SEQUENCES DEFINED BY h-VECTORS

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ABSTRACT. In this paper we consider the sequence whose n^{th} term is the number of h-vectors of length n. We show that the n^{th} term of this sequence is bounded above by the n^{th} Fibonacci number and bounded below by the number of integer partitions of n into distinct parts. Further we show embedded sequences that directly relate to integer partitions.

1. INTRODUCTION

Hilbert functions of graded rings have been well studied throughout the years and are known to relate to many different invariants such as dimension, multiplicity, and Betti numbers [BH93, Chapter 4]. In 1927, Macaulay showed that for every graded ideal there exists a LEX-segment with the same Hilbert function [Mac27]. Since then a wide range of research has accumulated generalizing this result [CL69, FR07, MM10, CK12]. These functions have a variety of uses in both algebra and combinatorics and are the subject of active research [PS09]. In particular, it is helpful giving necessary conditions for a ring to have the weak Lefschetz property [HMNW03].

In [Lin99], Linusson counted sequences and vectors associated to Hilbert functions. In particular, recursion formulas were given for the number of M-sequences (i.e. f-vectors for multicomplexes) in terms of the number of variables and a maximum degree. When the number of variables was restricted to 3, it was shown that the Bell numbers counted the number of M-sequences. In recent work of Whieldon [Whi13], given certain classes of monomial ideals, the sequence of Betti numbers satisfies nice recursion formulas. In particular, the Betti numbers of the resolution of $\Bbbk$ over $S = \Bbbk[x, y]/(x^2, xy)$ are given by the i^{th} Fibonacci number! The goal of this paper is to find recursion formulas related to Hilbert functions. We are mainly concerned with the sequence defined by the number of h-vectors of length n. We show that this sequence is bounded above (term-wise) by the sequence of Fibonacci numbers and below by the number of integer partitions of n into at least 2 distinct parts. As such, the sequence has exponential growth.

The rest of this section gives the necessary background and notation. In Section 2 we determine an upper bound for our sequence to be the Fibonacci numbers. The lower bound can be found in Section 3 as well as a one-to-one correspondence between integer partitions and LEX ideals in two variables. The rest of the paper generalizes these concepts.

1.1. **Basic Setup.** We first give some necessary background on Hilbert functions and h-vectors. Let $R = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring over a field $\Bbbk$ with the standard grading. In particular, $\deg x_i = 1$ for $1 \leqslant i \leqslant n$. If I is a graded ideal,

2010 *Mathematics Subject Classification.* Primary: 05E40; Secondary: 13D40.

the quotient ring R/I is also graded and we denote by $(R/I)_t$ the $\Bbbk$-vector space of all degree t homogeneous elements of R/I. The Hilbert function $H_{R/I} : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ is defined to be the $\Bbbk$-vector space dimension of each graded component, i.e. $H_{R/I}(t) := \dim_k(R/I)_t$.

If the Krull dimension of the graded quotient ring is zero, there exists an $s \geqslant 0$ such that $H_{R/I}(s) \neq 0$ but $H_{R/I}(t) = 0$ for all $t > s$. In this case, the h-vector of R/I is defined as

$$\mathbf{h}(R/I) = (H_{R/I}(0), H_{R/I}(1), H_{R/I}(2), \dots, H_{R/I}(s)).$$

Thus the h-vector of R/I has finitely many non-zero entries. The length of R/I is the k-vector space dimension of R/I, denoted $\lambda(R/I)$. In particular, $\lambda(R/I) = \sum_{i=0}^{s} H_{R/I}(i)$. Throughout this paper, we will also refer to $\lambda(R/I)$ as the length of $\mathbf{h}(R/I)$.

In [BH93, Chapter 4] a numerical constraint is given on the possible integer vectors that can be h-vectors. Given $d \in \mathbb{Z}_{\geqslant 0}$, each $a \in \mathbb{Z}_{\geqslant 0}$ has a unique representation as a sum of binomial coefficients

$$a = \binom{b_d}{d} + \binom{b_{d-1}}{d-1} + \cdots + \binom{b_j}{j}, \tag{1}$$

where $b_d > b_{d-1} > \cdots > b_j \geqslant j \geqslant 1$. Further, define

$$a^{\langle d \rangle} = \binom{b_d + 1}{d+1} + \binom{b_{d-1} + 1}{d} + \cdots + \binom{b_j + 1}{j+1}, \tag{2}$$

where $0^{\langle d \rangle} = 0$. For a map $h : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$, Macaulay's Theorem [BH93, Theorem 4.2.10] says the following conditions are equivalent:

(A) there exists a graded ideal I in R such that $H_{R/I}(t) = h(t)$ for all $t \geqslant 0$;
(B) there exists a monomial ideal I in R such that $H_{R/I}(t) = h(t)$ for all $t \geqslant 0$;
(C) one has $h(0) = 1$, and $h(t+1) \leqslant h(t)^{\langle t \rangle}$ for all $t \geqslant 1$.

Throughout this paper, for an arbitrary set Λ, we denote $|\Lambda|$ as the cardinality of Λ.

2. Fibonacci Bound

The main study of this paper is the sequence $\{\ell(n)\}_{n \geqslant 1}$ defined by the number of h-vectors of length n. In particular, for $n \geqslant 1$ we define

$$L(n) = \left\{ h = (h_0, h_1, \dots) \mid h \text{ is an } h\text{-vector and } \sum_i h_i = n \right\}.$$

and set $\ell(n) = |L(n)|$ for $n \geqslant 1$.

Using condition (C) in Macaulay's theorem above, we are able to construct all possible h-vectors of a given length. In Figure 1, we find the h-vectors of length at most 7 and that the first few terms of the sequence $\{\ell(n)\}_{n \geqslant 1}$ are: 1, 1, 2, 3, 5, 8, 12.

After seeing the first few terms of this sequence, a natural question to ask is whether or not it is related to the Fibonacci sequence. In Theorem 2.4 we show $\{\ell(n)\}_{n \geqslant 1}$ is bounded above by the Fibonacci sequence. To do this, we need to define the following family of integer vectors.

Definition 2.1. For $n \geqslant 1$, the set of integer vectors $B(n)$ is defined recursively as follows:

$\lambda =$	1	2	3	4	5	6	7
	1	11	111	1111	11111	111111	1111111
			12	121	1211	12111	121111
				13	122	1221	12211
					131	123	1231
					14	1311	1222
						132	13111
						141	1321
						15	133
							1411
							142
							151
							16
Total:	1	1	2	3	5	8	12

FIGURE 1. The h-vectors of length at most 7. We write $t_0t_1t_2\cdots t_s$ for the h-vector $(t_0, t_1, \ldots, t_s)$. E.g. 1221 is the h-vector $(1, 2, 2, 1)$.

(1) $B(1) = \{(1)\}$;
(2) $B(2) = \{(1, 1)\}$;
(3) For $n \geqslant 3$ define $B(n) := C(n) \cup D(n)$ where

$$C(n) := \{(1, t_1, \ldots, t_s, 1) \mid (1, t_1, \ldots, t_s) \in B(n-1)\},$$
$$D(n) := \{(1, t_1, \ldots, t_s + 1) \mid (1, t_1, \ldots, t_s) \in B(n-1), \text{ with } t_{s-1} > 1 \text{ or } s = 1\}.$$

Remark 2.2. It is worth noticing that the sets $C(n)$ and $D(n)$ of Definition 2.1 form a set partition of $B(n)$.

The first few sets $B(n)$ are

$$B(1) = \{(1)\};$$
$$B(2) = \{(1, 1)\};$$
$$B(3) = \{(1, 1, 1), (1, 2)\};$$
$$B(4) = \{(1, 1, 1, 1), (1, 2, 1), (1, 3)\};$$
$$B(5) = \{(1, 1, 1, 1, 1), (1, 2, 1, 1), (1, 3, 1), (1, 2, 2), (1, 4)\}.$$

Lemma 2.3. *The cardinality of $B(n)$ is the n^{th} Fibonacci number F_n.*

Proof. For notational convenience, we let $b_n = |B(n)|$ and observe that $b_1 = b_2 = 1$ and $b_3 = 2$. We need to show that for $n \geqslant 4$ this sequence satisfies the Fibonacci recurrence

$$\begin{aligned} b_n &= b_{n-1} + b_{n-2} \\ &= b_{n-1} + b_{n-1} - b_{n-3}. \end{aligned}$$

By Remark 2.2, we know $b_n = |C(n)| + |D(n)|$. Since $|C(n)| = b_{n-1}$, we need to show that $|D(n)| = b_{n-1} - b_{n-3}$. Partition the set $B(n-1) = E(n-1) \cup F(n-1)$ so that $E(n-1)$ is the set of vectors $v \in B(n-1)$ such that the last two entires of v equal 1, and $F(n-1)$ consists of the remaining vectors. A vector is in $E(n-1)$ if and only if it came from $B(n-3)$ by adjoining 1 twice in accordance to the recursion in Definition 2.1. In particular, we have that

$$|E(n-1)| = |C(n-2)| = b_{n-3}.$$

We claim that $|D(n)| = |F(n-1)|$, and hence $|D(n)| = b_{n-1} - b_{n-3}$. To see this, notice that $F(n-1) \subseteq B(n-1)$ consists of all vectors whose second to last term is greater than 1. Hence we can increase the last term of any vector in $F(n-1)$ by 1, and the resulting vector is in $B(n)$. As such, all the vectors in $D(n)$ will come from vectors in $F(n-1)$, hence $|D(n)| = |F(n-1)|$. □

Theorem 2.4. *For all $n \geqslant 1$, $L(n) \subseteq B(n)$. In particular, the sequence $\ell(n)$ is bounded above by the Fibonacci sequence.*

Proof. Notice that the set $B(n)$ consists of all integer vectors $(1, t_1, \ldots, t_s)$ with $1 + t_1 + t_2 + \cdots + t_s = n$ and the property that if $t_i = 1$, then $t_j = 1$ for all $j \geqslant i$. Let $\mathbf{h} = (1, h_1, \ldots, h_s) \in L(n)$ be an h-vector of length n. Using Macaulay's Theorem condition (C) it is not hard to see that if $h_i = 1$ for some $i \geqslant 1$, then $h_j = 1$ for all $j \geqslant i$. Thus $L(n) \subseteq B(n)$. □

Remark 2.5. For $n \geqslant 7$, there are elements of $B(n)$ that are not h-vectors. See $(1, 2, 4)$ for an example. Further analysis shows the first 20 terms of $\ell(n)$ shows this upper bound is not tight:

$$\{\ell(n)\}_{n \geqslant 1} = 1, 1, 2, 3, 5, 8, 12, 18, 27, 40, 57, 82, 116, 163, 227, 313, 428, 583, \ldots.$$

3. Integer Partitions

In this section we obtain a lower bound for the sequence $\{\ell(n)\}_{n \geqslant 1}$ by restricting our attention to zero-dimensional $\mathbb{k}$-algebras of the form $\mathbb{k}[x, y]/I$, where I is a homogeneous ideal of $\mathbb{k}[x, y]$. That is, we are concerned with h-vectors with $h_1 = 2$. The main result is Theorem 3.8 which shows that $\ell(n)$ is greater than or equal to the number of integer partitions of n into distinct parts. First, we develop the necessary background on integer partitions and LEX ideals.

For an (x, y)-primary monomial ideal I in $\mathbb{k}[x, y]$, we define

$$\lambda_i = \lambda_i(I) = |\{x^{i-1} y^j \notin I \mid j \geqslant 0\}|.$$

Notice that λ_i can also be viewed as a well-defined map from the set of monomial ideals in $\mathbb{k}[x, y]$ to the positive integers. Further, from the definition of λ_i, we are able to write

$$I = \left(y^{\lambda_1(I)}, x y^{\lambda_2(I)}, \ldots, x^{i-1} y^{\lambda_i(I)}, \ldots \right). \tag{3}$$

As such, we have the following results detailing the natural correspondence between monomial ideals in two variables and integer partitions. Proposition 3.2 is known (see [SP04]) but we prove it here for the convenience of the reader.

Lemma 3.1. *Let I be an (x, y)-primary monomial ideal in $R = \mathbb{k}[x, y]$. Then $(\lambda_1, \lambda_2, \cdots)$ is an integer partition of $\lambda(R/I)$.*

Proof. Since I is (x, y)-primary $\lambda(R/I)$ is finite. We need to show $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ and $\lambda(R/I) = \sum \lambda_i$. The first condition is true by the nature of monomial ideals. The second condition holds because both $\lambda(R/I)$ and $\sum \lambda_i$ count the number of monomials not in I. □

Proposition 3.2. *Let $R = \mathbb{k}[x, y]$. There exists a one-to-one correspondence between (x, y)-primary monomial ideals I and integer partitions.*

Proof. Let $\mathfrak{M}$ be the set of (x, y)-primary monomial ideals in R and $\mathfrak{P}$ be the set of integer partitions. Define the map $\Phi : \mathfrak{M} \to \mathfrak{P}$ by

$$\Phi(I) = (\lambda_1(I), \lambda_2(I), \cdots, \lambda_s(I)),$$

where s is the largest integer such that $\lambda_i(I) \neq 0$.

To show that Φ is one-to-one, let $I, J \in \mathfrak{M}$ such that $\Phi(I) = \Phi(J)$. This forces $\lambda_i = \lambda_i(I) = \lambda_i(J)$ for all i. In particular, for each i we have that

$$x^{i-1}y^0, x^{i-1}y^1, x^{i-1}y^2, \ldots, x^{i-1}y^{\lambda_i - 1} \notin I, J;$$
$$x^{i-1}y^{\lambda_i} \in I, J.$$

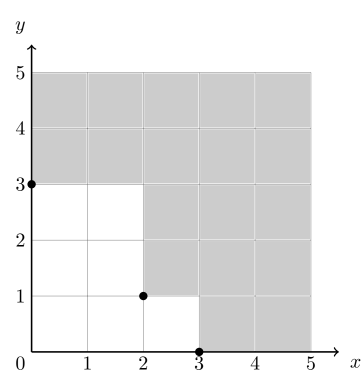
Hence $I = J$ as this completely defines the ideals.

To show Φ is onto, consider an integer partition $(v_1, \ldots, v_t) \in \mathfrak{P}$ and let $I = \big(y^{v_1}, xy^{v_2}, \ldots, x^{i-1}y^{v_t}\big)$. It is not hard to see that $\Phi(I) = (v_1, \ldots, v_t)$. $\square$

Example 3.3. Let $I = (y^3, x^2y, x^3) \subseteq \Bbbk[x, y]$. The monomials not in I are

$$1, y, y^2;$$
$$x, xy, xy^2;$$
$$x^2.$$

Therefore the partition is $(3, 3, 1)$. For those familiar with Ferrers diagrams, the diagram is bottom and left justified diagram in the plane where the parts are the number of boxes in the columns.



The term LEX represents the degree-lexicographical order of monomials. I.e., given a polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$,

$$x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} > x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}$$

if the first non-zero entry of $(\sum(a_i - b_i), a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n)$ is positive. This is a total order and as such, all monomials of degree t are totally ordered. A LEX-segment is the sequence of the first s monomial terms in a given degree (in descending order). We call an ideal generated by LEX-segments a LEX ideal. More precisely, we have the following definition.

Definition 3.4. A monomial ideal $I \subseteq \Bbbk[x_1, \ldots, x_n]$ is is called a LEX ideal if for each $j \geqslant 0$, $I \cap \Bbbk[x_1, \ldots, x_n]_j$ is generated as a $\Bbbk$-vector space by the first $\dim_{\Bbbk}(I \cap \Bbbk[x_1, \ldots, x_n]_j)$ monomials of degree j in descending lexicographical order.

Remark 3.5. In the ring $R = \Bbbk[x, y]$, an ideal $I \subseteq R$ is a LEX ideal if and only if it has the following property: if $x^i y^j \in I$ with $j \geq 1$, then $x^{i+1}y^{j-1} \in I$.

Given an h-vector, there are many ideals associated to it. However, by Macaulay's Theorem, a mapping $h : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ is the Hilbert function for some graded ideal if and only if it is the Hilbert function of an ideal generated by LEX-segments. In other words, for each h-vector, there exists a unique ideal generated by LEX-segments. The next proposition relates LEX ideals to integer partitions

Proposition 3.6. *Let $R = \Bbbk[x, y]$ and I be an (x, y)-primary monomial ideal. Then the integer partition $(\lambda_1(I), \lambda_2(I), \cdots)$ has distinct entries if and only if I is a* LEX *ideal.*

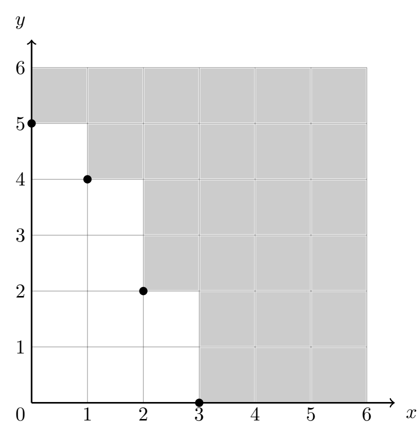
Proof. Assume that $(\lambda_1(I), \lambda_2(I), \ldots)$ is a partition with distinct parts and write

$$I = \left(y^{\lambda_1(I)}, xy^{\lambda_2(I)}, \ldots, x^{i-1}y^{\lambda_i(I)}, \ldots\right)$$

as noted in (3). We will use Remark 3.5 to show that I is LEX. Suppose $x^a y^b \in I$; then there is a $\lambda_i(I)$ such that $x^{i-1}y^{\lambda_i(I)}$ divides $x^a y^b$. As such, $a + 1 \geqslant i$ and $b - 1 \geqslant \lambda_{i+1}(I)$ since $b \geqslant \lambda_i(I) > \lambda_{i+1}(I)$. Therefore, $x^i y^{\lambda_{i+1}(I)}$ divides $x^{a+1}y^{b-1}$ and hence $x^{a+1}y^{b-1}$ is in I.

Conversely, let I be a LEX ideal and consider the partition $(\lambda_1(I), \lambda_2(I), \ldots)$. By definition of $\lambda_i(I)$, for each $i \geqslant 1$ the monomial $x^{i-1}y^{\lambda_i(I)}$ is in I but $x^{i-1}y^{\lambda_i(I)-1} \notin I$. Since I is a LEX ideal, we have $x^i y^{\lambda_i(I)-1} \in I$. However, by definition of $\lambda_{i+1}(I)$, $x^i y^{\lambda_{i+1}(I)-1} \notin I$. Therefore, $\lambda_i(I) > \lambda_{i+1}(I)$ and all the parts are distinct. □

Example 3.7. Consider partition $(5, 4, 2)$. The corresponding LEX ideal is $I = (y^5, xy^4, x^2y^2, x^3)$, with h-vector $(1, 2, 3, 3, 2)$, as can be seen by the lattice:



Notice that the i^{th} entry of the partition $(5, 4, 2)$ counts the lattice points not in the ideal on the line $x = i - 1$, for $i = 1, 2, 3$. Likewise, the j^{th} entry of the h-vector $(1, 2, 3, 3, 2)$ correspond to the number of lattice points not in the ideal on the line $y = -x + j$, for $0 \leqslant j \leqslant 4$.

Define the set $L_2(n) := \{(1, h_1, \ldots, h_s) \in L(n) \mid h_1 = 2\}$. Note that this set consists of all possible h-vectors of a zero-dimensional standard $\Bbbk$-algebra of the form $\Bbbk[x, y]/I$ (we are not allowing I to contain x or y). As such, we have the following.

Theorem 3.8. *The number of integer partitions of $n \in \mathbb{Z}_{\geqslant 1}$ into distinct parts is equal to $|L_2(n)|$.*

Proof. By Macaulay's Theorem [BH93, Theorem 4.2.10], each element of $L_2(n)$ corresponds uniquely to a LEX ideal. Hence by Propositions 3.6 and 3.2, every element of $L_2(n)$ is in one-to-one correspondence with an integer partition with distinct parts. □

Since $L_2(n) \subseteq L(n)$, we have a lower bound for the sequence $\{\ell(n)\}_{n\geqslant 1}$. Although the bounds for $\{\ell(n)\}_{n\geqslant 1}$ are not tight, they have nice combinatorial interpretations. Given this information, it is natural to ask the following questions: What are some better upper and lower bounds? Is it possible to write the sequence $\{\ell(n)\}_{n\geqslant 1}$ in a closed formula?

4. More Properties

In this section we refine the set $L(n)$ of h-vectors of length n in an attempt to obtain a closed form for $\ell(n)$. Let $L(n)$ be defined as in Section 2 and set

$$L_k(n) = \{(1, h_1, h_2, \dots) \in L(n) \mid h_1 = k\},$$

with $\ell_k(n) = |L_k(n)|$. Notice that the $L_k(n)$, $k \geqslant 1$ partition the set $L(n)$.

Proposition 4.1. *Fix $n \in \mathbb{Z}_{\geqslant 0}$. If $k \geqslant 1$ such that $\binom{k+2}{2} \geqslant n$, then $\ell_{k+1}(n+1) = \ell_k(n)$.*

Proof. Notice that if $(1, k, h_2, h_3, \dots) \in L_k(n)$, then $(1, k+1, h_2, h_3, \dots)$ satisfies Macaulay's condition (C). Let the map $\Psi : L_k(n) \to L_{k+1}(n+1)$ be defined by

$$\Psi(1, k, h_2, h_3, \dots) = (1, k+1, h_2, h_3, \dots).$$

We claim that Ψ is a bijection between $L_k(n)$ and $L_{k+1}(n+1)$ if $\binom{k+2}{2} \geqslant n$. As this map is certainly one-to-one for all $n \geqslant 1$, we only need to show it is onto. Let $\mathbf{h} = (1, k+1, h_2, h_3, \dots) \in L_{k+1}(n+1)$. Since the sum of the terms equals $n+1$, we have

$$h_2 \leqslant n + 1 - 1 - (k+1) = n - k - 1.$$

By Condition (C),

$$h_2 \leqslant h_1^{\langle 1 \rangle} = (k+1)^{\langle 1 \rangle} = \binom{k+2}{2}.$$

However, $\binom{k+2}{2} \geqslant n$ and thus

$$h_2 \leqslant n - k - 1 \leqslant \binom{k+2}{2} - k - 1 = \binom{k+1}{2} = k^{\langle 1 \rangle}.$$

This shows that $(1, k, h_1, h_2, \dots) \in L_k(n)$ and hence $\Psi(1, k, h_1, h_2, \dots) = \mathbf{h}$. □

Corollary 4.2. *For all $n, k \geqslant 1$, $\ell_k(n) \leqslant \ell_{k+1}(n+1)$.*

Proof. Follows from the fact that Ψ as defined in Proposition 4.1 is injective. □

Finding a recurrence relation for the sequence $\{\ell(n)\}_{n\geqslant 1}$ appears to be difficult. However, Proposition 4.1 allows us to give a recursion formula for a sequence giving lower bound of $\ell(n)$. Notice that $\{L_k(n)\}_{k\geqslant 1}$ form a partition of $L(n)$, and therefore $\ell(n) = \sum_{k\geqslant 1} \ell_k(n)$. Given the quadratic nature of $\binom{k+2}{2}$ versus the linear nature of n, we find that the recursion listed in Proposition 4.1 represents the "tail" of the summation $\sum_{k\geqslant 1} \ell_k(n)$. In particular, let $s(n) = \min \left\{ k \mid n \leqslant \binom{k+2}{2} \right\}$ and define the sequence

$$\tau(n) = \sum_{k=s(n)}^{n-1} \ell_k(n).$$

Corollary 4.3. *The sequence $\tau(n)$ is a lower bound of $\ell(n)$ and is defined by the following recurrence relation:*

$$\tau(1) = 1;$$
$$\tau(n) = \begin{cases} \tau(n-1) + \ell_{s(n)}(n) & \text{if } s(n) = s(n-1) \\ \tau(n-1) & \text{if } s(n) > s(n-1) \end{cases} \quad \text{for } n \geqslant 2.$$

Proof. It is clear that $\tau(n)$ is a lower bound of $\ell(n)$. If $s(n) = s(n-1)$, then by Proposition 4.1,

$$\tau(n-1) = \sum_{k=s(n-1)}^{n-2} \ell_k(n-1) = \sum_{k=s(n)}^{n-2} \ell_{k+1}(n) = \sum_{k=s(n)+1}^{n-1} \ell_k(n).$$

Hence $\tau(n) = \ell_{s(n)}(n) + \tau(n-1)$. If $s(n) > s(n-1)$, then $s(n) = s(n-1)+1$. Once again, by Proposition 4.1,

$$\tau(n-1) = \sum_{k=s(n-1)}^{n-2} \ell_k(n-1) = \sum_{k=s(n-1)}^{n-2} \ell_{k+1}(n) = \sum_{k=s(n)}^{n-1} \ell_k(n).$$

Therefore we have $\tau(n) = \tau(n-1)$. □

5. Further Directions

As noted in Section 3, $L_2(n)$ is a set whose cardinality represents the number of integer partitions of n into distinct parts. This was obtained by restricting to elements of $L(n)$ whose first two entries are 1,2. These are also the same h-vectors defined by 0-dimensional rings of the form $\Bbbk[x, y]/I$ where I is a graded ideal in $\Bbbk[x, y]$. In Section 4, this result was generalized with the sequences $|L_k(n)|$. Here the $L_k(n)$ are defined by h-vectors defined by 0-dimensional rings of the form $\Bbbk[x_1, \ldots, x_k]/I$. This raises the following question: What algebraic conditions $\mathfrak{C}$ give rise to sequences with interesting counting properties?

Definition 5.1. The h-sequence of a condition $\mathfrak{C}$ is the sequence whose n^{th} term is the number of h-vectors of length n that satisfy $\mathfrak{C}$.

One of the fundamental properties a 0-dimensional $\Bbbk$-algebra could have is the weak Lefschetz property (WLP). This property is geometric in origin, and is a current topic of study in algebra and combinatorics. As shown in [HMNW03, Proposition 3.5], given an integer vector $\mathbf{h} = (1, h_1, \ldots, h_s)$, $\mathbf{h}$ is the h-vector of a graded 0-dimensional $\Bbbk$-algebra having the WLP if and only if $\mathbf{h}$ is a unimodal h-vector such that the positive part of the first difference is also an h-vector. Thus if we let $\mathfrak{C}$ be the WLP, we are able to compute the h-sequence of $\mathfrak{C}$. We list this sequence in Figure 2 along with some other interesting conditions. Apart from the sequence $L_2(n)$, none of these sequences are found on the on-line encyclopedia of integer sequences [13].

$\mathfrak{C}$	$\ell(n)$	WLP	Unimodal	Symmetric	$L_2(n)$	$L_3(n)$	$L_4(n)$	$L_5(n)$
1	1	1	1	1	0	0	0	0
2	1	1	1	1	0	0	0	0
3	2	2	2	1	1	0	0	0
4	3	3	3	2	1	1	0	0
5	5	5	5	2	2	1	1	0
6	8	8	8	3	3	2	1	1
7	12	12	12	2	4	3	2	1
8	18	18	18	4	5	5	3	2
9	27	27	27	3	7	7	5	3
10	40	40	40	4	9	11	7	5
11	57	56	56	3	11	15	11	7
12	82	80	80	6	14	21	16	11
13	116	112	112	4	17	29	23	16
14	163	155	155	7	21	39	33	23
15	227	213	213	4	26	52	46	33
16	313	290	290	8	31	70	63	46
17	428	389	390	5	37	91	87	64
18	583	522	523	10	45	119	117	89
19	788	694	696	5	53	155	157	121
20	1059	915	920	13	63	199	210	164

FIGURE 2. h-sequences of various conditions

6. ACKNOWLEDGEMENTS

We would like to thank Craig Huneke for the initial motivation for the problem. Additionally, the calculations in this note were inspired by many Macaulay2 [Gra] computations. The interested reader should contact the authors if they would like Macaulay2 code for investigating these types of objects further.

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