Upper Bound for Critical Probability of Site Percolation on Triangular Lattice is $1/2^{\frac{3}{2}}$

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In triangular lattice, the upper bound for critical probability of site percolation is $\frac{1}{2^{\frac{3}{2}}} \approx 0.3535$.

I. INTRODUCTION AND DESCRIPTION OF $\mathbb{Z}^2$

In site percolation, vertices (sites) of a graph are open with probability p, and there is the smallest $p = p_H$, critical p, for which open vertices form an open path the long way across a graph, so a vertex at the origin is a part of an infinite connected open vertex set, [3]. Smirnov found that for triangular lattice $p_H = 1/2$, [1], but there is the traversal, from the origin upwards, so that an infinite connected open vertex set exists for $p_H = 1/2^{\frac{3}{2}} \approx 0.3535$.

In finite graph $\mathbb{Z}^2_k$, 2 pairs of opposite arcs are k edges away from the vertex at origin, Fig. 1. Basis $\mathcal{B}(\mathbb{Z}^2)$ of edges and the integers a_i assign the place to vertex $\mathbf{v} \in \mathbb{Z}^2_k$:

$$\mathbf{v}_0 = 0 = \text{vertex at origin of } \mathbb{Z}^2_k \subset \mathbb{Z}^2$$
$$\mathcal{B}\left(\mathbb{Z}^d\right) = \{\uparrow_1, \uparrow_2, \uparrow_3, .., \uparrow_d\},\ -\uparrow_1 = \downarrow_1\ \ \&\ -\uparrow_2 = \downarrow_2$$
$$\mathbf{v} = a_1\uparrow_1 + a_2\uparrow_2,\ \mathbf{v}_0 \to \mathbf{v} = \text{path from } \mathbf{v}_0 \text{ to } \mathbf{v}$$
$$|\mathbf{v}_0 \to \mathbf{v}| = \|\mathbf{v}\| = |a_1(\mathbf{v})| + |a_2(\mathbf{v})|$$

Places of neighbors of vertex $\mathbf{v} \in \mathbb{Z}^2$ can be partition into 2 up-step neighbors traversed via $\uparrow_1$ and $\uparrow_2$ and 2 down-step neighbors traversed via $\downarrow_1$ and $\downarrow_2$. For $\uparrow_1$, there is matching $\downarrow_1$, and for $\uparrow_2$, there is matching $\downarrow_2$:

$$\mathcal{N}(\mathbf{v}, \mathbb{Z}^2) = \text{neighbors of } \mathbf{v}$$
$$\mathcal{N}_u(\mathbf{v}, \mathbb{Z}^2) = \text{up-step neighbors of } \mathbf{v} = \{\mathbf{v}{+}\uparrow_1, \mathbf{v}{+}\uparrow_2\}$$

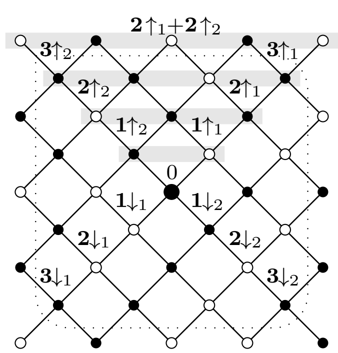

FIG. 1: In arcs, $\sum_i |a_i(\mathbf{v})| = 4$ (outside dotted curve). In the convenient pair of opposite arcs $\{\mathcal{A}_4(\mathbb{Z}^2), \mathcal{A}_{-4}(\mathbb{Z}^2)\}_1$, $\mathcal{A}_4(\mathbb{Z}^2) = \{4\uparrow_1, 3\uparrow_1{+}1\uparrow_2, 2\uparrow_1{+}2\uparrow_2, 1\uparrow_1{+}3\uparrow_2, 4\uparrow_2\}$.

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$\mathbf{v}_0$ is k edges away from the arcs and it can be any vertex in $\mathbb{Z}^2$. Two pairs of opposite arcs in $\mathbb{Z}^2_k$ look the same and any pair, by rotation of $\mathbb{Z}^2$, can be a convenient pair $\{\mathcal{A}_k(\mathbb{Z}^2), \mathcal{A}_{-k}(\mathbb{Z}^2)\}_1$:

$$\left\{\mathcal{A}_k(\mathbb{Z}^2), \mathcal{A}_{-k}(\mathbb{Z}^2)\right\}_1 = \text{one of 2 pairs of opposite arcs}$$
$$\mathcal{A}_k(\mathbb{Z}^2) \cap \mathcal{A}_{-k}(\mathbb{Z}^2) = \emptyset$$

$$\mathcal{A}_k(\mathbb{Z}^2) = \left\{ \begin{array}{c} \mathbf{v} : \|\mathbf{v}\| = k\ \&\ a_1(\mathbf{v}), a_2(\mathbf{v}) \geq 0 \\ \text{or} \\ \mathbf{v} \text{ built with } \uparrow_1\ \&\ \uparrow_2 \text{ edges only} \end{array} \right\}$$

The shortest traversal from $\mathbf{v}_0$ to $\mathcal{A}_k(\mathbb{Z}^2)$ is a traversal via vertices in $\mathcal{N}_u(\mathbf{v}, \mathbb{Z}^2)$, which are one edge closer to $\mathcal{A}_k(\mathbb{Z}^2)$:

$$1^{st}: \mathcal{A}_1\left(\mathbb{Z}^2\right) = \{\uparrow_1, \uparrow_2\}$$
$$2^{nd}: \mathcal{A}_2\left(\mathbb{Z}^2\right) = \bigcup_{\mathbf{v} \in \mathcal{A}_1(\mathbb{Z}^2)} \mathcal{N}_u(\mathbf{v}, \mathbb{Z}^2)$$
$$..$$
$$k^{th}: \mathcal{A}_k\left(\mathbb{Z}^2\right) = \bigcup_{\mathbf{v} \in \mathcal{A}_{k-1}(\mathbb{Z}^2)} \mathcal{N}_u(\mathbf{v}, \mathbb{Z}^2)$$

II. TRIANGULAR LATTICE PERCOLATION

Lattice $\mathbb{Z}^2$ is percolating when the open vertices form an open path connecting $\mathbf{v}_0$ with the vertices in the opposite sides of $\mathbb{Z}^2_k$ and $k \to \infty$, so we need to know the number of paths connecting $\mathbf{v}_0$ and vertices in $\mathcal{A}_k(\mathbb{Z}^2)$:

$$p = \text{probability of vertex being open}$$
$$\psi\left(\mathbb{Z}^2, p\right) = \text{number of percolating paths in } \mathbb{Z}^2$$

$\mathbb{Z}^2$ is embedded in $\mathbf{T}$, which has two pairs of opposite sides and two definitions of $\mathcal{B}(\mathbf{T})$, Fig. 2:

$$\mathbf{T} = \text{triangular lattice with embedded } \mathbb{Z}^2$$

$$\mathcal{B}(\mathbf{T}) = \left\{ \begin{array}{c} \uparrow_1, \uparrow_2, \uparrow_1{+}\uparrow_2{=}\uparrow_{1,2}{=}-\downarrow_{1,2} \\ \text{or} \\ \uparrow_1, \uparrow_2, \uparrow_1{+}\downarrow_2{=}\rightarrow_{1,2}{=}-\leftarrow_{1,2} \end{array} \right\}$$
$$\|\uparrow_{1,2}\| = \|\rightarrow_{1,2}\| = 1$$

$$\mathcal{N}_u(\mathbf{v}, \mathbf{T}) = \left\{ \begin{array}{c} \mathbf{v}{+}\uparrow_1, \mathbf{v}{+}\uparrow_2, \mathbf{v}{+}\uparrow_{1,2} \\ \text{or} \\ \mathbf{v}{+}\uparrow_1, \mathbf{v}{+}\uparrow_2 \end{array} \right\}$$

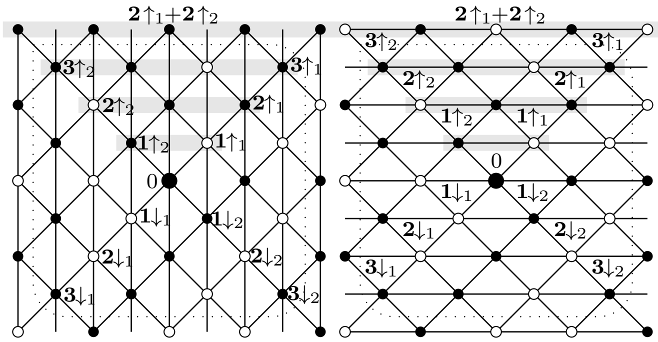

FIG. 2: Triangular lattice is built by adding $\uparrow_1+\uparrow_2$ and $\downarrow_1+\downarrow_2$ (left) or by adding $\uparrow_1+\downarrow_2$ and $\downarrow_1+\uparrow_2$ (right) to each $\mathbf{v}\in\mathbb{Z}^2$.

Arc $\mathcal{A}_k(\mathbf{T})$, k edges away from $\mathbf{v}_0\in\mathbf{T}$, contains the vertices in $\mathcal{A}_{k+i}(\mathbb{Z}^d)$ traversed via $\mathcal{N}_u(\mathbf{v},\mathbf{T})$:

$$\mathcal{A}_k(\mathbf{T})=\bigcup_{\mathbf{v}\in\mathcal{A}_{k-1}(\mathbf{T})}\mathcal{N}_u(\mathbf{v},\mathbf{T}) : \mathcal{A}_1(\mathbf{T})=\{\uparrow_1,\uparrow_2,\uparrow_{1,2}\}$$

$$n_k\left(\mathcal{A}_{k+i}(\mathbb{Z}^2),\mathbf{T}\right)=\text{number of paths to } \mathcal{A}_{k+i}\left(\mathbb{Z}^2\right)$$

$$n_k(\mathcal{A}_k(\mathbf{T}))=\sum_i n_k\left(\mathcal{A}_{k+i}\left(\mathbb{Z}^2\right),\mathbf{T}\right)$$

For vertices in arc $\mathcal{A}_1(\mathbf{T})$, 2 paths end in $\mathcal{A}_1(\mathbb{Z}^2)$ and 1 path ends in $\mathcal{A}_2(\mathbb{Z}^2)$:

$$\mathcal{A}_1(\mathbf{T})=\{\mathbf{v}_0+\uparrow_1,\mathbf{v}_0+\uparrow_2,\mathbf{v}_0+\uparrow_{1,2}\}\Rightarrow\begin{matrix}2 & \rightarrow_1\\ 1 & \rightarrow_2\end{matrix}\Rightarrow\begin{matrix}|\rightarrow_1|=2\\ |\rightarrow_2|=1\end{matrix}$$

$$\rightarrow_1=\text{path from } \mathbf{v}_0 \text{ to a vertex in embedded } \mathcal{A}_1(\mathbb{Z}^2)$$

$$|\rightarrow_1|=\text{number of paths from } \mathbf{v}_0 \text{ to } \mathcal{A}_1(\mathbb{Z}^2)$$

From $\mathcal{A}_1(\mathbf{T})$, the number of paths doubles for up-steps from $\mathcal{A}_1(\mathbb{Z}^2)$ to $\mathcal{A}_2(\mathbb{Z}^2)$ and from $\mathcal{A}_2(\mathbb{Z}^2)$ to $\mathcal{A}_3(\mathbb{Z}^2)$. The number of paths does not change for up-steps from $\mathcal{A}_1(\mathbb{Z}^2)$ to $\mathcal{A}_3(\mathbb{Z}^2)$ and from $\mathcal{A}_2(\mathbb{Z}^2)$ to $\mathcal{A}_4(\mathbb{Z}^2)$:

$$\mathcal{A}_2(\mathbf{T})=\bigcup_{\mathbf{v}\in\mathcal{A}_1(\mathbf{T})}\mathcal{N}_u(\mathbf{v},\mathbf{T})\Rightarrow\begin{matrix}2|\rightarrow_1| & .\rightarrow_2\\ 1|\rightarrow_1| & .\rightarrow_3\\ 2|\rightarrow_2| & .\rightarrow_3\\ 1|\rightarrow_2| & .\rightarrow_4\end{matrix}\Rightarrow\begin{matrix}|.\rightarrow_2|=1\cdot 2^2\\ |.\rightarrow_3|=2\cdot 2^1\\ |.\rightarrow_4|=1\cdot 2^0\end{matrix}$$

$$.\rightarrow_2=\text{path ending in } \mathcal{A}_2(\mathbb{Z}^2)$$

$$|.\rightarrow_2|=\text{number of paths ending in } \mathcal{A}_2(\mathbb{Z}^2)$$

When counting the paths from $\mathbf{v}_0$ to the arcs $\mathcal{A}_k(\mathbb{Z}^2)$, $\mathcal{A}_{k+1}(\mathbb{Z}^2)$, .., $\mathcal{A}_{2k}(\mathbb{Z}^2)$, the coefficients c_{i_k} follow the rule generated by Pascal's triangle, [4]:

$c_{i_k}\cdot 2^{k-i}$ = number of paths to $\mathcal{A}_{k+i}\left(\mathbb{Z}^2\right)$ after k^{th} step

1^{st}: $1,1\Rightarrow 1\cdot 2^1+1\cdot 2^0$ paths to $\mathcal{A}_1(\mathbf{T})$

2^{nd}: $1,2,1\Rightarrow 1\cdot 2^2+2\cdot 2^1+1\cdot 2^0$ paths to $\mathcal{A}_2(\mathbf{T})$

3^{rd}: $1,3,3,1\Rightarrow 1\cdot 2^3+3\cdot 2^2+3\cdot 2^1+1\cdot 2^0$ paths

4^{th}: $1=c_{0_4}, 4=c_{1_4}, 6=c_{2_4}, 4=c_{3_4}, 1=c_{4_4}$

..

$$\sum_{i=0}^{i=k}\binom{k}{i}\cdot 2^{k-i}=\text{number of paths to } \mathcal{A}_k(\mathbf{T})$$

$$\binom{k}{i}\cdot 2^{k-i}=\text{number of paths to } \mathcal{A}_{k+i}\left(\mathbb{Z}^2\right)$$

When odd or even $k\to\infty$,

$$p_H(\mathbf{T})\le \min p \text{ for which } \binom{k}{i}\cdot 2^{k-i}\cdot p^k\ge 1$$

$$\binom{k}{\frac{k-1}{2}}\cdot 2^{k-\left(\frac{k-1}{2}\right)}=\text{number of paths to } \mathcal{A}_{k+\frac{k-1}{2}}(\mathbb{Z}^2)$$

or

$$\binom{k}{\frac{k}{2}-1}\cdot 2^{k-\left(\frac{k}{2}-1\right)}=\text{number of paths to } \mathcal{A}_{k+\frac{k}{2}-1}(\mathbb{Z}^2)$$

$$\psi(\mathbf{T},p)=2^{\frac{1}{2}}\lim_{k\to\infty}\left(2^{\frac{1}{2}}\cdot\binom{k}{\frac{k-1}{2}}^{\frac{1}{k}}\cdot p\right)^k$$

or

$$\psi(\mathbf{T},p)=2\lim_{k\to\infty}\left(2^{\frac{1}{2}}\cdot\binom{k}{\frac{k}{2}-1}^{\frac{1}{k}}\cdot p\right)^k$$

$$\Rightarrow p_H(\mathbf{T})\le\begin{cases}\lim_{k\to\infty}\dfrac{1}{2^{\frac{1}{2}}\cdot\binom{k}{\frac{k-1}{2}}^{\frac{1}{k}}}\\ \text{or}\\ \lim_{k\to\infty}\dfrac{1}{2^{\frac{1}{2}}\cdot\binom{k}{\frac{k}{2}-1}^{\frac{1}{k}}}\end{cases}\approx\frac{1}{2^{\frac{3}{2}}}\approx 0.3535$$

$\lim_{k\to\infty}\binom{k}{\frac{k-1}{2}}^{\frac{1}{k}}\approx 2$, [2][5].

[1] Stanislav Smirnov. Critical percolation in the plane: Conformal invariance, cardy's formula, scaling limits. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 333(3):239–244, 2001.

[2] Herbert Robbins. A remark on stirling's formula. *The American Mathematical Monthly*, 62(1):26–29, 1955.

[3] Path is a walk via edges visiting each vertex only once.

[4] From *The On-Line Encyclopedia of Integer Sequences* at http://oeis.org.

[5] Limits at http://www.wolframalpha.com.