

Preventing Exceptions to Robins InEquality

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Abstract

For sufficiently large n Ramanujan gave a sufficient condition for the truth Robin's InEquality $X(n) := \frac{\sigma(n)}{n \ln \ln n} < e^\gamma$ (RIE). The largest known violation of RIE is $n_8 = 5040$. In this paper Robin's multipliers are split into logarithmic terms \mathcal{L} and relative divisor sums \mathcal{G} . A violation of RIE above n_8 is proposed to imply oscillations that cause \mathcal{G} to exceed \mathcal{L} . To this aim Alaoglu and Erdős's conjecture for the CA numbers algorithm is used and the paper could almost be reduced to section 4.3 on pages 11 to 16.

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1 Introduction

1.1 Outline

Robin's Inequality $\frac{\sigma(n)}{n \ln \ln n} < e^\gamma$ (RIE) for sufficiently large n can be derived from Ramanujan's Lost Notebook as necessary condition for RH. Unfortunately his work was not published until 1997. The inequality can be derived from an asymptotic expression that emerged from the study of generalised highly composite and generalised superior highly composite numbers. Alaoglu and Erdős coined the terms superabundant (SA) and colossally abundant (CA) in 1944 and mentioned the role of transcendental number theory in the process of finding CA numbers.

Proposition. (*Ramanujan [40, (382)]*)

If RH is true, [40, §56], it follows that

$$\sum_{-1}(N) = e^\gamma \left(\ln \ln N - \frac{2(\sqrt{2}-1)}{\sqrt{\ln n}} + S_1(\ln N) + \frac{O(1)}{\sqrt{\ln N \ln \ln N}} \right).$$

Conclusion. There is an n_0 such that $\frac{\sigma(n)}{n} < e^\gamma \ln \ln n$ for all $n > n_0$.

[Rf. notes at the end of [40].]

Robin clarified the meaning of “sufficiently large” in 1984 by finding

1. that the function $X(n) := \frac{\sigma(n)}{n \ln \ln n}$ takes maximal values on CA numbers.
2. It is sufficient for RH that RIE holds true for sufficiently large n , i.e. for $n > 5040$.
3. The oscillation theorem $X(n) = e^\gamma \cdot \left(1 + \Omega_\pm \left((\ln n)^{-b}\right)\right)$ in CA numbers if RH is false.
4. $X(n)$ has an unconditional bound $B(n)$ for some $B(n) = e^\gamma + o(1)$.

The major tool were estimations with Chebyshev's functions ψ and ϑ using the results of Rosser and Schoenfeld. In order to show that X takes maximal values on CA numbers Robin used a multiplier consisting of ratios of relative divisor sums $\sigma_{-1}(n)$ of consecutive CA numbers and iterated logarithms. The argument is iterated in this report which proves that X takes a greater value on a subsequent CA number if the product of ratios of relative divisor sums of intermediate CA numbers exceeds the respective product of ratios of iterated logarithms. But the CA numbers algorithm relies on the quotient of consecutive CA numbers to be prime which is not guaranteed unless Alaoglu and Erdős' special case of the Four Exponentials Conjecture is true.

The point of this investigation has been to find out if the minimal oscillations in case RH is false will force the products of ratios of relative divisor sums to become greater than the corresponding products of ratios of iterated logarithms. This has been achieved by finding a template for the quotient of maximal and minimal values of $X(n)$ as n proceeds in CA numbers and analysing the template with polar coordinates.

The paper is primarily organised as a chain of reductions that is summarised in the final Conclusion 4.32. Section 1.2 establishes the need to find multiples of every natural n on which X takes a greater value than it takes on n . Such multiples prevent n from being an exception. Section 2 demonstrates how the multipliers used by Robin work and how they can be split. This method is iterated in Section 3 to show the sufficiency of testing $\mathcal{G} > \mathcal{L}$ for $\mathcal{G} = \frac{\sigma_{-1}(nx)}{\sigma_{-1}(n)}$ and $\mathcal{L} = \frac{\ln \ln nx}{\ln \ln n}$. Similar conditions were found in [33, 35, 34] during the course of my investigation. The setup of the latter two reports is summarised using the present setup as a part of section 4 after presenting some numerical data. Then Mertens' theorem motivates expecting the truth of $\mathcal{G} > \mathcal{L}$ before Robin's oscillation theorem is used in section 4.3 to propose an indirect proof.

1.2 Preparation

Notation. Let $X(n) := \frac{\sigma(n)}{n \ln \ln n}$ with the sum of divisors σ , write RIE(n) short for Robin's InEquality $X(n) < e^\gamma$, [46], and denote the set of primes $\{p_n\}_{n=1}^\infty = 2, 3, 5, \dots$ by \mathbb{P} . The k th largest prime factor of an integer n is denoted by $P_k(n)$, [42, 5.17]. Also let $[a, b]_{\mathbb{N}} := [a, b] \cap \mathbb{N}$.

Grönwall [17] mentioned that the asymptotic behaviour of the function $Y(n) := \frac{\varphi(n)}{n} \cdot \ln \ln n$ had been studied by Landau, [25]. Then he proved Theorem 1.3 below. Rf. [36, 13] for Nicolas' inequality and [52, 53] for approaches with the Dedekind ψ function. Suppose

Condition 1.1. For every $n > 5040$ there is a number x such that $X(nx) > X(n)$.

Note. This section establishes

Claim 1.2. RIE(n) is true for all $n > 5040$.

If the opposite of Condition 1.1 was true for some $n > 5040$ the number n may be said to be **exceptional** since no such n is known so far. Without requiring $n > 5040$ this is called GA2 in [8, p. 2]. Known GA2 numbers are 3, 4, 5, 6, 8, 10, 12, 18, 24, 36, 48, 60, 72, 120, 180, 240, 360, 2520, and 5040. Recall

Theorem 1.3. (*Grönwall*) $\limsup_{n \rightarrow \infty} X(n) = e^\gamma.$

This is easily extended.

Theorem 1.4. $\limsup_{x \rightarrow \infty} X(n \cdot x) = e^\gamma = 1.78107\ 24179\ 90197\dots$

Proof. An adaption of [20, §22.9]. Rf. [24, App. A] for the numerical value. \square

Definition 1.5. $C := \left(\frac{7}{3} - e^\gamma \cdot \ln \ln 12\right) \cdot \ln \ln 12 \approx 0.64821365$, rf. [7, Theorem 1.1], [35, Eq. (1.4)], or [49, Lemma 13] and [50, Theorem 7].

Theorem 1.6. (*[46, Théorème 2]*)

$X(n) \leq B(n) := e^\gamma + C \cdot (\ln \ln n)^{-2}$ for all $n \in \mathbb{N} \setminus \{1, 2, 12\}$.

Thus, assuming Condition 1.1 it is easily seen that a minimal counterexample of RIE above 5040 contradicts Theorem 1.6 since for any number n Condition 1.1 implies the existence of a non-decreasing sequence of values of X that starts at $X(n)$. If $X(n) > e^\gamma$ choose $x_1 \cdots x_k$ such that

1. $X(n \cdot y_{i+1}) > X(n \cdot y_i)$ for all $i \in [0, k-1]_{\mathbb{N}}$ where $y_i := \prod \{x_j; j \in [1, i]_{\mathbb{N}}\}$ and
2. $C \cdot \ln \ln(n \cdot y_k)^{-2} < X(n) - e^\gamma$, i.e. $n \cdot y_k > \exp\left(\exp\left(\sqrt{\frac{C}{X(n) - e^\gamma}}\right)\right) = B^{-1}(X(n))$.

The contradiction $X(n \cdot y_k) > e^\gamma + C \cdot \ln \ln(n \cdot y_k)^{-2}$ to Theorem 1.6 follows. Thus

Theorem 1.7. *Claim 1.2 follows from Condition 1.1.*

Assuming Condition 1.1 a consequence of [8, Thm 5] is

Corollary 1.8. *There is no GA2 number $n > 5040$. In other words there is no exceptional number.*

Proving the absence of exceptional numbers seems to be just as difficult as proving Condition 1.1. This is no surprise because a one is an indirect proof of the other.

2 Colosally Abundant Numbers

Definition 2.1. Rf. [62, Superabundant and colosally abundant number]

1. n is SA if it meets $\sigma_{-1}(n) \geq \sigma_{-1}(k)$ for all $k < n$, rf. [64, p. 839], [51, A004394].
2. n is CA if $\sigma_{-1}(n) \cdot n^{-\varepsilon} \geq \sigma_{-1}(k) \cdot k^{-\varepsilon}$ for all k and an $\varepsilon > 0$, rf. [51, A004490].

By Theorem 1.3 there are infinitely many SA numbers but they are only mentioned here because the SA property suffices to determine the asymptotic behaviour of $P_1(n)$.

Fact 2.2.

1. By [16, p. 68] CA numbers are SA.
2. If n is SA and p the largest prime factor in n then $p \sim \ln n$ by [3, Theorem 7].

Rf. [37, 38] for more information.

“A superparticular number is when a great number contains a lesser number, to which it is compared, and at the same time one part of it.” Rf. [57, p.III.6.12,n.7].

Notation 2.3. Let $F(x, v) := \frac{1}{\ln x} \ln(G(x, v))$ with $G(x, v) := 1 + \frac{1}{g(x, v)}$; $g(x, v) := \sigma(x^v) - 1$. In virtue of assumption 2.9 below the parameters ε of CA numbers belong to the set $\mathcal{E} := \bigcup_{p \in \mathbb{P}} \{F(p, v); v \in \mathbb{N}\}$. Write E as decreasing sequence $\mathcal{E} = (\varepsilon_i)_{i=1}^{\infty}$ allowing for the definition $n_i := n(\varepsilon_i)$, rf. [40, 3, 16, 7]. Additionally put $q_{i+1} := \frac{n_{i+1}}{n_i}$ and $g_i := G(q_i, v_{q_i}(n_i))$ such that $\varepsilon_i = F(q_i, v_{q_i}(n_i)) = \log_{q_i}(g_i)$ and $q_i^{\varepsilon_i} = g_i = \frac{\sigma_{-1}(n_i)}{\sigma_{-1}(n_{i-1})}$ is superparticular. Let $q_0 := 1 =: n_0$, $\varepsilon_0 = 1$, and $L_i(n) := \frac{\ln(\ln(n))}{\ln(\ln(n_i))}$, too.

Definition 2.4. Let f_ε denote the function $x \mapsto \varepsilon \cdot x - \ln(\ln(x))$ and Robin's multiplier be $R_j(n, \varepsilon) := \exp(f_\varepsilon(\ln n) - f_\varepsilon(\ln n_j)) = \left(\frac{n}{n_j}\right)^\varepsilon \cdot (L_j(n))^{-1}$.

Note 2.5. 1. The derivatives of f_ε are given by

$$f'_\varepsilon(x) = \varepsilon - \frac{1}{x \cdot \ln(x)} \quad \text{and} \quad f''_\varepsilon(x) = \frac{1}{x^2 \ln(x)} + \frac{1}{(x \cdot \ln(x))^2} > 0$$

which implies that $f_\varepsilon \in C^2(1, \infty)$ is convex for every $\varepsilon > 0$.

2. $f_\varepsilon(x) \rightarrow \infty$ as $x \rightarrow 1$ or as $x \rightarrow \infty$ since logs grow slower than any power of x .
3. If $\varepsilon_1 < \varepsilon_2$ then $f_{\varepsilon_1}(x) < f_{\varepsilon_2}(x)$ for all x because $f_{\varepsilon_2}(x) - f_{\varepsilon_1}(x) = (\varepsilon_2 - \varepsilon_1) \cdot x > 0$. In particular f_{ε_1} and f_{ε_2} do not intersect if $\varepsilon_1 \neq \varepsilon_2$.
4. $R_i(n_{i+1}, \varepsilon_{i+1}) \cdot L_i(n_{i+1}) = g_{i+1}$ and $X(n_i) \cdot R_i(n_{i+1}, \varepsilon_{i+1}) = X(n_{i+1})$.
5. $\varepsilon = \varepsilon_{i+1}$ if $\sigma_{-1}(n_i) \cdot n_i^{-\varepsilon} = \sigma_{-1}(n_{i+1}) \cdot n_{i+1}^{-\varepsilon}$.

Proposition 2.6. ([46, §3, Prop 1], [44, p. 237])

Colosally abundant numbers maximise X , i.e. $X(n) \leq \max(X(n_i), X(n_{i+1}))$ if $n \in [n_i, n_{i+1}]_{\mathbb{N}}$.

Proof. $X(n) \leq X(n_j)$ follows from $R_j(n, \varepsilon) \leq 1$ since $X(n) \leq X(n_j) R_j(n, \varepsilon)$ follows from $\sigma_{-1}(n) \cdot n^{-\varepsilon} \leq \sigma_{-1}(n_j) \cdot n_j^{-\varepsilon}$ with Note 2.5, #4/5 if $n \in [n_i, n_{i+1}]_{\mathbb{N}}$, $\varepsilon = \varepsilon_{i+1}$, and $j \in \{i, i+1\}$. For $\frac{n^\varepsilon}{\ln \ln n} \leq \frac{n_j^\varepsilon}{\ln \ln n_j}$ it is sufficient to show the consequence $f_\varepsilon(\ln n) \leq \max(f_\varepsilon(\ln n_i), f_\varepsilon(\ln n_{i+1}))$ of Note 2.5, #1. \square

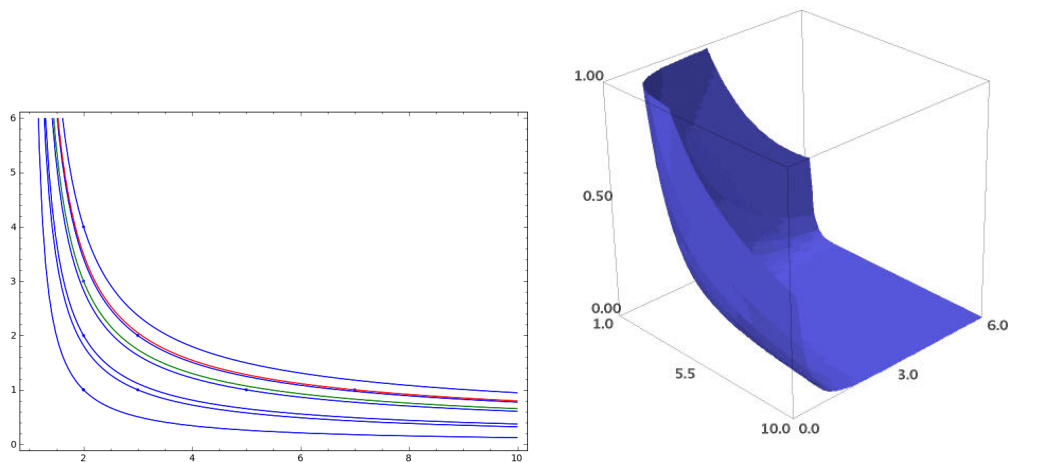


Figure 2.1: The surface $\varepsilon = F(x, v)$

Based on this setup the algorithm computing the sequence $(n_i)_i$ of CA numbers seems to be well-understood, [39, 3, 16, 46, 40, 37, 7, 38, 13, 8, 9, 51]. Nevertheless there is the open

Question 2.7.

1. Can the algorithm find a new violation of RIE? (Does one exist?)
2. Do two consecutive CA numbers exist whose quotient q_i is semiprime?

Item 2. has been given a negative answer under the following Conjecture, rf. [3, p. 455].

Conjecture 2.8. (*Four Exponentials*)

If x_1, x_2 and y_1, y_2 are two pairs of complex numbers, with each pair being linearly independent over the rational numbers, then at least one of the following four numbers is transcendental:

$$e^{x_1 y_1}, e^{x_1 y_2}, e^{x_2 y_1}, e^{x_2 y_2},$$

rf. [16, p. 71], [27, ch. 2], [60, ch. 2], [61, Section 1.3].

Semiprime quotients cause unexpected difficulties. Therefore I assume a special case of Conjecture 2.8.

Assumption 2.9. (*Alaoglu and Erdős*)

For any two distinct prime numbers p and q , the only real numbers t for which both p^t and q^t are rational are the positive integers.

Conclusion 2.10. q_i is prime for all i , rf. [62, Colossally abundant number].

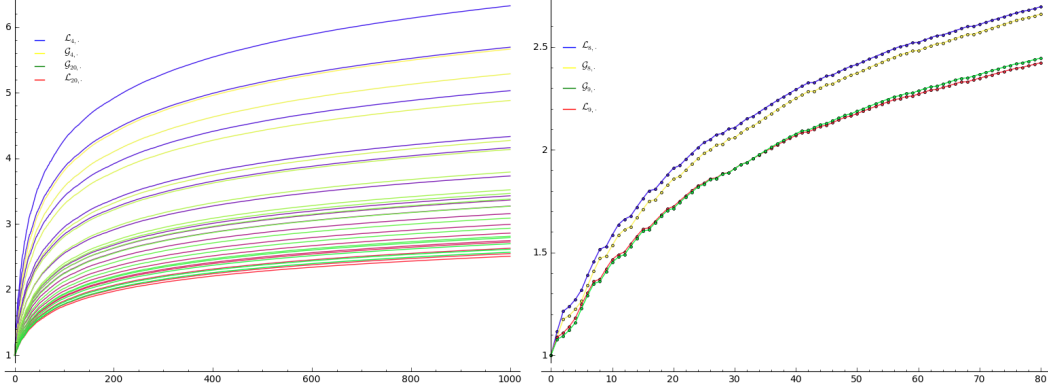


Figure 3.1: “The $\mathcal{G}_{i,\cdot}$ ’s are enclosed by the $\mathcal{L}_{i,\cdot}$ ’s”

3 Subsequent Maximisers

3.1 Extending Robin’s Method

Robin’s crucial argument was Proposition 2.6. A Transfer to Condition 1.1 follows.

Definition 3.1. Let $\mathcal{Q}_{i,k} := \prod_{j=1}^k q_{i+j}$, $\mathcal{R}_{i,k} := \prod_{j=1}^k R_{i+j-1}(n_{i+j}, \varepsilon_{i+j})$, $\mathcal{L}_{i,k} := \prod_{j=1}^k L_{i+j-1}(n_{i+j}) = \frac{\ln \ln n_{i+k}}{\ln \ln n_i}$, and $\mathcal{G}_{i,k} := \prod_{j=1}^k g_{i+j} = \frac{\sigma_{-1}(n_{i+k})}{\sigma_{-1}(n_i)}$. Contextually write $\mathcal{X}_{i,\cdot}$ for $(\mathcal{X}_{i,j})_{j \in \mathbb{N}}$ or $\{\mathcal{X}_{i,j}\}_{j \in \mathbb{N}}$ if $\mathcal{X} \in \{\mathcal{R}, \mathcal{G}, \mathcal{L}, \mathcal{Q}, \mathcal{D}\}$ where $\mathcal{D}_{i,k} := \mathcal{G}_{i,k} - \mathcal{L}_{i,k}$. Put $k_i := \inf \{k \in \mathbb{N}; \mathcal{R}_{i,k} \geq 1\}$.

Fact. $\frac{\mathcal{G}_{i,k}}{\mathcal{L}_{i,k}} = \frac{\sigma_{-1}(n_{i+k})}{\ln \ln n_{i+k}} \cdot \frac{\ln \ln n_i}{\sigma_{-1}(n_i)} = \frac{X(n_{i+k})}{X(n_i)}$.

Lemma 3.2. $\mathcal{R}_{i,k} \cdot \mathcal{L}_{i,k} = \mathcal{G}_{i,k}$ and $X(n_i) \cdot \mathcal{R}_{i,k} = X(n_{i+k})$.

Proof. By induction on k using Lemma 2.5, #4. □

Corollary 3.3. If $k_i < \infty$ then $\mathcal{G}_{i,k_i} \geq \mathcal{L}_{i,k_i}$ so $\mathcal{G}_{i,k} < \mathcal{L}_{i,k}$ for all k if $k_i = \infty$ but $k \geq k_i$ if $\mathcal{D}_{i,k} \geq 0$.

Theorem 3.4. n_i is exceptional if and only if $k_i = \infty$ and $i > 8$.

Proof. $X(n_i) > X(n_{i+k})$ for all k iff n_i is exceptional. On the the other hand $\mathcal{R}_{i,k} < 1$ for all k iff $k_i = \infty$ and the claim follows with Lemma 3.2. □

Condition 3.5. $k_i < \infty$ for all $i > 8$.

Theorem 3.6. Conditions 3.5 and 3.7 are equivalent to Condition 1.1 with $n = n_i$ and $x = \mathcal{Q}_{i,k_i}$.

Proof. For the bounds on i consider $n_8 = 5040$ and section 3.2.

All other statements follow from Lemma 3.2. □

Condition 3.7. For every n_i with $i > 143215$ there is some k such that $\mathcal{D}_{i,k} \geq 0$.

3.2 Number Crunching

1. Sage led me to my first results, [55]. My algorithm passed the CA numbers in table 1. According to [37, 38] T.D.Noë's form $(\alpha_v)_{v=1}^n$ represents $\prod_{v=1}^n \prod_{j=\pi(\alpha_{v+1})+1}^{\pi(\alpha_v)} p_j^v$ if α contains no zeros and $\alpha_{n+1} = 0 \wedge p_0 = 1$. Thus $n_8 = (7, 3, 0, 2)$, $n_{508} = (3257, 73, 19, 7, 5, 0^2, 3, 0^5, 2)$, $n_9 = (11, 3, 0, 2)$, $n_{42} = (101, 13, 5, 0, 3, 0^2, 2)$, $n_{2386} = (20359, 193, 37, 13, 7, 0, 5, 0^2, 3, 0^6, 2)$, $n_{143215} = (1911373, 1907, 173, 47, 23, 13, 0, 7, 0, 5, 0^4, 3, 0^8, 2)$, and $n_{13} = (13, 5, 3, 0, 2)$.

	n_8	n_{508}	n_9	n_{13}	n_{42}	n_{2386}	n_{143215}
$v_2(\cdot)$	4	14	4	5	8	17	24
\ln	8.5251	3274.0	10.9230	16.889	107.7176	20432.8	1912150.6
$\ln(P_1(\cdot))$	1.9459	8.0885	2.3978	2.5649	4.6151	9.9212	14.4633
$\ln\left(-\frac{1}{\varepsilon \ln(\varepsilon)}\right)$	1.9356	7.8588	2.1174	2.5342	4.3330	9.7132	14.2938
$\ln \ln$	2.1430	8.0937	2.3908	2.8266	4.6795	9.9249	14.4637
k_i	∞	1	33	1	1	1	1
\lg	3.7024	1421.9	4.7438	7.335	46.7811	8873.60	830436.46
$\lg \lg$	0.5684	3.1528	0.6761	0.8654	1.6700	3.9480	5.9193
σ_{-1}	3.8380	14.3887	4.1870	4.8559	8.1962	17.6663	25.7599
X	1.79097	1.7777	1.7512	1.7179	1.7515	1.7800001	1.781000003
B	1.9047	1.79096	1.8944	1.8621	1.8106	1.7877	1.7842
$\ln \ln B^{-1} \circ X$	8.0913	undefined since $X(n) < e^\gamma = 1.7811$					

Table 1: Statistics of some CA numbers

Theorem 3.8. For every CA n_i if $i \leq 143215$ there is a subsequent CA n_{i+j} such that $X(n_i) < X(n_{i+j})$.

Proof. RIE(n) was confirmed in every loop. (Not shown in Appendix A.) \square

2. Keith Briggs reported to me: "E.g. the following is a CA number:

$$n = 2^{41} \cdot 3^{25} \cdot 5^{17} \cdot 7^{14} \cdot 11^{11} \cdot 13^{10} \cdot 17^9 \dots 23^8 \dots 37^7 \dots 53^6 \dots 101^5 \dots \\ \dots 239^4 \dots 887^3 \dots 7789^2 \dots 562399^1 \dots 162216342187^0$$

with $\log \log n$ about 26." Denote it by n_{Br} and call the tuple $(2, 0^{15}, 3, 0^7, 5, 0^2, 7, 0^2, 11, 13, 17, 23, 37, 53, 101, 239, 887, 7789, 562399, 162216342187)$ *bottom-up form*. Since he reached about $10^{10^{10}}$ according to [7, §3]

$$\lg \lg n_{\text{Br}} \approx \lg(\lg(\exp(\exp(26)))) = \lg\left(\frac{\exp(26)}{\ln 10}\right) = \frac{1}{\ln 10} (26 - \ln \ln 10) \approx 10.9294$$

reveals $\ln \ln n_{\text{Br}} \approx 26$ in compliance with [56, Section 10.1.2] and Fact 2.2. This was significantly more than the theoretically obtained bound $10^{8576} < e^{19747}$ from [8, Corollary 1] which my rather short calculation capped, too. Noë's *top-down* representation is

$$(162216342179, 562361, 7759, 883, 233, 97, 47, 31, 19, 13, 11, 0^2, 7, 0^2, 5, 0^7, 3, 0^{15}, 2).$$

Theorem 3.9. For every CA n_i such that $\ln \ln n_i \leq 25 < 26$ there is a subsequent CA n_{i+j} such that $X(n_i) < X(n_{i+j})$.

$i+j$	v_2	q_{i+j}	p	\ln	$v_{i,j}$	g_{i+j}	$\mathcal{G}_{8,j}$	ε_{i+j}	ll	$\mathcal{L}_{8,j}$	$-\mathcal{D}_{9,j}$
8	4	2	7	8.5	4	31:30	1	$4.73d^2$	2.143	1	
9	4	11	11	10.9	1	12:11	1.090	$3.62d^2$	2.390	1.115	1
10	4	13	13	13.4	1	14:13	1.174	$2.88d^2$	2.601	1.214	$1.13d^2$
11	5	2	13	14.1	5	63:62	1.193	$2.31d^2$	2.651	1.237	$1.48d^2$
12	5	3	13	15.2	3	40:39	1.224	$2.30d^2$	2.726	1.272	$1.80d^2$
13	5	5	13	16.8	2	31:30	1.265	$2.03d^2$	2.826	1.319	$2.25d^2$
14	5	17	17	19.7	1	18:17	1.339	$2.01d^2$	2.981	1.391	$1.91d^2$
15	5	19	19	22.6	1	20:19	1.410	$1.74d^2$	3.120	1.456	$1.27d^2$
16	5	23	23	25.8	1	24:23	1.471	$1.35d^2$	3.250	1.516	$1.07d^2$
17	6	2	23	26.4	6	127:126	1.483	$1.12d^2$	3.276	1.529	$1.11d^2$
18	6	29	29	29.8	1	30:29	1.534	$1.00d^2$	3.396	1.584	$1.42d^2$
19	6	31	31	33.2	1	32:31	1.583	$9.24d^3$	3.505	1.635	$1.44d^2$
20	6	7	31	35.2	2	57:56	1.612	$9.09d^3$	3.562	1.662	$1.22d^2$
21	6	3	31	36.3	4	121:120	1.625	$7.55d^3$	3.592	1.676	$1.27d^2$
22	6	37	37	39.9	1	38:37	1.669	$7.38d^3$	3.687	1.720	$1.21d^2$
23	6	41	41	43.6	1	42:41	1.710	$6.48d^3$	3.776	1.762	$1.19d^2$
24	6	43	43	47.4	1	44:43	1.749	$6.11d^3$	3.859	1.800	$1.00d^2$
25	7	2	43	48.1	7	255:254	1.756	$5.66d^3$	3.873	1.807	$9.82d^3$
26	7	47	47	51.9	1	48:47	1.794	$5.46d^3$	3.950	1.843	$7.75d^3$
27	7	53	53	55.9	1	54:53	1.828	$4.70d^3$	4.024	1.877	$7.51d^3$
28	7	59	59	60.0	1	60:59	1.858	$4.12d^3$	4.094	1.910	$8.54d^3$
29	7	5	59	61.6	3	156:155	1.870	$3.99d^3$	4.121	1.923	$8.61d^3$
30	7	61	61	65.7	1	62:61	1.901	$3.95d^3$	4.185	1.953	$7.51d^3$
31	7	67	67	69.9	1	68:67	1.930	$3.52d^3$	4.247	1.982	$7.42d^3$
32	7	71	71	74.2	1	72:71	1.957	$3.28d^3$	4.306	2.009	$7.25d^3$
33	7	73	73	78.4	1	74:73	1.984	$3.17d^3$	4.363	2.035	$6.18d^3$
34	7	11	73	80.8	2	133:132	1.999	$3.14d^3$	4.393	2.049	$4.99d^3$
35	7	79	79	85.2	1	80:79	2.024	$2.87d^3$	4.445	2.074	$3.79d^3$
36	8	2	79	85.9	8	511:510	2.028	$2.82d^3$	4.453	2.078	$3.54d^3$
37	8	83	83	90.3	1	84:83	2.052	$2.71d^3$	4.503	2.101	$2.11d^3$
38	8	3	83	91.4	5	364:363	2.058	$2.50d^3$	4.516	2.107	$1.98d^3$
39	8	89	89	95.9	1	90:89	2.081	$2.48d^3$	4.563	2.129	$8.18d^4$
40	8	97	97	100	1	98:97	2.103	$2.24d^3$	4.610	2.151	$6.26d^4$
41	8	13	97	103	2	183:182	2.114	$2.13d^3$	4.635	2.163	$5.70d^4$
42	8	101	101	107	1	102:101	2.135	$2.13d^3$	4.679	2.183	$-3.05d^4$
\vdots							\vdots			\vdots	
507	14	3253	3253	3265	1	3254:3253	3.747	$3.80d^5$	8,091	3.775	$-5.12d^2$
508	14	3257	3257	3274	1	3258:3257	3.748	$3.79d^5$	8.093	3.776	$-5.12d^2$

Table 3: Quotients q_{i+j} of CA numbers, note $k_9 = 33$, [51, A073751] and abbreviate $p = P_1(n_{i+j})$, $ll = \ln \ln n_{i+j}$, $v_{i,j} = v_{q_{i+j}}(n_{i+j})$ and $d = 10^{-1}$.

4 The Question of Life

The next Lemma has a long track in my notes since the preprint of [13, Lemma 6.1] was not hard to complement in the present setup.

Proposition 4.1. $k_i = 1$ if

$$\left(\sum_{l=1}^{v_{i,1}} q_{i+1}^l \right) \ln q_{i+1} \leq \ln n_i \cdot \ln \ln n_i.$$

Proof. $\mathcal{G}_{i,1} = 1 + \left(\sum_{l=1}^{v_{i,1}} q_{i+1}^l \right)^{-1} \geq 1 + \frac{\ln \mathcal{Q}_{i,1}}{\ln n_i \cdot \ln \ln n_i}$ by assumption. A Taylor approximation of $\log_{\ln n_i}(\ln n_i + h)$ for $h = \ln(\mathcal{Q}_{i,k})$ has remainder term $-\frac{1}{2} \cdot \left(\frac{\ln \mathcal{Q}_{i,k}}{\ln n_i + \vartheta \ln \mathcal{Q}_{i,k}} \right)^2 \cdot \frac{1}{\ln \ln n_i} < 0$. Therefore the case $k = 1$ yields $\mathcal{D}_{i,1} \geq 0$. \square

Most recently, Morkotun demonstrated in [33, Theorem 2] how to include all prime factors of n_i without requiring the CA or SA property of n_i . The existence of a sequence on which X increases follows from Grönwall's theorem if exceptional numbers do not exist. But if there are exceptional numbers X will stop to take larger values because of Robin's unconditional bound. [33, (4)] is often met but once in a while abundant numbers have prime factors larger than $\ln n$ in which case it would have been possible to argue with Lemma 4.2 below. Likewise it is very possible that the sequence of i 's with $k_i > 1$ is infinite although the gaps between regions with $k_i > 1$ may be large.

Lemma 4.2. [13, Lemma 6.1]: If $\ln n < P_1(n)$ for a t -free n with $t \geq 2$ then $RIE(n)$.

However, applying either the Proposition above or Morkotun's condition of RIE it is sufficient to consider the greatest primes p with $v_p(n) = v$ for each valuation v between $1 = v_{P_1(n)}(n_i)$ and $v_2(n_i)$. This is reflected by Noe's representation of SA numbers in section 3.2. In each loop the CA numbers algorithm chooses the q_i for which $\varepsilon_{i+1} = \frac{1}{\ln q_{i+1}} \ln g_{i+1}$ is maximal when q_{i+1} varies over the primes p for which $n = n_i p$ meets $v_q(n) < v_p(n)$ if $q > p$.

4.1 Extremely Abundant Numbers

The recent papers [35, 34] will be summarised in the context of the present one. After some quotations from the follow-up paper this section employs the numbers of the text modules in [35].

Definition. (2.1): [34, Def. 1.2, 1.3, and 1.8]

1. $n \in XA$ iff $X(n) > X(m)$ for $m \in [10080, n]_{\mathbb{N}}$,
2. $n \in XA'$ iff $\frac{\sigma_{-1}(n)}{\sigma_{-1}(m)} > 1 + \frac{\ln n - \ln m}{\ln n \cdot \ln \ln m}$ for $m = \max\{k \in XA'; k < n\}$, and
3. $n \in XA''$ iff $\frac{\sigma_{-1}(n)}{\sigma_{-1}(m)} > 1 + 2 \frac{\ln n - \ln m}{(\ln n + \ln m) \cdot \ln \ln m}$ for $m = \max\{k \in XA''; k < n\}$.

Conclusion. [34, (6)] $XA \subseteq XA' \subseteq SA$.

Lemma. [34, Lemma 1.4] XA' is well-defined.

Theorem. [34, Theorem 1.7] $|XA'| = \infty$.

Theorem. (2.3): *The least $n > 5040$ such that $\text{RIE}(n)$ is false is in XA .*

Theorem. (2.4): $\text{RH} \iff |XA| = \infty$.

Theorem. (4.28): $\text{RH} \implies |CA \cap XA| = \infty$.

Theorem. (4.31): $|n \in CA; \ln n < P_1(n)| = \infty$.

Theorem. (4.32): $n \in XA \implies P_1(n) < \ln n$.

Theorem. (4.34): $|CA \setminus XA| = \infty$.

Essentially Theorem (2.4) asserts the necessity and sufficiency of Condition 1.1 for RH. Theorem (4.28) provides a necessary condition by restriction to CA numbers without mentioning the obvious reverse implication in virtue of Theorem (2.4). In particular, $|CA \setminus XA| = \infty$ and $|CA \cap XA| = \infty$ in case of RH make this case delicate. The advantage is the minimality condition of Theorem (2.3) at the cost of losing the availability of an algorithm that computes the sequence of hypothetical counterexamples of RIE.

4.2 Stronger Ingredients

The goal of this section is to show that Condition 3.7 is true. The subsection's title insinuates Assumption 4.5. The easiest step towards it was quoting Lemma 4.2.

Thus RIE is not violated unless the prime divisors of n_i cumulate too densely and n_9 is the only CA number in section 3.2 with $P_1(n) > \ln n$. On the other hand by [3, Thm 2] there must not be too many small prime divisors for RIE.

Theorem 4.3. *(Mertens, [32], [20, Thm 427-429]) The Meissel–Mertens constant is given by*

$$B_1 = \lim_{n \rightarrow \infty} \left(\sum_{p \leq n} \frac{1}{p} - \ln \ln n \right) = \gamma + \sum_p \left(\ln \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right) = 0.26149\ 72128\ 47642\dots$$

Note. Interesting additional references are [58] and [29, §1 (1)].

It can be considered reasonable to assume that $\mathcal{G}_{i,\cdot}$ grows at least as fast as $\mathcal{L}_{i,\cdot}$ for increasing k . This conjecture is based on Theorem 4.3 and the culmination of the work on the asymptotics of p_k , [48, 47, 45, 31] in P. Dusart's statement $p_k \geq k(\ln k + \ln \ln k - 1)$, [15] after $p_n \geq n(\ln n + \ln \ln n - 1 + o(\frac{\ln \ln n}{\ln n})) + k$ had become available in [41] without guaranteeing Dusart's lower bound, yet.

Lemma 4.4. $\mathcal{G}_{i,k} \rightarrow \infty$ and $\mathcal{L}_{i,k} \rightarrow \infty$ as $k \rightarrow \infty$ and $\mathcal{D}_{i,1} > 0$ if and only if $g_{i+1} - 1 > 2 \operatorname{artanh} \left(\frac{\ln q_{i+1}}{2 \ln n_i + \ln q_{i+1}} \right)$.

Proof. For every prime $p > P_1(n_i)$ there is some k such that $v_p(n_{i+k}) = 1$. Therefore $\frac{1}{p}$ occurs as a summand when expanding $\mathcal{G}_{i,k} := \prod_{j=1}^k \left(1 + \frac{1}{g_{v_{i,j}}(x)} \right)$ which must be a bound for the summed reciprocals of subsequent primes, i.e. $R_{i,k} := \sum \{p^{-1}; P_1(n_i) < p < P_1(n_{i+k})\} < \mathcal{G}_{i,k}$. But $R_{i,k} \rightarrow \infty$ as $k \rightarrow \infty$ by Theorem 4.3. With $y = \frac{1+x}{1-x}$ iff $x = \frac{y-1}{y+1}$ and $\ln(y) = 2 \operatorname{artanh}(x)$

$$\mathcal{L}_{i,k} = \log_{\ln n_i}(\ln n_{i+k}) = 1 + \log_{\ln n_i} \left(1 + \frac{\ln Q_{i,k}}{\ln n_i} \right)$$

can be expanded to

$$\log_a(a+z) - 1 = \frac{2}{\ln a} \left(\frac{z}{2a+z} + \frac{1}{3} \left(\frac{z}{2a+z} \right)^3 + \frac{1}{5} \left(\frac{z}{2a+z} \right)^5 + \dots \right)$$

for $a = \ln n_i > 0$ and $z = \ln \mathcal{Q}_{i,k} \geq -a$, rf. [1, 4.1.29]. The conclusion can be drawn by using $\mathcal{G}_{i,1} - 1 = g_{i+1} - 1$. (I came across the last formula in some book dealing with elliptic functions, too. Unfortunately I seem to be unable to find it again.) \square

Assumption 4.5. $\mathcal{D}_{i,\cdot}$ has at most one change of sign.

This is quite a strong assumption. Given Littlewood's theorem on the difference $\pi(x) - \text{Li}(x)$, [30] and Robin's theorem on $\sum \{p^{-1}; n \geq p \in \mathbb{P}\} - \ln \ln n - B_1$, [44, Théorème 2] it makes sense to assume the opposite. Viewing B_1 as safty buffer between $\sum \{p^{-1}; n \geq p \in \mathbb{P}\}$ and $\ln \ln n$ may render Assumption 4.5 reasonable and it could probably be deduced from [35, Thm 4.21] if its implied constant is not too large.

In virtue of Theorem 3.6 it is easy to derive Condition 1.1 from Assumption 4.5. For this the key is to derive

$$\begin{aligned} \forall k \geq k_i : \ln q_{i+k+1} &> (\ln n_{i+1})^{\mathcal{G}_{i+1,k}} - (\ln n_i)^{\mathcal{G}_{i,k}} \\ &> (\ln q_{i+1})^{\mathcal{G}_{i+1,k}} + (\ln n_i)^{\mathcal{G}_{i+1,k}} - (\ln n_i)^{\mathcal{G}_{i,k}} \end{aligned}$$

from $k_i < \infty$ and $k_{i+1} = \infty$. This in turn can be done with the equivalence of $\mathcal{D}_{i,k} > 0$ and $\ln \mathcal{Q}_{i,k} < (\ln n_i)^{G(q_{i+1}, v_{i,1}) \dots G(q_{i+k}, v_{i,k})} - \ln n_i$. Therefore if $k_{i+1} = \infty$ then small primes could occur only finitely often in the sequence $(q_i)_{i=1}^{\infty}$ which contradicts the CA numbers algorithm. It should have been possible to reduce Assumption 4.5 to requiring that $\mathcal{D}_{i,\cdot}$ has only finitely many changes of sign the last of which being from - to +. However, Assumption 4.5 remains undecided.

The idea that $\mathcal{L}_{i,k}$ converges faster to 1 than $\mathcal{G}_{i,k}$ as $i \rightarrow \infty$ - or equivalently $\mathcal{G}_{i,\cdot}$ does not grow slower than $\mathcal{L}_{i,\cdot}$ - was motivated by Figure 3.1 which covers a much too small part for a reasonable confidence level. Another way to express the higher speed of convergence is the next claim which is equivalent to Claim 1.2.

Claim 4.6. $i > 8$ and $k_{i+1} < \infty$ if $k_i < \infty$ for some $i > 2$.

By Theorem 3.6 this claim is sufficient for Claim 1.2. Conversely it is necessary as can be seen with Condition 3.5 and section 3.2, too.

4.3 Oscillation Theorems

Clearly, everything works fine if RH is true. An indirect proof with oscillation theorems like [36, Corollaire 1], [44, Sec. 4], or [18, 4] will be proposed by showing that the minimal oscillations force $\mathcal{G}_{i,k}$ above $\mathcal{L}_{i,k}$ for k sufficiently large. After all, Voros reported in [59, Ch. 11] that the amplitude of Keiper's sequence $(\lambda_n)_n$ grows exponentially, [22, 28, 6].

Definition 4.7. [19, 26]

1. $f(x) = \Omega_{\pm}(g(x))$ ($x \rightarrow \infty$) means that both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ ($x \rightarrow \infty$) are valid where

$$\begin{aligned} \bullet f(x) = \Omega_+(g(x)) \quad (x \rightarrow \infty) &\iff \limsup_{x \rightarrow \infty} \frac{f(x)}{g(x)} > 0 \text{ and} \\ \bullet f(x) = \Omega_-(g(x)) \quad (x \rightarrow \infty) &\iff \liminf_{x \rightarrow \infty} \frac{f(x)}{g(x)} < 0. \end{aligned}$$

2. For $\theta := \sup \{\Re(z); z \in \mathbb{C}, \zeta(z) = 0\}$ the set $[1 - \theta, \theta] + i\mathbb{R}$ will be called *very critical strip*.

Obviously RH is true if and only if the very critical strip coincides with the critical line.

Note 4.8.

1. By definition $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ hold if and only if $f(x) < o(g(x))$ and $f(x) > o(g(x))$ are respectively false whereas $f(x) = \Omega(g(x))$ means that $f(x) \neq o(g(x))$ is false, i.e. on some sequence of x 's f is at least of order g .

2. D. E. Knuth preferred $f(x) = \Omega(g(x)) \Leftrightarrow g(x) = O(f(x))$ which is not quite the same, [23].

Assumption 4.9. For the rest of the section let $b < \frac{1}{2}$ be in the very critical strip and n_i the least exceptional number which fixes an index i for the remaining section.

Theorem 4.10. [46, §4, Proposition]

Under Assumption 4.9 the following holds true for CA numbers n .

$$X(n) = e^\gamma \cdot \left(1 + \Omega_\pm \left((\ln n)^{-b} \right)\right)$$

Corollary 4.11. $\frac{X(n)}{e^\gamma} - 1 < o\left((\ln n)^{-b}\right)$ and $\frac{X(n)}{e^\gamma} - 1 > o\left((\ln n)^{-b}\right)$ are false, i.e. there are $\varepsilon_1, \varepsilon_2 > 0$ such that for every natural N there are CA numbers $n_1, n_2 > N$ for which $\left(\frac{\sigma(n_1)}{e^\gamma \cdot n_1 \cdot \ln \ln n_1} - 1\right) \cdot (\ln n_1)^b > \varepsilon_1$ and $\left(\frac{\sigma_{-1}(n_2)}{\ln \ln n_2} - e^\gamma\right) \frac{(\ln n_2)^b}{e^\gamma} < -\varepsilon_2$ are true.

Definition 4.12. Let $\delta_i := \frac{1}{2} \cdot \min(-\beta_i, \alpha_i) > 0$ for

$$\begin{aligned} \alpha_i &:= \limsup_{k \rightarrow \infty} \left(\left(\frac{\sigma_{-1}(n_{i+k})}{\lambda_i(k)} - e^\gamma \right) \cdot \frac{(\ln n_{i+k})^b}{e^\gamma} \right) > 0, \\ \beta_i &:= \liminf_{k \rightarrow \infty} \left(\left(\frac{\sigma_{-1}(n_{i+k})}{\lambda_i(k)} - e^\gamma \right) \cdot \frac{(\ln n_{i+k})^b}{e^\gamma} \right) < 0. \end{aligned}$$

The *oscillation quotient* is $f_i(c, d) := \frac{\lambda_i(c)(1 + \delta_i e^{-b\lambda_i(c)})}{\lambda_i(d)(1 - \delta_i e^{-b\lambda_i(d)})}$ for $\lambda_i(k) := \ln \ln n_{i+k}$.

Note 4.13. Under Assumption 4.9 infinitely many changes of sign of $\left(\frac{\sigma_{-1}(n)}{\ln \ln n} - e^\gamma\right) \cdot e^{b \ln \ln n - \gamma}$ were established by Robin referring to the contributions of Nicolas, Landau, and Grönwall.

Condition 4.14. There are indices c and d with $c < d$ and $f_i(c, d) > 1$.

Theorem 4.15. There is no exceptional number under Condition 4.14.

Proof. $\left(\frac{\sigma(n_1)}{e^\gamma \cdot n_1 \cdot \ln \ln n_1} - 1\right) \cdot (\ln n_1)^b > \varepsilon_1$ and $\left(\frac{\sigma_{-1}(n_2)}{\ln \ln n_2} - e^\gamma\right) \frac{(\ln n_2)^b}{e^\gamma} < -\varepsilon_2$ for some indices c and d follow from Corollary 4.11. Therefore

$$\begin{aligned} \sigma_{-1}(n_{i+c}) &> e^\gamma \left(1 + \delta_i e^{-b\lambda_i(c)}\right) \lambda_i(c) \quad \text{and} \\ \sigma_{-1}(n_{i+d}) &< e^\gamma \left(1 + \delta_i e^{-b\lambda_i(d)}\right) \lambda_i(d) \end{aligned}$$

contradict $c < d$ as claimed by Condition 4.14 because a consequence after dividing the two inequalities is

$$1 < f_i(c, d) \leq \frac{\sigma_{-1}(n_{i+c})}{\sigma_{-1}(n_{i+d})} = \begin{cases} (\mathcal{G}_{i+c,d})^{-1} < 1 & ; c < d, \\ \mathcal{G}_{i+d,c} > 1 & ; d < c. \end{cases}$$

□

Lemma 4.16. The oscillation quotient can be written as $f_i(c, d) = g(\lambda_i(c), \lambda_i(d))$ for

$$g(\mu, \nu) := \frac{\mu(e^{b(\mu+\nu)} + \delta_i e^{b\nu})}{\nu(e^{b(\mu+\nu)} - \delta_i e^{b\mu})} = \frac{\mu}{\nu} \cdot (1 + \delta_i e^{-b\mu}) \cdot \frac{e^{b\nu}}{e^{b\nu} - \delta_i}. \quad (4.1)$$

Proof. Verify the factorisation by expanding the product on RHS. What remains follows from

$$\begin{aligned} f_i(c, d) &= \frac{\lambda_i(c)(1 + \delta_i e^{-b\lambda_i(c)})}{\lambda_i(d)(1 - \delta_i e^{-b\lambda_i(d)})} \\ &= \frac{\lambda_i(c) \left((e^{\lambda_i(c)} \cdot e^{\lambda_i(d)})^b + \delta_i e^{b\lambda_i(d)} \right)}{\lambda_i(d) \left((e^{\lambda_i(c)} \cdot e^{\lambda_i(d)})^b - \delta_i e^{b\lambda_i(c)} \right)} = g(\lambda_i(c), \lambda_i(d)). \end{aligned}$$

□

Definition 4.17. Define three sets of points $(\mu, \nu) \in \mathbb{R}^2$:

1. *Eligible points* $(\mu, \nu) \in E_i$ meet $\mu = \lambda_i(c)$ and $\nu = \lambda_i(d)$ for some $(c, d) \in K_1 \times K_2$,
2. the *upper part* is $U = \{\nu > \mu\}$, and
3. the *big points* are those in $B = \{g > 1\}$.

Claim 4.18. The margin $M := U \cap B$ contains at least one eligible point.

Theorem 4.19. *Condition 4.14 follows from Claim 4.18.*

Proof. By Claim 4.18 there is at least one $(\mu, \nu) \in M \cap E_i$. Then there are indices c and d with $\mu = \lambda_i(c)$, $\nu = \lambda_i(d)$ because of $(\mu, \nu) \in E_i$ and $(\mu, \nu) \in U$ implies $\nu > \mu$. Lemma 4.16 is invoked to obtain $f_i(c, d) = g(\mu, \nu)$ which is greater than 1 because of $(\mu, \nu) \in B$. □

Definition 4.20. Let

1. $\varphi_0 := \arctan(e^{\epsilon_\infty})$ for $\epsilon_\infty := \ln\left(\frac{1+\delta_i}{1-\delta_i}\right)$,
2. $\epsilon_{\mu,\nu} := \ln\left(\frac{1+\delta_i e^{-b\mu}}{1-\delta_i e^{-b\nu}}\right)$ and $\epsilon_{r,\varphi} := \ln\left(\frac{1+\delta_i e^{-br \cos \varphi}}{1-\delta_i e^{-br \sin \varphi}}\right)$, a context-sensitive notation.

Fact 4.21. If $\mu = r \cos \varphi$ and $\nu = r \sin \varphi$ then $\epsilon_{r,\varphi} = \epsilon_{\mu,\nu}$ is positive because $\frac{e^{br \cos \varphi} + \delta_i}{e^{br \sin \varphi} - \delta_i} > \frac{e^{br \cos \varphi}}{e^{br \sin \varphi}}$. With $\frac{d}{dr} e^{\epsilon_{r,\varphi}} = -\left(\frac{1}{1-\delta_i e^{-br}} b \delta_i e^{-b\mu} \cos \varphi + \frac{(1+\delta_i e^{-b\mu})}{(1-\delta_i e^{-br})^2} b \delta_i e^{-b\nu} \sin \varphi\right) < 0$ it turns out $\epsilon_{r,\varphi}$ is decreasing in r for all φ such that $\epsilon_\infty < \epsilon_{r,\varphi}$ and $\epsilon_{r,\varphi} \rightarrow \epsilon_\infty$ as $r \rightarrow \infty$. Note $\epsilon_\infty = 2 \operatorname{artanh} \delta_i > 2\delta_i > 0$, cf. [21, §0.7 16.] or [1, 4.6.22].

Lemma 4.22. 1. $g(\mu, \mu) = \frac{e^{b\mu} + \delta_i}{e^{b\mu} - \delta_i} = e^{\epsilon_{r,\varphi}}$ if $r = \mu\sqrt{2}$ and $\varphi = \frac{\pi}{4}$.

2. If $\mu = r \cos \varphi$ and $\nu = r \sin \varphi$ and $r \rightarrow \infty$ for constant φ then

$$g(\mu, \nu) = e^{\epsilon_{\mu,\nu}} \cdot \cot(\operatorname{atan2}(\mu, \nu)) \searrow e^{\epsilon_\infty} \cdot \cot \varphi > \cot \varphi.$$

3. $\nabla g(\mu, \nu) = \frac{1}{\nu} \cdot \frac{e^{b\nu}}{e^{b\nu} - \delta_i} \cdot \left(1 + (1 - \mu b) \cdot \delta_i e^{-b\mu}, -\mu \cdot \left(\frac{1}{\nu} + \frac{b \cdot \delta_i}{e^{b\nu} - \delta_i}\right) \cdot (1 + \delta_i e^{-b\mu})\right)$.

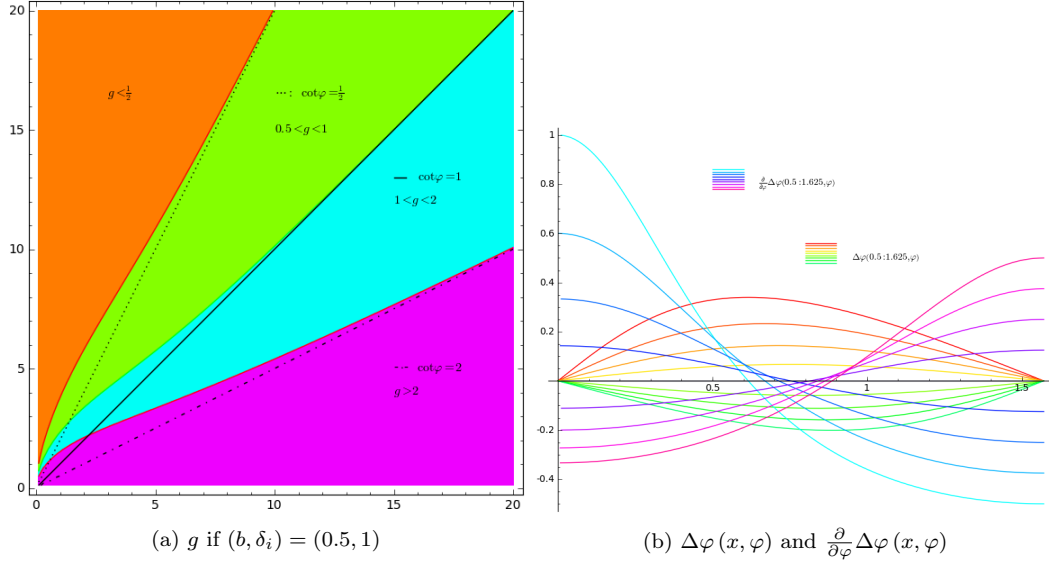
Proof. For $g(\mu, \nu) = \frac{\mu}{\nu} \cdot (1 + \delta_i e^{-b\mu}) \cdot \frac{e^{b\nu}}{e^{b\nu} - \delta_i}$ from Lemma 4.16 it holds true that

1. Both sides are equal to $\frac{\mu}{\nu} \cdot \frac{e^{b\mu} + \delta_i}{e^{b\mu}} \cdot \frac{e^{b\nu}}{e^{b\nu} - \delta_i}$, and
2. $g(\mu, \nu) = \frac{r \cos \varphi}{r \sin \varphi} \cdot \frac{e^{br \cos \varphi} + \delta_i}{e^{br \cos \varphi}} \cdot \frac{e^{br \sin \varphi}}{e^{br \sin \varphi} - \delta_i}$ such that $e^{\epsilon_{r,\varphi}} e^{br(\cos \varphi - \sin \varphi)} = \frac{e^{br \cos \varphi} + \delta_i}{e^{br \sin \varphi} - \delta_i}$ causes

$$g(\mu, \nu) = \cot \varphi \cdot e^{br(\sin \varphi - \cos \varphi)} \cdot \frac{e^{br \cos \varphi} + \delta_i}{e^{br \sin \varphi} - \delta_i} = \cot \varphi \cdot e^{br(\sin \varphi - \cos \varphi)} \cdot e^{\epsilon_{r,\varphi}} e^{br(\cos \varphi - \sin \varphi)} = e^{\epsilon_{r,\varphi}} \cdot \cot \varphi.$$
3. $\frac{\partial}{\partial \mu} g(\mu, \nu) = \frac{1}{\nu} \cdot \frac{e^{b\nu}}{e^{b\nu} - \delta_i} \cdot (1 + \delta_i e^{-b\mu} - \mu \cdot b \cdot \delta_i e^{-b\mu})$
 $= \frac{1}{\nu} \cdot \frac{e^{b\nu}}{e^{b\nu} - \delta_i} \cdot (1 + (1 - \mu b) \cdot \delta_i e^{-b\mu})$ and
 $\frac{\partial}{\partial \nu} g(\mu, \nu) = \mu \cdot (1 + \delta_i e^{-b\mu}) \cdot \left(-\frac{1}{\nu^2} \cdot \frac{e^{b\nu}}{e^{b\nu} - \delta_i} + \frac{1}{\nu} \cdot \left(\frac{be^{b\nu} \cdot (e^{b\nu} - \delta_i) - e^{b\nu} \cdot be^{b\nu}}{(e^{b\nu} - \delta_i)^2} \right) \right)$
 $= -\frac{\mu}{\nu} \cdot \left(\frac{1}{\nu} + \frac{b \cdot \delta_i}{e^{b\nu} - \delta_i} \right) \cdot \frac{e^{b\nu}}{e^{b\nu} - \delta_i} \cdot (1 + \delta_i e^{-b\mu}) < 0.$

□

Figure 4.1: Contour Plot of g and Cartesian Plots of $\Delta\varphi(x, \varphi)$, rf. Appendix



Multiplying $\cot \varphi$ by $e^{\epsilon_{r,\varphi}}$ can be realised by adding $\Delta\varphi(x, \varphi) := \operatorname{arccot}(x \cdot \cot \varphi) - \varphi$ to φ presuming $x = e^{\epsilon_{r,\varphi}} > 0$ and $\varphi \in (0, \frac{\pi}{2})$ s.t. $\Delta\varphi(x, \varphi) < 0$ iff $x > 1$. With arccot $\Delta\varphi(x, \varphi)$ is decreasing in x and $\varphi \mapsto \frac{\partial}{\partial \varphi} \Delta\varphi(x, \varphi) = \frac{x \cdot \csc(\varphi)^2}{x^2 \cot(\varphi)^2 + 1} - 1$ has one change of sign. So $\Delta\varphi(x, \varphi)$ as a function of φ has one minimal and one maximal turning point for $x > 1$ and for $x < 1$, resp.

Corollary. $\cot(\varphi + \Delta\varphi(x, \varphi)) = \cot(\varphi + \operatorname{arccot}(x \cdot \cot \varphi) - \varphi) = x \cdot \cot(\varphi)$ and if $\mu = r \cos \varphi$ and $\nu = r \sin \varphi$ then $g(\mu, \nu) = \cot(\varphi + \Delta\varphi(e^{\epsilon_{r,\varphi}}, \varphi))$.

Proposition 4.23. $g(\mu, \nu) = 1$ for $\mu = r \cos \varphi$ and $\nu = r \sin \varphi$ iff $\tan \varphi = \frac{1 + \delta_i e^{-br \cos \varphi}}{1 - \delta_i e^{-br \sin \varphi}}$. Moreover $g(\mu, \nu) > 1$ iff $\tan \varphi < e^{\epsilon_{r,\varphi}}$ such that $e^{\epsilon_{r,\varphi}} > 1$ for $\varphi = \frac{1}{4}\pi$.

Proof. $1 = g(\mu, \nu) = \cot(\varphi + \Delta\varphi(e^{\epsilon_{r,\varphi}}, \varphi))$ holds if and only if $\frac{\pi}{4} = \operatorname{arccot}(e^{\epsilon_{r,\varphi}} \cdot \cot \varphi)$ which is true if and impossible unless $\tan \varphi = e^{\epsilon_{r,\varphi}}$. Likewise $\frac{\pi}{4} > \operatorname{arccot}(e^{\epsilon_{r,\varphi}} \cdot \cot \varphi)$ holds if and only if $\tan \varphi < e^{\epsilon_{r,\varphi}}$. A special case is $\tan \varphi = 1 < \frac{e^{br\sqrt{2}/2 + \delta_i}}{e^{br\sqrt{2}/2 - \delta_i}}$ for $\varphi = \frac{1}{4}\pi$. \square

Proposition 4.24. *Let $(a_n)_n$ be a sequence in \mathbb{R} with $H < \infty$ for $H := \liminf_{n \rightarrow \infty} a_{n+1} - a_n$ and $a_n \rightarrow \infty$. Then $\arctan \frac{a_{1+n_k}}{a_{n_k}} \rightarrow \frac{\pi}{4}$ as $k \rightarrow \infty$ for the indices $(n_k)_k$ of a suitable subsequence. Moreover $a_{n_k} < a_{1+n_k}$ and $\arctan \frac{a_{1+n_k}}{a_{n_k}} > \frac{\pi}{4}$ for all natural k .*

Proof. A sequence $(n_k)_k$ of indices with $a_{1+n_k} - a_{n_k} \rightarrow H$ and $a_{n_k} \rightarrow \infty$ can be chosen. $(a_{n_k}, a_{1+n_k}) \in \mathbb{R}^2$ in polar coordinates has the angle φ_k with $\tan \varphi_k = 1 + \frac{a_{1+n_k} - a_{n_k}}{a_{n_k}} \leq 1 + \frac{H + \epsilon}{a_{n_k}} \rightarrow 1$ for an arbitrarily fixed $\epsilon > 0$ if k is sufficiently large. $(a_{n_k})_k$ has infinitely many members with $a_{n_k} < a_{1+n_k}$ since $a_{n_k} \rightarrow \infty$. Other members are not suitable. \square

Proposition 4.25. *If $(a_n)_n$ is an increasing sequence in \mathbb{R} with $\varphi_{n+1} := \arctan \frac{a_{1+n}}{a_n} \rightarrow \frac{\pi}{4}$ as $n \rightarrow \infty$ then there is an index n with $(a_n, a_{1+n}) \in M$.*

Proof. If $(a_n, a_{1+n}) \notin M$ was true for all n then each n would meet either $a_n \geq a_{1+n}$ or $g(a_n, a_{1+n}) \leq 1$. Therefore $\tan \varphi_n \geq e^{\epsilon_{r,\varphi_n}} > e^{\epsilon_\infty} > 1$ follows from Proposition 4.23 for all n where $a_n = r_n \sin \varphi_n$ and $a_{1+n} = r_n \sin \varphi_n$. But this contradicts the assumption $\tan \varphi_n \rightarrow 1$ as $n \rightarrow \infty$. \square

Theorem 4.26. [65]: $\liminf_{n \rightarrow \infty} p_{n+1} - p_n \leq H$ for $H = 70 \cdot 10^6$.

Proposition. *Polymath8:*

The last theorem holds true with $H = 5414$.

Corollary 4.27. *M contains infinitely many pairs (p_{n_k}, p_{1+n_k}) of consecutive primes.*

Proof. $\varphi_k := \arctan \frac{p_{1+n_k}}{p_{n_k}} \rightarrow \frac{\pi}{4}$ with $\varphi_k > \frac{\pi}{4}$ and $\frac{p_{1+n_k}}{p_{n_k}} \rightarrow 1$ for a suitable sequence of indices n_k follow from Proposition 4.24 because its requirements are fulfilled by Theorem 4.26. Proposition 4.25 shows that there is a pair $(p_n, p_{1+n}) \in M$ for some index n . The argument can be applied to the sequence of primes above p_{1+n} , too. \square

Figure 4.1a seems to show that the margin M is essentially a bulge that only allows eligible points with small coordinates. But the so-called bulge depends on the choice of δ_i and disappears as δ_i approaches zero. The factor $e^{\epsilon_{r,\varphi}}$ has a positive lower limit because of which the contour $g = 1$ diverges away from the bisecting line. Because of the convergence $\epsilon_{r,\varphi} \searrow \epsilon_\infty$ the asymptote is given by $\operatorname{atan2}(\mu, \nu) = \frac{\pi}{4} + \Delta\varphi(e^{-\epsilon_\infty}, \frac{\pi}{4}) > \frac{\pi}{4}$.

Lemma 4.28. $\frac{\ln \ln nx}{\ln \ln n} \rightarrow 1$ as $n \rightarrow \infty$ for every fixed real number x .

Proof. Put $a_n := \ln n + \ln x$. Since $\ln \ln nx \rightarrow \infty \leftarrow \ln \ln n$ as $n \rightarrow \infty$ l'Hôpital's rule implies

$$\frac{\ln \ln nx}{\ln \ln n} \sim \frac{n \ln n}{n \ln nx} = 1 - \frac{\ln x}{a_n} \rightarrow 1 \quad (n \rightarrow \infty).$$

\square

Corollary 4.29. *If Assumption 2.9 is true then $\liminf_{c \rightarrow \infty} \mathcal{L}_{i+c,1} = 1$ and $\lambda_i(k) \rightarrow \infty$ as $k \rightarrow \infty$.*

Proof. Requiring $\varepsilon_{i+c_k+1} = F(p, v)$ for a fixed prime p and $v > v_p(n_i)$ determines a sequence $(c_k)_k$ such that $q_{i+c_k+1} = p$ for all k . Then $\liminf_{k \rightarrow \infty} \mathcal{L}_{i+c_k, 1} = 1 \geq \liminf_{c \rightarrow \infty} \mathcal{L}_{i+c, 1} \geq 1$ follows with Lemma 4.28. $\lambda_i(k) = \ln \ln n_{i+k} \rightarrow \infty$ as $k \rightarrow \infty$ follows from Theorem 1.3. \square

Proposition 4.30. *Let $(a_n)_n$ be an increasing sequence in \mathbb{R} with $\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$. Then $\arctan \frac{a_{1+n_k}}{a_{n_k}} \rightarrow \frac{\pi}{4}$ as $k \rightarrow \infty$ for the indices $(n_k)_k$ of a suitable subsequence. Moreover $a_{n_k} < a_{1+n_k}$ and $\arctan \frac{a_{1+n_k}}{a_{n_k}} > \frac{\pi}{4}$ for all natural k .*

Proof. A sequence $(n_k)_k$ of indices with $\frac{a_{n+1}}{a_n} \rightarrow 1$ can be chosen. $a_{n_k} < a_{1+n_k}$ holds for infinitely many members of $(a_{n_k})_k$ because $a_n \rightarrow \infty$. In polar coordinates the points $(a_{n_k}, a_{1+n_k}) \in \mathbb{R}^2$ have the angle φ_k with $\tan \varphi_k = \frac{a_{1+n_k}}{a_{n_k}} \leq 1 + \epsilon$ for an arbitrarily fixed $\epsilon > 0$ if k is sufficiently large. Members with $a_{n_k} \geq a_{1+n_k}$ are not suitable. \square

Corollary 4.31. *M contains eligible points.*

Proof. $\varphi_k := \arctan \frac{\lambda_i(1+n_k)}{\lambda_i(n_k)} \rightarrow \frac{\pi}{4}$ as $k \rightarrow \infty$ is true with $\varphi_k > \frac{\pi}{4}$ for a suitable sequence of indices n_k as Proposition 4.30 can be applied because of Corollary 4.29. Proposition 4.25 shows that there is a pair $(\lambda_i(k), \lambda_i(1+k)) \in M$ for some index k . It should also be possible to apply the argument to the sequence of CA numbers above n_{i+k} but one eligible point is sufficient. \square

Conclusion 4.32. Claim 1.2 follows from Assumption 2.9.

Proof. A chain of reductions:

1. Claim 1.2 is reduced to Condition 1.1 by Theorem 1.7,
2. which is reduced to Condition 3.7 by Theorem 3.6,
3. which is reduced to Condition 4.14. The contradiction to Assumption 4.9 in Theorem 4.15 is achieved with Assumption 2.9 and the methods of section 3.
4. Condition 4.14 is reduced to Claim 4.18 by Theorem 4.19 and
5. Claim 4.18 is established by Corollary 4.31 for which Assumption 2.9 has been used in Corollary 4.29, too.

\square

5 Final Remarks

Recently it has already been pointed out in [12, 10, 11] that RIE holds for all $n > 5040$ without requiring Assumption 2.9. Independently of this approach Conclusion 4.32 will have many consequences once Assumption 2.9 is established. A few of them are mentioned.

1. RH follows with the original papers [44, 46] while GRH remains undecided.
2. The weakened version $M(x) = O(x^{\frac{1}{2}+\epsilon})$ of the disproved Mertens conjecture.
3. The only extraordinary number is 4, [8, 9].
4. There are infinitely many extremely abundant numbers, [35, Thm 2.4].

5. The status of Cramér’s conjecture is still undecided but with Cramér’s work $O(\sqrt{p} \cdot \ln p)$ can be deduced for every gap.
6. A recent result is Hypothesis P in [14] according to Proposition 40 in that paper.
7. The Riesz criterion, Nicolas’ inequality for φ , Weil’s and Li’s criterions, and Speiser’s statement on ζ' , [43, 36, 5, 63, 28, 54].

Approaches related to the present one are [35, 34] as well as [33, 2]. The former led to [34, Thm 1.7] and a sequence of increasing values of X whose existence follows from Grönwall’s theorem with Robin’s Oscillation theorems. The latter pointed out that increasing values of $\frac{\sigma(n)}{n \cdot \ln \ln n}$ on superabundant numbers are sufficient. CA numbers do not allow for the minimality condition. Exceptional numbers cause oscillations whereas the explicit formulas are more precise under RH. If oscillations prevent exceptional numbers RH could be said to be hoist by its own petard.

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A Implementation

Results have already been mentioned above. Sage code follows below. My first version computed for two weeks last year on a MacBook Air until $X(n) > 1.781$, i.e. n_{143215} was reached. A revised implementation did the job in a bit more than half an hour (without standby phases). There are two reasons for the difference:

1. Pre-Computation of a list of primes,
2. Consequent exploitation of the factorisation of CA numbers.

Because of assumption 2.9 it is only necessary to select **the** next prime in every loop, determine new primes that may follow next, and to compute the values of ε associated with the (in virtue of [40, §59] and [3, Thm 1] at most 2) additional new primes.

The following functions compute CA numbers as they were represented in section 3.2. Noe’s top-down form of primes triggering the next valuation is used to store n_i . When n_i has been computed the primes p such that $n_i p$ does not violate the *basic SA condition* are called *candidates for q_{i+1}* , cf. [3, Theorem 2]. It is convenient that the candidates for q_{i+1} are the primes that occur in the bottom-up form of n_i .

TODO: in `triggers` and `candidates` store indices in `sieve` instead of elements of `sieve`, endow `sieve` with logs of primes, in `addSievedPrimeToTriggers()` avoid searching `newprime` in `sieve`.

Compute CA Numbers in top-down form, a potentially not so big CA number is to be given, e.g. counter = 4 and triggers = [5, 2], seems not to work with the known smaller CA numbers

```
sieve = prime_range(2, 50000000)
def getSubsequentCAnumber(counter, triggers, number, sieve):
    k = 0
    candidates = getBottomUpTriggers(triggers)
    epsilons = map(lambda i: getCAparameter(candidates[i], i+1),
                   range(len(candidates)))
    while k < number:
        k = k + 1
        vmax = selectNextTrigger(epsilons)
        triggers = addSievedPrimeToTriggers(candidates[vmax], triggers,
                                             candidates, epsilons, sieve)
    return triggers
```

Convert CA Numbers to bottom-up form

```
def getBottomUpTriggers(triggers):
    l = len(triggers); i = 0
    result = []
    while i < l:
        p = triggers[i]; j = 1
        if p!=0:
            result.append(next_prime(p))
        else:
            p = triggers[i+j]
            while p==0:
                j=j+1
                p = triggers[i+j]
            result.append(next_prime(p))
            result.extend([0 for dummy in range(j)])
            j=j+1
        i=i+j
    result.append(2)
    return result
```

Choose the next candidate

```
def selectNextTrigger(E):
    emax = max(E)
    vmax = [v for v in range(len(E)) if E[v] >= emax]
    if len(vmax) > 1:
        print "FOUR EXPONENTIALS DISPROVED!"
    else:
        return vmax[0]
```

Enter the next candidate in the list of triggering primes and update candidates and epsilons

```

def addSievedPrimeToTriggers(newprime, triggers, candidates, epsilons, sieve):
    vmax = candidates.index(newprime)+1
    npi = sieve.index(newprime)
    i = vmax-1
    l = len(triggers)
    if i < l:
        triggers[i] = newprime
        if candidates[i+1] == 0:
            candidates[i+1] = newprime
            epsilons[i+1] = getCAparameter(newprime, vmax+1)
        if triggers[i-1] == newprime:
            triggers[i-1] = 0
            candidates[i] = 0
            epsilons[i] = 0
        else:
            candidates[i] = sieve[npi+1]
            epsilons[i] = getCAparameter(candidates[i], vmax)
    else:
        triggers.append(newprime)
        candidates.append(newprime)
        vmax = vmax + 1
        epsilons.append(getCAparameter(newprime, vmax))
        if triggers[i-1] > 0:
            triggers[i-1] = 0
            candidates[i] = 0
            epsilons[i] = 0
    return triggers

```

Plotting $\Delta\varphi(x, \varphi)$ and $\frac{\partial}{\partial\varphi}\Delta\varphi(x, \varphi)$ in Figure 4.1b.

```

DeltaPhi(x, phi) = arccot(x*cot(phi))-phi
P = sum([plot(DeltaPhi(i/8,phi), (phi, 0, pi/2),
             rgbcolor = hue(((i+16)%20)/20)) for i in range(4, 13)])
P = P + sum([line([(0.8, 0.6-i/100), (0.9, 0.6-i/100)],
                 rgbcolor = hue(((i+16)%20)/20)) for i in range(4, 13)])
P = P + text('$\Delta\varphi\left(0.5:1.625,\varphi\right)$',
            (0.95, 0.5), color="black", horizontal_alignment='left')
P = P + sum([plot(DeltaPhi.diff(phi)(i/8,phi), (phi, 0, pi/2),
                 rgbcolor = hue(((i+6)%20)/20)) for i in range(4, 13)])
P = P + sum([line([(0.5, 0.9-i/100), (0.6, 0.9-i/100)],
                 rgbcolor = hue(((i+6)%20)/20)) for i in range(4, 13)])
(P+text('$\frac{\partial}{\partial\varphi}\Delta\varphi\left(0.5:1.625,\varphi\right)$',
        (0.65, 0.8), color="black", horizontal_alignment='left')).show()

```

Compute $\varepsilon = F(x, v)$

```
def getCAparameter(x, v):  
    if x == 0:  
        return 0  
    else:  
        if v == 0:  
            return log(1+1/x)/log(x)  
        else:  
            return log((1-x^(v+1))/(x-x^(v+1))) / log(x)
```

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