. \tag{2.4}$$

Proof. For any integer a, it is well known that

$$\begin{aligned}a^{(p-1)/2}\equiv 1 \pmod p \iff& \left(\frac{a}{p}\right)=1\\ \iff& a\equiv x^2 \pmod p \quad \text{for some } x=1,\ldots,\frac{p-1}{2}.\end{aligned}$$

Therefore

$$\prod_{x=1}^{(p-1)/2}(z-x^2)\equiv z^{(p-1)/2}-1 \pmod p$$

and hence

$$\prod_{x=1}^{(p-1)/2}(y+d-x^2)\equiv(y+d)^{(p-1)/2}-1\pmod p. \tag{2.5}$$

Comparing the constant terms of both sides of the congruence (2.5), we obtain

$$\prod_{x=1}^{(p-1)/2}(d-x^2)\equiv d^{(p-1)/2}-1\equiv-2\pmod p$$

and hence (2.4 follows. □

Remark 2.1. Under the condition of Lemma 2.3, we could also prove the following congruences

$$\sum_{x=1}^{(p-1)/2}\frac{1}{x^2-d}\equiv\frac{1}{4d}\pmod p\ \ \text{and}\ \ \sum_{x=1}^{(p-1)/2}\frac{1}{(x^2-d)^2}\equiv-\frac{5}{16d^2}\pmod p \tag{2.6}$$

by comparing coefficients of y and y^2 in the congruence (2.5).

Proof of Theorem 1.3. (i) Set $n=(p-1)/2$. Clearly

$$\left|\frac{(\frac{i+j}{p})}{i+j}\right|_{1\leqslant i,j\leqslant(p-1)/2}\equiv|(i+j)^{n-1}|_{1\leqslant i,j\leqslant n}\pmod p.$$

By Lemma 2.1,

$$\begin{aligned}|(i+j)^{n-1}|_{1\leqslant i,j\leqslant n}=&\prod_{k=0}^{n-1}\binom{n-1}{k}\times\prod_{1\leqslant i<j\leqslant n}(i-j)(j-i)\\ =&\frac{(n-1)!^n}{\prod_{k=0}^{n-1}k!(n-1-k)!}(-1)^{n(n-1)/2}\prod_{1\leqslant i<j\leqslant n}(j-i)^2\\ =&(-1)^{n(n-1)/2}(n-1)!^n\\ =&(-1)^{(p-1)(p-3)/8}\left(\frac{2}{p-1}\right)^{(p-1)/2}\left(\frac{p-1}{2}!\right)^{(p-1)/2}.\end{aligned}$$

Therefore

$$\left|\frac{(\frac{i+j}{p})}{i+j}\right|_{1\leqslant i,j\leqslant(p-1)/2}\equiv\left(\frac{p-1}{2}!\right)^{(p-1)/2}\pmod p. \tag{2.7}$$

In the case $p \equiv 1 \pmod 4$, this yields

$$\left|\frac{(\frac{i+j}{p})}{i+j}\right|_{1\leqslant i,j\leqslant (p-1)/2} \equiv (-1)^{(p-1)/4} = \left(\frac{2}{p}\right) \pmod p.$$

If $p \equiv 3 \pmod 4$, then by (2.7) we have

$$\left|\frac{(\frac{i+j}{p})}{i+j}\right|_{1\leqslant i,j\leqslant (p-1)/2} \equiv \frac{p-1}{2}!\left(\frac{p-1}{2}!\right)^{2(p-3)/4} \equiv \frac{p-1}{2}! \pmod p.$$

So (1.20) always holds.

(ii) It is known (cf. [K05, (5.5)]) that

$$\left|\frac{1}{x_i+y_j}\right|_{1\leqslant i,j\leqslant n} = \frac{\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)(y_j-y_i)}{\prod_{i=1}^n\prod_{j=1}^n(x_i+y_j)}.$$

Taking $n=(p-1)/2$ and $x_i=y_i=i^2$ for $i=1,\ldots,n$, we get

$$\left|\frac{1}{i^2+j^2}\right|_{1\leqslant i,j\leqslant (p-1)/2} = \frac{\prod_{1\leqslant i<j\leqslant (p-1)/2}(j^2-i^2)^2}{\prod_{i=1}^{(p-1)/2}\prod_{j=1}^{(p-1)/2}(i^2+j^2)}. \tag{2.8}$$

Observe that

$$\begin{aligned}\prod_{1\leqslant i<j\leqslant (p-1)/2}(j^2-i^2)^2 &= \prod_{j=1}^{(p-1)/2}((j-1)!(j+1)\cdots(2j-1))^2\\ &= \prod_{j=1}^{(p-1)/2}\frac{(2j-1)!^2}{j^2} = \frac{\prod_{j=1}^{(p-1)/2}(2j-1)!(p-2j)!}{((p-1)/2)!^2}\\ &= \frac{\binom{p-1}{(p-1)/2}}{(p-1)!}\prod_{j=1}^{(p-1)/2}\frac{(p-1)!}{\binom{p-1}{2j-1}}\\ &\equiv \frac{(-1)^{(p-1)/2}}{-1}\prod_{j=1}^{(p-1)/2}\frac{-1}{(-1)^{2j-1}} = 1 \pmod p\end{aligned}$$

with the help of Wilson's theorem. Also,

$$\begin{aligned}\prod_{i=1}^{(p-1)/2}\prod_{j=1}^{(p-1)/2}(i^2+j^2) &= \prod_{i=1}^{(p-1)/2}\left(\prod_{j=1}^{(p-1)/2} i^2\left(1+\frac{j^2}{i^2}\right)\right)\\ &\equiv \prod_{i=1}^{(p-1)/2}\left(i^{p-1}\prod_{x=1}^{(p-1)/2}(1+x^2)\right) \pmod p.\end{aligned}$$

As -1 is a quadratic non-residue modulo p, applying (2.4) with $d=-1$ we get

$$\prod_{x=1}^{(p-1)/2}(x^2+1)\equiv(-1)^{(p+1)/2}2=2 \pmod p.$$

Therefore

$$\prod_{i=1}^{(p-1)/2}\prod_{j=1}^{(p-1)/2}(i^2+j^2)\equiv\prod_{i=1}^{(p-1)/2}2=2^{(p-1)/2}\equiv\left(\frac{2}{p}\right) \pmod p.$$

So the desired congruence (1.21) follows from (2.8). We are done. □

3. Some open conjectures on determinants

Wilson's theorem implies that

$$\frac{p-1}{2}!\equiv\pm1 \pmod p \quad \text{for any prime } p\equiv3 \pmod 4.$$

Conjecture 3.1 (2013-08-05)**.** *Let p be an odd prime. Then we have*

$$\left|\left(\frac{i^2-((p-1)/2)!j}{p}\right)\right|_{1\leqslant i,j\leqslant(p-1)/2}=0 \iff p\equiv3 \ (\mathrm{mod}\ 4). \tag{3.1}$$

Remark 3.1. See [Su13, A226163] for the sequence

$$\left|\left(\frac{i^2-((p_n-1)/2)!j}{p_n}\right)\right|_{1\leqslant i,j\leqslant(p_n-1)/2} \quad (n=2,3,\ldots),$$

where p_n denotes the nth prime. In 1961 L. J. Mordell [M61] proved that for any prime $p>3$ with $p\equiv3$ (mod 4) we have

$$\frac{p-1}{2}!\equiv(-1)^{(h(-p)+1)/2} \pmod p, \tag{3.2}$$

where $h(-p)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$.

Conjecture 3.2 (2013-07-18)**.** *Let p be an odd prime. For $d\in\mathbb{Z}$ let $S(d,p)$ be given by* (1.12)*. Then*

$$\left(\frac{-S(d,p)}{p}\right)=1 \quad \text{if} \ \left(\frac{d}{p}\right)=1. \tag{3.3}$$

Remark 3.2. See [Su13, A227609] for the sequence $S(1,p_n)$ $(n=2,3,\ldots)$. Let p be any odd prime and let $d\in\mathbb{Z}$ with $p\nmid d$. The sum of entries in each row or column of the determinant $S(d,p)$ actually equals $-(1+(\frac{d}{p}))/2$. Indeed, for any $i_0,j_0=1,\ldots,(p-1)/2$ we have

$$\sum_{j=1}^{(p-1)/2}\left(\frac{i_0^2+dj^2}{p}\right)=\sum_{i=1}^{(p-1)/2}\left(\frac{i+dj_0^2}{p}\right)=\begin{cases}0 & \text{if } (\frac{d}{p})=-1,\\ -1 & \text{if } (\frac{d}{p})=1.\end{cases}\tag{3.4}$$

To see this we note that

$$\begin{aligned}\sum_{j=1}^{(p-1)/2}\left(\frac{i_0^2+dj^2}{p}\right)\equiv&\sum_{j=1}^{(p-1)/2}(i_0^2+dj^2)^{(p-1)/2}\\ \equiv&\frac{p-1}{2}i_0^{p-1}+\sum_{k=1}^{(p-1)/2}\binom{(p-1)/2}{k}i_0^{p-1-2k}\frac{d^k}{2}\sum_{j=1}^{p-1}j^{2k}\\ \equiv&-\frac{1}{2}+\frac{1}{2}\left(\frac{d}{p}\right)(p-1)\equiv-\frac{1+(\frac{d}{p})}{2}\pmod p.\end{aligned}$$

The following conjecture can be viewed as a supplement to Conjecture 3.2.

Conjecture 3.3 (2013-08-07)**.** *Let p be an odd prime, and let $c,d\in\mathbb{Z}$ with $p\nmid cd$. Define*

$$S_c(d,p)=\left|\left(\frac{i^2+dj^2+c}{p}\right)\right|_{1\leqslant i,j\leqslant(p-1)/2}.\tag{3.5}$$

Then

$$\left(\frac{S_c(d,p)}{p}\right)=\begin{cases}1 & \text{if } (\frac{c}{p})=1\ \&\ (\frac{d}{p})=-1,\\ (\frac{-1}{p}) & \text{if } (\frac{c}{p})=(\frac{d}{p})=-1,\\ (\frac{-2}{p}) & \text{if } (\frac{-c}{p})=(\frac{d}{p})=1,\\ (\frac{-6}{p}) & \text{if } (\frac{-c}{p})=-1\ \&\ (\frac{d}{p})=1.\end{cases}\tag{3.6}$$

Remark 3.3. See [Su13, A228005] for the sequence $S_1(1,p_n)$ $(n=2,3,\ldots)$. Let p be an odd prime and let $b,c,d\in\mathbb{Z}$ with $p\nmid bcd$. It is easy to see that

$$\begin{aligned}\left|\left(\frac{i^2+dj^2+b^2c}{p}\right)\right|_{1\leqslant i,j\leqslant(p-1)/2}=&\left|\left(\frac{(bi)^2+d(bj)^2+b^2c}{p}\right)\right|_{1\leqslant i,j\leqslant(p-1)/2}\\ =&\left|\left(\frac{i^2+dj^2+c}{p}\right)\right|_{1\leqslant i,j\leqslant(p-1)/2}.\end{aligned}$$

Conjecture 3.4 (2013-08-12). *Let p be an odd prime. For $c,d\in\mathbb{Z}$ define*

$$(c,d)_p := \left|\left(\frac{i^2+cij+dj^2}{p}\right)\right|_{1\leqslant i,j\leqslant p-1}. \tag{3.7}$$

(i) *If d is nonzero, then there are infinitely many odd primes q with $(c,d)_q=0$. Also,*

$$\left(\frac{d}{p}\right)=-1 \implies (c,d)_p=0. \tag{3.8}$$

When $(c,d)_p$ is nonzero, its p-adic valuation (i.e., p-adic order) must be even.

(ii) *We have*

$$\begin{aligned}(6,1)_p=0 &\quad \text{if } p\equiv 3 \pmod 4,\\ (3,2)_p=(4,2)_p=0 &\quad \text{if } p\equiv 7 \pmod 8,\\ (3,3)_p=0 &\quad \text{if } p\equiv 11 \pmod{12},\\ (10,9)_p=0 &\quad \text{if } p\equiv 5 \pmod{12}.\end{aligned} \tag{3.9}$$

Remark 3.4. See [Su13, A225611] for the sequence $(6,1)_{p_n}$ $(n=2,3,\ldots)$. It is easy to see that $(-c,d)_p=(\frac{-1}{p})(c,d)_p$ for any odd prime p and integers c and d.

Conjecture 3.5 (2013-08-12). *Let p be an odd prime. For $c,d\in\mathbb{Z}$ define*

$$[c,d]_p := \left|\left(\frac{i^2+cij+dj^2}{p}\right)\right|_{0\leqslant i,j\leqslant p-1}. \tag{3.10}$$

(i) *If d is nonzero, then there are infinitely many odd primes q with $[c,d]_q=0$. When $[c,d]_p$ is nonzero, its p-adic valuation (i.e., p-adic order) must be even.*

(ii) *If $p\nmid d$ and $(c,d)_p\neq 0$, then*

$$\frac{[c,d]_p}{(c,d)_p}=\begin{cases}(p-1)/2 & \text{if } p\nmid c^2-4d,\\ (1-p)/(p-2) & \text{if } p\mid c^2-4d.\end{cases} \tag{3.11}$$

(iii) *We have*

$$\begin{aligned}[6,1]_p=[3,2]_p=0 &\quad \text{if } p\equiv 3 \pmod 4,\\ [3,3]_p=0 &\quad \text{if } p\equiv 5 \pmod 6,\\ [4,2]_p=0 &\quad \text{if } p\equiv 5,7 \pmod 8,\\ [5,5]_p=0 &\quad \text{if } p\equiv 13,17 \pmod{20}.\end{aligned} \tag{3.12}$$

Remark 3.5. See [Su13, A228095] for the sequence $[3,3]_{p_n}$ $(n=2,3,\ldots)$. It is easy to see that $[-c,d]_p=(\frac{-1}{p})[c,d]_p$ for any odd prime p and integers c and d.

Let p be any odd prime. For $a,b,c\in\mathbb{Z}$ with $p\nmid a$, it is known (cf. [BEW]) that

$$\sum_{x=0}^{p-1}\left(\frac{ax^2+bx+c}{p}\right)=\begin{cases}-(\frac{a}{p}) & \text{if } p\nmid b^2-4ac,\\ (p-1)(\frac{a}{p}) & \text{if } p\mid b^2-4ac.\end{cases}$$

Thus, for any $c,d\in\mathbb{Z}$ we can easily calculate the sum of all entries in a row or a column of $(c,d)_p$ or $[c,d]_p$.

Conjecture 3.6 (2013-08-11)**.** *Let $p>5$ be a prime with $p\equiv 1\pmod 4$. Define*

$$D_p^+:=\left|(i+j)\left(\frac{i+j}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}\ \text{and}\ D_p^-:=\left|(j-i)\left(\frac{j-i}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}. \tag{3.13}$$

Then

$$\left(\frac{D_p^+}{p}\right)=\left(\frac{D_p^-}{p}\right)=1. \tag{3.14}$$

Remark 3.6. It is known that a skew-symmetric $2n\times 2n$ determinant with integer entries is always a square (cf. [St90] and [K99]).

Conjecture 3.7 (2013-08-20)**.** *For any prime $p>3$, we have*

$$\left|(i^2+j^2)\left(\frac{i^2+j^2}{p}\right)\right|_{0\leqslant i,j\leqslant (p-1)/2}\equiv 0\pmod p. \tag{3.15}$$

Furthermore, for any integer $n>2$, both

$$\frac{(-1)^{n(n-1)/2}|(i+j)^n|_{0\leqslant i,j<n}}{(n-2)!n\prod_{k=1}^n k!}\quad\text{and}\quad\frac{(-1)^{n(n-1)/2}|(i^2+j^2)^n|_{0\leqslant i,j<n}}{2\prod_{k=1}^n(k!(2k-1)!)}$$

are positive integers.

Remark 3.7. Note that for any prime $p=2n-1$ and $a\in\mathbb{Z}$ we have $a(\frac{a}{p})\equiv a^n\pmod p$.

Conjecture 3.8 (2013-08-12)**.** *Let $p\equiv 5\pmod 6$ be a prime. Then*

$$\operatorname{ord}_p\left|\frac{1}{i^2-ij+j^2}\right|_{1\leqslant i,j\leqslant (p-1)/2}=\frac{p+1}{6}, \tag{3.16}$$

where $\operatorname{ord}_p x$ denotes the p-adic order of a rational number x. Also, we have

$$\left|\frac{1}{i^2-ij+j^2}\right|_{1\leqslant i,j\leqslant p-1}\equiv 2x^2\pmod p \tag{3.17}$$

for some $x \in \{1, \ldots, (p-1)/2\}$.

Remark 3.8. Compare this conjecture with Theorem 1.3(ii).

The $(n+1)\times(n+1)$ Hankel determinant associated with a sequence $a_0, a_1, \ldots$ of numbers is defined by $|a_{i+j}|_{0\leqslant i,j\leqslant n}$. The evaluation of this determinant is known for some particular sequences including Catalan numbers and Bell numbers (cf. [K99]).

Conjecture 3.9 (2013-08-17). *For any positive integers m and n, we have*

$$(-1)^n \left|H_{i+j}^{(m)}\right|_{0\leqslant i,j\leqslant n} > 0, \tag{3.18}$$

where $H_k^{(m)}$ denotes the m-th order harmonic number $\sum_{0<j\leqslant k} 1/j^m$.

Remark 3.9. The author also conjectured that for any prime $p \equiv 1 \pmod 4$ and $m = 2, 4, 6, \ldots$ we have

$$\left|H_{i+j}^{(m)}\right|_{0\leqslant i,j\leqslant (p-1)/2} \equiv 0 \pmod p. \tag{3.19}$$

This was later confirmed by C. Krattenthaler.

Conjecture 3.10. (i) (2013-08-14) *For Franel numbers $f_n := \sum_{k=0}^n \binom{n}{k}^3$ $(n = 0, 1, \ldots)$, the number $6^{-n}|f_{i+j}|_{0\leqslant i,j\leqslant n}$ is always a positive odd integer. In general, for any integer $r > 1$ and the r-th order Franel numbers $f_n^{(r)} := \sum_{k=0}^n \binom{n}{k}^r$ $(n = 0, 1, \ldots)$, the number $2^{-n}|f_{i+j}^{(r)}|_{0\leqslant i,j\leqslant n}$ is always a positive odd integer.*

(ii) (2013-08-20) *For any prime $p \equiv 1 \pmod 4$ with $p \not\equiv 1 \pmod{24}$, we have*

$$|f_{i+j}|_{0\leqslant i,j\leqslant (p-1)/2} \equiv 0 \pmod p. \tag{3.20}$$

Remark 3.10. See [Su13, A225776] for the sequence $|f_{i+j}|_{0\leqslant i,j\leqslant n}$ $(n = 0, 1, 2, \ldots)$.

Conjecture 3.11. (i) (2013-08-14) *For two kinds of Apéry numbers*

$$b_n := \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} \text{ and } A_n := \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \quad (n = 0, 1, 2, \ldots),$$

both

$$\frac{|b_{i+j}|_{0\leqslant i,j\leqslant n}}{10^n} \quad \text{and} \quad \frac{|A_{i+j}|_{0\leqslant i,j\leqslant n}}{24^n}$$

are always positive integers.

(ii) (2013-08-20) *For any prime p with $2 \nmid \lfloor p/10 \rfloor$ and $p \not\equiv 31, 39 \pmod{40}$, we have*

$$|b_{i+j}|_{0\leqslant i,j\leqslant (p-1)/2} \equiv 0 \pmod p. \tag{3.21}$$

Remark 3.11. See [Su13, A228143] for the sequence $|A_{i+j}|_{0\leqslant i,j\leqslant n}$ $(n = 0, 1, 2, \ldots)$.

Conjecture 3.12 (2013-08-20)**.** *For $n = 0, 1, 2, \ldots$ define*

$$c_n := \sum_{k=0}^{n} (-1)^k \binom{n}{k}^4 \quad \text{and} \quad d_n := \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}.$$

Then, for any odd prime p we have

$$|c_{i+j}|_{0 \leqslant i,j \leqslant p-1} \equiv \left(\frac{-1}{p}\right) \pmod{p} \quad \text{and} \quad |d_{i+j}|_{0 \leqslant i,j \leqslant p-1} \equiv 1 \pmod{p}. \tag{3.22}$$

Remark 3.12. See [Su13, A228304] for the sequence c_n $(n = 0, 1, 2, \ldots)$.

Conjecture 3.13. (2013-08-14) *For Catalan-Larcombe-French numbers*

$$P_n := \sum_{k=0}^{n} \frac{\binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2}{\binom{n}{k}} = 2^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k}^2 4^{n-2k} \ (n = 0, 1, \ldots),$$

the number $2^{-n(n+3)} |P_{i+j}|_{0 \leqslant i,j \leqslant n}$ is always a positive odd integer.

Remark 3.13. See [Sl, A053175] for some basic properties of Catalan-Larcombe-French numbers. We are able to prove for any odd prime p the supercongruence

$$|P_{i+j}|_{0 \leqslant i,j \leqslant p-1} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}. \tag{3.23}$$

Conjecture 3.14. (i) (2013-08-14) *For Domb numbers*

$$D_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k} \ (n = 0, 1, \ldots),$$

the number $12^{-n} |D_{i+j}|_{0 \leqslant i,j \leqslant n}$ is always a positive odd integer.

(ii) (2013-08-20) *For any prime p, we have*

$$|D_{i+j}|_{0 \leqslant i,j \leqslant p-1} \equiv \begin{cases} (\frac{-1}{p})(4x^2 - 2p) \pmod{p^2} & \text{if } p = x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \tag{3.24}$$

Remark 3.14. See [Sl, A002895] for some basic properties of Domb numbers, and [Su13, A228289] for the sequence $|D_{i+j}|_{0 \leqslant i,j \leqslant p_n - 1}$ $(n = 1, 2, 3, \ldots)$. It is known that any prime $p \equiv 1 \pmod{3}$ can be written uniquely in the form $x^2 + 3y^2$ with x and y positive integers.

Conjecture 3.15 (2013-08-22). *For $n = 0, 1, 2, \ldots$ let*

$$s_n := \sum_{k=0}^{n} \binom{n}{k}^2 C_k \quad \text{and} \quad S(n) = |s_{i+j}|_{0\leqslant i,j\leqslant n},$$

where C_k denotes the Catalan number $\binom{2k}{k}/(k+1) = \binom{2k}{k} - \binom{2k}{k+1}$.

(i) *$S(n)$ is always positive and odd, and not congruent to 7 modulo 8.*

(ii) *Let p be an odd prime. If $p \equiv 1 \pmod 3$ and $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 3$, then*

$$S(p-1) \equiv \left(\frac{-1}{p}\right)\left(2x - \frac{p}{2x}\right) \pmod{p^2}. \tag{3.25}$$

If $p \equiv 2 \pmod 3$, then

$$S(p-1) \equiv -\left(\frac{-1}{p}\right)\frac{3p}{\binom{(p+1)/2}{(p+1)/6}} \pmod{p^2}. \tag{3.26}$$

Remark 3.15. See [Sl, A086618] for the sequence s_n $(n = 0, 1, 2, \ldots)$, and [Su13, A228456] for the sequence $S(n)$ $(n = 0, 1, 2, \ldots)$.

Conjecture 3.16 (2013-08-21). *For $n = 0, 1, 2, \ldots$ let*

$$w_n := \sum_{k=0}^{\lfloor n/3\rfloor} (-1)^k 3^{n-3k} \binom{n}{3k}\binom{2k}{k}\binom{3k}{k} \text{ and } W(n) = |w_{i+j}|_{0\leqslant i,j\leqslant n}.$$

When $n \equiv 0, 2 \pmod 3$, the number $(-1)^{\lfloor (n+1)/3\rfloor} W(n)/6^n$ is always a positive odd integer. For any prime $p \equiv 1 \pmod 3$, if we write $4p = x^2 + 27y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 3$, then

$$W(p-1) \equiv \left(\frac{-1}{p}\right)\left(\frac{p}{x} - x\right) \pmod{p^2}.$$

Remark 3.16. See [Sl, A006077] for the sequence w_n $(n = 0, 1, 2, \ldots)$. The author's conjecture that $W(3n+1) = 0$ for all $n = 0, 1, 2, \ldots$, was confirmed by C. Krattenthaler.

Conjecture 3.17 (2013-08-15). *For any positive integer n, we have*

$$\left|B_{i+j}^2\right|_{0\leqslant i,j\leqslant n} < 0 \quad \text{and} \quad \left|E_{i+j}^2\right|_{0\leqslant i,j\leqslant n} > 0, \tag{3.28}$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers and $E_0, E_1, E_2, \ldots$ are Euler numbers.

Remark 3.17. We have many similar conjectures with Bernoulli or Euler numbers replaced by some other classical numbers.

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