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Linear Sigma Models With Strongly Coupled Phases
— One Parameter Models —

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Abstract

We systematically construct a class of two-dimensional $(2, 2)$ supersymmetric gauged linear sigma models with phases in which a continuous subgroup of the gauge group is totally unbroken. We study some of their properties by employing a recently developed technique. The focus of the present work is on models with one Kähler parameter. The models include those corresponding to Calabi-Yau threefolds, extending three examples found earlier by a few more, as well as Calabi-Yau manifolds of other dimensions and non-Calabi-Yau manifolds. The construction leads to predictions of equivalences of D-brane categories, systematically extending earlier examples. There is another type of surprise. Two distinct superconformal field theories corresponding to Calabi-Yau threefolds with different Hodge numbers, $h^{2,1} = 23$ versus $h^{2,1} = 59$, have exactly the same quantum Kähler moduli space. The strong-weak duality plays a crucial rôle in confirming this, and also is useful in the actual computation of the metric on the moduli space.

1　Introduction

$(2,2)$ superconformal field theories in $1+1$ dimensions are interesting systems of study. Those with $c = 9$ and integral R-charges are of special importance for supersymmetric string compactifications. There have been a lot of surprises and our understanding has increased considerably, but there is a lot to be understood, both in individual systems and in the grasp of the whole picture. We expect to encounter new surprises ahead and we look forward to face them with sincerity.

A driving force of the progress in our understanding has been the linear sigma models [1]. The earlier study was mostly centered on models with Abelian gauge groups, in which there are as many Fayet-Iliopoulos (FI) parameters as the dimension of the gauge groups. The FI parameter space is decomposed into "phases", and the gauge symmetry is broken to a finite subgroup in each of them. There, the classical analysis can be used to find the nature of the low energy theory. On the other hand, in theories with non-Abelian gauge groups, a continuous subgroup of the gauge group is typically unbroken in some of the phases. In such a phase, we need to have some understanding of the low energy dynamics of the strongly coupled systems with unbroken gauge symmetry. This makes the analysis difficult but at the same time interesting. In some cases, it is possible to understand the low energy dynamics to the extent that we can find the nature of the theory [2, 3], but that is not always the case. Recently, some new techniques to study $(2,2)$ supersymmetric gauge theories have been developed [4–14], and a natural idea would be to apply them to the strongly coupled phases.

In this paper, we systematically construct a class of linear sigma models with strongly coupled phases, extending the examples found in [2, 3], and then employ the recent techniques to study some of their properties. We find two series of theories, (A) and (S). The gauge group G is of the form

$$G \sim U(1) \times H \tag{1.1}$$

where H is (A) a symplectic group or (S) an orthogonal group, and the matter consists of a number of H-singlets $p^1, p^2, \dots$ and a number of H-fundamentals $x_1, x_2, \dots$. The superpotential is of the form

$$(\text{A}): \quad \sum_{i,j} A^{ij}(p)[x_i x_j], \quad (\text{S}): \quad \sum_{i,j} S^{ij}(p)(x_i x_j), \tag{1.2}$$

where $[x_i x_j]$ or $(x_i x_j)$ are respectively the symplectic or orthogonal bilinear invariants and $A^{ij}(p)$ or $S^{ij}(p)$ are antisymmetric or symmetric matrices with polynomial entries. The system has two phases, $r \gg 0$ and $r \ll 0$, where r is the FI parameter of the $U(1)$ factor.

$r \ll 0$ is the strongly coupled phase where H factor is totally unbroken, while $r \gg 0$ is generically weakly coupled. Each system has a dual description [3], in which $r \ll 0$ is generically weakly coupled and $r \gg 0$ is strongly coupled.

The models include those in which the understanding of low energy dynamics [2, 3] can be used to find the nature of the theory at the strongly coupled phase. In particular, there are models in which the $r \ll 0$ phase corresponds to a smooth compact Calabi-Yau threefolds, as in the following list:

model	$h^{1,1}$	$h^{2,1}$	$r \ll 0$ phase	$r \gg 0$ phase	ref
$(\mathrm{A}^2_{(-2)^7,1^7})$	1	50	Pfaffian in $\mathbb{P}^6$	Int. in $G(2,7)$	[15, 2]
$(\mathrm{S}^{8,0}_{(-2)^{32},1^8})$	1	65	Sym-det in $\mathbb{P}^{31}$	$\mathbb{Z}_2$ quad(8) on $\mathbb{P}^3$	[16, 3], $*$
$(\mathrm{S}^{2,+}_{(-2)^5,1^5})$	1	26	Double sym-det in $\mathbb{P}^4$	Reye congruence	[17, 3]
$(\mathrm{A}^2_{(-1)^4,(-2)^3,1^5})$	1	51	Pfaffian in $\mathbb{P}^6_{1111222}$	$\mathbb{Z}_2$ quad(4) on B_5	[18], $*$
$(\mathrm{A}^2_{(-1)^6,(-2),1^4,0})$	1	59	Pfaffian in $\mathbb{P}^6_{1111112}$	$\mathbb{Z}_2$ quad(4) on B_4	[18], $*$
$(\mathrm{A}^2_{(-2)^7,3,1^4})$	1	61	Pfaffian in $\mathbb{P}^6$	$\mathbb{Z}_2$ quad(4) $\mathbb{P}^3$	[19, 18], $*$
$(\mathrm{A}^2_{(-2)^5,(-4)^2,3^2,1^3})$	1	61	Pfaffian in $\mathbb{P}^6_{1111122}$	Pseudo Hybrid	[18], $*$
$(\mathrm{S}^{2,+}_{(-1)^2,(-2)^3,1^4})$	1	23	Double sym-det in $\mathbb{P}^4_{11222}$	$\mathbb{Z}_2$ quad(2) on B/τ	$*$

$$(1.3)$$

Let us briefly explain the ingredients of the table. The name of each model is shown as (A^k_q) or $(\mathrm{S}^{k,\bullet}_q)$: the superscript k encodes the rank of the group H and the subscript q shows the $U(1)$ charges of the matters ($\bullet = \pm, 0$ labels the type of the group, see the main text). $h^{1,1}$ and $h^{2,1}$ are the numbers of exactly marginal twisted chiral and chiral operators respectively. "Pfaffian" or "Sym-det" (for "symmetric determinantal variety") is the locus of p in which the rank of $A(p)$ or $S(p)$ is restricted. "Double sym-det" is a certain double cover of the symmetric determinantal variety. "$\mathbb{Z}_2$ quad(n) on M" means a hybrid model which is a fibration over M of a $\mathbb{Z}_2$ Landau-Ginzburg orbifold of n variables with quadratic superpotential. B_5, B_4 and B are certain Fano threefolds (B_5 is the intersection of three hyperplanes in $G(2,5)$, B_4 is a hyperplane in $G(2,4)$ and B is the intersection of three symmetric bilinears in $\mathbb{CP}^3 \times \mathbb{CP}^3$; τ is the exchange of the two $\mathbb{CP}^3$'s). The references show where the Calabi-Yau manifold is first constructed and studied, or where the linear sigma model is first constructed. "$*$" stands for the present work.

The other phase $r \gg 0$ of some of these models is not a simple non-linear sigma model but a hybrid model, as was just mentioned. Semiclassical analysis can give us some hint concerning the nature of the model [16, 3] but that cannot access the actual content of the quantum theory. In this situation, we computed the two sphere partition functions [4, 5] (with a correction [8, 10]), and learned of the asymptotic behavior of the metric on the

Kähler moduli space, assuming the conjecture of [6]. We found that the metrics for all hybrids of the type "$\mathbb{Z}_2$ quad(2m) on M" behave the same way as in the geometric phase, where the limiting point is a cusp singularity at infinite distance. One hybrid which is not of that type behaves differently and the limiting point is at a finite distance. This is the pattern found in [20] where the former is called the *true hybrid* and the latter is called the *pseudo-hybrid*. (The former is a special case of *good hybrid* in the sense of [21].)

A surprise was waiting for us after the careful computation of the metric on the Kähler moduli space. Two of the above models, $(A^2_{(-1)^6,(-2),1^4,0})$ and $(S^{2,+}_{(-1)^2,(-2)^3,1^4})$, have exactly the same moduli space as a Kähler manifold! The parameters are related by $t_A = -t_S +$ constant: the geometric phase of one model corresponds to the true hybrid phase of the other. The two models correspond to completely different superconformal field theories as the numbers of exactly marginal chiral operators are different, 59 versus 23, and yet they have exactly the same Kähler metric on the twisted chiral moduli space. It would be interesting to see if the equivalence extends to more refined structure, for example by examining the hemisphere partition function [8–10].

We also study models in which $A(p)$ and $S(p)$ are linear in p, more systematically than in [3]. We find that the dual model can be simplified in such a case, and the two phases can be described in a symmetric manner. The model with the data $A : \mathbb{C}^M \hookrightarrow \wedge^2 V^*$ or $S : \mathbb{C}^M \hookrightarrow \mathrm{Sym}^2 V^*$ is dual to the model with the orthogonal data $A^\vee : \mathbb{C}^{M^\vee} \hookrightarrow \wedge^2 V$ or $S^\vee : \mathbb{C}^{M^\vee} \hookrightarrow \mathrm{Sym}^2 V$. This clearly shows that the gauge theory duality has a close connection to projective duality. Our construction implies the equivalences of D-brane categories between the two phases, and they seem to correspond to the equivalences of derived categories in the framework of "homological projective duality" [22]. Such a relation had been observed in sporadic cases [2, 23, 24, 16]. The present work extend these examples systematically.

2 The linear sigma models (A) and (S)

2.1 Low energy dynamics of symplectic and orthogonal gauge theories

Here we describe the essential part of the low energy dynamics of symplectic and orthogonal gauge theories with fundamental matter fields, claimed in [3] with some non-trivial evidence. We shall say that a theory is *regular* when there is no non-compact quantum Coulomb branch. This is a condition to have a well behaved theory with discrete spectrum when the Higgs branch is lifted or compactified by the superpotential.

Let us first consider the $USp(k)$ gauge theory with N massless fundamentals, $x_1, \ldots, x_N$, and vanishing superpotential ($k = 2, 4, 6, \ldots$ and $N = 0, 1, 2, 3, \ldots$). It is regular if and only if N is odd. For $N \leq k$, there is no zero energy state — supersymmetry is broken. For $N = k + 1$, the low energy theory is the free conformal field theory of the mesons, $[x_j x_j] = \sum_{a,b=1}^{k} x_i^a J_{ab} x_j^b$, where J_{ab} is the symplectic structure defining the gauge group. For higher odd $N \geq k + 3$, there is a duality: the theory flows to the same fixed point as the theory with the dual gauge group

$$USp(k) \quad \longleftrightarrow \quad USp(N - k - 1), \tag{2.1}$$

with N fundamentals $\widetilde{x}^1, \ldots, \widetilde{x}^N$ and $\frac{N(N-1)}{2}$ singlets $a_{ij} = -a_{ji}$ with the superpotential

$$W = \sum_{i,j=1}^{N} a_{ij} [\widetilde{x}^i \widetilde{x}^j]. \tag{2.2}$$

The mesons in the original theory correspond to the singlets in the dual, $[x_i x_j] = a_{ij}$.

Next, let us consider the $O(k)$ or $SO(k)$ gauge theory with N massless fields in the fundamental representation, $x_1, \ldots, x_N$, and vanishing superpotential ($k = 1, 2, 3, \ldots$ and $N = 0, 1, 2, 3, \ldots,$). For $k \geq 3$, we have a mod 2 theta angle associated to the $\mathbb{Z}_2$ fundamental group of the gauge group. An $O(k)$ theory can be regarded as a $\mathbb{Z}_2$ orbifold of an $SO(k)$ theory, and there are two possibilities, denoted by $O_+(k)$ and $O_-(k)$, from the choice of orbifold group action. The theory with $k \geq 2$ is regular when $N - k$ is odd and the mod 2 theta angle is turned off, or when $N - k$ is even and the mod 2 theta angle is turned on. For $N \leq k - 2$, there is no zero energy state — supersymmetry is broken. The rest applies only to regular theories. For $N = k - 1$, the $SO(k)$ and $O_-(k)$ theories flow in the infra-red limit to the free theory of the mesons, $(x_i x_j) = \sum_{a=1}^{k} x_i^a x_j^a$. The $O_+(k)$ theory flows to two copies of such a free theory. For $N \geq k$, there is a duality: the theory flows to the same infra-red fixed point as the theory with the dual gauge group

$$O_+(k) \quad \longleftrightarrow \quad SO(N - k + 1)$$

$$SO(k) \quad \longleftrightarrow \quad O_+(N - k + 1) \tag{2.3}$$
$$O_-(k) \quad \longleftrightarrow \quad O_-(N - k + 1),$$

with N vectors $\widetilde{x}^1, \ldots, \widetilde{x}^N$ and $\frac{N(N+1)}{2}$ singlets $s_{ij} = s_{ji}$ with the superpotential

$$W = \sum_{i,j=1}^{N} s_{ij}(\widetilde{x}^i \widetilde{x}^j). \tag{2.4}$$

The mesons in the original theory correspond to the singlets in the dual, $(x_i x_j) = s_{ij}$. The symmetry $\mathbb{Z}_2 = O(k)/SO(k)$ in the $SO(k)$ theory corresponds to the quantum $\mathbb{Z}_2$ symmetry of the dual $O_+(N - k + 1)$ theory (regarded as a $\mathbb{Z}_2$ orbifold). In particular, the baryons $[x_{i_1} \cdots x_{i_k}] = \det(x_{i_b}^a)$ in the $SO(k)$ theory correspond to twist operators in the dual $O_+(N - k + 1)$ theory. There are similar correspondences between $\mathbb{Z}_2$ symmetries in the other dual pairs.

2.2   Description of the model

We now introduce the linear sigma model which we will consider in this paper. The gauge group is

$$G = \frac{U(1) \times H}{\Gamma} \quad \text{with } H = USp(k), \ O_\pm(k), \text{ or } SO(k), \tag{2.5}$$

and the matter contents are M fields, $p^1, \ldots, p^M$, which are charged only under the $U(1)$ factor, and N fields, $x_1, \ldots, x_N$, which are in the fundamental representation of H but can also carry $U(1)$ charges. Γ is a finite normal subgroup of $U(1) \times H$ to make the matter representation faithful: When all the $U(1)$ charges of the p^i's are even and all the $U(1)$ charges of the x_j's are odd, and if $H \neq SO(k)$ with odd k, then $\Gamma = \{(\pm 1, \pm \mathbf{1}_k)\} \cong \mathbb{Z}_2$. Otherwise, Γ is trivial. We assume that the $U(1)$ charges are negative for the p^i's and non-negative for the x_i's. We consider a superpotential of the following form

$$W = \begin{cases} \displaystyle\sum_{i,j=1}^{N} A^{ij}(p)[x_i x_j] & \text{(symplectic)}, \\ \displaystyle\sum_{i,j=1}^{N} S^{ij}(p)(x_i x_j) & \text{(orthogonal)}, \end{cases} \tag{2.6}$$

where $A^{ij}(p)$ and $S^{ij}(p)$ are respectively antisymmetric and symmetric matrices with polynomial entries of degree at least one. Of course we need W to be gauge invariant — the $U(1)$ charges must cancel between $A^{ij}(p)$ and $[x_i x_j]$ or between $S^{ij}(p)$ and $(x_i x_j)$.

Finally and most importantly, we require regularity. We shall describe the criterion momentarily. Let us call this model (A_q^k) or $(S_q^{k,\pm,0})$ if the group H is symplectic or orthogonal respectively. The subscript q shows the $U(1)$ charge assignment, q_{p^i} to p^i and q_{x_j} to x_j. The superscript "$\pm, 0$" in $(S_q^{k,\pm,0})$ refers to the type $O_\pm(k), SO(k)$ of the orthogonal gauge group.

The model has a dual description, with the gauge group

$$\widetilde{G} = \frac{U(1) \times \widetilde{H}}{\widetilde{\Gamma}}, \tag{2.7}$$

where $\widetilde{H}$ is found via (2.2) or (2.4), and the matter contents are M fields $p^1, \ldots, p^M$, with the same charge as in the original theory, N fields, $\widetilde{x}^1, \ldots, \widetilde{x}^N$, in the fundamental of $\widetilde{H}$ and with the opposite $U(1)$ charges compared to $x_1, \ldots, x_N$, and fields $a_{ij} = -a_{ji}$ or $s_{ij} = s_{ji}$ with the same $U(1)$ charges as $[x_i x_j]$ or $(x_i x_j)$. $\widetilde{\Gamma}$ is a finite normal subgroup of $U(1) \times \widetilde{H}$ to make the matter representation faithful: it is either trivial or isomorphic to $\mathbb{Z}_2$. The superpotential is

$$W = \begin{cases} \displaystyle\sum_{i,j=1}^N \left([\widetilde{x}^i \widetilde{x}^j] + A^{ij}(p)\right) a_{ij} & \text{(symplectic)}, \\ \displaystyle\sum_{i,j=1}^N \left((\widetilde{x}^i \widetilde{x}^j) + S^{ij}(p)\right) s_{ij} & \text{(orthogonal)}. \end{cases} \tag{2.8}$$

There is a subtle rule concerning the correspondence $G \leftrightarrow \widetilde{G}$, as described below.

Regularity

The criterion of regularity is:

gauge group	k	$(N-k)$ odd	$(N-k)$ even
$U(1) \times USp(k)$	(even)	regular	not regular
$U(1) \times O_\pm(k)$	any	regular iff $\theta_D = 0$	regular iff $\theta_D = \pi$
$U(1) \times SO(k)$	any	regular iff $\theta_D = 0$	regular iff $\theta_D = \pi$
$\frac{U(1) \times USp(k)}{\{(\pm 1, \pm\mathbf{1}_k)\}}$	(even)	regular	not regular
$\frac{U(1) \times O_\pm(k)}{\{(\pm 1, \pm\mathbf{1}_k)\}}$	even	regular	not regular
$\frac{U(1) \times SO(k)}{\{(\pm 1, \pm\mathbf{1}_k)\}}$	even	regular iff $\theta_D = 0$	regular iff $\theta_D = \pi$

$$\tag{2.9}$$

In this table, $\theta_D \in \{0, \pi\}$ is the mod 2 discrete theta angle. The regularity criterion for the group $G = U(1) \times H$ is copied from the previous subsection. The one for the

group $G = (U(1) \times H)/\{(\pm 1, \pm \mathbf{1}_k)\}$ will be explained in the next subsection. Note that $(U(1) \times O(k))/\{(\pm 1, \pm \mathbf{1}_k)\}$ with k odd is isomorphic to $U(1) \times SO(k)$ which is included in the table. Also, $(U(1) \times SO(k))/\{(\pm 1, \pm \mathbf{1}_k)\}$ with k odd does not make sense because $-\mathbf{1}_k$ is not an element of $SO(k)$.

The rule of $G \leftrightarrow \widetilde{G}$

The above description of $G \leftrightarrow \widetilde{G}$ includes some ambiguity. The ambiguity exists only if all the $U(1)$ charges of the p^i are even and all the $U(1)$ charges of the x_j are odd. Otherwise, there is no ambiguity: we have $U(1) \times H \leftrightarrow U(1) \times \widetilde{H}$, with $H \leftrightarrow \widetilde{H}$ given by (2.2) or (2.4). For the group $G = (U(1) \times H)/\{(\pm 1, \pm \mathbf{1}_k)\}$ with k even, there is no ambiguity either: the dual group is $\widetilde{G} = (U(1) \times \widetilde{H})/\{(\pm 1, \pm \mathbf{1}_{k^\vee})\}$ where $H \leftrightarrow \widetilde{H}$ is given by (2.2) or (2.4), and $k^\vee = N - k \mp 1$. Note that the nonsensical group $(U(1) \times SO(k^\vee)/\{(\pm 1, \pm \mathbf{1}_{k^\vee})\}$ with $k^\vee$ odd does not appear as the dual group since the theory with gauge group $(U(1) \times O_+(k))/\{(\pm 1, \pm \mathbf{1}_k)\}$ with k and $(N - k)$ both even is not regular. The ambiguity arises when k is odd because of the group isomorphism

$$\frac{U(1) \times O(k)}{\{(\pm 1, \pm \mathbf{1}_k)\}} \cong U(1) \times SO(k). \tag{2.10}$$

There are three ways to interpret the gauge group: the left hand side with $O(k) = O_+(k)$, the left hand side with $O(k) = O_-(k)$ and the right hand side. A different choice corresponds to a different dual. We claim that only one choice is acceptable and yields the correct dual. The right choice and the dual is

$$N \text{ even}: \qquad \frac{U(1) \times O_+(k)}{\{(\pm 1, \pm \mathbf{1}_k)\}} \quad \longleftrightarrow \quad \frac{U(1) \times SO(N - k + 1)}{\{(\pm 1, \pm \mathbf{1}_{N-k+1})\}}, \tag{2.11}$$

$$N \text{ odd}: \qquad \frac{U(1) \times O_-(k)}{\{(\pm 1, \pm \mathbf{1}_k)\}} \quad \longleftrightarrow \quad \frac{U(1) \times O_-(N - k + 1)}{\{(\pm 1, \pm \mathbf{1}_{N-k+1})\}}. \tag{2.12}$$

If we had chosen $O_-(k)$ for N even, we would have a non-regular dual theory. If we had chosen $O_+(k)$ for N odd, we would have a nonsensical gauge group in the dual. If we had chosen $U(1) \times SO(k)$, the matter representation of the dual group would be non-faithful. We shall take $(S_q^{k,+})$ and $(S_q^{k,-})$ as the names of the the systems (2.11) and (2.12) respectively.

2.3   Theta angle and regularity

In general, the theta angle specifies a way to sum over different topological sectors in the path integral. In two dimensions, the topological type of principal G-bundles is

labeled by an element of the fundamental group of G. Therefore, the theta angle assigns a phase to each element of $\pi_1(G)$. For locality, it is natural to require the assignment to be a group homomorphism $\pi_1(G) \to U(1)$. We also require that it is invariant under the adjoint action, a condition which is non-trivial only for elements which are not in the identity component. Therefore, the theta angle is a parameter of the group

$$\Theta_G := \mathrm{Hom}(\pi_1(G), U(1))^{\pi_0(G)}. \tag{2.13}$$

For example, for $G = U(1)$, we have $\pi_1(G) \cong \mathbb{Z}$ and $\Theta_G \cong U(1)$. Its element $\mathrm{e}^{i\theta} \in \Theta_G$ assigns a phase $\mathrm{e}^{i\theta n}$ to a $U(1)$ bundle over the spacetime Σ with magnetic flux n. It can be placed as a factor $\exp\left(i\theta \int_\Sigma \frac{i}{2\pi} F_A\right)$ in the path-integral (F_A is the $i\mathbb{R}$ valued fieldstrength of a $U(1)$ gauge potential A). For the product group $G = U(1) \times H$, the theta angle is just a combination of the ones for the two factors, $\Theta_G \cong \Theta_{U(1)} \times \Theta_H$. Note that $\Theta_{USp(k)} = \{1\}$ since $USp(k)$ is simply connected. $\Theta_{SO(k)} = \Theta_{O(k)}$ is isomorphic to $\mathbb{Z}_2$ (see [3] for example) and this is what we meant by the discrete mod 2 theta angle, $\mathrm{e}^{i\theta_D} \in \{\pm 1\}$. For $G = (U(1) \times H)/\{(\pm 1, \pm \mathbf{1}_k)\}$, we must look into it more carefully:

Let us start with $H = USp(k)$. We note that the group G can be written as

$$G = \frac{U(1) \times USp(k)}{\{(\pm 1, \pm \mathbf{1}_k)\}} \cong \frac{\mathbb{R} \times USp(k)}{\langle(\pi, -\mathbf{1}_k)\rangle}, \tag{2.14}$$

where $\langle(\pi, -\mathbf{1}_k)\rangle$ is the free Abelian group generated by the element $(\pi, -\mathbf{1}_k)$ (we consider $U(1)$ as $\mathbb{R}/2\pi\mathbb{Z}$ here). Therefore $\pi_1(G) \cong \mathbb{Z}$ and hence $\Theta_G \cong U(1)$. We define $\mathrm{e}^{i\theta} \in \Theta_G$ to be the one that assigns $\mathrm{e}^{i\theta} \in U(1)$ to the generator $(\pi, -\mathbf{1}_k)$.

Let us next consider $H = SO(k)$, k even. We have

$$G = \frac{U(1) \times SO(k)}{\{(\pm 1, \pm \mathbf{1}_k)\}} \cong \frac{\mathbb{R} \times Spin(k)}{\langle(\pi, \gamma_k), (0, -1)\rangle}, \tag{2.15}$$

where $-1 \in Spin(k)$ projects to $\mathbf{1}_k \in SO(k)$ and $\pm\gamma_k \in Spin(k)$ projects to $-\mathbf{1}_k \in SO(k)$. (π, γ_k) and $(0, -1)$ commute with each other and $(0, -1)$ is of order two. Thus, $\pi_1(G) \cong \mathbb{Z} \times \mathbb{Z}_2$ and $\Theta_G \cong U(1) \times \mathbb{Z}_2$. The element $(\mathrm{e}^{i\theta}, \mathrm{e}^{i\theta_D}) \in \Theta_G$ assigns $\mathrm{e}^{i\theta} \in U(1)$ and $\mathrm{e}^{i\theta_D} \in \{\pm 1\}$ to (π, γ_k) and $(0, -1)$ respectively.

Finally, we consider $H = O(k)$, k even. We only have to impose the $\pi_0(G) \cong \mathbb{Z}_2$ invariance to the theta angle for the case of $H = SO(k)$. Note that the non-trivial element of $\pi_0(G)$ acts on $\pi_1(G)$ as $(\pi, \gamma_k) \mapsto (\pi, -\gamma_k)$ and $(0, -1) \mapsto (0, -1)$. Invariance under this imposes the constraint $\mathrm{e}^{i\theta} \mathrm{e}^{i\theta_D} = \mathrm{e}^{i\theta}$, that is, $\mathrm{e}^{i\theta_D} = 1$. Thus, we have $\Theta_G \cong U(1)$. The discrete theta angle does not survive.

The theta angle enters into the twisted superpotential on the Coulomb branch. For example, in a $U(1)$ theory it enters as $\widetilde{W}_\Theta = i\theta\sigma$, where σ is the fieldstrength superfield.

Note that $2\pi i\sigma$ in $\widetilde{W}$ can be regarded as zero as long as σ is normalized in the correct way. (The F-component of σ has a fieldstrength and obeys a quantization condition.) In this way, the periodicity $\theta \equiv \theta + 2\pi$ is realized. For the group $G = U(1) \times H$, we parametrize the Coulomb branch by $\sigma_0, \sigma_1, \ldots, \sigma_l$ where σ_0 is the $U(1)$ component and the latter $l := [\frac{k}{2}]$ are the components for the maximal torus of H given by

$$\sigma_{\mathfrak{h}} := \begin{cases} \mathrm{diag}(\sigma_1, \ldots, \sigma_l, -\sigma_1, \ldots, -\sigma_l) & \mathfrak{h} = \mathfrak{usp}(2l), \\ \mathrm{diag}(R(\sigma_1), \ldots, R(\sigma_l)) & \mathfrak{h} = \mathfrak{so}(2l) \\ \mathrm{diag}(R(\sigma_1), \ldots, R(\sigma_l), 0) & \mathfrak{h} = \mathfrak{so}(2l+1) \end{cases}$$

$$R(\sigma) := \begin{pmatrix} 0 & -i\sigma \\ i\sigma & 0 \end{pmatrix}.$$

These are correctly normalized — $2\pi i$ times any of these coordinates is equivalent to zero. The above can also be used for the group $G = (U(1) \times H)/\{(\pm 1, \pm\mathbf{1}_k)\}$ but they are not correctly normalized. The correct normalization can be found from the isomorphism

$$\begin{array}{rcl} \dfrac{U(1) \times U(1)^l}{\{(\pm 1, (\pm 1)^l)\}} & \cong & U(1) \times U(1)^l \\ (g, h_1, \ldots, h_l) & \mapsto & (g^2, gh_1, \ldots, gh_l) \end{array} \qquad (2.16)$$

The fieldstrength fields $\sigma_0', \sigma_1', \ldots, \sigma_l'$ on the right hand side are the correctly normalized ones. They are related to the above by

$$\sigma_0' = 2\sigma_0, \qquad \sigma_a' = \sigma_0 + \sigma_a, \quad a = 1, \ldots, l. \qquad (2.17)$$

Let us now write down the theta angle term in each theory.

$$U(1) \times USp(k) \qquad \widetilde{W}_\Theta = i\theta\sigma_0, \qquad (2.18)$$

$$U(1) \times SO(k), \quad U(1) \times O(k) \qquad \widetilde{W}_\Theta = i\theta\sigma_0 + i\theta_D(\sigma_1 + \cdots + \sigma_l), \qquad (2.19)$$

$$\frac{U(1) \times USp(k)}{\{(\pm 1, \pm\mathbf{1}_k)\}} \qquad \widetilde{W}_\Theta = i\theta 2\sigma_0, \qquad (2.20)$$

$$\frac{U(1) \times SO(k)}{\{(\pm 1, \pm\mathbf{1}_k)\}} \; (k \text{ even}) \qquad \widetilde{W}_\Theta = i\theta 2\sigma_0 + i\theta_D\{(\sigma_1 - \sigma_0) + \cdots + (\sigma_l - \sigma_0)\}, \; (2.21)$$

$$\frac{U(1) \times O(k)}{\{(\pm 1, \pm\mathbf{1}_k)\}} \; (k \text{ even}) \qquad \widetilde{W}_\Theta = i\theta 2\sigma_0. \qquad (2.22)$$

We have $i\theta 2\sigma_0$ in the latter three cases and we have the combination $(\sigma_a - \sigma_0)$ in (2.21) since $(\pi, \gamma_k) \in \pi_1(G)$ can be realized as a connection of the maximal torus in which each $U(1)$ factor of the numerator of $(U(1) \times U(1)^l)/\{(\pm 1, (\pm 1)^l)\}$ has magnetic flux $\frac{1}{2}$. The sum $i\theta_D(\sigma_1 + \cdots + \sigma_l)$ is in (2.19) and (2.21) because $-1 \in \pi_1(SO(k))$ can be realized as

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a connection where one $U(1)$ factor of the maximal torus has magnetic flux 1. Note that each of the above $\widetilde{W}_\Theta$ is Weyl invariant. For example, the Weyl group for $SO(k)$ with k even includes the sign flip of an even number of σ_a's, say σ_1 and σ_2. This changes (2.21) by

$$\Delta\widetilde{W}_\Theta = -2i\theta_D(\sigma_1 + \sigma_2) = -2i\theta_D(\sigma_1' + \sigma_2' - \sigma_0') \equiv 0 \qquad (2.23)$$

since $\theta_D \in \{0, \pi\}$. Note that this $\widetilde{W}_\Theta$ would not have been invariant under the sign flip of an odd number of σ_a's, say just σ_1, because then the change is

$$\Delta\widetilde{W}_\Theta = -2i\theta_D\sigma_1 = -i\theta_D(2\sigma_1' - \sigma_0') \equiv i\theta_D\sigma_0' \not\equiv 0. \qquad (2.24)$$

Thus, it cannot be $O(k)$ Weyl invariant. This is another way to see that the mod 2 theta angle is gone for the group $(U(1) \times O(k))/\{(\pm 1, \pm \mathbf{1}_k)\}$. On the other hand, (2.19) is fine for $U(1) \times O(k)$ because σ_a's are correctly normalized for that group. This also means the following for the theory with gauge group $(U(1) \times SO(k))/\{(\pm 1, \pm \mathbf{1}_k)\}$ (k even): It has a $\mathbb{Z}_2$ symmetry associated with $O(k)/SO(k) \cong \mathbb{Z}_2$, but that shifts θ by π. This is related to an ambiguity in the choice of γ_k: we could have taken $(\pi, -\gamma_k)$ instead of (π, γ_k) in the definition of $e^{i\theta}$. In fact, (2.21) corresponds to one choice of γ_k. The swap will change it, say, as $(\sigma_1 - \sigma_0) \to (\sigma_1 + \sigma_0)$ in one of the terms, and that results in the π shift of θ.

To examine the regularity, let us write down the full effective twisted superpotential on the Coulomb branch. First, the Fayet-Iliopoulos (FI) parameter r is combined with the theta angle θ into a complex parameter $t = r - i\theta$. The tree level twisted superpotential is therefore given by

$$\widetilde{W}_{tree}(\sigma) = \begin{cases} -r\sigma_0 + \widetilde{W}_\Theta(\sigma) & \text{if } G = U(1) \times H, \\ -2r\sigma_0 + \widetilde{W}_\Theta(\sigma) & \text{if } G = \dfrac{U(1) \times H}{\{(\pm 1, \pm \mathbf{1}_k)\}}, \end{cases} \qquad (2.25)$$

The effective twisted superpotential is

$$\widetilde{W}_{eff}(\sigma) = \widetilde{W}_{tree}(\sigma) + 2\pi i\rho(\sigma) - \sum_\chi \chi(\sigma)\left(\log(\chi(\sigma)) - 1\right). \qquad (2.26)$$

The term $2\pi i\rho(\sigma)$ comes from the W-boson integral [3], where ρ is half the sum of positive roots of G. The last sum is over the weights of the matter representation. Let us write down $\rho(\sigma)$ more explicitly. For $G = U(1) \times H$, it is

$$2\pi i\rho(\sigma) = \begin{cases} 0 & \text{if } \mathfrak{h} = \mathfrak{usp}(2l), \mathfrak{so}(2l), \\ \pi i(\sigma_1 + \cdots + \sigma_l) & \text{if } \mathfrak{h} = \mathfrak{so}(2l+1), \end{cases} \qquad (2.27)$$

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For $G = (U(1) \times H)/\{(\pm 1, \pm \mathbf{1}_k)\}$, $k = 2l$, it is

$$2\pi i \rho(\sigma) = \begin{cases} -\dfrac{l(l+1)}{2}\pi i\sigma_0' & \text{if } \mathfrak{h} = \mathfrak{usp}(2l), \\ -\dfrac{l(l-1)}{2}\pi i\sigma_0' & \text{if } \mathfrak{h} = \mathfrak{so}(2l), \end{cases} \tag{2.28}$$

To find the regularity in the H sector, we only need to look at the region in the Coulomb branch where σ_a's are much bigger than σ_0. In this limit, the matter contribution is

$$-\sum_\chi \chi(\sigma)\left(\log\chi(\sigma) - 1\right) \;\sim\; N\pi i(\sigma_1 + \cdots + \sigma_l). \tag{2.29}$$

The theory is regular if and only if $\partial_{\sigma_a}\widetilde{W}_{\text{eff}}$ (for $G = U(1) \times H$) or $\partial_{\sigma_a'}\widetilde{W}_{\text{eff}}$ (for $G = (U(1) \times H)/\{(\pm 1. \pm \mathbf{1}_k)\}$) does not vanish modulo $2\pi i\mathbb{Z}$, $a = 1, \ldots, l$. This leads to the criterion (2.9).

2.4 Strongly coupled phase

This theory has two phases, $r \gg 0$ and $r \ll 0$. In the phase $r \gg 0$, the gauge group G is broken to a finite subgroup at a generic solution to the D-term equation, while in the $r \ll 0$ phase the H factor of the gauge group is completely unbroken at any solution of the D-term and F-term equations. In this sense, the $r \gg 0$ phase is generically weakly coupled while the $r \ll 0$ phase is strongly coupled. The situation is opposite in the dual description: $r \gg 0$ is a strongly coupled phase while $r \ll 0$ is generically weakly coupled.

Let us look at the $r \ll 0$ phase of the model (A_q^k) (*resp.* $(S_q^{k,\pm,0})$) in some detail. In this phase, $p = (p^1, \ldots, p^M)$ acquires non-zero values by the D-term equation, breaking the $U(1)$ factor of G to a finite subgroup, and spans a weighted projective space of dimension $(M-1)$, which we denote by $\mathbb{P}$. On the other hand, the x fields all vanish at any solution to the D-term and F-term equations. (See Appendix A concerning this point.) Therefore, the H part of the gauge group is completely unbroken, and the classical analysis is totally invalid. We may regard the theory as a fibration over $\mathbb{P}$ of the theory of gauge group H with N fundamentals. The latter is of the type discussed in Section 2.1, and we may employ some understanding of its low energy dynamics which is summarized there.

The fundamentals have mass matrix $A(p) = (A^{ij}(p))$ (*resp.* $S(p) = (S^{ij}(p))$) by the superpotential (2.6), and the number of massless fundamentals is determined by its rank. If the rank is too big, the number is too small for the theory to have a zero energy state. There must be at least $(k+1)$ (*resp.* $(k-1)$) massless fundamentals. Therefore, the low

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energy dynamics localizes near

$$Y_A \;=\; \Big\{\, p \in \mathbb{P} \;\Big|\; \mathrm{rank}\,A(p) \,\leq\, N - k - 1 \,\Big\}, \tag{2.30}$$

$$\Big(resp. \quad Y_S \;=\; \Big\{\, p \in \mathbb{P} \;\Big|\; \mathrm{rank}\,S(p) \,\leq\, N - k + 1 \,\Big\}.\Big) \tag{2.31}$$

$Y = Y_A$ (*resp.* Y_S) is called the *Pfaffian variety* (*resp. symmetric determinantal variety*) and has codimension $\frac{(k+1)k}{2}$ (*resp.* $\frac{(k-1)k}{2}$) in $\mathbb{P}$. Suppose there is no point p with $\mathrm{rank}\,A(p) < N - k - 1$ (*resp.* $\mathrm{rank}\,S(p) < N - k + 1$). Then the variety Y is smooth except at the intersection with the orbifold singularity of the ambient space $\mathbb{P}$. In that case, the mass matrix has constant rank so that the H theory has $(k+1)$ (*resp.* $(k-1)$) massless fundamentals, x_i^L, everywhere on Y. As described above, the H-theory flows to the free theory of $\frac{(k+1)k}{2}$ mesons $[x_i^L x_j^L]$ (*resp.* $\frac{(k-1)k}{2}$ mesons $(x_i^L x_j^L)$). The superpotential (2.6) is a perfect pairing between these mesons and the coordinates transverse to Y in $\mathbb{P}$. Therefore, the low energy theory is simply the non-linear sigma model with Y as the target, except for the theory with $H = O_+(k)$ where the target is an unramified double cover $\widetilde{Y}_S$ of $Y = Y_S$. Suppose now that there is a locus with $\mathrm{rank}\,A(p) < N - k - 1$ (*resp.* $\mathrm{rank}\,S(p) < N - k + 1$). Then, the variety Y has a singularity there in addition to the orbifold singularity from $\mathbb{P}$. In that case, the number of massless fundamentals jumps at that locus. Unfortunately, we do not always know the nature of the low energy dynamics of such a theory.

Let us look at the dual description in the corresponding phase, which is $\widetilde{r} \ll 0$. A part of the F-term equations is

$$[\widetilde{x}^i \widetilde{x}^j] + A^{ij}(p) = 0 \quad \forall i, j \quad \Big(resp. \quad (\widetilde{x}^i \widetilde{x}^j) + S^{ij}(p) = 0 \quad \forall i, j\Big). \tag{2.32}$$

The field p is forbidden to vanish by these equations and the D-term equations. It therefore breaks the $U(1)$ factor to a finite subgroup and spans $\mathbb{P}$ again. Since $\widetilde{x}^i$'s take values in $\mathbb{C}^{N-k-1}$ (*resp.* $\mathbb{C}^{N-k+1}$), the equations (2.32) can be solved only if $\mathrm{rank}\,A(p) \leq N - k - 1$, i.e. $p \in Y_A$ (*resp.* $\mathrm{rank}\,S(p) \leq N - k + 1$, i.e. $p \in Y_S$). If the inequality is saturated everywhere, then the $\widetilde{H}$ factor of the gauge group is broken everywhere. In particular, the gauge group $\widetilde{G}$ is broken to a finite subgroup. It is a weakly coupled phase and we can trust the classical analysis. We find the non-linear sigma model whose target space is the vacuum manifold. Let us see what that manifold is. For a given $p \in Y$, the equation (2.32) has a unique solution for $\widetilde{x}$ up to gauge, except in the case $\widetilde{H} = SO(N - k + 1)$ where there are *two* solutions distinguished by the sign of the baryons $\det(\widetilde{x}_{a_j}^i)$. Thus, the target space is Y in (2.30) (*resp.* (2.31)) or a double cover of Y_S. We reproduce the same

low energy theory as in the original. In the model $(S_q^{k,+})$, this specifies the unramified double cover

$$\widetilde{Y}_S = \Big\{ (p, \widetilde{x}) \;\Big|\; \text{stability}, \;\; (\widetilde{x}^i \widetilde{x}^j) + S^{ij}(p) = 0 \;\Big\} \Big/ \widetilde{G}_{\mathbb{C}} \tag{2.33}$$

of Y_S which was not even constructed in the original model. What if the rank of $A(p)$ (*resp.* $S(p)$) can go lower at some locus? For Model (A), it is lowered by an even number since $A(p)$ is antisymmetric, and a continuous subgroup of $\widetilde{H}$ will be unbroken at such a locus. Therefore, a classical analysis is not valid. On the other hand, in Model (S), the rank can drop just by one. In such a case, the unbroken part of $\widetilde{H}$ is not continuous, and the classical analysis can be employed. We have the non-linear sigma model on the quotient $\widetilde{Y}_S$ given by (2.33), which in this case is a double cover of Y_S ramified along the rank $(N-k)$ locus. The dual description has an advantage over the original in these cases.

In general, neither the original nor the dual description can be used to find the nature of the $r \ll 0$ phase. However, this does not mean that the theory is sick. Rather, we know that the theory is regular in the sense that there is no Coulomb branch. We simply do know not how to directly analyze such a strongly coupled system at the moment. It would be interesting to employ recently developed techniques [4–14] to study such linear sigma models.

2.5   Examples

Some of the models have been studied in the past.

$(A_{(-2)^7,1^7}^2)$ This was introduced in [2] to understand Rødland's work [15] by a linear sigma model. The $r \gg 0$ phase is the geometric phase corresponding to the complete intersection X_A of seven hypersurfaces in the Grassmannian $G(2,7)$. It predicted equivalences of the derived category, $D^b(X_A) \cong D^b(Y_A)$, despite the fact that X_A and Y_A are not birationally equivalent. The equivalence was later proved by Borisov-Caldararu and Kuznetsov [23, 24].

$(S_{(-2)^4,1^8}^{1,+})$ This was studied in [16] as an Abelian analog of [2]. The $r \gg 0$ phase is the geometric phase corresponding to the complete intersection X_S of four quadrics in $\mathbb{CP}^7$. Y_S is singular because of the rank two degeneration of $S(p)$. The category of B-branes can be described as a noncommutative resolution of Y_S, and is shown to be equivalent to $D^b(X_S)$.

$(S_{(-2)^5,1^5}^{2,+})$ This was introduced in [3] to put the work of Hosono and Takagi [17] in the

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framework of the linear sigma model. The $r \gg 0$ phase is the geometric phase corresponding to the Reye congruence X_S which is not birationally equivalent to Y_S. It predicted the derived equivalences, $D^b(X_S) \cong D^b(Y_S)$, which was later proved by Hosono-Takagi themselves [25].

All these examples involve Calabi-Yau threefolds, and $A(p)$ and $S(p)$ are all linear in the p's.

In this paper we would like to find and study other examples in which the variety Y_A, Y_S or $\widetilde{Y}_S$ is a smooth Calabi-Yau threefold. The Calabi-Yau condition amounts to

$$\sum_{i=1}^{M} q_{p^i} + k \sum_{j=1}^{N} q_{x_j} = 0. \tag{2.34}$$

The condition on the dimension is

$$M - 1 - \frac{k(k \pm 1)}{2} = 3. \tag{2.35}$$

If we exclude those which are equal to or dual to models with Abelian gauge groups, one can only find surprisingly small number of examples. We found five models:

$$(A^2_{(-1)^4,(-2)^3,1^5}), \ (A^2_{(-1)^6,(-2),1^4,0}), \ (A^2_{(-2)^7,3,1^4}), \ (A^2_{(-2)^5,(-4)^2,3^2,1^3}), \ (S^{2,+}_{(-1)^2,(-2)^3,1^4})$$

The Pfaffian varieties Y_A in the first four models had been studied in Kanazawa's paper [18] which aimed at generalization of Rødland's example. (The fourth had been constructed earlier by Tonoli [19].) They correspond respectively to X_5, X_{10}, X_{13} and X_7 in the notation of [18]. The double cover $\widetilde{Y}_S$ of the symmetric determinantal variety in the last model is a new Calabi-Yau threefold which nobody has studied, as far as we are aware of. In the next two sections, we study these examples in detail, with the focus on the other phase $r \gg 0$.

To be precise, the search also found three other models,

$$(A^4_{(-2)^{14},1^7}), \ (S^{8,0}_{(-2)^{32},1^8}), \ (S^{4,0}_{(-2)^{10},1^5})$$

in which the Pfaffian Y_A and the symmetric determinants Y_S are smooth Calabi-Yau threefolds. However, it turns out they are not new. These are nothing but the $X_{A^\vee}$, $X_{S^\vee}$, $X_{S^\vee}$ respectively of the above three models, i.e. the intersection of seven hyperplanes in $G(2,7)$, the intersection of four quadrics in $\mathbb{CP}^7$, and the Reye congruence, where $A^\vee$ and $S^\vee$ are the orthogonals of A and S in a certain sense. This has led us to notice that, when $A(p)$ and $S(p)$ are linear in p, the dual model can be simplified to take the same form

as the original, but with different number of p fields and of course with the dual gauge group. This will be discussed in Section 5 in a more general context, including models not limiting to threefolds nor Calabi-Yaus. The simplification allows us to describe the $r \gg 0$ and the $r \ll 0$ phases symmetrically.

3   Pfaffian Calabi-Yau threefolds from (A) series

In this section, we study the four models in (A) series in detail. As mentioned above, the Pfaffian Calabi-Yau threefolds Y_A that appear in the $r \ll 0$ phase are those studied by Kanazawa [18]. The main focus here is the nature of the $r \gg 0$ phase. We shall study it by looking at the classical Lagrangian and also by looking at the metric on the moduli space, found by computing the two sphere partition function. The classical analysis shows that they are all in hybrid phase. To be precise, in each of the first three models, it may be regarded as a fibration over a Fano threefold of a four-variable $\mathbb{Z}_2$ Landau-Ginzburg orbifold with quadratic superpotential, while the last model has no such interpretation. The metric of the moduli space in the first three models behave in the same way as the geometric phase, where $r \to +\infty$ is at infinite distance, while the one of the last model is different, where $r \to +\infty$ is at finite distance. This relation between the classical picture and quantum Kähler metric was observed in a work of Aspinwall and Plesser [20]. In their language, the first three are in true hybrid phase while the last one is in pseudo-hybrid.

Before studying the individual cases, let us describe what can be said uniformly in all four models. The gauge group is $G = U(1) \times SU(2)$ or $\frac{U(1) \times SU(2)}{\{(\pm 1, \pm \mathbf{1}_2)\}} \cong U(2)$, and the matter consists of seven $SU(2)$ singlets $p^1, \ldots, p^7$ which are negatively charged under $U(1)$, and five $SU(2)$ doublets $x_1, \ldots, x_5$ which are mostly positively charged under $U(1)$. The D-term equations are

$$\sum_{i=1}^{7} q_{p^i} |p^i|^2 + \sum_{j=1}^{5} q_{x_j} ||x_j||^2 = r_{\mathfrak{u}(1)}, \tag{3.1}$$

$$xx^\dagger = \frac{1}{2} ||x||^2 \mathbf{1}_2, \tag{3.2}$$

and the F-term equation are

$$\sum_{j=1}^{5} A^{ij}(p) x_j^1 = \sum_{j=1}^{5} A^{ij}(p) x_j^2 = 0, \quad i = 1, \ldots, 5 \tag{3.3}$$

$$\sum_{1 \le i < j \le 5} \frac{\partial}{\partial p^k} A^{ij}(p) x_i^1 x_j^2 = 0, \quad k = 1, \ldots, 7. \tag{3.4}$$

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In (3.1), $r_{\mathrm{u}(1)}$ is the FI parameter for the $U(1)$ factor. It is related to the correctly normalized FI parameter r by $r_{\mathrm{u}(1)} = r$ or $r_{\mathrm{u}(1)} = 2r$ depending on whether the gauge group is $U(1) \times SU(2)$ or $U(2)$. See (2.25). We used $||x_j|| = \sum_{a=1}^{2} |x_j^a|^2$ and $||x||^2 = \sum_{j=1}^{5} ||x_j||^2$. "$x$" in (3.2) stands for the 2×5 matrix (x_i^a).

We assume that $A(p)$ is generic so that $A(p) = 0$ has no solution other than $p = 0$. Then the Pfaffian Y_A is the locus of rank $A(p) = 2$. In that case, Y_A is smooth unless it intersects the orbifold singularity of the ambient space. In fact, in all cases, the intersection is impossible under further genericity, since Y_A is of codimension three in the six-dimensional ambient space $\mathbb{P}$ while the codimensions of the orbifold locus is at least four. As discussed in the previous section, in the $r \ll 0$ phase, the theory reduces to the non-linear sigma model whose target space is this smooth Pfaffian Y_A.

The nature of the $r \gg 0$ phase depends on the model, but two things can be said on all four models:

(i) *It is a weakly coupled phase; the gauge group is broken to a finite subgroup at any solution to the D-term equations (3.1)-(3.2),*

(ii) *All p^i's must vanish at any solution to the D-term and F-term equations (3.1)-(3.4), given the genericity assumption that Y_A is smooth.*

For $r \gg 0$, i.e. $r_{\mathrm{u}(1)} \gg 0$, the $U(1)$ D-term equation (3.1) forces some x_j to have non-zero values. Then, the $SU(2)$ D-term equation (3.2) requires that the matrix x must have rank two. This means that two of x_i's, say x_i and x_j, are linearly independent. Then, a stabilizer $(u, U) \in U(1) \times SU(2)$ must satisfy the equations, $u^{q_{x_i}} U x_i = x_i$ and $u^{q_{x_j}} U x_j = x_j$. The infinitesimal version can be written as

$$\delta U \cdot V + V \cdot \begin{pmatrix} q_{x_i} \delta u & 0 \\ 0 & q_{x_j} \delta u \end{pmatrix} = 0, \tag{3.5}$$

where V is the invertible 2×2 matrix made of x_i and x_j. Multiplying V^{-1} from left and taking the trace, we find $(q_{x_i} + q_{x_j})\delta u = 0$. Since $q_{x_i} + q_{x_j}$ is positive, we find $\delta u = 0$ and hence $\delta U = 0$. Thus, no continuous subgroup of the gauge group remains unbroken. This proves (i). Next let us show (ii). The first set of F-term equations (3.3) means that $x^1 = (x_i^1)$ and $x^2 = (x_i^2)$ are both in the kernel of $A(p)$. If $A(p)$ has rank four, the kernel of $A(p)$ is one-dimensional and hence the rank of (x_i^a) is at most one, failing to satisfy the requirement from the D-term equation. If $A(p)$ has rank two, in which case p represents a point of Y_A, x^1 and x^2 are linearly independent vectors in the three-dimensional kernel of $A(p)$. Then, the second set of F-term equations (3.4) contradicts the smoothness of

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the Pfaffian Y_A at p. This can be shown as follows. (In this proof, we fix $p = p_*$ with rank $A(p_*) = 2$.) We take a basis $(x^1, x^2, x^3, x^4, x^5)$ of $\mathbb{C}^5$, adding three vectors to x^1 and x^2, so that the first three elements span the kernel of $A(p_*)$. In this basis, the matrix $A(p_*)$ is of the form

$$A(p_*) = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \lambda \\ & & & -\lambda & 0 \end{pmatrix}, \tag{3.6}$$

with $\lambda \neq 0$. In a neighborhood of p_*, the Pfaffian Y_A can be defined as the set of solutions to the three equations (See Appendix A),

$$\mathrm{Pf}_1 A(p) = \mathrm{Pf}_2 A(p) = \mathrm{Pf}_3 A(p) = 0. \tag{3.7}$$

Here $\mathrm{Pf}_i A$ is defined to be the Pfaffian of the 4×4 minor of A obtained by deleting the i-th row and i-th column. Smoothness of Y_A at p_* requires that the 7×3 matrix of differentials

$$\left(\frac{\partial}{\partial p^i} \mathrm{Pf}_1 A, \frac{\partial}{\partial p^i} \mathrm{Pf}_2 A, \frac{\partial}{\partial p^i} \mathrm{Pf}_3 A \right)_{i=1,\dots,7} \tag{3.8}$$

must be of rank three at p_*. Since $A^{ij}(p_*)$ vanishes unless $i, j = 4, 5$, we have

$$\frac{\partial}{\partial p^i} \mathrm{Pf}_3 A(p_*) = \frac{\partial}{\partial p^i} A^{12}(p_*) A^{45}(p_*). \tag{3.9}$$

These all vanish if we use F-term equations (3.4) which reads $\frac{\partial}{\partial p^i} A^{12}(p) = 0$ in this basis. Therefore, the matrix (3.8) has rank two or less, violating the smoothness condition at p_*. Thus, the only option left for us is that $A(p)$ has rank zero, which means $p = 0$ by the assumption of genericity. This proves (ii).

Now we move on to the research in the individual cases. A note on notation: We shall write $A^{lm}_{ij\dots}$ for the coefficient of $p^i p^j \cdots$ in $A^{lm}(p)$.

3.1 $(\mathbf{A}^2_{(-1)^4, (-2)^3, 1^5})$

The model has gauge group $G = U(1) \times SU(2)$ and the matter consists of $p^1, \dots, p^4$ of charge -1, p^5, p^6, p^7 of charge -2 and doublets $x_1, \dots, x_5$ of charge 1. The matrix $A(p)$ must have charge -2 and hence is quadratic in $p^1, \dots, p^4$ and linear in p^5, p^6, p^7.

First, let us identify the location of the singular points. Parametrizing the maximal torus of $SU(2)$ in the standard way, $\sigma_{\mathfrak{su}(2)} = \mathrm{diag}(\sigma_1, -\sigma_1)$, the effective twisted superpo-

tential on the Coulomb branch is

$$\widetilde{W}_{\textit{eff}} = -t\sigma_0 - 4(-\sigma_0)(\log(-\sigma_0) - 1) - 3(-2\sigma_0)(\log(-2\sigma_0) - 1)$$
$$-5(\sigma_0 + \sigma_1)(\log(\sigma_0 + \sigma_1) - 1) - 5(\sigma_0 - \sigma_1)(\log(\sigma_0 - \sigma_1) - 1). \quad (3.10)$$

The singularities are at the locus where the true Coulomb branch exists, i.e. where there is a non-compact space of solutions to $\partial_{\sigma_0}\widetilde{W}_{\textit{eff}} \equiv 0$, $\partial_{\sigma_1}\widetilde{W}_{\textit{eff}} \equiv 0$ (mod $2\pi i\mathbb{Z}$), which read

$$\mathrm{e}^{-t} = \frac{(\sigma_0 + \sigma_1)^5(\sigma_0 - \sigma_1)^5}{(-\sigma_0)^4(-2\sigma_0)^6}, \quad 1 = \frac{(\sigma_0 + \sigma_1)^5}{(\sigma_0 - \sigma_1)^5}. \quad (3.11)$$

Note that $\sigma_0 = 0, \sigma_0 \pm \sigma_1 = 0$ and $\sigma_1 = 0$ should be excluded since they correspond to the loci where some of the matter or the W-boson is massless, in which the analysis is invalid. Then, the admissible solutions are $(\sigma_1/\sigma_0)^2 = -5\pm\sqrt{20}$. There are four solutions in total but we need to mod out by the Weyl group action $\sigma_1 \mapsto -\sigma_1$. We therefore have two singular points at

$$\mathrm{e}^{-t} = 984 \pm 440\sqrt{5}. \quad (3.12)$$

The Pfaffian Y_A in the $r \ll 0$ phase is nothing but X_5 in Kanazawa's work [18]. It is a Calabi-Yau threefold with the following topological data

$$h^{1,1}(X_5) = 1, \quad h^{2,1}(X_5) = 51, \quad \int_{X_5} H^3 = 5, \quad \int_{X_5} c_2(X_5)H = 38, \quad (3.13)$$

where H is a generator of $H^2(X_5, \mathbb{Z})$. In that paper, the mirror family was proposed and the Picard-Fuchs equation was identified with the Calabi-Yau equation No. 238 in [26][1]. The singular points of that equation agrees with (3.12) if we identify the parameter ϕ in [18] with our $-\mathrm{e}^t$.

Now let us look into the $r \gg 0$ phase. We first determine the vacuum manifold. We already know that $p = 0$ is required. Then, the remaining equations are

$$xx^\dagger = \frac{1}{2}r\,\mathbf{1}_2, \quad (3.14)$$

$$\sum_{i,j=1}^{5} A_k^{ij}[x_ix_j] = 0, \quad k = 5, 6, 7. \quad (3.15)$$

If the gauge group were $U(2)$, the vacuum manifold would be a complete intersection of three hypersurfaces in the Grassmannian $G(2,5)$. This is a Fano threefold which we denote by $M_{A'}$, where $A' = (A_k^{ij})$. However, the actual gauge group is $U(1) \times SU(2)$ and

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[1]In older versions of the database it had the number 302.

hence the vacuum manifold is something like $M_{A'}/\mathbb{Z}_2$ where $\mathbb{Z}_2$ is generated by $(-1, -\mathbf{1}_2)$ and acts trivially on $M_{A'}$. The fields p^5, p^6, p^7 are used up in imposing the equations (3.15) and the remaining fields are $p_{(4)} = (p^1, \ldots, p^4)$. They are not always massive and hence we have a hybrid model. The target space is the quotient

$$\mathfrak{X}_{A'} = \left\{ (p_{(4)}, x) \in \mathbb{C}^4 \oplus \mathrm{Hom}(\mathbb{C}^5, \mathbb{C}^2) \;\middle|\; \mathrm{rank}(x) = 2, \;\; (3.15) \right\} \Big/ \mathbb{C}^\times \times SL(\mathbb{C}^2), \quad (3.16)$$

and the superpotential is

$$W_{A''} = \sum_{i,j=1}^{5} A^{ij}(p_{(4)})[x_i x_j] = \sum_{l,m=1}^{4} A_{lm}(x) p^l p^m, \quad (3.17)$$

where $A_{lm}(x) := \sum_{i,j=1}^{5} A_{lm}^{ij}[x_i x_j]$ and $A'' = (A_{lm}^{ij})$. The space $\mathfrak{X}_{A'}$ is the total space of an orbifold vector bundle of rank 4 over $M_{A'}/\mathbb{Z}_2$. It is an orbifold with a $\mathbb{Z}_2$ orbifold singularity at the zero section $p_{(4)} = 0$. The model $(\mathfrak{X}_{A'}, W_{A''})$ may be regarded as a fibration over $M_{A'}$ of a $\mathbb{Z}_2$ Landau-Ginzburg orbifold of four variables with a quadratic superpotential. Since $M_{A'}$ is three dimensional, the corank of the mass matrix $A''(x) = (A_{lm}(x))$ can be $0, 1$ or 2. The situation is similar to the $r \ll 0$ phase of the model $(\mathrm{S}_{(-1)^4,1^8}^{1,+})$ studied in [16]. There are two possibilities [3]. If the $\mathbb{Z}_2$ is of the type $O_+(1)$, then, the situation is really like [16], where we have a double cover of $M_{A'}$, ramified along the corank $A''(x) = 1$ locus, and have an unresolvable conifold singularity at the points with corank $A''(x) = 2$. If the $\mathbb{Z}_2$ is of the type $O_-(1)$, then, we do not have the double cover.

As an alternative to the above classical analysis, we may employ a recently developed technique. That is, we compute the sphere partition function after [4, 5] and study the behavior of the Kähler metric on the moduli space in the $r \gg 0$ regime, assuming the conjecture of [6]. (Such a computation in the hybrid phase has been done recently in [27, 28].) The calculation of the partition function for the $r \gg 0$ phase of this model is straightforward. Details about the calculation can be found in appendix B. The result is

$$Z_{S^2} = -\frac{(z\bar{z})^q}{2} \oint \frac{\mathrm{d}\varepsilon_1 \mathrm{d}\varepsilon_2}{(2\pi i)^2} (z\bar{z})^{-\frac{\varepsilon_1}{2} - \varepsilon_2} \pi^3 \frac{\left[\cos \pi \left(\frac{\varepsilon_1}{2} + \varepsilon_2\right)\right]^4 \left[\sin \pi \left(\varepsilon_1 + 2\varepsilon_2\right)\right]^3}{\left[\sin \pi \varepsilon_1\right]^5 \left[\sin 2\pi \varepsilon_2\right]^5} \times$$

$$\times \left| \sum_{\substack{k,l=0 \\ k+l=even}}^{\infty} (k - l - \varepsilon_1 + 2\varepsilon_2) z^{\frac{k+l}{2}} \frac{\left[\Gamma\left(\frac{1}{2} + \frac{k}{2} + \frac{l}{2} - \frac{\varepsilon_1}{2} - \varepsilon_2\right)\right]^4 \left[\Gamma\left(1 + k + l - \varepsilon_1 - 2\varepsilon_2\right)\right]^3}{\left[\Gamma\left(1 + k - \varepsilon_1\right)\right]^5 \left[\Gamma\left(1 + l - 2\varepsilon_1\right)\right]^5} \right|^2, \quad (3.18)$$

where $z := \mathrm{e}^{-t}$. We define $|f(z, \varepsilon_1, \varepsilon_2)|^2 := f(z, \varepsilon_1, \varepsilon_2) f(\bar{z}, \varepsilon_1, \varepsilon_2)$. The same will be used in what follows. Evaluating the residue integrals to lowest order, the correspondence

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$Z_{S^2} = e^{-K}$ gives the following leading behavior

$$e^{-K} = -\frac{5}{96}(z\bar{z})^q \left(\log^3 \frac{z\bar{z}}{2^{16}} - 240\zeta(3)\right) + \dots \tag{3.19}$$

The leading behavior of the Kähler metric near $z = 0$ is thus

$$g_{z\bar{z}} = \frac{3}{z\bar{z}(\log \frac{z\bar{z}}{2^{16}})^2} + \dots. \tag{3.20}$$

This shows that the limit $r \to +\infty$ is at infinite distance in the moduli space. Thus, we are in the true hybrid phase in the sense of [20].

This suggests a possible geometric interpretation of the model. In [6] it has been observed in examples that the sphere partition function and the correctly normalized Kähler potential are related by a Kähler transformation

$$e^{-K} \sim \frac{Z_{S^2}}{X^0(z)\overline{X^0(z)}}, \tag{3.21}$$

where $X^0(z)$ was found to be the fundamental period up to some rescalings. Therefore we can attempt to extract the fundamental period using this property. Given the general form of the A-model Kähler potential, the normalization factor is the coefficient of the $\log^3 z\bar{z}$-term of the partition function. After multiplication with a constant and rescaling[2] $z \to -256z$ we obtain

$$X^0(z) = 1 - 76z + 45036z^2 - 41983600z^3 + 47990065900z^4 - 61620234426576z^5 + \dots \tag{3.22}$$

Consulting the database of Calabi-Yau equations [26], this is the fundamental period associated to the Picard-Fuchs operator

$$\begin{aligned}
\mathcal{L} &= \theta^4 + z(2000\theta^4 + 3904\theta^3 + 2708\theta^2 + 756\theta + 76) \\
&\quad + z^2(63488\theta^4 + 63488\theta^3 - 21376\theta^2 - 18624\theta - 2832) \\
&\quad + z^3(512000\theta^4 + 24576\theta^3 - 37888\theta^2 + 6144\theta + 3072) + z^4(4096(2\theta + 1)^4)
\end{aligned} \tag{3.23}$$

where $\theta = z\frac{d}{dz}$. This is precisely the differential operator No. 238 derived in [18] via analytic continuation of the Picard-Fuchs operator of the Pfaffian Calabi-Yau.

We can further check this result using mirror symmetry. Since the Picard-Fuchs operator in the $r \ll 0$-phase is known, we can determine the Picard-Fuchs operator at $r \gg 0$ from it and compute the periods. The Kähler potential can be determined using a procedure described in [29]. The result matches the localization computation. Further details are discussed in appendix E.

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[2]Such a scaling factor is also observed in [18].

It would also be interesting to employ more recent techniques, such as the one in [9, 10, 8], to the original and the dual linear sigma models, to obtain further information about the system.

3.2 $(\mathbf{A}^2_{(-1)^6,(-2),1^4,0})$

The model has gauge group $G = U(1) \times SU(2)$ and the matter consists of $p^1, \ldots, p^6$ of charge -1, p^7 of charge -2, doublets $x_1, \ldots, x_4$ of charge 1 and a doublet x_5 of charge 0. The matrix $A(p)$ must have charge -2 in the first 4×4 block (quadratic in $p^{1,\ldots,6}$ and linear in p^7) and charge -1 in the remaining off-diagonals (linear in $p^{1,\ldots,6}$).

The singular points are found in the same way as in the previous subsection. The vacuum equations read

$$\mathrm{e}^{-t} = \frac{(\sigma_0 + \sigma_1)^4 (\sigma_0 - \sigma_1)^4}{(-\sigma_0)^6 (-2\sigma_0)^2}, \quad 1 = \frac{(\sigma_0 + \sigma_1)^4 \sigma_1}{(\sigma_0 - \sigma_1)^4 (-\sigma_1)}. \tag{3.24}$$

Taking the admissible solutions and modding out by the Weyl group, we find that there are two singular points at

$$\mathrm{e}^{-t} = 272 \pm 192\sqrt{2}. \tag{3.25}$$

The Pfaffian Y_A in the $r \ll 0$ phase is nothing but X_{10} in Kanazawa's work [18]. It is a Calabi-Yau threefold with the following topological data

$$h^{1,1}(X_{10}) = 1, \quad h^{2,1}(X_{10}) = 59, \quad \int_{X_{10}} H^3 = 10, \quad \int_{X_{10}} c_2(X_{10}) H = 52. \tag{3.26}$$

In that paper, the mirror family was proposed and the Picard-Fuchs equation was identified with the Calabi-Yau equation No. 210 (formerly 263) in [26]. The singular points of that equation agrees with (3.25) if we identify the parameter ϕ in [18] with our $-\mathrm{e}^t$.

Let us now look at the $r \gg 0$ phase. First, we determine the vacuum manifold. The F-term equations that remain after $p = 0$ are

$$\sum_{l,m=1}^{4} A_7^{lm}[x_l x_m] = 0, \tag{3.27}$$

$$\sum_{l=1}^{5} A_j^{l5}[x_l x_5] = 0, \qquad j = 1, \ldots, 6. \tag{3.28}$$

By genericity of A, the second set of equations yields $[x_l x_5] = 0$ for $l = 1, \ldots, 4$. If $x_5 \neq 0$, this means that $x_1, \ldots, x_4$ are all proportional to x_5, violating the D-term constraint

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rank$(x) = 2$. Thus, x_5 must vanish. If the gauge group were $U(2)$, then the vacuum manifold would be the hypersurface (3.27) of the Grassmannian $G(2,4)$ spanned by $x_{(4)} = (x_1, \ldots, x_4)$. This is a Fano threefold which we denote by M_{A_7} where $A_7 = (A_7^{lm})$. Since the gauge group is actually $U(1) \times SU(2)$, the vacuum manifold is something like $M_{A_7}/\mathbb{Z}_2$ where $\mathbb{Z}_2 = \{(\pm 1, \pm \mathbf{1}_2)\}$. Since p^7 is used up, the superpotential at this stage is

$$W = \sum_{i,j=1}^{6} \sum_{l,m=1}^{4} A_{ij}^{lm} p^i p^j [x_l x_m] + \sum_{j=1}^{6} \sum_{l=1}^{4} A_j^{l5} p^j [x_l x_5]$$

$$= \sum_{i,j=1}^{6} A_{ij}(x_{(4)}) p^i p^j + \sum_{j=1}^{6} \sum_{a=1}^{2} A_{j,a}^{5}(x_{(4)}) p^j x_5^a, \qquad (3.29)$$

where $A_{ij}(x_{(4)}) = \sum_{l,m} A_{ij}^{lm} [x_l x_m]$ and $A_{j,a}^{5}(x_{(4)}) = \sum_{l,b} A_j^{l5} x_l^b \epsilon_{ba}$ in which ϵ_{ab} is the symplectic form defining $SU(2)$. Under the genericity assumption, the 6×2 matrix $(A_{j,a}^{5}(x_{(4)}))$ is always of rank 2 since (x_l^b) is always of rank 2. Therefore, the second term on the right hand side of (3.29) gives mass to (x_5^1, x_5^2) and two of $p_{(6)} = (p^1, \ldots, p^6)$. Integrating them out, we have the constraint

$$\sum_{j=1}^{6} A_{j,a}^{5}(x_{(4)}) p^j = 0, \qquad a = 1, 2. \qquad (3.30)$$

The remaining fields are not always massive. Thus, we have the hybrid model with the target space

$$\mathfrak{X}_{A'} = \frac{\left\{ (p_{(6)}, x_{(4)}) \in \mathbb{C}^6 \oplus \mathrm{Hom}(\mathbb{C}^4, \mathbb{C}^2) \ \middle|\ \mathrm{rank}\, x_{(4)} = 2,\ (3.27),\ (3.30) \right\}}{\mathbb{C}^\times \times SL(\mathbb{C}^2)}, \qquad (3.31)$$

and the superpotential

$$W_{A''} = \sum_{i,j=1}^{6} A_{ij}(x_{(4)}) p^i p^j. \qquad (3.32)$$

In the above expressions, $A' = (A_7^{lm}, A_j^{l5})$ and $A'' = (A_{ij}^{lm})$. The space $\mathfrak{X}_{A'}$ is the total space of an orbifold vector bundle of rank 4 over $M_{A_7}/\mathbb{Z}_2$. It is an orbifold with a $\mathbb{Z}_2$ orbifold singularity at the zero section. The model $(\mathfrak{X}_{A'}, W_{A''})$ may be regarded as a fibration over M_{A_7} of a $\mathbb{Z}_2$ Landau-Ginzburg orbifold of four variables with a quadratic superpotential, as in the $r \gg 0$ phase of the model $(\mathrm{A}^2_{(-1)^4,(-2)^3,1^5})$ studied in Section 3.1 and as in the $r \ll 0$ phase of the model $(\mathrm{S}^{1,+}_{(-1)^4,1^8})$ studied in [16]. It would be interesting to find the type of the $\mathbb{Z}_2$ orbifold, $O_+(1)$ or $O_-(1)$, in order to see whether there is a double cover or not.

Let us compute the sphere partition function in order to find the behavior of the Kähler metric on the moduli space in the limit $r \to +\infty$. The calculation is very similar to X_5. The result is

$$
Z_{S^2} = -\frac{(z\bar{z})^q}{2} \oint \frac{d\varepsilon_1 d\varepsilon_2}{(2\pi i)^2} (z\bar{z})^{-\frac{\varepsilon_1}{2}-\varepsilon_2} \pi^3 \frac{\left[\cos \pi \left(\frac{\varepsilon_1}{2}+\varepsilon_2\right)\right]^6 \left[\sin \pi \left(\varepsilon_1 + 2\varepsilon_2\right)\right]}{\left[\sin \pi \varepsilon_1\right]^4 \left[\sin 2\pi \varepsilon_2\right]^4} \times
$$

$$
\times \left| \sum_{\substack{k,l=0 \\ k+l=even}}^{\infty} (k-l-\varepsilon_1+2\varepsilon_2)(-z)^{\frac{k+l}{2}} \frac{\left[\Gamma \left(\frac{1}{2}+\frac{k}{2}+\frac{l}{2}-\frac{\varepsilon_1}{2}-\varepsilon_2\right)\right]^6 \left[\Gamma \left(1+k+l-\varepsilon_1-2\varepsilon_2\right)\right]}{\left[\Gamma \left(1+k-\varepsilon_1\right)\right]^4 \left[\Gamma \left(1+l-2\varepsilon_1\right)\right]^4} \right|^2 ,
$$
(3.33)

where $z := e^{-t}$. The Kähler potential to lowest order is

$$
e^{-K} = -\frac{\pi^2}{48}(z\bar{z})^q \left[\log^3 \frac{z\bar{z}}{2^{24}} - 264\zeta(3)\right] + \dots
$$
(3.34)

The leading behavior of the metric is thus

$$
g_{z\bar{z}} = \frac{3}{z\bar{z} \log \left(\frac{z\bar{z}}{2^{24}}\right)^2} + \dots .
$$
(3.35)

Thus, the limit $r \to +\infty$ is at infinite distance in the moduli space. We are in the true hybrid phase.

This result can be confirmed independently by a mirror symmetry calculation. The fundamental period can be extracted from the sphere partition function. After rescaling $z \to -2^{12}z$ one obtains

$$
X^0(z) = 1-208z+531216z^2-2168300800z^3+10900554288400z^4-61672477170302208z^5+\dots
$$
(3.36)

In agreement with [18], this is annihilated by the Picard-Fuchs operator No. 211 (formerly 271) of [26]:

$$
\begin{aligned}
\mathcal{L} = {} & \theta^4 + z(208 + 2368\theta + 9792\theta^2 + 14848\theta^3 + 11264\theta^4) \\
& -z^2(495616 + 3981312\theta + 6684672\theta^2 - 19267584\theta^3 - 23986176\theta^4) \\
& +z^3(36700160 + 1258291200\theta - 3145728000\theta^2 + 5033164800\theta^3 + 14428405760\theta^4) \\
& +z^4(6710886400(2\theta + 1)^4).
\end{aligned}
$$
(3.37)

3.3 $(A^2_{(-2)^7,3,1^4})$

The model has gauge group $G \cong U(2)$ and the matter given in the following table

	$p^{1,\dots,7}$	x_1	$x_{2,\dots,5}$
	$\det^{-1}$	$\det \otimes \mathbf{2}$	$\mathbf{2}$

(3.38)

The matrix $A(p)$ must transform as $\det^{-1}$ in the last 4×4 block (linear in p) and $\det^{-2}$ in the remaining off diagonal block (quadratic in p).

Let us identify the singular loci. Parametrizing the maximal torus as $\sigma = \operatorname{diag}(\sigma_1, \sigma_2)$, the effective twisted superpotential is

$$\widetilde{W}_{\mathit{eff}} = -t(\sigma_1 + \sigma_2) + \pi i(\sigma_1 - \sigma_2) - \sum_\chi \chi(\sigma)(\log \chi(\sigma) - 1), \tag{3.39}$$

where t is the correctly normalized FI-theta parameter, $\pi i(\sigma_1 - \sigma_2)$ is from the W-boson integral and the last sum, coming from the matter one loop integral, is over the weights,

$$\begin{array}{c|ccc}
 & p^{1,\ldots,7} & x_1 & x_{2,\ldots,5} \\
\hline
\chi(\sigma) & -\sigma_1 - \sigma_2 & \begin{pmatrix} 2\sigma_1 + \sigma_2 \\ \sigma_1 + 2\sigma_2 \end{pmatrix} & \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}
\end{array}. \tag{3.40}$$

The vacuum equation reads

$$- \mathrm{e}^{-t} = \frac{(2\sigma_1 + \sigma_2)^2(\sigma_1 + 2\sigma_2)\sigma_1^4}{(-\sigma_1 - \sigma_2)^7} = \frac{(2\sigma_1 + \sigma_2)(\sigma_1 + 2\sigma_2)^2\sigma_2^4}{(-\sigma_1 - \sigma_2)^7}. \tag{3.41}$$

We need to exclude the solutions such as $\sigma_1 + \sigma_2 = 0$ and $\sigma_1 = \sigma_2$ where some of the matter or the W-bosons are massless. Then, there are four admissible solutions but we need to mod out by the Weyl group action $\sigma_1 \leftrightarrow \sigma_2$. Thus, we find two singular points at

$$\mathrm{e}^{-t} = -\frac{1}{2}(349 \pm 85\sqrt{17}). \tag{3.42}$$

The Pfaffian Y_A in the $r \ll 0$ phase is nothing but the Pfaffian constructed by Tonoli in [19] (it is denoted by X_{13} in [18]). It is a Calabi-Yau threefold with the following topological data

$$h^{1,1}(X_{13}) = 1, \quad h^{2,1}(X_{13}) = 61, \quad \int_{X_{13}} H^3 = 13, \quad \int_{X_{13}} c_2(X_{13})H = 58. \tag{3.43}$$

In [18], the mirror family was proposed and the Picard-Fuchs equation was identified with the Calabi-Yau equation No. 99 in [26]. The singular points of that equation agrees with (3.42) if we identify the parameter ϕ in [18] with our $-\mathrm{e}^t$.

Let us look into the $r \gg 0$ phase. After $p = 0$ the remaining F-term equations are

$$\sum_{\alpha,\beta=2}^{5} A_i^{\alpha\beta}[x_\alpha x_\beta] = 0 \qquad i = 1, \ldots, 7. \tag{3.44}$$

There are seven equations for six $[x_\alpha x_\beta]$'s. By genericity, this implies that they are all zero, which leads to the conclusion that the matrix $x_{(4)} = (x_2, \ldots, x_5)$ has rank one or less.

Since x must have rank two, x_1 is non-zero, $x_{(4)}$ has rank one, and the two are linearly independent. Using the left action of $GL(2,\mathbb{C})$ we can fix $x_1 = (1,0)^T$. The stabilizer of this is determined as follows.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto (ad - bc) \begin{pmatrix} a \\ c \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.45}$$

This fixes $c = 0, d = a^{-2}$. Using what is left of the symmetry we can furthermore simplify the shape of $x_{(4)}$. Since this matrix has rank 1 and linearly independent of $x_1 = (1,0)^T$, $x_{(4)}$ can be brought to the form $(\lambda w, w)^T$ with $w = (w_2, \ldots, w_5)$ non-zero. Now we apply the residual gauge symmetry

$$\begin{pmatrix} a & b \\ 0 & a^{-2} \end{pmatrix} \begin{pmatrix} \lambda w \\ w \end{pmatrix} = \begin{pmatrix} a\lambda w + bw \\ a^{-2}w \end{pmatrix}. \tag{3.46}$$

We see that we can eliminate the upper components of $x_{(4)}$. Thus, the vacuum manifold is the space $\mathbb{C}^4 \setminus \{0\}$ of w modulo $w \mapsto a^{-2}w$ for $a \in \mathbb{C}^\times$. This is nothing but $\mathbb{CP}^3/\mathbb{Z}_2$ or $\mathbb{P}^3_{2222}$. Let us write down the superpotential in the gauge $x_1 = (1,0)^T$ with the residual gauge symmetry parametrized by $(a,b) \in \mathbb{C}^\times \times \mathbb{C}$: Denoting $x_{(4)} = (z,w)^T$, it reads

$$\begin{aligned} W &= \sum_{i,j=1}^{7}\sum_{\alpha=2}^{5} A^{1\alpha}_{ij} p^i p^j [x_1 x_\alpha] + \sum_{i=1}^{7}\sum_{\alpha,\beta=2}^{5} A^{\alpha\beta}_i p^i [x_\alpha x_\beta] \\ &= \sum_{i,j=1}^{7} A_{ij}(w) p^i p^j + \sum_{i=1}^{7}\sum_{\alpha=2}^{5} A^\alpha_{\ i}(w) z_\alpha p^i \end{aligned} \tag{3.47}$$

where $A_{ij}(w) = \sum_\alpha A^{1\alpha}_{ij} w_\alpha$ and $A^\alpha_{\ i}(w) = \sum_\beta A^{\alpha\beta}_i w_\beta$. Note that $\sum_\alpha A^\alpha_{\ i}(w) w_\alpha = 0$ (as it should be since $z \to z + bw$ is a gauge symmetry), and hence the 4×7 matrix $A'(w) = (A^\alpha_{\ i}(w))$ has rank three or less. Under the genericity assumption, it is of rank 3 for any $w \neq 0$. [1] Integrating out z (modulo shift by w), we have the constraint

$$A'(w)p = 0. \tag{3.48}$$

This leaves us $7 - 3 = 4$ dimensional space of p's, and none of them are massive everywhere. Thus, we have the hybrid model with the target space

$$\mathfrak{X}_{A'} = \frac{\left\{ (p,w) \in \mathbb{C}^7 \oplus \mathbb{C}^4 \;\middle|\; w \neq 0,\ A'(w)p = 0. \right\}}{\mathbb{C}^\times \ni a : (p,w) \mapsto (ap, a^{-2}w)} \tag{3.49}$$

---

[1] The codimension of the space of rank $\leq k$ matrices in the space of $m \times n$ matrices is $(m-k)(n-k)$. In the present case, if we apply $m = 3$, $n = 7$, $k = 2$ the codimension is $(3-2)(7-2) = 5$ which is too big compared to the dimension 3 of the space of w's.

and the superpotential

$$W_{A''} = \sum_{i,j=1}^{7} A_{ij}(w)p^i p^j,$$ (3.50)

where $A'' = (A_{ij}^{1\alpha})$. The space $\mathfrak{X}_{A'}$ is the total space of an orbifold vector bundle of rank 4 over $\mathbb{CP}^3/\mathbb{Z}_2$, with a $\mathbb{Z}_2$ orbifold singularity at the zero section. The model $(\mathfrak{X}_{A'}, W_{A''})$ may be regarded as a fibration over $\mathbb{CP}^3$ of a $\mathbb{Z}_2$ Landau-Ginzburg orbifold of four variables with a quadratic superpotential, as in the $r \gg 0$ phase of the models studied in Section 3.1 and 3.2 and as in the $r \ll 0$ phase of the model $(S_{(-1)^4,1^8}^{1,+})$ studied in [16]. It would be interesting to find the type of the $\mathbb{Z}_2$ orbifold, $O_+(1)$ or $O_-(1)$, in order to see whether there is a double cover or not.

Let us compute the sphere partition function in order to find the behavior of the Kähler metric on the moduli space in the limit $r \to +\infty$. The detail is outlined in the appendix, and here we only show the result:

$$
\begin{aligned}
Z_{S^2} &= -\frac{(z\bar{z})^q}{32} \oint \frac{d\varepsilon_1 d\varepsilon_2}{(2\pi i)^2} \pi^3 (z\bar{z})^{\frac{1}{2}-\frac{\varepsilon_1}{4}-\varepsilon_2} \times \\
&\quad \times \frac{\left[\cos\pi\left(\frac{\varepsilon_1}{4}+\varepsilon_2\right)\right]^5 \left[\sin\pi\left(\frac{\varepsilon_1}{2}+2\varepsilon_2\right)\right]^2}{\left[\sin\pi\frac{\varepsilon_1}{2}\right]^2 \left[\cos\pi\left(\frac{\varepsilon_1}{4}+3\varepsilon_2\right)\right]^2 \left[\cos\pi\left(\frac{\varepsilon_1}{4}-\varepsilon_2\right)\right]^3 \left[\sin 2\pi\varepsilon_2\right]^3} \times \\
&\quad \times \left| \sum_{\substack{k,l=0 \\ k+l=even}}^{\infty} (2-2k+6l+\varepsilon_1-12\varepsilon_2)(-z)^{\frac{k+l}{2}} \times \right. \\
&\quad \times \frac{\left[\Gamma\left(\frac{1}{2}(1+k+l)-\frac{\varepsilon_1}{4}-\varepsilon_2\right)\right]^5}{\left[\Gamma\left(1+k-\frac{\varepsilon_1}{2}\right)\right]^2 \left[\Gamma\left(\frac{3}{2}+\frac{k}{2}+\frac{3l}{2}-\frac{\varepsilon_1}{4}-3\varepsilon_2\right)\right]^2} \times \\
&\quad \times \left. \frac{\left[\Gamma\left(1+k+l-\frac{\varepsilon_1}{2}+2\varepsilon_2\right)\right]^2}{\left[\Gamma\left(\frac{1}{2}+\frac{k}{2}-\frac{l}{2}-\frac{\varepsilon_1}{4}+\varepsilon_2\right)\right]^3 \left[\Gamma(1+l-2\varepsilon_2)\right]^3} \right|^2,
\end{aligned}
$$ (3.51)

where $z := -e^{-t}$. The leading term is

$$e^{-K} = -\frac{1}{96}(z\bar{z})^{2q+\frac{1}{2}} \left[\left(\log\frac{z\bar{z}}{2^{32}}\right)^3 + 576\zeta(3)\right] + \dots$$ (3.52)

and the Kähler metric behaves as

$$g_{z\bar{z}} = \frac{3}{z\bar{z}\log\left(\frac{z\bar{z}}{2^{32}}\right)^2} + \dots.$$ (3.53)

Thus, the limit $r \to +\infty$ is at infinite distance in the moduli space. We are in the true hybrid phase.

Rescaling $z \to -2^{16}z$, the fundamental period is

$$X^0(z) = 1 - 2320z + 57601296z^2 - 2373661139200z^3 + 121665506430000400z^4 + \dots \quad (3.54)$$

This is annihilated by the Picard-Fuchs operator No. 207 (formerly 225) of [26] in agreement with [18]:

$$
\begin{aligned}
\mathcal{L} = \ & \theta^4 - z(17152\theta^4 - 285184\theta^3 - 174208\theta^2 - 31616\theta - 2320) \\
& - z^2(6696206336\theta^4 + 15252586496\theta^3 - 5932843008\theta^2 - 1864892416\theta - 183107584) \\
& + z^3(2551081724805120\theta^4 + 53601191854408\theta^3 - 17029545328640\theta^2 \\
& + 13400297963520\theta + 338497110016) - z^4(2973079441506304(2\theta+1)^4) \quad (3.55)
\end{aligned}
$$

3.4 $(\mathbf{A}^2_{(-2)^5,(-4)^2,3^2,1^3})$

The model has gauge group $G \cong U(2)$ and the matter given in the following table

$$
\begin{array}{c|cccc}
 & p^{1,\dots,5} & p^{6,7} & x_{1,2} & x_{3,\dots,5} \\
\hline
 & \det^{-1} & \det^{-2} & \det \otimes \mathbf{2} & \mathbf{2}
\end{array}. \quad (3.56)
$$

where det is the determinant representation and $\mathbf{2}$ is the fundamental doublet. The matrix $A(p)$ must transform as $\det^{-3}$ in the first 2×2 block (cubic in $p^{1,\dots,5}$ and bilinear in $(p^{1,\dots,5}, p^{6,7})$), $\det^{-1}$ in the last 3×3 block (linear in $p^{1,\dots,5}$), and $\det^{-2}$ in the off diagonal block (quadratic in $p^{1,\dots,5}$ and linear in $p^{6,7}$).

Singular points are found in the same way as in the previous subsection. The vacuum equations read

$$-e^{-t} = \frac{(2\sigma_1 + \sigma_2)^4(\sigma_1 + 2\sigma_2)^2\sigma_1^3}{(-\sigma_1 - \sigma_2)^5(-2\sigma_1 - 2\sigma_2)^4} = \frac{(2\sigma_1 + \sigma_2)^2(\sigma_1 + 2\sigma_2)^4\sigma_2^3}{(-\sigma_1 - \sigma_2)^5(-2\sigma_1 - 2\sigma_2)^4}. \quad (3.57)$$

Taking the admissible solutions and modding out by the Weyl group, we find that there are two singular points at

$$e^{-t} = 540 \pm 312\sqrt{3}. \quad (3.58)$$

The Pfaffian Y_A in the $r \ll 0$ phase is nothing but X_7 in Kanazawa's work [18]. It is a Calabi-Yau threefold with the following topological data

$$h^{1,1}(X_7) = 1, \quad h^{2,1}(X_7) = 61, \quad \int_{X_7} H^3 = 7, \quad \int_{X_7} c_2(X_7)H = 46. \quad (3.59)$$

In that paper, the mirror family was proposed and the Picard-Fuchs equation was identified with the Calabi-Yau equation No. 109 in [26]. The singular points of that equation agrees with (3.58) if we identify the parameter ϕ in [18] with our $-e^t$.

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Now let us look at the $r \gg 0$ phase. First what is the vacuum manifold? Regarding $p = 0$, the remaining F-term equations are

$$\sum_{\alpha,\beta=3}^{5} A_i^{\alpha\beta} [x_\alpha x_\beta] = 0 \qquad i = 1, \ldots, 5$$

$$\sum_{l=1}^{2} \sum_{\alpha=3}^{5} A_\mu^{l\alpha} [x_l x_\alpha] = 0 \qquad \mu = 6, 7. \tag{3.60}$$

From genericity of A, the first set of equations imply $[x_\alpha x_\beta] = 0$ for $\alpha, \beta = 3, 4, 5$. This is equivalent to the requirement that x_3, x_4, x_5 are proportional to each other. Thus the dimension is cut only by two. The other two equations are independent, and the vacuum manifold has dimension $10 - 2 - 2 - 4 = 2$. We do not know the details about this variety other than the fact that there is one point with $x_{3,4,5} = 0$ with the orbifold group $\mathbb{Z}_3$ and a line $\mathbb{CP}^1$ with the orbifold group $\mathbb{Z}_2$. We are unable to identify any massive direction, and we only have a poor description of the model: It is a hybrid model with the target space given by the solution space to D-term equation modulo gauge group and the original superpotential. It is similar to the pseudo-hybrid model discussed in [20] in that the vacuum manifold is two-dimensional (and even in that there is one point with $\mathbb{Z}_3$ orbifold group and a curve with $\mathbb{Z}_2$ orbifold group).

Failing to obtain any idea on the nature of the low energy physics from the classical analysis, we now look at the behavior of the Kähler metric of the moduli space via the two sphere partition function. Details can be found in appendix B. The leading terms of the partition function are

$$\mathrm{e}^{-K} = \frac{2}{\sqrt{3}\pi} \frac{\Gamma\left(\frac{1}{3}\right)^{10}}{\Gamma\left(\frac{2}{3}\right)^{8}} (z\bar{z})^{2q+\frac{1}{3}} - 4(z\bar{z})^{2q+\frac{1}{2}}(36 + 8\log 4 - 3\log z\bar{z}) + \ldots, \tag{3.61}$$

and hence the Kähler metric behaves as

$$g_{z\bar{z}} = -\frac{\pi\Gamma\left(\frac{2}{3}\right)^{8}\log^{3}\left(\frac{z\bar{z}}{2^{\frac{16}{3}}}\right)}{6\sqrt{3}\Gamma\left(\frac{1}{3}\right)^{10}(z\bar{z})^{\frac{5}{6}}}. \tag{3.62}$$

where $z := -\mathrm{e}^{-t}$. This shows that the limiting point actually lies at a finite distance in the moduli space. The very same leading behavior of the Kähler metric has been observed in the pseudo-hybrid model in [20] mentioned above.

4 A new Calabi-Yau threefold from (S) series

We now study the model $(S^{2,+}_{(-1)^2,(-2)^3,1^4})$ in detail. The gauge group is $G = U(1) \times O_+(2)$ and the matter consists of five $O(2)$ singlets — p^1, p^2 and p^3, p^4, p^5 of $U(1)$ charges -1

and -2 respectively — and four $O(2)$ doublets, $x_1, \ldots, x_4$, of $U(1)$ charge 1. The matrix $S(p)$ must have charge -2. It must be linear in $p^{3,4,5}$ and quadratic in $p^{1,2}$.

Let us identify the singular loci. The effective twisted superpotential is

$$\widetilde{W}_{\mathit{eff}} = -t\sigma_0 + \pi i \sigma_1 - \sum_\chi \chi(\sigma)(\log \chi(\sigma) - 1), \tag{4.1}$$

where t is the correctly normalized FI-theta parameter, $\pi i \sigma_1$ is the discrete theta angle needed for the regularity, and the last sum is over the weights of the matter representation,

$$
\begin{array}{c|ccc}
 & p^{1,2} & p^{3,4,5} & x_{1,\ldots,4} \\
\hline
\chi(\sigma) & -\sigma_0 & -2\sigma_0 & \left(\begin{array}{c} \sigma_0 + \sigma_1 \\ \sigma_0 - \sigma_1 \end{array} \right)
\end{array}. \tag{4.2}
$$

The vacuum equations read

$$e^{-t} = \frac{(\sigma_0 + \sigma_1)^4 (\sigma_0 - \sigma_1)^4}{(-\sigma_0)^2(-2\sigma_0)^6}, \qquad -1 = \frac{(\sigma_0 + \sigma_1)^4}{(\sigma_0 - \sigma_1)^4}. \tag{4.3}$$

The second equation is solved by $(\sigma_1/\sigma_0)^2 = -3 \pm 2\sqrt{2}$. The four solutions are all admissible, but we need to mod out by the Weyl group action $\sigma_1 \mapsto -\sigma_1$. Thus, we find two singular points at

$$e^{-t} = 17 \mp 12\sqrt{2}. \tag{4.4}$$

Let us count the number of parameters that enter into the matrix $S(p)$. It is a 4×4 symmetric matrix and hence has 10 independent entries, each of which is a linear combination of $(p^1)^2, p^1 p^2, (p^2)^2, p^3, p^4, p^5$. Thus, it has $10 \cdot 6 = 60$ parameters. On the other hand, reparametrizations of $x_{1,\ldots,4}$, $p^{1,2}$ and $p^{3,4,5}$ have $4^2 = 16$, $2^2 = 4$ and $3 \cdot 6 = 18$ parameters respectively. Since one of these is the gauge symmetry, the net number of reparametrization is $16 + 4 + 18 - 1 = 37$. Therefore, we found that

$$\text{the number of parameters in } W = 60 - 37 = 23. \tag{4.5}$$

The elliptic genus of the model behaves as

$$Z_{T^2}(\tau, z) = -22 \left(e^{\pi i z} + e^{-\pi i z} \right) + O\left(e^{2\pi i \tau} \right) \qquad \text{as } \tau \to +i\infty. \tag{4.6}$$

See Appendix C for the computation. In particular, the Witten index is

$$\text{Tr} \, (-1)^F = -44. \tag{4.7}$$

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4.1 $r \ll 0$: the double cover of symmetric determinantal variety

Let us start with the $r \ll 0$ phase. As discussed in Section 2.4, we have a non-linear sigma model whose target space is the double cover $\widetilde{Y}_S$ of the symmetric determinantal variety $Y_S = \{\, p \in \mathbb{P} \,|\, \mathrm{rank}\, S(p) \leq 3 \}$ which is ramified along the curve

$$C_S = \left\{\, p \in \mathbb{P} \,\bigg|\, \mathrm{rank}\, S(p) = 2 \,\right\}. \tag{4.8}$$

Note that $\mathrm{rank}\, S(p) = 1$ is impossible for dimensional reasons as long as $S(p)$ is generic. Y_S has an A_1 singularity (i.e. $\mathbb{C}^2/\{\pm 1\}$ singularity) along C_S which is unfolded by the double cover $\widetilde{Y}_S$. Therefore, $\widetilde{Y}_S$ is guaranteed to be smooth if Y_S is away from the orbifold locus of the ambient space $\mathbb{P}$. Unfortunately, the orbifold locus of $\mathbb{P} = \mathbf{WCP}^4_{11222}$, which is $\mathfrak{S} = \{p^1 = p^2 = 0\}$, is of codimension 2 and hence cannot be avoided by Y_S. The intersection is along a curve

$$\Sigma_S = \mathfrak{S} \cap Y_S. \tag{4.9}$$

This is again a curve of A_1 singularity in Y_S (it is away from the other singular curve C_S). A closer inspection is needed to see if $\widetilde{Y}_S$ is smooth or not.

The double cover $\widetilde{Y}_S$ is defined to be the classical vacuum manifold of the dual theory,

$$\widetilde{Y}_S = \frac{\left\{\, (p, \widetilde{x}) \in \mathbb{C}^5 \oplus \mathrm{Hom}(\mathbb{C}^4, \mathbb{C}^3) \,\bigg|\, \text{stability},\ S(p) + (\widetilde{x}\widetilde{x}) = 0 \,\right\}}{\mathbb{C}^\times \times SO(\mathbb{C}^3)}. \tag{4.10}$$

The covering map $\widetilde{Y}_S \to Y_S$ is simply the forgetful map $(p, \widetilde{x}) \mapsto p$. We know that the only possible singularity is above Σ_S. Let $(p^{1,2}, p^{3,4,5}) \in Y_S$ be in a neighborhood of Σ_S. In Y_S there is an identification $(p^{1,2}, p^{3,4,5}) \equiv (-p^{1,2}, p^{3,4,5})$, and this causes the A_1 singularity at $p^{1,2} = 0$. In $\widetilde{Y}_S$, there are two preimages of this point: $(p^{1,2}, p^{3,4,5}, \widetilde{x}) \equiv (-p^{1,2}, p^{3,4,5}, -\widetilde{x})$ and $(-p^{1,2}, p^{3,4,5}, \widetilde{x}) \equiv (p^{1,2}, p^{3,4,5}, -\widetilde{x})$. They are different points if $p^{1,2} \neq 0$ but coalesce as $p^{1,2} \to 0$. A choice of $\widetilde{x}$ provides one sheet over Y_S and the deck transform of the covering map $\widetilde{Y}_S \to Y_S$ can be described as

$$(p^{1,2}, p^{3,4,5}, \widetilde{x}) \longmapsto (-p^{1,2}, p^{3,4,5}, \widetilde{x}). \tag{4.11}$$

We see that the $\mathbb{C}^2/\{\pm 1\}$ singularity along Σ_S is unfolded by the cover $\widetilde{Y}_S \to Y_S$.

To summarize, $\widetilde{Y}_S$ is smooth and the covering map $\widetilde{Y}_S \to Y_S$ provides a simultaneous unfolding of the A_1 singularity of Y_S along the two disjoint curves, C_S and Σ_S.

$\widetilde{Y}_S$ is a Calabi-Yau manifold with the following topological data[1]

$$h^{1,1}(\widetilde{Y}_S) = 1, \quad h^{2,1}(\widetilde{Y}_S) = 23, \quad \int_{\widetilde{Y}_S} H^3 = 2, \quad \int_{\widetilde{Y}_S} c_2(\widetilde{Y}_S) H = 20, \tag{4.12}$$

---

[1]We thank Hiromichi Takagi for this information. See Appendix D for the outline.

where H is some element of $H^2(\widetilde{Y}_S, \mathbb{Z})$. The Hodge numbers confirm that the one FI-theta parameter is the only exactly marginal twisted chiral parameter and that, in view of (4.5), the parameters entering in $S(p)$ are all the exactly marginal chiral parameters. The Euler number of $\widetilde{Y}_S$ is -44. This is consistent with the Witten index (4.7).

Let us compute the two sphere partition function in order to see if the metric of the Kähler moduli space is of the expected form in the geometric phase. The details of the calculation are again given in the appendix. The result is

$$
\begin{aligned}
Z_{S^2}^{r\ll 0} &= \frac{1}{2} \lim_{\delta\to 0} \oint \frac{d\varepsilon_1 d\varepsilon_2}{(2\pi i)^2} (z\bar{z})^{q-\frac{1}{2}-\varepsilon_2} \pi^5 \frac{[\sin\pi(\varepsilon_1 + 2\varepsilon_2)]^4}{[\sin\pi(\varepsilon_2)]^2 [\sin 2\pi\varepsilon_2]^3 [\sin\pi\varepsilon_1]^4} \times \\
&\quad \times \left| \sum_{k,l=0}^{\infty} (-e^{-\delta})^k (-z)^l \frac{[\Gamma(1+k+2l+\varepsilon_1 - 2\varepsilon_2)]^4}{[\Gamma(1+l-\varepsilon_2)]^2 [\Gamma(1+2l-2\varepsilon_2)]^2 [\Gamma(1+k+\varepsilon_1)]^4} \right|^2 (4.13)
\end{aligned}
$$

Here, $e^{-\delta}$ is a convergence factor which is introduced following [6]. It is hoped by the presence of $(-1)^k$ that the result does not depend on how to take the $\delta \to 0$ limit. Evaluating this partition function is harder than the previous examples of $r \gg 0$ phases because not all the summation variables appear as exponents of $z, \bar{z}$. The consequence is that the coefficients at a given order in $z, \bar{z}$ are infinite sums which have to be evaluated order by order.

At this moment, we are unable to find even the leading behavior of the metric. A similar difficulty was encountered in [6] (and we indeed followed their procedure) in the strongly coupled phase of the Rødland model ($A_{(-2)^7, 1^7}^2$). This difficulty seems to be correlated to the presence of continuous unbroken gauge group. We shall find a resolution to this problem momentarily.

Even though we are unable to find the metric, we can extract the fundamental period by looking at the coefficient of $\log^3 z\bar{z}$. After an overall rescaling and after setting $z \to -\frac{2^8}{z}$ the fundamental period is

$$
X^0 = 1 - 208z + 531216z^2 - 2168300800z^3 + 10900554288400z^4 - 61672477170302208z^5 + \ldots
$$
(4.14)

This is nothing but the fundamental period of the weakly coupled phase ($r \gg 0$) of the model ($A_{(-1)^6, (-2), 1^4, 0}^2$)! It is also worth noting that $\widetilde{Y}_S$ has the right property (4.12) for the Calabi-Yau threefold predicted in [30] for the Picard-Fuchs operator 211 (formerly 271) that annihilates (4.14). We will say more on this in a moment.

4.2 $r \gg 0$: true hybrid phase

Let us now move on to the study of the opposite phase, $r \gg 0$. We parametrize the doublets as $x_j = \frac{1}{\sqrt{2}}(u_j + v_j, -iu_j + iv_j)^T$, $j = 1, \ldots, 4$. We also write the gauge group element as (g, h, ε) where g is the $U(1)$ element, h is the element of $SO(2) \cong U(1)$ and $\varepsilon = 1, \tau$ labels the disconnected part $O(2)/SO(2) \cong \mathbb{Z}_2$. Note that τ acts as the exchange of u_j and v_j.

The D-term equations read

$$-\sum_{\mu=1}^{2} |p^\mu|^2 - 2\sum_{m=3}^{5} |p^m|^2 + ||u||^2 + ||v||^2 = r, \qquad ||u||^2 = ||v||^2, \qquad (4.15)$$

and the F-term equations read

$$\sum_{j=1}^{4} S^{ij}(p)u_j = \sum_{j=1}^{4} S^{ij}(p)v_j = 0, \qquad i = 1, \ldots, 4, \qquad (4.16)$$

$$\sum_{i,j=1}^{4} \sum_{\nu=4}^{5} S_{\mu\nu}^{ij} p^\nu u_i v_j, \qquad \mu = 1, 2, \qquad (4.17)$$

$$\sum_{i,j=1}^{4} S_m^{ij} u_i v_j = 0, \qquad m = 3, 4, 5. \qquad (4.18)$$

The D-term equations with $r \gg 0$ require that u and v are both non-zero. Therefore, the gauge group is broken at most to $\mathbb{Z}_2 \times \langle \tau \rangle$ where the first $\mathbb{Z}_2$ is $g = h = \pm 1$ and the second $\langle \tau \rangle$ is $O(2)/SO(2) \cong \mathbb{Z}_2$. Thus, $r \gg 0$ is a weakly coupled phase.

Next, we show that $p = 0$ follows from the D-term and the F-term equations. Suppose u and v are proportional to each other. The three equations (4.18) fix $u \propto v$ to a number of points up to constant multiplication. (There are eight such points.) If $S(p)$ is chosen generic, the 2×2 matrix $(S_{\mu\nu}(u)) = (\sum_i S_{\mu\nu}^{ij} u_i u_j)$ is invertible at those points and thus $p^1 = p^2 = 0$ is enforced by (4.17). Similarly the 4×3 matrix $(S_m^i(u)) = (\sum_j S_m^{ij} u_j)$ has rank 3 at those points and thus $p^3 = p^4 = p^5 = 0$ is also enforced by (4.16). Next, suppose u and v are not proportional to each other. Then, (4.16) are solvable only when $S(p)$ has rank two or less. As discussed above, for a generic $S(p)$, rank one is impossible and rank zero means $p = 0$. Thus, we only have to exclude the rank two case. This can be done in the same way as in the (A) series in the previous section: The equations (4.18)-(4.17) are in conflict with the smoothness of the rank two curve C_S. Choose a point $p_* \in C_S$ and

choose a basis of $\mathbb{C}^4$ so that $S(p_*)$ is represented by a matrix of the form

$$S(p_*) = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & a & b \\ & & b & d \end{pmatrix}. \tag{4.19}$$

with the last 2×2 block invertible. In a neighborhood of p_*, the rank two curve C_S can be defined as the set of solutions to the three equations (See Appendix A),

$$\Delta_{11}(p) = \Delta_{22}(p) = \Delta_{12}(p) = 0, \tag{4.20}$$

where $\Delta_{ij}(p)$ is the determinant of the minor of $S(p)$ obtained by deleting the i-th row and j-th column. Smoothness of C_S at p_* requires that the 5×3 matrix of differentials

$$\left(\frac{\partial}{\partial p^k} \Delta_{11}, \ \frac{\partial}{\partial p^k} \Delta_{22}, \ \frac{\partial}{\partial p^k} \Delta_{12} \right)_{k=1,\dots,5} \tag{4.21}$$

must be of rank three at p_*. Since $S^{ij}(p_*)$ vanishes unless $i, j = 3, 4$. we have

$$\frac{\partial}{\partial p^k} \Delta_{11} = \frac{\partial}{\partial p^k} S^{22}(p_*) \det \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \quad \text{etc.} \tag{4.22}$$

A linear combination of these vanishes for all k by the equations (4.18)-(4.17). Therefore the matrix (4.21) has rank two or less, violating the smoothness of C_S at p_*. This completes the proof that all p must vanish.

Now let us determine the vacuum manifold. It is the space of (u, v), both non-vanishing and satisfying (4.18), modulo the action of the complexified gauge group. If the gauge group were $(U(1) \times SO(2))/\{(\pm 1, \pm \mathbf{1}_2)\}$, it would be the complete intersection $M_{S'}$ of the three hypersurfaces (4.18) in $\mathbb{CP}^3 \times \mathbb{CP}^3 = \{([u], [v])\}$, where $S' = (S_m^{ij})$. But the actual gauge group is $G = U(1) \times O(2)$ and hence the vacuum manifold is like $M_{S'}/\mathbb{Z}_2 \times \langle \tau \rangle$ where the first $\mathbb{Z}_2$ acts trivially and τ is the exchange of the two $\mathbb{CP}^3$'s.

The fields p^3, p^4, p^5 are used up in imposing the equations (4.18) and the remaining fields $p_{(2)} = (p^1, p^2)$ are not always massive. Therefore, we have a hybrid model with the target space

$$\mathfrak{X}_{S'} = \frac{\left\{ (p_{(2)}, u, v) \in \mathbb{C}^2 \oplus \mathbb{C}^4 \oplus \mathbb{C}^4 \ \middle| \ u \neq 0, \ v \neq 0, \ (4.18) \right\}}{\mathbb{C}^\times \times (\mathbb{C}^\times \rtimes \langle \tau \rangle)}, \tag{4.23}$$

and the superpotential

$$W_{S''} = \sum_{\mu,\nu=1}^{2} S_{\mu\nu}(u, v) p^\mu p^\nu, \tag{4.24}$$

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where $S_{\mu\nu}(u,v) = \sum_{i,j} S_{\mu\nu}^{ij} u_i v_j$ and $S'' = (S_{\mu\nu}^{ij})$. The space $\mathfrak{X}_{S'}$ is the total space of an orbifold vector bundle of rank two over $M_{S'}/\mathbb{Z}_2 \times \langle\tau\rangle$. It is an orbifold with a $\mathbb{Z}_2$ orbifold singularity at the zero section and $\mathbb{Z}_2 \times \mathbb{Z}_2$ singularity at the eight fixed points of $\tau : M_{S'} \to M_{S'}$. The model $(\mathfrak{X}_{S'}, W_{S''})$ may be regarded as a fibration over $M_{S'}/\langle\tau\rangle$ of a $\mathbb{Z}_2$ Landau-Ginzburg orbifold of two variables with a quadratic superpotential, as in the $r \gg 0$ phase of the first three models studied in Section 3 and as in the $r \ll 0$ phase of the model $(S^{1,+}_{(-2)^4,1^8})$ studied in [16]. It would be interesting to find the type of the $\mathbb{Z}_2$ orbifold, $O_+(1)$ or $O_-(1)$, in order to see whether there is a double cover or not.

Let us compute the sphere partition function in order to find the behavior of the metric on the moduli space in the limit $r \to +\infty$. The result is

$$
\begin{aligned}
Z_{S^2}^{r\gg0} &= \frac{(z\bar z)^q}{2} \oint \frac{d\varepsilon_1 d\varepsilon_2}{(2\pi i)^2} (z\bar z)^{-\frac{\varepsilon_1}{2}-\varepsilon_2} \pi^3 \frac{\left[\cos\pi\left(\frac{\varepsilon_1}{2}+\varepsilon_2\right)\right]^2 \left[\sin\pi\left(\varepsilon_1+2\varepsilon_2\right)\right]^3}{\left[\sin\pi\varepsilon_1\right]^4 \left[\sin 2\pi\varepsilon_2\right]^4} \times \\
&\quad \times \left| \sum_{\substack{k,l=0 \\ k+l=even}}^{\infty} (-z)^{\frac{k+l}{2}} \frac{\left[\Gamma\left(\frac{1}{2}+\frac{k}{2}+\frac{l}{2}-\frac{\varepsilon_1}{2}-\varepsilon_2\right)\right]^2 \left[\Gamma\left(1+k+l-\varepsilon_1-2\varepsilon_2\right)\right]^3}{\left[\Gamma\left(1+k-\varepsilon_1\right)\right]^4 \left[\Gamma\left(1+l-2\varepsilon_1\right)\right]^4} \right|^2
\end{aligned}
\tag{4.25}
$$

The leading term of the Kähler potential associated to this is

$$
e^{-K} = -\frac{5}{24}\log^3\frac{z\bar z}{2^8} + 29\zeta(3) + \dots
\tag{4.26}
$$

Therefore the metric behaves as

$$
g_{z\bar z} = \frac{3}{z\bar z \log\left(\frac{z\bar z}{2^8}\right)^2} + \dots .
\tag{4.27}
$$

Thus, the limit $r \to +\infty$ is at infinite distance in the moduli space. We are in the true hybrid phase.

We may also extract the fundamental period as in the earlier examples. After rescaling $z \to -16z$ it is expanded as

$$
X_{r\gg0}^0 = 1 - 28z + 4716z^2 - 1226800z^3 + 389349100z^4 - 138518544528z^5 + \dots
\tag{4.28}
$$

This is annihilated by the Picard-Fuchs operator No. 210 (formerly 263) of [26]:

$$
\begin{aligned}
\mathcal{L}^{r\gg0} = {} & 25\theta^4 + z(700 + 6100\theta + 19620\theta^2 + 27040\theta^3 + 13760\theta^4) \\
& - z^2(4240 + 20160\theta - 1536\theta^2 - 112128\theta^3 - 93696\theta^4) \\
& + z^3(5120 + 307200\theta + 70656\theta^2 + 122880\theta^3 + 180224\theta^4) \\
& + z^4(4096(2\theta+1)^4).
\end{aligned}
\tag{4.29}
$$

Comparing with [18], this is the same Picard-Fuchs operator as in the strongly coupled Pfaffian phase ($r \ll 0$) of the model $(A^2_{(-1)^6,(-2),1^4,0})$!

4.3 A surprise

We have encountered a relation between the present model $(S^{2,+}_{(-1)^2,(-2)^3,1^4})$ and the model $(A^2_{(-1)^6,(-2),1^4,0})$ (For simplicity, we call them (S) and (A) respectively only here):

- The fundamental period in the weakly coupled $r \gg 0$ phase of (A) is the same as the fundamental period for the strongly coupled $r \ll 0$ phase of (S). Compare (3.36) and (4.14).

- The Picard-Fuchs operator in the strongly coupled $r \ll 0$ phase of (A) matches the Picard-Fuchs operator in the weakly coupled $r \gg 0$ phase of (S). Compare [18] and (4.29).

More fundamentally, we may also add

- The singular points match. Compare (3.25) and (4.4):

$$272 \pm 192\sqrt{2} = 16(17 \pm 12\sqrt{2}) = \frac{16}{17 \mp 12\sqrt{2}}. \tag{4.30}$$

All these agreements hold under the relation

$$\mathrm{e}^{-t_A} = 16 \cdot \mathrm{e}^{t_S}. \tag{4.31}$$

These observations suggest that the Kähler moduli spaces of the two models are the same. However, it appears to be difficult to check — we have seen that the computation is difficult in the strongly coupled phase, and there is no phase where both are weakly coupled. At this point, we make use of the duality, which maps the strongly coupled phase to the weakly coupled phase and vice versa. We compare the partition functions of the model (A) and the dual model of (S).[1] In this comparison, as we will see, the equality holds at the level of the integrand and we actually do not need to evaluate the integral in either phase.

The dual of (S) has the gauge group $U(1) \times SO(3)$ and the matter contents as follows:

	$p^{1,2}$	$p^{3,4,5}$	$\widetilde{x}^{1,\dots,4}$	$s_{ij} = s_{ji}$
$U(1) \times SO(3)$	$(-1, \mathbf{1})$	$(-2, \mathbf{1})$	$(-1, \mathbf{3})$	$(2, \mathbf{1})$
$U(1)_V$	$1 - \epsilon$	$2 - 2\epsilon$	$1 - \epsilon$	2ϵ

$$(4.32)$$

[1]The same comparison can be done also between the dual model of (A) and the model (S).

where **1** and **3** stand for the $SO(3)$ singlet and triplet respectively. The two sphere partition function of this model with $\epsilon = 0$ is

$$Z_{S^2}^{(\widetilde{S})}(r,\theta) = \sum_{(m_0,m_1)\in\mathbb{Z}^{\oplus 2}} \int_{(\mathbb{R}-i\cdot 0)\times\mathbb{R}} d\sigma_0 \, d\sigma_1 \, e^{2ir\sigma_0 + i\theta m_0 + \pi i m_1} \left((2\sigma_1)^2 + (m_1)^2 \right) Z_{\text{matter}}^{(\widetilde{S})}(\vec{\sigma}_0, \vec{\sigma}_1).$$

(4.33)

We have shifted the σ_0 contour slightly, in order to avoid the pole that came down to $\sigma_0 = 0$ in the limit $\epsilon \searrow 0$. The factor $e^{\pi i m_1}$ is the effect of the W-boson integral $e^{2\pi i \rho(m)}$, see (B.1) and (2.27). The last factor is

$$
\begin{aligned}
Z_{\text{matter}}^{(\widetilde{S})}(\vec{\sigma}_0, \vec{\sigma}_1) &= f_{-1,1}(\vec{\sigma}_0)^2 f_{-2,2}(\vec{\sigma}_0)^3 f_{1,1}(-\vec{\sigma}_0 + \vec{\sigma}_1)^4 f_{1,1}(-\vec{\sigma}_0 - \vec{\sigma}_1)^4 f_{1,1}(-\vec{\sigma}_0)^4 f_{2,0}(\vec{\sigma}_0)^{10} \\
&= f_{-1,1}(\vec{\sigma}_0)^6 f_{2,0}(\vec{\sigma}_0)^7 f_{1,1}(-\vec{\sigma}_0 + \vec{\sigma}_1)^4 f_{1,1}(-\vec{\sigma}_0 - \vec{\sigma}_1)^4,
\end{aligned}
$$

(4.34)

where we used the notation $\vec{\sigma} = (\sigma, m)$ and

$$f_{Q,R}(\vec{\sigma}) = \frac{\Gamma\left(iQ\sigma - \frac{Qm}{2} + \frac{R}{2}\right)}{\Gamma\left(1 - iQ\sigma - \frac{Qm}{2} - \frac{R}{2}\right)}.$$

(4.35)

In (4.34), we used

$$f_{-2,2}(\vec{\sigma}_0)f_{2,0}(\vec{\sigma}_0) = \frac{\Gamma(-2i\sigma_0 + m_0)}{\Gamma(1 + 2i\sigma_0 + m_0)} \cdot \frac{\Gamma(2i\sigma_0 + m_0 + 1)}{\Gamma(-2i\sigma_0 + m_0)} = 1.$$

(4.36)

Let us revisit the two sphere partition function of the model (A). The gauge group is $U(1) \times SU(2)$ and the matter content is

	$p^{1,\dots,6}$	p^7	$x_{1,\dots,4}$	x_5
$U(1) \times SU(2)$	$(-1, \mathbf{1})$	$(-2, \mathbf{1})$	$(1, \mathbf{2})$	$(0, \mathbf{2})$
$U(1)_V$	$1 - \epsilon$	$2 - 2\epsilon$	ϵ	ϵ

(4.37)

The partition function with $\epsilon = 1$ is

$$Z_{S^2}^{(A)}(r,\theta) = \sum_{(m_0,m_1)\in\mathbb{Z}^{\oplus 2}} \int_{(\mathbb{R}+i\cdot 0)\times\mathbb{R}} d\sigma_0 \, d\sigma_1 \, e^{2ir\sigma_0 + i\theta m_0} \left((2\sigma_1)^2 + (m_1)^2 \right) Z_{\text{matter}}^{(A)}(\vec{\sigma}_0, \vec{\sigma}_1).$$

(4.38)

We have shifted the σ_0 contour slightly, in order to avoid the pole that came up to $\sigma_0 = 0$ in the limit $\epsilon \nearrow 1$. The matter factor of the integrand is

$$Z_{\text{matter}}^{(A)}(\vec{\sigma}_0, \vec{\sigma}_1) = f_{-1,0}(\vec{\sigma}_0)^6 f_{-2,0}(\vec{\sigma}_0) f_{1,1}(\vec{\sigma}_0 + \vec{\sigma}_1)^4 f_{1,1}(\vec{\sigma}_0 - \vec{\sigma}_1)^4 f_{1,1}(\vec{\sigma}_1) f_{1,1}(-\vec{\sigma}_1).$$

(4.39)

Note that

$$f_{1,1}(\vec{\sigma}_1)f_{1,1}(-\vec{\sigma}_1) = \frac{\Gamma\left(i\sigma_1 - \frac{m_1}{2} + \frac{1}{2}\right)}{\Gamma\left(-i\sigma_1 - \frac{m_1}{2} + \frac{1}{2}\right)} \cdot \frac{\Gamma\left(-i\sigma_1 + \frac{m_1}{2} + \frac{1}{2}\right)}{\Gamma\left(i\sigma_1 + \frac{m_1}{2} + \frac{1}{2}\right)} = (-1)^{m_1}.$$

(4.40)

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and

$$f_{-1,1}(\vec{\sigma}_0)^6 f_{2,0}(\vec{\sigma}_0)^6|_{\vec{\sigma}_0 \to -\vec{\sigma}_0} = 2^{-24i\sigma_0 - 6} f_{-1,0}(\vec{\sigma}_0)^6 \tag{4.41}$$

where we have used the identity $\Gamma(2z) = \frac{1}{2\sqrt{\pi}} 2^{2z} \Gamma(z) \Gamma(z + \frac{1}{2})$. It follows that

$$Z_{\text{matter}}^{(\widetilde{S})}(-\vec{\sigma}_0, \vec{\sigma}_1) = 2^{-24i\sigma_0 - 6}(-1)^{m_1} Z_{\text{matter}}^{(A)}(\vec{\sigma}_0, \vec{\sigma}_1). \tag{4.42}$$

This means that

$$Z_{S^2}^{(\widetilde{S})}(r, \theta) = 2^{-6} Z_{S^2}^{(A)}(-r - \log 2^{12}, -\theta) \tag{4.43}$$

The relation between the FI-theta parameter of the model (S) and the one of its dual $(\widetilde{S})$ can be read off from the singular loci: $\mathrm{e}^{-t_{\widetilde{S}}} = 2^8 \, \mathrm{e}^{-t_S}$. Taking this duality for granted, we have

$$\begin{aligned} Z_{S^2}^{(S)}(r_S, \theta_S) &\sim Z_{S^2}^{(\widetilde{S})}(r_S - \log 2^8, \theta_S) = 2^{-6} Z_{S^2}^{(A)}(-r_S - \log 2^4, -\theta_S) \\ &= 2^{-6} Z_{S^2}^{(A)}(r_A, \theta_A) \end{aligned} \tag{4.44}$$

where the relation (4.31) is used. Assuming the conjecture of [6], this proves that the two moduli spaces are exactly the same as Kähler manifolds.

This is a surprise. The Hodge numbers of the corresponding Calabi-Yau threefolds are different, $h^{2,1} = 59$ for (A) and $h^{2,1} = 23$ for (S). Therefore, the two families of superconformal field fixed points cannot be the same, as the number of exactly marginal chiral operators are different. And yet they have exactly the same moduli space of twisted chiral parameters. What we have observed is perhaps stronger than just the equivalence as Kähler manifold: assuming that the observation (3.21) of [6] extends here, we have also read off the Picard-Fuchs operators of the two models and they also match. This would mean the equivalence of the $U(1)_V$ R-charge zero sector of the vacuum vector bundle over the moduli space. It would be tempting to conjecture that the category of B-branes are also the same. But that cannot be the case since $H^{0,*}(X, \wedge^* T_X)$ is an invariant of the derived category of a variety X and two Calabi-Yau threefolds of different $h^{2,1}$'s cannot have the same $H^{0,*}(X, \wedge^* T_X)$. At this moment, we do not know how to interpret this observation. Is this just a special case? and is there some particular reason behind the equivalence? Or is this a generic feature of (2,2) superconformal field theories? We believe that it is worth doing further study in order to clarify the situation.

4.4 A practical use of duality

Recall that we have encountered a difficulty in evaluating the integral for the sphere partition function in the strongly coupled phase $r \ll 0$. But the same phase is weakly

coupled in the dual theory, and we expect that the difficulty disappears. Indeed, in the present example, we have just seen that the partition function of the dual theory is identical (up to a parameter change and irrelevant overall multiplication) to the one for the model $(A^2_{(-1)^6,(-2),1^4,0})$ and the computation in its weakly coupled phase $r_A \gg 0$ (corresponding to $r = r_S \ll 0$) has been done already in Section 3.2. Inserting the parameter relation (4.31) into the result (3.34), we find

$$e^{-K} \sim \left[\log^3 \frac{w\bar{w}}{2^{16}} - 264\zeta(3) \right] + \dots \tag{4.45}$$

for $w = e^t = z^{-1}$, which yields

$$g_{w\bar{w}} = \frac{3}{w\bar{w}(\log(w\bar{w}/2^{16}))^2} + \dots . \tag{4.46}$$

The dual theory must also be useful in evaluating the integral in the strongly coupled phase of other models, such as the Pfaffian phase in Rødland model $(A^2_{(-2)^7,1^7})$.

5 Systems with linear $A(p)$ and $S(p)$

In this section, we study the systems in which $A(p)$ and $S(p)$ are linear in p. The charges of the p^i's are all -2 and the charges of the x_j's are all $+1$. We shall call the systems $(A^k_{M,N})$ and $(S^{k,\bullet}_{M,N})$ rather than $(A^k_{(-2)^M,1^N})$ and $(S^{k,\bullet}_{(-2)^M,1^N})$. It turns out that the dual theory can be simplified to take the same form as the original, but with different number of p fields and of course with the dual gauge group. The dual pairs are:

$(A^k_{M,N})$ and $(A^{k^\vee}_{M^\vee,N})$, $k + k^\vee = N - 1$, $M + M^\vee = \frac{N(N-1)}{2}$,   [N odd];

$(S^{k,+}_{M,N})$ and $(S^{k^\vee,0}_{M^\vee,N})$, $k + k^\vee = N + 1$, $M + M^\vee = \frac{N(N+1)}{2}$,   [$(N - k)$ odd];

$(S^{k,-}_{M,N})$ and $(S^{k^\vee,-}_{M^\vee,N})$, $k + k^\vee = N + 1$, $M + M^\vee = \frac{N(N+1)}{2}$,   [N odd].

5.1 Simplifying the dual theory

5.1.1 (A) series

The basic data to specify the model $(A^k_{M,N})$ is a generic $N \times N$ antisymmetric matrix $A(p)$ which is linear in p. This can be regarded as an embedding

$$A : \mathbb{C}^M \; \hookrightarrow \; \wedge^2 V^*, \tag{5.1}$$

where V is a complex vector space of dimension N. The dual theory can be simplified into the model $(A^{k^\vee}_{M^\vee,N})$ associated with the orthogonal embedding

$$A^\vee : \mathbb{C}^{M^\vee} \; \hookrightarrow \; \wedge^2 V. \tag{5.2}$$

"Orthogonal" means that the image of A and the image of $A^\vee$ are orthogonal complement of each other under the perfect pairing $\langle \, , \, \rangle : \wedge^2 V^* \times \wedge^2 V \to \mathbb{C}$.

Before explaining how it works, let us write down the dual pair of the original system with the data A and the dual pair of the system with the data $A^\vee$.

gauge group	matter	superpotential
$\dfrac{U(1) \times USp(k)}{\{(\pm 1, \pm \mathbf{1}_k)\}}$	$p \in \mathbb{C}^M(-2), \ x \in \mathbb{C}^k \otimes V$	$\langle A(p), [xx] \rangle$
$\dfrac{U(1) \times USp(k^\vee)}{\{(\pm 1, \pm \mathbf{1}_{k^\vee})\}}$	$p \in \mathbb{C}^M(-2), \ \widetilde{x} \in \mathbb{C}^{k^\vee} \otimes V^*, \ a \in \wedge^2 V$	$\langle [\widetilde{x}\widetilde{x}] + A(p), a \rangle$
$\dfrac{U(1) \times USp(k^\vee)}{\{(\pm 1, \pm \mathbf{1}_{k^\vee})\}}$	$p^\vee \in \mathbb{C}^{M^\vee}(2), \ x^\vee \in \mathbb{C}^{k^\vee} \otimes V^*$	$\langle [x^\vee x^\vee], A^\vee(p^\vee) \rangle$
$\dfrac{U(1) \times USp(k)}{\{(\pm 1, \pm \mathbf{1}_k)\}}$	$p^\vee \in \mathbb{C}^{M^\vee}(2), \ \widetilde{x}^\vee \in \mathbb{C}^k \otimes V, \ a^\vee \in \wedge^2 V^*$	$\langle a^\vee, [\widetilde{x}^\vee \widetilde{x}^\vee] + A^\vee(p^\vee) \rangle$

$$(5.3)$$

The number in the parenthesis shows the $U(1)$ charges, and we assigned $U(1)$ charges 1 to V. Note that we take the non-standard convention in the dual pair for $A^\vee$ where we have assigned positive $U(1)$ charges to $p^\vee$ and negative $U(1)$ charges to $x^\vee$.

Let us now show how the theory with the data $A^\vee$ (the third theory) arises from the dual theory (the second theory). Integrating out all the p fields in the second theory, we obtain a constraint that a is orthogonal to the image of $A : \mathbb{C}^M \hookrightarrow \wedge^2 V^*$. Then, we can write it as $a = A^\vee(p^\vee)$ for some $p^\vee \in \mathbb{C}^{M^\vee}(2)$. This leaves us with the third theory. Similarly, the first theory is obtained from the fourth by integrating out all the $p^\vee$ fields. The correspondence of the chiral variables is therefore

$$
\begin{array}{ccccccc}
[xx] & \longleftrightarrow & a & \longleftrightarrow & A^\vee(p^\vee) & \longleftrightarrow & -[\widetilde{x}^\vee \widetilde{x}^\vee], \\
-A(p) & \longleftrightarrow & [\widetilde{x}\widetilde{x}] & \longleftrightarrow & [x^\vee x^\vee] & \longleftrightarrow & a^\vee.
\end{array}
\tag{5.4}
$$

Note that, if we keep the relation $[\widetilde{x}\widetilde{x}] = [x^\vee x^\vee]$ then we must have a sign in the relation $[xx] = -[\widetilde{x}^\vee \widetilde{x}^\vee]$. The top-left and the bottom-right arrows are the relations in the quantum duality, whereas the top-right and the bottom left arrows are the F-term equations. These quantum relations and the classical relations are mapped to each other under the exchange of A and $A^\vee$.

Recall that the $r \ll 0$ phase is, apart from the possible presence of singularity, the non-linear sigma model on the Pfaffian

$$
Y_A = \left\{ p \in \mathbb{CP}^{M-1} \ \middle| \ \operatorname{rank} A(p) \leq k^\vee \right\}.
\tag{5.5}
$$

If we use the simplified dual description, we find that the $r \gg 0$ phase can also be regarded in the same way, with the target space being the linearly dual Pfaffian

$$Y_{A^\vee} = \left\{ p^\vee \in \mathbb{CP}^{M^\vee - 1} \,\middle|\, \mathrm{rank}\, A^\vee(p^\vee) \le k \right\}. \tag{5.6}$$

This simplification can also be applied even if $A(p)$ is not totally linear. In the dual theory, we may integrate out the p's which enter linearly in $A(p)$. Then, a can be written as $A'^\vee(p'^\vee)$ for the orthogonal $A'^\vee$ to the linear part A' of A, and the dual can be written as the theory with the superpotential $W = \langle A''(p'') + [\widetilde{x}\widetilde{x}], A'^\vee(p'^\vee)\rangle$ where $A''(p'')$ is the non-linear part of $A(p)$.

5.1.2   (S) series

Let us next consider the (S) series. The basic data to specify the model $(\mathrm{S}_{M,N}^{k,+,0,-})$, a generic $N \times N$ symmetric matrix $S(p)$ which is linear in p, can be regarded as an embedding

$$S : \mathbb{C}^M \hookrightarrow \mathrm{Sym}^2 V^*, \tag{5.7}$$

where V is a complex vector space of dimension N. The dual theory can be simplified into the model $(\mathrm{S}_{M^\vee,N}^{k^\vee,0,+,-})$ associated with the orthogonal embedding

$$S^\vee : \mathbb{C}^{M^\vee} \hookrightarrow \mathrm{Sym}^2 V. \tag{5.8}$$

"Orthogonal" means that the image of S and the image of $S^\vee$ are orthogonal complement of each other under the perfect pairing $\langle\, ,\, \rangle : \mathrm{Sym}^2 V^* \times \mathrm{Sym}^2 V \to \mathbb{C}$.

Let us write down the dual pair of the original system with data S the the dual pair of the system with the data $S^\vee$ ($N - k$ must be odd here).

gauge group	matter	superpotential
$\dfrac{U(1) \times O_+(k)}{\{(\pm 1, \pm \mathbf{1}_k)\}}$	$p \in \mathbb{C}^M(-2),\ x \in \mathbb{C}^k \otimes V$	$\langle S(p), (xx)\rangle$
$\dfrac{U(1) \times SO(k^\vee)}{\{(\pm 1, \pm \mathbf{1}_{k^\vee})\}}$	$p \in \mathbb{C}^M(-2),\ \widetilde{x} \in \mathbb{C}^{k^\vee} \otimes V^*,\ s \in \mathrm{Sym}^2 V$	$\langle (\widetilde{x}\widetilde{x}) + S(p), s\rangle$
$\dfrac{U(1) \times SO(k^\vee)}{\{(\pm 1, \pm \mathbf{1}_{k^\vee})\}}$	$p^\vee \in \mathbb{C}^{M^\vee}(2),\ x^\vee \in \mathbb{C}^{k^\vee} \otimes V^*$	$\langle (x^\vee x^\vee), S^\vee(p^\vee)\rangle$
$\dfrac{U(1) \times O_+(k)}{\{(\pm 1, \pm \mathbf{1}_k)\}}$	$p^\vee \in \mathbb{C}^{M^\vee}(2),\ \widetilde{x}^\vee \in \mathbb{C}^k \otimes V,\ s^\vee \in \mathrm{Sym}^2 V^*$	$\langle a^\vee, (\widetilde{x}^\vee \widetilde{x}^\vee) + S^\vee(p^\vee)\rangle$

$$\tag{5.9}$$

There is also a version where all the orthogonal groups are O_- (N must be odd for that). We have assigned $U(1)$ charges 1 to V.

The third theory is obtained from the second by simply integrating out the p fields, while the first is obtained from the fourth by integrating out the $p^\vee$ fields. The correspondence of the chiral variables is

$$
\begin{array}{ccccccc}
(xx) & \longleftrightarrow & s & \longleftrightarrow & S^\vee(p^\vee) & \longleftrightarrow & -(\widetilde{x}^\vee \widetilde{x}^\vee), \\
-S(p) & \longleftrightarrow & (\widetilde{x}\widetilde{x}) & \longleftrightarrow & (x^\vee x^\vee) & \longleftrightarrow & s^\vee.
\end{array}
\tag{5.10}
$$

The top-left and the bottom-right arrows are the relations in the quantum duality, whereas the top-right and the bottom left arrows are the F-term equations. These quantum relations and the classical relations are mapped to each other under the exchange of S and $S^\vee$.

Recall that the $r \ll 0$ phase is, apart from the possible presence of a singularity, the non-linear sigma model on a double cover $\widetilde{Y}_S$ of the symmetric determinantal variety

$$
Y_S = \left\{ p \in \mathbb{CP}^{M-1} \,\middle|\, \operatorname{rank} S(p) \le k^\vee \right\}.
\tag{5.11}
$$

If we use the simplified dual description, we find that the $r \gg 0$ phase can also be regarded in the similar way, with the target space being the linearly dual symmetric determinantal variety

$$
Y_{S^\vee} = \left\{ p^\vee \in \mathbb{CP}^{M^\vee-1} \,\middle|\, \operatorname{rank} S^\vee(p^\vee) \le k \right\}.
\tag{5.12}
$$

In the O_- version, we have Y_S itself in the $r \ll 0$ phase.

This simplification can also be applied even if $S(p)$ is not totally linear. In the dual theory, we may integrate out the p's which enter linearly in $S(p)$. Then, s can be written as $S'^\vee(p'^\vee)$ for the orthogonal $S'^\vee$ to the linear part S' of S, and the dual can be written as the theory with the superpotential $W = \langle S''(p'') + (\widetilde{x}\widetilde{x}), S'^\vee(p'^\vee) \rangle$ where $S''(p'')$ is the non-linear part of $S(p)$. In fact, we have already encountered such a simplification in the computation of the two sphere partition function. See (4.34) and (4.36).

5.2 Coulomb branch analysis

Let us study the effective theory on the Coulomb branch, in order to identify the location of the singularity (Calabi-Yau case) and the massive vacua (non-Calabi-Yau case). We shall see that the dual pair yield the same results, under a certain relationship among the FI-theta parameters. In what follows, by "the dual theory" we mean the simplified dual theory. We denote the FI-theta parameters of the original theory and the

(simplified) dual theory by t and $t^\vee$. For completeness, we shall also briefly mention the relation to the FI-theta parameters of the dual theories before the simplification, denoted by $\widetilde{t}$ and $\widetilde{t}^\vee$.

5.2.1   (A) series

The effective twisted superpotential (2.26) of the model $(A_{M,N}^k)$ is

$$
\begin{aligned}
\widetilde{W}_{\mathit{eff}} \;=\; & -2t\sigma_0 - \frac{l(l+1)}{2}\pi i\sigma_0' - M(-2\sigma_0)(\log(-2\sigma_0) - 1) \\
& -N\sum_{a=1}^{l}(\sigma_0 + \sigma_a)(\log(\sigma_0 + \sigma_a) - 1) - N\sum_{a=1}^{l}(\sigma_0 - \sigma_a)(\log(\sigma_0 - \sigma_a) - 1) \\
\;=\; & -t\sigma_0' - \frac{l(l+1)}{2}\pi i\sigma_0' - M\sigma_0'(\log(-\sigma_0') - 1) \\
& -N\sum_{a=1}^{l}\sigma_a'(\log\sigma_a' - 1) - N\sum_{a=1}^{l}(\sigma_0' - \sigma_a')(\log(\sigma_0' - \sigma_a') - 1).
\end{aligned}
\tag{5.13}
$$

The vacuum equation yields

$$
(-1)^{\frac{l(l+1)}{2}+M}\,e^{-t} = \frac{\prod_{a=1}^{l}(\sigma_0' - \sigma_a')}{(\sigma_0')^M}, \qquad (\sigma_0' - \sigma_a')^N = (\sigma_a')^N \quad (a = 1, \dots, l).
\tag{5.14}
$$

The latter equation is solved by $\sigma_0'/\sigma_a' = 1 + \omega_a$ where $\omega_a^N = 1$ and we find

$$
(-1)^{\frac{l(l+1)}{2}+M}\,e^{-t} = (\sigma_0')^{Nl-M}\prod_{a=1}^{l}(1 + \omega_a)^{-N}.
\tag{5.15}
$$

Note that the region with $\sigma_a = -\sigma_b$ (any a, b) and $\sigma_a = \sigma_b$ $(a \neq b)$ is excluded and that the Weyl group acts as permutations and sign flips of the σ_a's. This corresponds to excluding solutions with $\omega_a = \omega_b^{-1}$ (any a, b) and $\omega_a = \omega_b$ $(a \neq b)$, and modding out the solutions by permutations and inversions of of ω_a's. Then, there are

$$
n(k, N) = \binom{\frac{N-1}{2}}{\frac{k}{2}}
\tag{5.16}
$$

inequivalent choices of $\{\omega_a\}$. In the Calabi-Yau case, $Nl = M$, this is the number of singular points in the FI-theta parameter space and (5.15) shows the location. In the non-Calabi-Yau case, for each such $\{\omega_a\}$, the equation (5.15) has $|Nl - M|$ solutions for σ_0'. Thus, the number of massive vacua is

$$
\frac{1}{2}|Nk - 2M|\,n(k, N).
\tag{5.17}
$$

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The value of the twisted superpotential at such a vacuum is

$$\widetilde{W}_{e\!f\!f} = \frac{1}{2}(Nk - 2M)\sigma_0'. \tag{5.18}$$

Computation in the dual theory is similar. The vacuum equation yields

$$(-1)^{\frac{l^\vee(l^\vee+1)}{2}+M^\vee}\,\mathrm{e}^{t^\vee} = (-\sigma_0^{\vee\prime})^{Nl^\vee-M^\vee}\prod_{a=1}^{l^\vee}(1+\omega_a)^{-N}, \tag{5.19}$$

and the value of the twisted superpotential at each solution is

$$\widetilde{W}_{e\!f\!f} = -\frac{1}{2}(Nk^\vee - 2M^\vee)\sigma_0^{\vee\prime}. \tag{5.20}$$

We need to choose $l^\vee = \frac{k^\vee}{2}$ distinct roots among the $\frac{N-1}{2}$ possibilities. Recalling that $l + l^\vee = \frac{N-1}{2}$, we find that a one to one correspondence between the solutions is given by complementary choices of the roots. Noting that $Nk - 2M = -(Nk^\vee - 2M^\vee)$, we find a one to one correspondence between the Coulomb branch vacua, with the same values of the twisted superpotential (non Calabi-Yau case) and the same location of the singular points (Calabi-Yau case), up to an overall normalization. The matching is perfect if the FI-theta parameters of the dual pair are related by:

$$\mathrm{e}^{-t} = \mathrm{e}^{-t^\vee}(-1)^{\frac{l(l+1)}{2}+\frac{l^\vee(l^\vee+1)}{2}+M^\vee+Nl}\prod_{a=1}^{\frac{N-1}{2}}(1+\mathrm{e}^{\frac{2\pi i a}{N}})^{-N}. \tag{5.21}$$

This is the duality map between the parameters. In the non-Calabi-Yau case, this is to be regarded as the relation at the same scale. The FI-theta parameters after and before the simplification are related by $\mathrm{e}^{-t} = (-1)^{M^\vee}\,\mathrm{e}^{\widetilde{t}^\vee}$ and $\mathrm{e}^{-t^\vee} = (-1)^M\,\mathrm{e}^{-\widetilde{t}}$.

Note that the number $n(k, N)$ in (5.16) agrees with the Witten index for the $USp(k)$ gauge theory with N fundamentals with generic twisted masses [3]. This is not a coincidence: the problem of solving the above equations for σ_a's with fixed σ_0 is exactly the same as the problem of finding the vacua in that theory with twisted masses $\widetilde{m}_i = -\sigma_0$. This observation will be useful when we discuss the more complicated (S) series.

5.2.2 (S) series

We only show the results and sketch the outline, since it is lengthy to describe all the detail of the analysis that depends on the cases. We denote by $n(k, N)$ the Witten index of the H theory with N fundamentals with a generic twisted masses, where H is the

orthogonal group factor in the gauge group $G = (U(1) \times H)/\{(\pm 1, \pm \mathbf{1}_k)\}$ of the theory $(\mathrm{S}_{M,N}^{k,\pm})$. Concretely, it is given in the following table: (It is an extract from [3]. Not all is needed here, because of the constraint discussed in Section 2.)

group	k	N	$n(k,N)$	group	k	N	$n(k,N)$
$O_+(k)$	even	even	$\binom{\frac{N}{2}}{\frac{k}{2}}$	$O_+(k)$	odd	even	$2\binom{\frac{N}{2}}{\frac{k-1}{2}}$
$O_-(k)$	even	even	$\binom{\frac{N}{2}}{\frac{k}{2}}$	$O_-(k)$	odd	even	$\binom{\frac{N}{2}}{\frac{k-1}{2}}$
$SO(k)$	even	even	$2\binom{\frac{N}{2}}{\frac{k}{2}}$	$SO(k)$	odd	even	$\binom{\frac{N}{2}}{\frac{k-1}{2}}$
$O_+(k)$	even	odd	$\binom{\frac{N-1}{2}}{\frac{k}{2}} + 2\binom{\frac{N-1}{2}}{\frac{k}{2}-1}$	$O_+(k)$	odd	odd	$\binom{\frac{N-1}{2}}{\frac{k-1}{2}}$
$O_-(k)$	even	odd	$\binom{\frac{N+1}{2}}{\frac{k}{2}}$	$O_-(k)$	odd	odd	$2\binom{\frac{N-1}{2}}{\frac{k-1}{2}}$
$SO(k)$	even	odd	$2\binom{\frac{N-1}{2}}{\frac{k}{2}} + \binom{\frac{N-1}{2}}{\frac{k}{2}-1}$	$SO(k)$	odd	odd	$\binom{\frac{N-1}{2}}{\frac{k-1}{2}}$

$$(5.22)$$

First, we describe an overview of the results. If the theory is not Calabi-Yau, $kN \neq 2M$, the number of massive vacua is

$$\frac{1}{2}\,|kN - 2M|\,n(k,N). \tag{5.23}$$

If the theory is Calabi-Yau, $kN = 2M$, the number of singular points in the FI-theta parameter space, including the multiplicity, is

$$n(k,N), \tag{5.24}$$

except in the series $(\mathrm{S}_{M,N}^{k,+})$ with k odd where the number is one half of this.

If the group H is not $O_\pm(k)$ with k odd, the reason we get these numbers is the same as in the (A) series. We find $n(k,N)$ vacua from the H sector. And we find $\frac{1}{2}|kN - 2M|$ vacua from the $U(1)$ sector in non-Calabi-Yau case. If $H = O_\pm(k)$ with k odd, the gauge group G is isomorphic to $U(1) \times SO(k)$ and we need to use the $SO(k)$ results. The Witten index of the $SO(k)$ theory is one half of $n(k,N)$ in these cases (i.e. $O_+(k)$ with k odd and N even as well as $O_-(k)$ with k and N both odd). But the number of vacua from the $U(1)$ sector is twice as much, $|kN - 2M|$, getting (5.23). In the Calabi-Yau case with $H = O_+(k)$ with k odd and N even, the number of singular points is one half of (5.24), while it is (5.24) itself in the dual, which has $H = SO(k^\vee)$ with $k^\vee$ even. Let us describe the results in more detail.

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$(\mathrm{S}_{M,N}^{k,\pm})$ with k even (N odd): The vacuum equation yields

$$(-1)^{\frac{l(l-1)}{2}+M}\,\mathrm{e}^{-t} = (\sigma_0')^{Nl-M}\prod_{a=1}^{l}(1+\omega_a)^{-N},\tag{5.25}$$

where $\omega_a^N = 1$, $\omega_a \neq \omega_b^{\pm 1}$ for $a \neq b$; ω_a's related by permutations and inversions are regarded as the same. Those involving the fixed point $\omega_a = 1$ of the inversion supports two vacua (*resp.* one vacuum) in the theory $(\mathrm{S}_{M,N}^{k,+})$ (*resp.* $(\mathrm{S}_{M,N}^{k,-})$). The count is $\binom{\frac{N-1}{2}}{l} + 2\binom{\frac{N-1}{2}}{l-1} = n(k,N)$ (*resp.* $\binom{\frac{N-1}{2}}{l} + \binom{\frac{N-1}{2}}{l-1} = n(k,N)$). The value of the twisted superpotential at each vacuum is

$$\widetilde{W}_{\text{eff}} = \frac{1}{2}(Nk - 2M)\sigma_0'.\tag{5.26}$$

Computations in the dual theory yield exactly the same results, provided the parameters are related by the duality map

$$\mathrm{e}^{-t} = \mathrm{e}^{-t^{\vee}}(-1)^{\frac{l(l-1)}{2}+\frac{l^{\vee}(l^{\vee}-1)}{2}+M^{\vee}+Nl}\prod_{a=0}^{\frac{N-1}{2}}(1+\mathrm{e}^{\frac{2\pi i a}{N}})^{-N}.\tag{5.27}$$

Note also the relations $\mathrm{e}^{-t} = (-1)^{M^{\vee}}\mathrm{e}^{\widetilde{t}^{\vee}}$ and $\mathrm{e}^{-t^{\vee}} = (-1)^{M}\mathrm{e}^{-\widetilde{t}}$.

$(\mathrm{S}_{M,N}^{k,+})$ with k odd (N even): The theory has gauge group $G \cong U(1) \times SO(k)$ and the vacuum equation is

$$\mathrm{e}^{-t} = \frac{\sigma_0^N \prod_{a=1}^{l}(\sigma_0+\sigma_a)^N(\sigma_0-\sigma_a)^N}{(-2\sigma_0)^{2M}},$$
$$(\sigma_0+\sigma_a)^N = -(\sigma_0-\sigma_a)^N \quad (a=1,\dots,l).\tag{5.28}$$

We require $\sigma_a \neq \pm\sigma_b$ for $a \neq b$ and identify solutions related by permutations and arbitrary sign flips of the σ_a's. The count is $\binom{\frac{N}{2}}{l} = \frac{1}{2}n(k,N)$. The value of the twisted superpotential at each vacuum is

$$\widetilde{W}_{\text{eff}} = (Nk - 2M)\sigma_0.\tag{5.29}$$

On the other hand, the gauge group of the dual theory is $G^{\vee} = (U(1)\times SO(k^{\vee}))/\{(\pm 1, \pm \mathbf{1}_{k^{\vee}})\}$ and the vacuum equation is

$$(-1)^{\frac{l^{\vee}(l^{\vee}+1)}{2}}\,\mathrm{e}^{-t^{\vee}} = \left[\frac{\prod_{a=1}^{l^{\vee}}(\sigma_0^{\vee}-\sigma_a^{\vee})^N}{(2\sigma_0^{\vee})^{M^{\vee}}}\right]^{-1},$$
$$(\sigma_0^{\vee}+\sigma_a^{\vee})^N = -(\sigma_0^{\vee}-\sigma_a^{\vee})^N \quad (a=1,\dots,l^{\vee}).\tag{5.30}$$

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We require $\sigma_a^\vee \neq \pm\sigma_b^\vee$ for $a \neq b$ and identify solutions related by permutations and an even number of sign flips of the $\sigma_a^\vee$'s. The count is $2\binom{\frac{N}{2}}{l^\vee} = n(k, N)$. The value of the twisted superpotential at each vacuum is

$$\widetilde{W}_{\textit{eff}} = -(Nk^\vee - 2M^\vee)\sigma_0^\vee. \tag{5.31}$$

We see that the two theories yield the same results concerning the massive vacua in non-Calabi-Yau cases but different results concerning the singular points in the Calabi-Yau case. In fact, there is a good reason for the mismatch. An odd number of sign flips of $\sigma_a^\vee$'s corresponds to the $\mathbb{Z}_2$ symmetry associated with $O(k^\vee)/SO(k^\vee) \cong \mathbb{Z}_2$, and, as discussed in Section 2.3, that symmetry induces a shift of the theta angle by π. In particular, the theories with $e^{-t^\vee}$ and $-e^{-t^\vee}$ are physically equivalent. Indeed, there is a $2 : 1$ map from $e^{-t^\vee}$ to e^{-t},

$$e^{-t} = e^{-2t^\vee}(-1)^{l^\vee}2^{-N(N+1)}\prod_{a=1}^{N}(1 + \omega_a)^N, \tag{5.32}$$

where ω_a are the N solutions to $(1 + \omega)^N = -(1 - \omega)^N$. The FI-theta parameters after and before the simplification are related by $e^{-t} = e^{\tilde{t}^\vee}$ and $e^{-t^\vee} = (-1)^M e^{-\tilde{t}}$.

$(\underline{S_{M,N}^{k,-}})$ with k odd (N odd): The gauge group is $G \cong U(1) \times SO(k)$ and the vacuum equation is

$$e^{-t} = \frac{\sigma_0^N \prod_{a=1}^{l}(\sigma_0 + \sigma_a)^N(\sigma_0 - \sigma_a)^N}{(-2\sigma_0)^{2M}},$$
$$(\sigma_0 + \sigma_a)^N = (\sigma_0 - \sigma_a)^N \quad (a = 1, \ldots, l). \tag{5.33}$$

We require $\sigma_a \neq \pm\sigma_b$ for $a \neq b$ and $\sigma_a \neq 0$ for any a, and identify solutions related by permutations and arbitrary sign flips of σ_a's. The count is $\binom{\frac{N-1}{2}}{l} = \frac{1}{2}n(k, N)$. The count for the $U(1)$ part is $|kN - 2M|$ yielding (5.23). The value of the twisted superpotential at each vacuum is

$$\widetilde{W}_{\textit{eff}} = (Nk - 2M)\sigma_0. \tag{5.34}$$

The dual theory yield exactly the same results, provided the parameters at the same scale are related by

$$e^{-t} = e^{-t^\vee}2^{-N(N+1)}\prod_{a=1}^{N}(1 + \omega_a)^{2N}. \tag{5.35}$$

where ω_a are the solutions to $(1+\omega)^N = (1-\omega)^N$. We also have $e^{-t} = e^{\tilde{t}^\vee}$ and $e^{-t^\vee} = e^{-\tilde{t}}$. There is no Calabi-Yau case, since $kN - 2M$ can never vanish if k and N are both odd.

5.3 Category of B-branes

The category of B-type D-branes (B-branes for short) is believed to be invariant under renormalization group flow as well as under deformation of the twisted chiral parameters of the theory. If the resulting target variety Y is smooth, the category of B-branes in the series (A) or (S^+) is the same as the derived category $D^b(Y)$ of coherent sheaves on Y. Even if they are singular, it is possible that the quantum gauge theory itself is perfectly fine. Although we do not have a proof, the absence of a Coulomb branch may be regarded as one indication for that. If the theory is indeed fine, there must be a good category of B-branes. Such a good category has been constructed in some cases under the name of "non-commutative resolution". Here we assume that the theory is fine and denote the good category of B-branes simply by $D(Y)$. We also denote the category for the (S^-) series by $D_{(-1)^{F_s}}(Y)$.

The above results then yield the following prediction concerning equivalences of the categories.

$$
\begin{aligned}
D_- &\cong D_+, & \text{if } kN = 2M, \\
D_- + \cdots &\cong D_+, & \text{if } kN > 2M, \\
D_- &\cong D_+ + \cdots, & \text{if } kN < 2M,
\end{aligned}
\tag{5.36}
$$

with

	D_-	D_+
(A)	$D(Y_A)$	$D(Y_{A^\vee})$
(S^+)	$D(\widetilde{Y}_S)$	$D(Y_{S^\vee})$
(S^-)	$D_{(-1)^{F_s}}(Y_S)$	$D_{(-1)^{F_s}}(Y_{S^\vee})$

$$\tag{5.37}$$

$+\cdots$ is the collection of $\frac{1}{2}|kN - 2M|\, n(k,N)$ objects associated to the massive vacua on the Coulomb branch.

Such equivalences of categories seem to fit with the framework of "homological projective duality" by A. Kuznetsov [22]. Interestingly, the latter seems to have something to say about the (A) series with even N. It would be interesting to understand the reason of the similarity between the quantum duality in gauge theory and the homological projective duality in the study of derived categories.

We should also note that we expect equivalences of categories $D_- \cong D_+$ in the first three systems of Section 3 and in the system of Section 4. D_- is the derived category of the smooth compact Calabi-Yau threefolds (the Pfaffian or the double cover of the symmetric determinantal variety). D_+ is the category of B-branes in the true hybrid

model. It would be interesting to provide a useful description of D_+, such as modules over sheaves of algebras on the base Fano threefold, as in [16].

Obviously, it is an interesting problem to determine and apply the grade restriction rule [31] in the above situations, in order to physically find the equivalences of categories as well as monodromy along the closed loops in the FI-theta parameter space. We would like to mention that there is a recent mathematical progress in the classical grade restriction rule [32–35]. More recently a way to determine the quantum grade restriction rule in non-Abelian gauged linear sigma model was suggested in [8].

5.4 Smooth Calabi-Yau examples

Let us see when we have a smooth Calabi-Yau manifold as either or both of the Pfaffians Y_A and $Y_{A^\vee}$, or of the (double cover of) symmetric determinants, $\widetilde{Y}_S$ and $Y_{S^\vee}$. Calabi-Yau condition reads

$$kN - 2M = 0, \tag{5.38}$$

which can also be stated as $k^\vee N - 2M^\vee = 0$. The dimension of the variety Y_A (*resp.* Y_S) is

$$d = M - 1 - \frac{k(k+1)}{2} \quad \left(resp.\ M - 1 - \frac{k(k-1)}{2}\right) = \frac{kk^\vee}{2} - 1. \tag{5.39}$$

Y_A is smooth when the corank $k + 3$ degeneration of $A(p)$ is absent, which is the case when the dimension $M - 1$ of the ambient space is smaller than $\frac{(k+3)(k+2)}{2}$. That is, $N \leq k + 5 + \frac{6}{k}$. Recalling also $N > k + 1$ is needed for $d \geq 0$, we find that Y_A is smooth when

$$k = 2: \quad N = 5,\ 7,\ 9 \ \ (d = 1,\ 3,\ 5),$$
$$k \geq 4: \quad N = k + 3,\ k + 5 \ (d = k - 1,\ 2k - 1).$$

The smoothness condition for $Y_{A^\vee}$ is obtained by the replacement $k \to k^\vee$. Both are smooth in the cases below:

	$k^\vee = 2$	$k^\vee = 4$	$k^\vee = 6$
$k = 2$	$N = 5,\ M = M^\vee = 5$ $d = 1$	$N = 7,\ M = 7,\ M^\vee = 14$ $d = 3$	$N = 9,\ M = 9,\ M^\vee = 27$ $d = 5$
$k = 4$	$-$	$N = 9,\ M = M^\vee = 18$ $d = 7$	Y_A not smooth
$k = 6$	$-$	$Y_{A^\vee}$ not smooth	both not smooth

$$\tag{5.40}$$

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We do not write the lower triangular part by the redundancy from $Y_A \leftrightarrow Y_{A^\vee}$. The threefold pair is the pair in Rødland's work. The pair of elliptic curves is in the same family. In fact, from the equivalence $D^b(Y_A) \cong D^b(Y_{A^\vee})$ and from the fact that they are elliptic curves, they must be biholomorphic, $Y_A \cong Y_{A^\vee}$. In the fivefold case, $Y_{A^\vee}$ can also be regarded as the complete intersection of nine hypersurfaces in the Grassmannian $G(2,9)$. ($k = 2$ case is always like that.) The ninefold pair are in the same family. We do not know if they are biholomorphic to each other. At this moment, the derived equivalences $D^b(Y_A) \cong D^b(Y_{A^\vee})$ for these fivefold and ninefold pairs seem to be an interesting prediction.

$\widetilde{Y}_S$ is smooth when the corank $(k+1)$ degeneration is absent, which is the case when $M - 1$ is smaller than $\frac{(k+1)(k+2)}{2}$. That is, $N \leq k + 3 + \frac{2}{k}$. Recalling also $N \geq k$ for $d \geq 0$ and that $N - k$ is odd, we find that $\widetilde{Y}_S$ is smooth when

$$
\begin{aligned}
k = 1 : \quad & N = 2,\, 4,\, 6 \quad (d = 0,\, 1,\, 2), \\
k \geq 2 : \quad & N = k + 1,\, k + 3 \ (d = k - 1,\, 2k - 1).
\end{aligned}
$$

$Y_{S^\vee}$ is smooth when the corank $k^\vee$ degeneration is absent, i.e., $M^\vee - 1 \leq \frac{k^\vee(k^\vee+1)}{2} - 1$. Recalling also that $N \geq k^\vee$, we find that $Y_{S^\vee}$ when

$$
N = k^\vee,\, k^\vee + 1, \quad \text{i.e. } k = 1,\, 2. \tag{5.41}
$$

We should also bear in mind that $k^\vee$ must be even for regularity. Both are smooth and regularity is satisfied in the cases below:

	$k^\vee = 2$	$k^\vee = 4$	$k^\vee = 6$
$k = 1$	$N = 2,\ M = 1,\ M^\vee = 2$ $d = 0$	$N = 4,\ M = 2,\ M^\vee = 8$ $d = 1$	$N = 6,\ M = 3,\ M^\vee = 18$ $d = 2$
$k = 2$	$N = 3,\ M = 3,\ M^\vee = 3$ $d = 1$	$N = 5,\ M = 5,\ M^\vee = 10$ $d = 3$	$\widetilde{Y}_S$ not smooth

$$\tag{5.42}$$

The two cases with $d = 1$ (elliptic curve) must be such that $\widetilde{Y}_S$ and $Y_{S^\vee}$ are biholomorphic to each other. The threefold pair are the pair in Hosono-Takagi's work.

Finally, among the model involving $O_-(k)$ gauge group, there is only one example where both Y_S and $Y_{S^\vee}$ are smooth. This is the case where $k = k^\vee = 2$ and the varieties the the elliptic curves.

5.5 The partition function and the fundamental period in $d = 1$ examples

Let us comment on the results of the computation of the two sphere partition function and the fundamental periods in two of the examples in which the elliptic curve appears in both phases.

5.5.1 $(A_{5,5}^2)$

This model is the one-dimensional version of the Rødland model $(A_{7,7}^2)$. As discussed above, both phases correspond to elliptic curves and they must be biholomorphic to each other.

The calculation of the sphere partition function is the same as for the Rødland example. Therefore we can use the results of [6]. The partition function in the $r \gg 0$-phase is

$$
\begin{aligned}
Z_{S^2}^{r \gg 0} &= \frac{(z\bar{z})^{2q}}{2} \oint \frac{d^2\varepsilon}{(2\pi i)^2} \frac{\pi^5 \sin^5 \pi(\varepsilon_1 + \varepsilon_2)}{\sin^5 \pi\varepsilon_1 \sin^5 \pi\varepsilon_2} (z\bar{z})^{\varepsilon_1 + \varepsilon_2} \times \\
&\quad \times \left| \sum_{K=0}^{\infty} (-z)^K \sum_{k=0}^{K} (2k - K + \varepsilon_1 - \varepsilon_2) \frac{\Gamma(1 + K + \varepsilon_1 + \varepsilon_2)^5}{\Gamma(1 + k + \varepsilon_1)^5 \Gamma(1 + K - k + \varepsilon_2)^5} \right|^2
\end{aligned}
\tag{5.43}
$$

The residue integrals can be evaluated and the the fundamental period can be extracted from the $\log z\bar{z}$-term of the result. We get

$$
\begin{aligned}
X_{r \gg 0}^0 &= 1 + 3z + 19z^2 + 147z^3 + 1251z^4 + 11253z^5 + \dots \\
&= \sum_{n=0}^{\infty} \sum_{\alpha=0}^{n} \binom{n}{\alpha}^2 \binom{n+\alpha}{\alpha} z^n.
\end{aligned}
\tag{5.44}
$$

Note that the expansion coefficients are the Apéry numbers. The fundamental period is annihilated by the Picard-Fuchs operator

$$
\mathcal{L} = \theta^2 - z(11\theta^2 + 11\theta + 3) - z^2(\theta + 1)^2
\tag{5.45}
$$

which also appears in the database [26]. The second solution of the Picard-Fuchs equation is

$$
X_{r \gg 0}^1 = \log z + (3\log z + 5)z + \left(19\log z + \frac{75}{2}\right)z^2 + \left(147\log z + \frac{1855}{6}\right)z^3 + \dots
\tag{5.46}
$$

We can compute the complexified Kähler modulus of the elliptic curve,

$$
\rho = \frac{B}{2\pi} + i\text{Area}
\tag{5.47}
$$

via the "mirror map" $2\pi i\rho = \frac{X^1}{X^0}$. With the definition $q = e^{2\pi i\rho}$ we obtain

$$q(z) = z + 5z^2 + 35z^3 + 280z^4 + 2410z^5 + \dots \tag{5.48}$$

The inverse of this series is

$$q - 5q^2 + 15q^3 - 30q^4 + 40q^5 + \dots \tag{5.49}$$

Actually this can be given in a closed form (see for instance [36]):

$$q\prod_{n=1}^{\infty}(1 - q^n)^{5\left(\frac{5}{n}\right)}, \tag{5.50}$$

where $\left(\frac{m}{n}\right)$ denotes the Jacobi symbol. This is a modular function with respect to the congruence subgroup $\Gamma_1(5)$ of $SL(2, \mathbb{Z})$.

Let us compare the result with the computation of the two sphere partition function for the non-linear sigma model whose target space is the elliptic curve. Note that there is no instanton effect as there is no topologically non-trivial map from the two sphere to the elliptic curve. Note also that the contribution from the non-zero modes is insensitive to the moduli. Therefore, the non-trivial part is just the zero mode integral. If we take the usual path-integral measure induced from the (Kähler) metric of the elliptic curve, the result is the area[1]

$$Z_{S^2}^{\text{nlsm}} = \text{Area} = \text{Im}\,\rho. \tag{5.51}$$

Note that this is indeed e^{-K} for the the Kähler moduli space which is known to have the metric $ds^2 = |d\rho|^2/(\text{Im}\,\rho)^2$. We would like to see if the partition function (5.43) of the gauged linear sigma model reproduces this result. Let us perform the following Kähler transform,

$$e^{-K'} = \frac{1}{40\pi} \frac{Z_{S^2}^{r\gg 0}}{X_{r\gg 0}^0(z)\bar{X}_{r\gg 0}^0(\bar{z})}. \tag{5.52}$$

After expanding in z and $\bar{z}$ we obtain

$$e^{-K'} = \frac{1}{2i}\frac{1}{2\pi i}\left[\left(\log z + 5z + \frac{45}{2}z^2 + \frac{440}{3}z^3 + \dots\right) + c.c.\right]. \tag{5.53}$$

where we used $\log z\bar{z} = \log z + \log \bar{z}$. We compare this to the solutions of the Picard-Fuchs equation, where we computed

$$2\pi i\rho = \frac{X^1(z)}{X^0(z)} = \log z + 5z + \frac{45}{2}z^2 + \frac{440}{3}z^3 + \dots \tag{5.54}$$

---

[1]We thank Hirosi Ooguri and Yuji Tachikawa for discussions including this result.

Using this, we find

$$e^{-K'} = \frac{1}{2i}(\rho - \bar{\rho}) = \operatorname{Im} \rho, \tag{5.55}$$

which is indeed the same as the result (5.51) of the non-linear sigma model.

The sphere partition function in the $r \ll 0$ phase is

$$
\begin{aligned}
Z_{S^2}^{r \ll 0} &= \frac{(z\bar{z})^{2q-1}}{2} \lim_{\delta \to 0} \oint \frac{d^2\varepsilon}{(2\pi i)^2} \frac{\pi^5 \sin^5 \pi(\varepsilon_1 + \varepsilon_2)}{\sin^5 \pi\varepsilon_1 \sin^5 \pi\varepsilon_2} (z\bar{z})^{-\varepsilon_1} \times \\
&\quad \times \left| \sum_{K,k \geq 0} (-e^{-\delta})^k (-z)^{-K} (1 + K + k + \varepsilon_1 + 2\varepsilon_2) \frac{\Gamma(1 + K + k + \varepsilon_1 + 2\varepsilon_2)^5}{\Gamma(1 + K + \varepsilon_1)^5 \Gamma(1 + k + \varepsilon_2)^5} \right|^2
\end{aligned}
\tag{5.56}
$$

Since the coefficients to a given order in z are infinite sums, the partition function is hard to evaluate. Alternatively, one can also use the fact that the mirror on the $r \ll 0$ phase is known [37] to calculate the period directly or one can compute the Picard-Fuchs operator at $r \ll 0$ by transforming the Picard-Fuchs operator of the $r \gg 0$ phase. Details about this can be found in appendix F. In any case, one finds that the period in the $r \ll 0$ phase is the same as in the $r \gg 0$ phase.

5.5.2 $(\mathbf{S}_{3,3}^{2,+})$

This is the one-dimensional analogue on the Hosono-Takagi model $(\mathrm{S}_{5,5}^{2,+})$. Both phases correspond to elliptic curves and they must be biholomorphic to each other.

The sphere partition function in the $r \gg 0$ phase is

$$
\begin{aligned}
Z_{S^2}^{r \gg 0} &= (z\bar{z})^{2q} \oint \frac{d^2\varepsilon}{(2\pi i)^2} \frac{\pi^3 \sin^3 \pi(\varepsilon_1 + \varepsilon_2)}{\sin^3 \pi\varepsilon_1 \sin^3 \pi\varepsilon_2} (z\bar{z})^{\varepsilon_1 + \varepsilon_2} \times \\
&\quad \times \left| \sum_{K=0}^{\infty} (-z)^K \sum_{k=0}^{K} \frac{\Gamma(1 + K - \varepsilon_1 - \varepsilon_2)^3}{\Gamma(1 + k - \varepsilon_1)^3 \Gamma(1 + K - k - \varepsilon_2)^3} \right|^2
\end{aligned}
\tag{5.57}
$$

If we replaced the 3's by 5's everywhere, we would get the partition function for the Hosono-Takagi model. Again, we can extract the fundamental period from the $\log z\bar{z}$ term. The result is (after a shift $z \to -z$):

$$
\begin{aligned}
X_{r \gg 0}^0 &= 1 + 2z + 10z^2 + 56z^3 + 346z^4 + \dots \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}^3 z^n
\end{aligned}
\tag{5.58}
$$

Since the mirror is a free $\mathbb{Z}_2$-quotient of complete intersection of codimension 3 in $\mathbb{P}^2 \times \mathbb{P}^2$ we can use toric methods to confirm this results. The procedure is completely analogous to the mirror construction of the Hosono-Takagi model given in [17]. Some more details can be found in appendix F. The Picard-Fuchs operator which annihilates the fundamental period is

$$\mathcal{L} = \theta^2 - z(7\theta^2 + 7\theta + 2) - 8z^2(\theta + 1)^2. \tag{5.59}$$

Solving the Picard-Fuchs equation, we can determine the second period and the mirror map. Its inverse has following power series expansion in terms of $q = e^{2\pi i\rho}$:

$$q - 3q^2 + 3q^3 + 5q^4 - 18q^5 + 15q^6 + \dots. \tag{5.60}$$

This series can be obtained as the expansion around $q = 0$ of [38]:

$$\frac{q}{\left(\frac{\chi(-q^3)}{\chi(-q)}\right)^3}, \tag{5.61}$$

where $\chi(q)$ is the Ramanujan theta function given by

$$\chi(q) = \prod_{k \geq 0}(1 + q^{2k+1}). \tag{5.62}$$

It would be interesting to find out more about the modular properties of this function.

Let us also check if the Kähler transform of (5.57) computes the area of the elliptic curve. For this purpose we define

$$e^{-K'} = \frac{1}{24\pi} \frac{Z_{S^2}^{r \gg 0}}{X_{r \gg 0}^0(z) \bar{X}_{r \gg 0}^0(\bar{z})} \tag{5.63}$$

Expanding in z and $\bar{z}$ we get

$$e^{-K'} = \frac{1}{2i} \frac{1}{2\pi i} \left[\left(\log z - 3z + \frac{21}{2}z^2 - 49z^3 + \dots \right) + c.c. \right]. \tag{5.64}$$

Comparing with the solutions of the Picard-Fuchs equation and setting $z \to -z$, one finds that the expression in the parentheses is indeed the quotient of the periods $\frac{X^1(z)}{X^0(z)} = 2\pi i\rho$, which gives the expected result (5.51).

In the $r \ll 0$ regime the sphere partition function is

$$
\begin{aligned}
Z_{S^2}^{r \ll 0} &= (z\bar{z})^{2q-1} \lim_{\delta \to 0} \oint \frac{d^2\varepsilon}{(2\pi i)^2} \frac{\sin^2 \pi(\varepsilon_1 + \varepsilon_2)}{\sin^3 \pi\varepsilon_1 \sin^3 \pi\varepsilon_2} (z\bar{z})^{-\varepsilon_1} \times \\
&\quad \times \left| \sum_{K,k \geq 0} (-e^{-\delta})^k (-z)^{-K} \frac{\Gamma(1 + K + k + \varepsilon_1 + \varepsilon_2)^3}{\Gamma(1 + K + \varepsilon_1)^3 \Gamma(1 + k + \varepsilon_2)^3} \right|^2
\end{aligned} \tag{5.65}
$$

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Extracting the period is awkward due to the involvement of infinite sums but we can nevertheless read off the period from the coefficient of the $\log z\bar{z}$ term. The result is:

$$X^0_{r \ll 0} \sim -\frac{3}{2^{14}} \left(1 + \frac{z}{4} + \frac{5z^2}{32} + \frac{7z^3}{64} + \dots \right) \tag{5.66}$$

Redefining $z \to 8z$ the expression in the parentheses is the same is the period in the $r \gg 0$ phase. Alternatively, one can transform the Picard-Fuchs operator to the $r \ll 0$ phase which immediately shows that $\mathcal{L}$ transforms into itself.

Appendix

A Some linear algebra

In this appendix, we show (i) in the $r \ll 0$ phase of the linear sigma model (A) or (S), the D-term and F-term equations require $x_j = 0$ for all $j = 1, \ldots, N$. (ii) the Pfaffian Y_A for $k = 2$, $M = 7$, $N = 5$ is locally defined by $\mathrm{Pf}_i A(p) = 0$, $i = 1, 2, 3$ in a neighborhood of a point p_* with (3.6), and (iii) the rank two curve C_S that appears in the model $(\mathrm{S}^{2,+}_{(-1)^2,(-2)^3,1^4})$ is defined by $\Delta_{ij}(p) = 0$ for $(i, j) = (1,1), (2,2), (1,2)$ in a neighborhood of a point p_* with (4.19).

(i) We first consider a model in the (A) series. Let us write down the F-term equations:

$$\sum_{j=1}^{N} A(p)^{ij} x_j^a = 0, \quad i = 1, \ldots, N, \tag{A.1}$$

$$\sum_{i,j=1}^{N} \frac{\partial}{\partial p^k} A^{ij}(p)[x_i x_j], \quad k = 1, \ldots, M. \tag{A.2}$$

We know from the D-term equation for $U(1)$ that p cannot vanish and span a weighted projective space $\mathbb{P}$. For an odd number l, let $Y_A^{(l)}$ be the locus of $p \in \mathbb{P}$ such that $A(p)$ has rank $N - l$ or less. It has codimenion $\frac{l(l-1)}{2}$ in $\mathbb{P}$ and dimension $M - 1 - \frac{l(l-1)}{2}$. If the latter number is negative, $Y_A^{(l)}$ is empty. The dimension of the kernel of $A(p)$ is l, and we consider the matrix of first derivatives $\frac{\partial}{\partial p^k} A^{ij}(p)$ in which (i,j) is restricted to the kernel direction. This may be regarded as an $M \times \binom{l}{2}$ matrix. It has generically rank $\binom{l}{2}$ because we consider l such that $M - 1 - \frac{l(l-1)}{2} \geq 0$. Its rank drops from maximal if we tune $M - (\binom{l}{2} - 1)$ parameters but that is too big compared to the dimension of $Y_A^{(l)}$. Therefore, the rank of $\frac{\partial}{\partial p^k} A^{ij}(p)$ stays maximal, i.e. $\binom{l}{2}$. By (A.1), $x^a = (x_j^a)_{j=1}^N$ belongs to the kernel of $A(p)$ for each $a = 1, \ldots, k$ and by (A.2) and by the observation on the rank of the matrix of first derivatives, we find $[x_i x_j] = 0$ for all (i, j). By the D-term equation for $USp(k)$, this means $x_i^a = 0$ for all (a, i). The argument is essentially the same for a model in the (S) series.

(ii) In the situation described above, we would like to show that $\mathrm{Pf}_i A(p) = 0$ for $i = 1, 2, 3$ imply $\mathrm{Pf}_j A(p) = 0$ for any j. Suppose $\mathrm{Pf}_5 A(p) \neq 0$. This means that $A^1(p), \ldots, A^4(p)$ are linearly independent vectors in $\mathbb{C}^N$, where $A^i(p) = (A^{ij}(p))_{j=1}^N$. On the other hand, $A^4(p)$ and $A^5(p)$ are also linearly independent by the assumption. By the antisymmetry of $A^{ij}(p)$, $A^1(p), \ldots, A^5(p)$ cannot be linearly independent. Thus, there is $(c_1, \ldots, c_5) \neq 0$ such that $\sum_{i=1}^5 c_i A^i(p) = 0$. By the linear independence of $A^1(p), \ldots, A^4(p)$, we have

$c_5 \neq 0$. By the linear independence of $A^4(p)$ and $A^5(p)$, we have $(c_1, c_2, c_3) \neq 0$, say, $c_1 \neq 0$. $A^1(p)$ can then be written as a linear combination of $A^2(p), \ldots, A^5(p)$ which then mean that $A^2(p), \ldots, A^5(p)$ are linearly independent. That is, $\mathrm{Pf}_1 A(p)$ is not zero. A contradiction.

(iii) Under the situation described above, we have

$$
\begin{pmatrix} S^{11} \\ S^{31} \\ S^{41} \end{pmatrix} = c_{13} \begin{pmatrix} S^{13} \\ S^{33} \\ S^{43} \end{pmatrix} + c_{14} \begin{pmatrix} S^{14} \\ S^{34} \\ S^{44} \end{pmatrix}, \quad \begin{pmatrix} S^{22} \\ S^{32} \\ S^{42} \end{pmatrix} = c_{23} \begin{pmatrix} S^{23} \\ S^{33} \\ S^{43} \end{pmatrix} + c_{24} \begin{pmatrix} S^{24} \\ S^{34} \\ S^{44} \end{pmatrix},
$$

$$
\begin{pmatrix} S^{12} \\ S^{32} \\ S^{42} \end{pmatrix} = d_{23} \begin{pmatrix} S^{13} \\ S^{33} \\ S^{43} \end{pmatrix} + d_{24} \begin{pmatrix} S^{14} \\ S^{34} \\ S^{44} \end{pmatrix}, \quad \begin{pmatrix} S^{21} \\ S^{31} \\ S^{41} \end{pmatrix} = d_{13} \begin{pmatrix} S^{23} \\ S^{33} \\ S^{43} \end{pmatrix} + d_{14} \begin{pmatrix} S^{24} \\ S^{34} \\ S^{44} \end{pmatrix}.
$$

Since the last 2×2 block of $S(p)$ is invertible, we find $c_{ij} = d_{ij}$. This means that

$$
\begin{pmatrix} S^{11} \\ S^{21} \\ S^{31} \\ S^{41} \end{pmatrix} = c_{13} \begin{pmatrix} S^{13} \\ S^{23} \\ S^{33} \\ S^{43} \end{pmatrix} + c_{14} \begin{pmatrix} S^{14} \\ S^{24} \\ S^{34} \\ S^{44} \end{pmatrix}, \quad \begin{pmatrix} S^{12} \\ S^{22} \\ S^{32} \\ S^{42} \end{pmatrix} = c_{23} \begin{pmatrix} S^{13} \\ S^{23} \\ S^{33} \\ S^{43} \end{pmatrix} + c_{24} \begin{pmatrix} S^{14} \\ S^{24} \\ S^{34} \\ S^{44} \end{pmatrix}.
$$

$$\tag{A.3}$$

This clearly shows that $S(p)$ is of rank 2.

B   Details on the two sphere partition function

In this appendix we give details on the computation of the two sphere partition function in the hybrid-like phases of the Pfaffian GLSMs, as well as both phases of the $U(1) \times O(2)$ model.

The two sphere partition function of the $(2, 2)$ supersymmetric gauge theory has been computed in [4, 5]. There is a correction noticed in [8, 39] (and also in [10] for a special gauge group). When applied to a linear sigma model of Calabi-Yau type, it reads as follows:

$$
Z_{S^2} = (\ell \Lambda)^{\widehat{c}} \sum_{m \in Q^\vee} \int_{i\mathfrak{t}} \mathrm{d}^{l_G} \sigma \, \exp\Big(2ir(\sigma) + i(\theta + 2\pi\rho)(m) \Big) \tag{B.1}
$$

$$
\times \prod_{\alpha > 0} \left(\frac{\alpha(m)^2}{4} + \alpha(\sigma)^2 \right) \prod_i \frac{\Gamma\left(i\chi_i(\sigma) - \frac{\chi_i(m)}{2} + \frac{R_i}{2} \right)}{\Gamma\left(1 - i\chi_i(\sigma) - \frac{\chi_i(m)}{2} - \frac{R_i}{2} \right)}.
$$

ℓ is the radius of the two sphere, Λ is an energy scale introduced in the renormalization, $\widehat{c}$ is one third of the central charge of the infra-red fixed point, $\mathfrak{t}$ is the Lie algebra of a maximal torus of the gauge group G ($i\mathfrak{t}$ is the real part of its complexification $\mathfrak{t}_{\mathbb{C}}$), $Q^\vee \subset i\mathfrak{t}$ is the coroot lattice of G (i.e., the dual to the weight lattice $P \subset i\mathfrak{t}^*$ of G), l_G is the rank of G, r is the FI parameter, θ is the theta parameter, ρ is half the sum of the positive roots (for a choice of Weyl chamber), $\prod_{\alpha>0}$ is the product over the positive roots, $\prod_i$ is the product over the weights of the matter representation, and R_i is the vector R-charge of the i-th chiral multiplet. For the above contour, we need to choose $0 < R_i < 2$. The factor $e^{2\pi i \rho(m)}$ is the correction noticed in [8, 39]. Since 2ρ is always a weight, it is a sign factor. It may affect the sum over topological types of the G bundles over the sphere. It is therefore trivial if G is simply connected, but can be non-trivial otherwise.

In this appendix, we take a non-standard sign convention for the theta angle. What is written below as θ is actually $-\theta$ in the main text and also in Eqn (B.1). This does not matter since the final result is invariant under $\theta \to -\theta$. Also, what is written as $2\pi r$ below stands for r in the main text and in Eqn (B.1). We take simple names of the models as $(A^2_{(-1)^4,(-2)^3,1^5}) = X_5$, $(A^2_{(-1)^6,(-2),1^4,0}) = X_{10}$, $(A^2_{(-2)^7,3,1^4}) = X_{13}$, $(A^2_{(-2)^5,(-4)^2,3^2,1^3}) = X_7$, after the name of the Pfaffian Calabi-Yau given in [18].

B.1 X_5

This example has gauge group $G = U(1) \times SU(2)$. The weights and the R-charges of the fields are

	$p^{1,\dots,4}$	$p^{5,6,7}$	$x_{1,\dots,5}$
$\chi(\sigma)$	$-\sigma_0$	$-2\sigma_0$	$\begin{pmatrix} \sigma_0 + \sigma_1 \\ \sigma_0 - \sigma_1 \end{pmatrix}$
$U(1)_V$	$1 - 2q$	$2 - 4q$	$2q$

(B.2)

We will assume from now on that $0 < q < \frac{1}{2}$ such that all the charges are positive. Inserting into the general formula for the sphere partition function one gets

$$Z_{gauge} = m_1^2 + 4\sigma_1^2 \tag{B.3}$$

$$Z_{class} = e^{-4\pi i r \sigma_0 - i\theta m_0} \tag{B.4}$$

$$Z_{p^{1,\dots,4}} = \left[\frac{\Gamma\left(\frac{1}{2} - q + i\sigma_0 + \frac{m_0}{2}\right)}{\Gamma\left(\frac{1}{2} + q - i\sigma_0 + \frac{m_0}{2}\right)} \right]^4 \qquad Z_{p^{5,\dots,7}} = \left[\frac{\Gamma\left(1 - 2q + 2i\sigma_0 + m_0\right)}{\Gamma\left(2q - 2i\sigma_0 + m_0\right)} \right]^3 \tag{B.5}$$

$$Z_{x_{1,\dots,5}} = \left[\frac{\Gamma\left(q - i\sigma_0 - i\sigma_1 - \frac{(m_0+m_1)}{2}\right)\Gamma\left(q - i\sigma_0 + i\sigma_1 - \frac{(m_0-m_1)}{2}\right)}{\Gamma\left(1 - q + i\sigma_0 + i\sigma_1 - \frac{(m_0+m_1)}{2}\right)\Gamma\left(1 - q - i\sigma_0 - i\sigma_1 - \frac{(m_0-m_1)}{2}\right)} \right]^5 \tag{B.6}$$

Introducing the coordinates $\tau_0 = q - i\sigma_0$, $\tau_1 = -i\sigma_1$ the partition function becomes

$$
\begin{aligned}
Z_{S^2} \;=\; & \frac{1}{2} \sum_{m_0, m_1 = -\infty}^{\infty} \int_{q-i\infty}^{q+i\infty} \frac{d\tau_0}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d\tau_1}{2\pi i} \left(m_1^2 - 4\tau_1^2 \right) e^{-4\pi r q - i\theta m_0} e^{4\pi r \tau_0} \times \\
& \left[\frac{\Gamma\left(\frac{1}{2} - \tau_0 + \frac{m_0}{2}\right)}{\Gamma\left(\frac{1}{2} + \tau_0 + \frac{m_0}{2}\right)} \right]^4 \left[\frac{\Gamma\left(1 - 2\tau_0 + m_0\right)}{\Gamma\left(2\tau_0 + m_0\right)} \right]^3 \times \\
& \left[\frac{\Gamma\left(\tau_0 + \tau_1 - \frac{(m_0 + m_1)}{2}\right)}{\Gamma\left(1 - \tau_0 - \tau_1 - \frac{(m_0 + m_1)}{2}\right)} \frac{\Gamma\left(\tau_0 - \tau_1 - \frac{(m_0 - m_1)}{2}\right)}{\Gamma\left(1 - \tau_0 + \tau_1 - \frac{(m_0 - m_1)}{2}\right)} \right]^5 .
\end{aligned}
\tag{B.7}
$$

We aim to compute the sphere partition function in the $r \gg 0$ phase. For the result to be convergent, the τ_0 contour has to be closed in the left half-plane. For the τ_1-integration we can close the contour either to the left or to the right. The result will be independent of the choice since the integrand is symmetric under the exchange $\tau_1 \to -\tau_1, m_1 \to -m_1$. We start with the τ_1-integration and close the contour to the left. The possible poles come from the numerator of $Z_{x_1, \ldots, 5}$. The factor $\Gamma\left(\tau_0 + \tau_1 - \frac{(m_0 + m_1)}{2}\right)$ has a pole if

$$
\tau_1 = -k - \tau_0 + \frac{(m_0 + m_1)}{2} \qquad k \in \mathbb{Z} \geq 0.
\tag{B.8}
$$

The poles lie inside the contour if $m_0 + m_1 \leq 2k$ (τ_0 is at this point just an imaginary parameter, so it does not enter the condition). These poles may be canceled by poles in the denominator. The factor $\Gamma\left(1 - \tau_0 - \tau_1 - \frac{(m_0 + m_1)}{2}\right)$ has a pole for $\tau_1 = 1 + k_1 - \tau_0 - \frac{(m_0 + m_1)}{2}$ with $k_1 \geq 0$. The poles will cancel if

$$
-k + \frac{(m_0 + m_1)}{2} = 1 + k_1 - \frac{(m_0 + m_1)}{2} \quad \Rightarrow \quad k_1 = -k - 1 + (m_0 + m_1).
\tag{B.9}
$$

Since $k_1 \geq 0$, cancellation is not possible for $m_0 + m_1 \leq k$. This happens when the pole in the numerator is within the contour, and therefore there will a non-zero contribution to the residue integral if

$$
\tau_1 = -k - \tau_0 + \frac{(m_0 + m_1)}{2} \qquad k \geq 0 \qquad m_0 + m_1 \leq k.
\tag{B.10}
$$

The second quotient in $Z_{x_1, \ldots, 5}$ does not contribute. $\Gamma\left(\tau_0 - \tau_1 - \frac{(m_0 - m_1)}{2}\right)$ has a pole inside the contour if $\tau_1 = l + \tau_0 - \frac{(m_0 - m_1)}{2}$ with $l \geq 0$ and $m_0 - m_1 \geq 2l$. This is canceled against a pole of $\Gamma\left(1 - \tau_0 + \tau_1 - \frac{(m_0 - m_1)}{2}\right)$ in the denominator if $m_0 - m_1 \geq l + 1$. Thus, there are no cancellations if $m_0 - m_1 \leq 0$. This is always outside the contour, which shows that there is indeed no contribution. Therefore the expression for the partition function

becomes

$$
\begin{aligned}
Z_{S^2} &= \frac{1}{2} \sum_{k=0}^{\infty} \sum_{\substack{m_0, m_1 \\ m_0 + m_1 \le k}} \int_{q-i\infty}^{q+i\infty} \frac{d\tau_0}{2\pi i} \oint \frac{d\varepsilon_1}{2\pi i} \left[m_1^2 - 4\left(-k - \tau_0 + \frac{(m_0 + m_1)}{2} + \varepsilon_1 \right)^2 \right] \times \\
&\quad \times e^{-4\pi r q - i\theta m_0} e^{4\pi r \tau_0} \left[\frac{\Gamma\left(\frac{1}{2} - \tau_0 + \frac{m_0}{2}\right)}{\Gamma\left(\frac{1}{2} + \tau_0 + \frac{m_0}{2}\right)} \right]^4 \left[\frac{\Gamma\left(1 - 2\tau_0 + m_0\right)}{\Gamma\left(2\tau_0 + m_0\right)} \right]^3 \times \\
&\quad \times \left[\frac{\Gamma\left(-k + \varepsilon_1\right)}{\Gamma\left(1 + k - (m_0 + m_1) - \varepsilon_1\right)} \frac{\Gamma\left(2\tau_0 + k - m_0 - \varepsilon_1\right)}{\Gamma\left(1 - 2\tau_0 - k + m_1 + \varepsilon_1\right)} \right]^5
\end{aligned}
\tag{B.11}
$$

No poles of $Z_{p^{1,\dots,4}}$ and $Z_{p^{5,\dots,7}}$ contribute to the τ_0-integral. The only poles which are inside the contour are

$$
\tau_0 = \frac{1}{2}(-l - k + m_0 + \varepsilon_1) \qquad l \in \mathbb{Z}_{\ge 0}
\tag{B.12}
$$

These poles can be canceled by poles in the denominator. A calculation like the one for the τ_1-integration shows that the poles of the denominator of $Z_{x_{1,\dots,5}}$ cancel the poles in the numerator, unless $m_0 - m_1 \le l$. There are additional cancellations from the poles of the denominators of $Z_{p^{1,\dots,4}}$ and $Z_{p^{5,\dots,7}}$. In order to cancel the pole in the numerator both factors in the denominator have to have a pole. At first we show that the denominator of $Z_{p^{5,\dots,7}}$ always reduces the pole order of the numerator. It has a pole if $\tau_0 = -\frac{l_1}{2} - \frac{m_0}{2}$ with integer $l_1 \ge 0$. Therefore there is a cancellation of poles if

$$
l_1 = k + l - m_0.
\tag{B.13}
$$

Inside the contour the right-hand side is always a positive integer and the equality can always be satisfied. The denominator of $Z_{p^{1,\dots,4}}$ has a pole if $\tau_0 = -\frac{1}{2} - l_1 - \frac{m_0}{2}$ This cancels the pole in the numerator if

$$
l_1 = \frac{1}{2}(-1 + k + l) - m_0.
\tag{B.14}
$$

The condition implies that l_1 is only an integer if $k + l$ is odd. In this case the pole in the numerator is canceled. There is no cancellation only if $k + l$ is an even integer. Therefore we only have a contribution to the residue integral if

$$
\tau_0 = \frac{1}{2}(-l - k + m_0 + \varepsilon_1) \qquad l \ge 0, \quad m_0 - m_1 \le l, \quad k + l = even
\tag{B.15}
$$

Defining

$$
k' = k - (m_0 + m_1) \qquad l' = l - (m_0 - m_1),
\tag{B.16}
$$

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the partition function reduces to

$$
Z_{S^2} = \frac{1}{2} \sum_{\substack{k,k',l,l'=0 \\ k(k')+l(l')=even}}^{\infty} \oint \frac{d\varepsilon_1 d\varepsilon_2}{(2\pi i)^2} \left[-(k-l-\varepsilon_1+2\varepsilon_2)(k'-l'-\varepsilon_1+2\varepsilon_2) \right] e^{-4\pi rq} \times
$$

$$
\times e^{2\pi r \frac{1}{2}(-k-l-k'-l'+2\varepsilon_1+4\varepsilon_2)} e^{-i\theta \frac{1}{2}(k-k'+l-l')} \left[\frac{\Gamma\left(\frac{1}{2}+\frac{k}{2}+\frac{l}{2}-\frac{\varepsilon_1}{2}-\varepsilon_2\right)}{\Gamma\left(\frac{1}{2}-\frac{k'}{2}-\frac{l'}{2}+\frac{\varepsilon_1}{2}+\varepsilon_2\right)} \right]^4 \times
$$

$$
\times \left[\frac{\Gamma\left(1+k+l-\varepsilon_1-2\varepsilon_2\right)}{\Gamma\left(-k'-l'+\varepsilon_1+2\varepsilon_2\right)} \right]^3 \left[\frac{\Gamma\left(-k+\varepsilon_1\right)}{\Gamma\left(1+k'-\varepsilon_1\right)} \frac{\Gamma\left(-l+2\varepsilon_2\right)}{\Gamma\left(1+l'-2\varepsilon_1\right)} \right]^4. \qquad \text{(B.17)}
$$

Finally, we use the identity $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$ and define $z = e^{-2\pi r + i\theta}$. With the condition that $k+l$ and $k'+l'$ are even we can remove all the dependence on the summation variables from the sin-factors and the partition function factorizes nicely. The final result is

$$
Z_{S^2} = -\frac{(z\bar{z})^q}{2} \oint \frac{d\varepsilon_1 d\varepsilon_2}{(2\pi i)^2} (z\bar{z})^{-\varepsilon_1-2\varepsilon_2} \pi^3 \frac{\left[\cos \pi \left(\frac{\varepsilon_1}{2}+\varepsilon_2\right)\right]^4 \left[\sin \pi \left(\varepsilon_1+2\varepsilon_2\right)\right]^3}{\left[\sin \pi \varepsilon_1\right]^5 \left[\sin 2\pi \varepsilon_2\right]^5} \times
$$

$$
\times \left| \sum_{\substack{k,l=0 \\ k+l=even}}^{\infty} (k-l-\varepsilon_1+2\varepsilon_2) z^{\frac{k+l}{2}} \frac{\left[\Gamma\left(\frac{1}{2}+\frac{k}{2}+\frac{l}{2}-\frac{\varepsilon_1}{2}-\varepsilon_2\right)\right]^4 \left[\Gamma\left(1+k+l-\varepsilon_1-2\varepsilon_2\right)\right]^3}{\left[\Gamma\left(1+k-\varepsilon_1\right)\right]^5 \left[\Gamma\left(1+l-2\varepsilon_1\right)\right]^5} \right|^2.
$$

$$
\text{(B.18)}
$$

B.2 X_{10}

Up to a tiny subtlety the localization computation for this model is almost identical to the X_5-case. The gauge group is $U(1) \times SU(2)$ and the weights and the R-charges of the fields are

	$p^{1,\dots,6}$	p^7	$x_{1,\dots,4}$	x_5
$\chi(\sigma)$	$-\sigma_0$	$-2\sigma_0$	$\begin{pmatrix} \sigma_0+\sigma_1 \\ \sigma_0-\sigma_1 \end{pmatrix}$	$\begin{pmatrix} \sigma_1 \\ -\sigma_1 \end{pmatrix}$
$U(1)_V$	$1-2q$	$2-4q$	$2q$	1

(B.19)

($0 < q < \frac{1}{2}$ is assumed as usual.) The partition function is thus

$$
Z_{S^2} = \frac{1}{2} \sum_{m_0,m_1=-\infty}^{\infty} \int_{q-i\infty}^{q+i\infty} \frac{d\tau_0}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d\tau_1}{2\pi i} \left(m_1^2 - 4\tau_1^2\right) e^{-4\pi rq - i\theta m_0} e^{4\pi r \tau_0} Z_{p^{1,\dots,6}} Z_{p^7} Z_{x_{1,\dots,4}} Z_{x_5}
$$

$$
= \frac{1}{2} \sum_{m_0,m_1=-\infty}^{\infty} \int_{q-i\infty}^{q+i\infty} \frac{d\tau_0}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d\tau_1}{2\pi i} \left(m_1^2 - 4\tau_1^2\right) e^{-4\pi rq - i\theta m_0} e^{4\pi r \tau_0} \left[\frac{\Gamma\left(\frac{1}{2}-\tau_0+\frac{m_0}{2}\right)}{\Gamma\left(\frac{1}{2}+\tau_0+\frac{m_0}{2}\right)} \right]^6 \times
$$

$$\times \left[\frac{\Gamma \left(1 - 2\tau_0 + m_0 \right)}{\Gamma \left(2\tau_0 + m_0 \right)} \right] \left[\frac{\Gamma \left(\tau_0 + \tau_1 - \frac{(m_0+m_1)}{2} \right)}{\Gamma \left(1 - \tau_0 - \tau_1 - \frac{(m_0+m_1)}{2} \right)} \frac{\Gamma \left(\tau_0 - \tau_1 - \frac{(m_0-m_1)}{2} \right)}{\Gamma \left(1 - \tau_0 + \tau_1 - \frac{(m_0-m_1)}{2} \right)} \right]^4 \times$$

$$\times \frac{\Gamma \left(\frac{1}{2} + \tau_1 - \frac{m_1}{2} \right) \Gamma \left(\frac{1}{2} - \tau_1 + \frac{m_1}{2} \right)}{\Gamma \left(\frac{1}{2} - \tau_1 - \frac{m_1}{2} \right) \Gamma \left(\frac{1}{2} + \tau_1 + \frac{m_1}{2} \right)}. \tag{B.20}$$

Z_{x_5} does not contribute any poles. To see this, use the identity $\Gamma \left(\frac{1}{2} + x \right) \Gamma \left(\frac{1}{2} - x \right) = \frac{\pi}{\cos \pi x}$ to show that $Z_{x_5} = \frac{\cos \pi (y+m_1)}{\cos \pi y} = (-1)^{m_1}$ with $y = \tau_1 - \frac{m_1}{2}$. Z_{x_5} thus reduces to the contribution of a theta angle of π. From this point onward the calculation carries over line by line from the X_5-example. The result is

$$Z_{S^2} = -\frac{(z\bar{z})^q}{2} \oint \frac{d\varepsilon_1 d\varepsilon_2}{(2\pi i)^2} (z\bar{z})^{-\varepsilon_1 - 2\varepsilon_2} \pi^3 \frac{\left[\cos \pi \left(\frac{\varepsilon_1}{2} + \varepsilon_2 \right) \right]^6 \left[\sin \pi \left(\varepsilon_1 + 2\varepsilon_2 \right) \right]}{\left[\sin \pi \varepsilon_1 \right]^4 \left[\sin 2\pi \varepsilon_2 \right]^4} \times$$

$$\times \left| \sum_{\substack{k,l=0 \\ k+l=even}}^{\infty} (k - l - \varepsilon_1 + 2\varepsilon_2)(-z)^{\frac{k+l}{2}} \frac{\left[\Gamma \left(\frac{1}{2} + \frac{k}{2} + \frac{l}{2} - \frac{\varepsilon_1}{2} - \varepsilon_2 \right) \right]^6 \left[\Gamma \left(1 + k + l - \varepsilon_1 - 2\varepsilon_2 \right) \right]}{\left[\Gamma \left(1 + k - \varepsilon_1 \right) \right]^4 \left[\Gamma \left(1 + l - 2\varepsilon_1 \right) \right]^4} \right|^2$$

$$\tag{B.21}$$

The sphere partition function for the $r \ll 0$-phase of X_{10} is

$$Z_{S^2}^{r \ll 0} = \frac{1}{2} \lim_{\delta \to 0} \oint \frac{d\varepsilon_1 d\varepsilon_2}{(2\pi i)^2} (z\bar{z})^{q - \frac{1}{2} - \varepsilon_2} \pi^7 \frac{\left[\sin \pi (\varepsilon_1 + 2\varepsilon_2) \right]^4}{\left[\sin \pi \varepsilon_2 \right]^6 \left[\sin 2\pi \varepsilon_2 \right] \left[\sin \pi \varepsilon_1 \right]^4} \times$$

$$\times \left| \sum_{k,l=0}^{\infty} (-e^{-\delta})^k (-z)^{-l} (1 + 2k + 2l + 2\varepsilon_1 + 2\varepsilon_2) \times \right.$$

$$\times \left. \frac{\left[\Gamma (1 + k + 2l + \varepsilon_1 + 2\varepsilon_2) \right]^4}{\left[\Gamma (1 + l + \varepsilon_2) \right]^6 \left[\Gamma (1 + 2l + 2\varepsilon_2) \right] \left[\Gamma (1 + k + \varepsilon_1) \right]^4} \right|^2, \tag{B.22}$$

B.3 X_{13}

While the localization procedure for X_5 and X_{10} is very similar to other examples in the literature, X_{13} is more challenging at the technical level. The gauge group for this model is $G = U(2)$ and the weights and R-charges of the matter fields are

	$p^{1,...,7}$	x_1	$x_{2,...,5}$
$\chi(\sigma)$	$-(\sigma_1 + \sigma_2)$	$\begin{pmatrix} 2\sigma_1 + \sigma_2 \\ \sigma_1 + 2\sigma_2 \end{pmatrix}$	$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$
$U(1)_V$	$-4q$	$1 + 6q$	$1 + 2q$

$$\tag{B.23}$$

$(-\frac{1}{6} < q < 0$ is assumed.) Defining $\tau_{1,2} = q - i\sigma_{1,2}$ the partition function can be written as

$$
Z_{S^2} = \frac{1}{2} \sum_{m_1, m_2 = -\infty}^{\infty} \int_{q-i\infty}^{q+i\infty} \frac{d\tau_1 d\tau_2}{(2\pi i)^2} \left[\frac{(m_1 - m_2)^2}{4} - (\tau_1 - \tau_2)^2 \right] \times
$$
$$
\times e^{4\pi r(\tau_1 + \tau_2)} e^{-8\pi r q} e^{-i\theta(m_1 + m_2)} e^{-i\pi(m_1 - m_2)} \times
$$
$$
\times Z_{p^1,\dots,7} Z_{x_1} Z_{x_2,\dots,5}
$$
(B.24)

where

$$
Z_{p^1,\dots,7} = \left[\frac{\Gamma\left(-\tau_1 - \tau_2 + \frac{m_1 + m_2}{2}\right)}{\Gamma\left(1 + \tau_1 + \tau_2 + \frac{m_1 + m_2}{2}\right)} \right]^7
$$
(B.25)

$$
Z_{x_1} = \left[\frac{\Gamma\left(\frac{1}{2} + 2\tau_1 + \tau_2 - \frac{2m_1 + m_2}{2}\right) \Gamma\left(\frac{1}{2} + \tau_1 + 2\tau_2 - \frac{m_1 + 2m_2}{2}\right)}{\Gamma\left(\frac{1}{2} - 2\tau_1 - \tau_2 - \frac{2m_1 + m_2}{2}\right) \Gamma\left(\frac{1}{2} - \tau_1 - 2\tau_2 - \frac{m_1 + 2m_2}{2}\right)} \right]
$$
(B.26)

$$
Z_{x_2,\dots,5} = \left[\frac{\Gamma\left(\frac{1}{2} + \tau_1 - \frac{m_1}{2}\right) \Gamma\left(\frac{1}{2} + \tau_2 - \frac{m_2}{2}\right)}{\Gamma\left(\frac{1}{2} - \tau_1 - \frac{m_1}{2}\right) \Gamma\left(\frac{1}{2} - \tau_2 - \frac{m_2}{2}\right)} \right]^4 .
$$
(B.27)

The following transformation tremendously simplifies the calculation:

$$
a = \tau_1 + \tau_2 \qquad b = \tau_1 - \tau_2.
$$
(B.28)

The partition function in the new coordinates is

$$
Z_{S^2} = \frac{1}{4} \sum_{m_1, m_2 = -\infty}^{\infty} \int_{2q-i\infty}^{2q+i\infty} \frac{da}{2\pi i} \int_{-i\infty}^{i\infty} \frac{db}{2\pi i} \left[\frac{(m_1 - m_2)^2}{4} - b^2 \right] \times
$$
$$
\times e^{4\pi r a} e^{-8\pi q r} e^{-i\theta(m_1 + m_2)} e^{-i\pi(m_1 - m_2)} \times
$$
$$
\times \left[\frac{\Gamma\left(-a + \frac{m_1 + m_2}{2}\right)}{\Gamma\left(1 + a + \frac{m_1 + m_2}{2}\right)} \right]^7 \left[\frac{\Gamma\left(\frac{1}{2} + \frac{3a}{2} + \frac{b}{2} - \frac{2m_1 + m_2}{2}\right) \Gamma\left(\frac{1}{2} + \frac{3a}{2} - \frac{b}{2} - \frac{m_1 + 2m_2}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{3a}{2} - \frac{b}{2} - \frac{2m_1 + m_2}{2}\right) \Gamma\left(\frac{1}{2} - \frac{3a}{2} + \frac{b}{2} - \frac{m_1 + 2m_2}{2}\right)} \right] \times
$$
$$
\times \left[\frac{\Gamma\left(\frac{1}{2} + \frac{a}{2} + \frac{b}{2} - \frac{m_1}{2}\right) \Gamma\left(\frac{1}{2} + \frac{a}{2} - \frac{b}{2} - \frac{m_2}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{a}{2} - \frac{b}{2} - \frac{m_1}{2}\right) \Gamma\left(\frac{1}{2} - \frac{a}{2} + \frac{b}{2} - \frac{m_2}{2}\right)} \right]^4
$$
(B.29)

The structure of the transformed integral is that of a partition function for a model with gauge group $G = U(1) \times SU(2)$. It is strongly advised to do the the b-integration first. We choose to choose the contour on the left half-plane. An analysis of the pole structure shows that there are two types of poles, stemming from the first Γ-factors in Z_{x_1} and $Z_{x_2,\dots,5}$, respectively. They are

$$
\text{pole 1:} \qquad b = -2k - 1 - 3a + (2m_1 + m_2) \qquad 2m_1 + m_2 \le k \qquad \text{(B.30)}
$$
$$
\text{pole 2:} \qquad b = -2k - 1 - a + m_1 \qquad m_1 \le k. \qquad \text{(B.31)}
$$

Since the m_i-dependence is different the poles cannot be dealt with at once and the partition function splits into contributions from the two types of poles: $Z_{S^2} = Z_{S^2}^{(1)} + Z_{S^2}^{(2)}$, where the first term is

$$
Z_{S^2}^{(1)} = \frac{1}{4} \sum_{k=0}^{\infty} \sum_{\substack{m_1,m_2 \\ 2m_1+m_2 \leq k}} \int \frac{da}{2\pi i} \oint \frac{d\varepsilon_1}{2\pi i} \left[\frac{(m_1-m_2)^2}{4} - (-1 - 3a - 2k + 2m_1 + m_2 + \varepsilon_1)^2 \right] \times
$$

$$
\times e^{4\pi r a} e^{-8\pi r q} e^{-i\theta(m_1+m_2)} e^{-i\pi(m_1-m_2)} \left[\frac{\Gamma\left(-a + \frac{m_1+m_2}{2}\right)}{\Gamma\left(1 + a + \frac{m_1+m_2}{2}\right)} \right]^7 \times
$$

$$
\times \left[\frac{\Gamma\left(-k + \frac{\varepsilon_1}{2}\right)}{\Gamma\left(1 + k - 2m_1 - m_2 - \frac{\varepsilon_1}{2}\right)} \frac{\Gamma\left(1 + 3a + k - \frac{3(m_1+m_2)}{2} - \frac{\varepsilon_1}{2}\right)}{\Gamma\left(-3a - k + \frac{m_1+m_2}{2} + \frac{\varepsilon_1}{2}\right)} \right]
$$

$$
\times \left[\frac{\Gamma\left(-a - k + \frac{m_1+m_2}{2} + \frac{\varepsilon_1}{2}\right)}{\Gamma\left(1 + a + k - \frac{3m_1}{2} - \frac{m_2}{2} - \frac{\varepsilon_1}{2}\right)} \frac{\Gamma\left(1 + 2a + k - m_1 - m_2 - \frac{\varepsilon_1}{2}\right)}{\Gamma\left(-2a - k + m_1 + \frac{\varepsilon_1}{2}\right)} \right]^4 \qquad \text{(B.32)}
$$

The second term is

$$
Z_{S^2}^{(2)} = \frac{1}{4} \sum_{k=0}^{\infty} \sum_{\substack{m_1,m_2 \\ m_1 \leq k}} \int \frac{da}{2\pi i} \oint \frac{d\varepsilon_1}{2\pi i} \left[\frac{(m_1-m_2)^2}{4} - (-1 - a - 2k + m_1 + \varepsilon_1)^2 \right] \times
$$

$$
\times e^{4\pi r a} e^{-8\pi r q} e^{-i\theta(m_1+m_2)} e^{-i\pi(m_1-m_2)} \left[\frac{\Gamma\left(-a + \frac{m_1+m_2}{2}\right)}{\Gamma\left(1 + a + \frac{m_1+m_2}{2}\right)} \right]^7 \times
$$

$$
\times \left[\frac{\Gamma\left(a - k - \frac{m_1+m_2}{2} + \frac{\varepsilon_1}{2}\right)}{\Gamma\left(1 - a + k - \frac{3m_1}{2} - \frac{m_2}{2} - \frac{\varepsilon_1}{2}\right)} \frac{\Gamma\left(1 + 2a + k - m_1 - m_2 - \frac{\varepsilon_1}{2}\right)}{\Gamma\left(-2a - k - m_2 + \frac{\varepsilon_1}{2}\right)} \right] \times
$$

$$
\times \left[\frac{\Gamma\left(-k + \frac{\varepsilon_1}{2}\right)}{\Gamma\left(1 + k - m_1 - \frac{\varepsilon_1}{2}\right)} \frac{\Gamma\left(1 + a + k - \frac{(m_1+m_2)}{2} - \frac{\varepsilon_1}{2}\right)}{\Gamma\left(-a - k + \frac{m_1-m_2}{2} + \frac{\varepsilon_1}{2}\right)} \right]^4 \qquad \text{(B.33)}
$$

Next, we take care of the a-integration. All the possible poles except for those of $Z_{p^1,\dots,7}$ contribute. Therefore each of the two expressions will split into three pieces. Let us start with $Z_{S^2}^{(1)} = Z_{S^2}^{(1a)} + Z_{S^2}^{(1b)} + Z_{S^2}^{(1c)}$. The three poles are

pole 1a: $\quad a = \frac{1}{3}\left(-1 - k - l + \frac{3(m_1+m_2)}{2} + \frac{\varepsilon_1}{2}\right) \quad m_1 + 2m_2 \leq l, \mathrm{mod}(-2 + k + l, 3) \neq 0$

pole 1b: $\quad a = l - k + \frac{m_1+m_2}{2} + \frac{\varepsilon_1}{2} \quad m_1 \leq l, l \leq k, \dots$

pole 1c: $\quad a = \frac{1}{2}\left(-1 - l - k + m_1 + m_2 + \frac{\varepsilon_1}{2}\right) \quad m_2 \leq l, k + l = even \qquad \text{(B.34)}$

For poles 1a and 1c the restrictions in k, l are due to extra cancellations of poles coming from the denominator of $Z_{p^1,\dots,7}$. For pole 1b the condition that it is enclosed by the

contour results in an extra condition $k \leq l$. There may be further conditions which we do not check because the contribution will cancels completely. Inserting we get for the first term

$$
\begin{aligned}
Z_{S^2}^{(1a)} \;=\; & \frac{1}{4} \sum_{\substack{k,l,k',l'=0 \\ \mathrm{mod}(-2+k(k')+l(l'),3)\neq 0}}^{\infty} \oint \frac{\mathrm{d}\varepsilon_1 \mathrm{d}\varepsilon_2}{(2\pi i)^2} \left[-\frac{1}{4}\left(2(k-l)-\varepsilon_1+6\varepsilon_2\right)\left(2(k'-l')-\varepsilon_1+6\varepsilon_2\right) \right] \times \\
& \times e^{-8\pi q r} e^{4\pi r \frac{1}{6}(-2-k-k'-l-l'+\varepsilon_1+6\varepsilon_2)} e^{-i\theta \frac{1}{3}(k-k'+l-l')} e^{-i\pi(k-k'-l+l')} \times \\
& \times \left[\frac{\Gamma\left(\frac{1}{3}+\frac{k}{3}+\frac{l}{3}-\frac{\varepsilon_1}{6}-\varepsilon_2\right)}{\Gamma\left(\frac{2}{3}-\frac{k'}{3}-\frac{l'}{3}+\frac{\varepsilon_1}{6}+\varepsilon_2\right)} \right]^7 \left[\frac{\Gamma\left(-k+\frac{\varepsilon_1}{2}\right)}{\Gamma\left(1+k'-\frac{\varepsilon_1}{2}\right)} \frac{\Gamma\left(-l+3\varepsilon_2\right)}{\Gamma\left(1+l'-3\varepsilon_2\right)} \right] \times \\
& \times \left[\frac{\Gamma\left(\frac{1}{3}-\frac{2k}{3}+\frac{l}{3}+\frac{\varepsilon_1}{3}-\varepsilon_2\right)}{\Gamma\left(\frac{2}{3}+\frac{2k'}{3}-\frac{l'}{3}-\frac{\varepsilon_1}{3}+\varepsilon_2\right)} \frac{\Gamma\left(\frac{1}{3}+\frac{k}{3}-\frac{2l}{3}-\frac{\varepsilon_1}{6}+2\varepsilon_2\right)}{\Gamma\left(\frac{2}{3}-\frac{k'}{3}+\frac{2l'}{3}+\frac{\varepsilon_1}{6}-2\varepsilon_2\right)} \right]^4 ,
\end{aligned} \tag{B.35}
$$

where we defined $k' = k - 2m_1 - m_2$ and $l' = l - m_1 - 2m_2$. For the second term one gets

$$
\begin{aligned}
Z_{S^2}^{(1b)} \;=\; & \frac{1}{4} \sum_{\substack{k,l,k',l'=0 \\ l \leq k}}^{\infty} \oint \frac{\mathrm{d}\varepsilon_1 \mathrm{d}\varepsilon_2}{(2\pi i)^2} \left[-\frac{1}{4}\left(2-2k+6l+\varepsilon_1+6\varepsilon_2\right)\left(2-2k'+6l'+\varepsilon_1+6\varepsilon_2\right) \right] \times \\
& \times e^{-8\pi q r} e^{4\pi r \frac{1}{2}(-k-k'+l+l'+\varepsilon_1+2\varepsilon_2)} e^{-i\theta(k-k'-l+l')} e^{-i\pi(-k+k'+3l-3l')} \times \\
& \times \left[\frac{\Gamma\left(k-l-\frac{\varepsilon_1}{2}-\varepsilon_2\right)}{\Gamma\left(1-k'+l'+\frac{\varepsilon_1}{2}+\varepsilon_2\right)} \right]^7 \left[\frac{\Gamma\left(-k+\frac{\varepsilon_1}{2}\right)}{\Gamma\left(1+k'-\frac{\varepsilon_1}{2}\right)} \frac{\Gamma\left(1-2k+3l+\varepsilon_1+3\varepsilon_2\right)}{\Gamma\left(2k'-3l'-\varepsilon_1-3\varepsilon_2\right)} \right] \times \\
& \times \left[\frac{\Gamma\left(-l-\varepsilon_2\right)}{\Gamma\left(1+l'+\varepsilon_2\right)} \frac{\Gamma\left(1-k+2l+\frac{\varepsilon_1}{2}+2\varepsilon_2\right)}{\Gamma\left(k'-2l'-\frac{\varepsilon_1}{2}-2\varepsilon_2\right)} \right]^4
\end{aligned} \tag{B.36}
$$

with $k' = k - 2m_1 - m_2$, $l' = l - m_1$ and finally

$$
\begin{aligned}
Z_{S^2}^{(1c)} \;=\; & \frac{1}{4} \sum_{\substack{k,l,k',l'=0 \\ k(k')+l(l')=even}}^{\infty} \oint \frac{\mathrm{d}\varepsilon_1 \mathrm{d}\varepsilon_2}{(2\pi i)^2} \left[-\frac{1}{16}\left(2-2k+6l+\varepsilon_1-12\varepsilon_2\right)\left(2-2k'+6l'+\varepsilon_1-12\varepsilon_2\right) \right] \times \\
& \times e^{-8\pi q r} e^{4\pi r \frac{1}{4}(-2-k-k'-l-l'+\varepsilon_1+4\varepsilon_2)} e^{-i\theta \frac{1}{2}(k-k'+l-l')} e^{-i\pi \frac{1}{2}(k-k'-l+l')} \times \\
& \times \left[\frac{\Gamma\left(\frac{1}{2}+\frac{k}{2}+\frac{l}{2}-\frac{\varepsilon_1}{4}-\varepsilon_2\right)}{\Gamma\left(\frac{1}{2}-\frac{k'}{2}-\frac{l'}{2}+\frac{\varepsilon_1}{4}+\varepsilon_2\right)} \right]^7 \left[\frac{\Gamma\left(-k+\frac{\varepsilon_1}{2}\right)}{\Gamma\left(1+k'-\frac{\varepsilon_1}{2}\right)} \frac{\Gamma\left(-\frac{1}{2}-\frac{k}{2}-\frac{3l}{2}+\frac{\varepsilon_1}{4}+3\varepsilon_2\right)}{\Gamma\left(\frac{3}{2}+\frac{k'}{2}+\frac{3l'}{2}-\frac{\varepsilon_1}{4}-3\varepsilon_2\right)} \right] \times \\
& \times \left[\frac{\Gamma\left(\frac{1}{2}-\frac{k}{2}+\frac{l}{2}+\frac{\varepsilon_1}{4}-\varepsilon_2\right)}{\Gamma\left(\frac{1}{2}+\frac{k'}{2}-\frac{l'}{2}-\frac{\varepsilon_1}{4}+\varepsilon_2\right)} \frac{\Gamma\left(-l+2\varepsilon_2\right)}{\Gamma\left(1+l'-2\varepsilon_2\right)} \right]^4 ,
\end{aligned} \tag{B.37}
$$

where $k' = k - 2m_1 - m_2$ and $l' = l - m_2$. The evaluation of $Z_{S^2}^{(2)}$ proceeds in exactly the same way. The two poles are

$$
\text{pole 2a}: \quad a = -l + k + \frac{m_1 + m_2}{2} - \frac{\varepsilon_1}{2} \qquad 2m_1 + m_2 \leq l, k \leq l, \ldots
$$

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$$\text{pole 2b:} \qquad a = \frac{1}{2}\left(-1 - l - k + m_1 + m_2 + \frac{\varepsilon_1}{2}\right) \qquad m_1 + 2m_2 \leq l, k + l = even$$

$$(\text{B.38})$$

The numerator has a additional pole at $a = -1 - l - k + \frac{m_1 + m_2}{2} + \frac{\varepsilon_1}{2}$ coming from $Z_{x_2,\ldots,5}$, which is however always canceled by poles in the denominator. From the first pole we get

$$
Z_{S^2}^{(2a)} = \frac{1}{4}\sum_{\substack{k,l,k',l'=0 \\ k \leq l}}^{\infty}\oint\frac{d\varepsilon_1 d\varepsilon_2}{(2\pi i)^2}\left[-\frac{1}{4}\left(2 + 6k - 2l - 3\varepsilon_1 + 2\varepsilon_2\right)\left(2 + 6k' - 2l' - 3\varepsilon_1 + 2\varepsilon_2\right)\right] \times
$$
$$
\times e^{-8qr}e^{4\pi r\frac{1}{2}(k+k'-l-l'-\varepsilon_1+2\varepsilon_2)}e^{-i\theta(-k+k'+l-l')}e^{-i\pi(k-k'-l+l')} \times
$$
$$
\times \left[\frac{\Gamma\left(-k+l+\frac{\varepsilon_1}{2}-\varepsilon_2\right)}{\Gamma\left(1+k'-l'+\frac{\varepsilon_1}{2}+\varepsilon_2\right)}\right]^7\left[\frac{\Gamma\left(-l+\varepsilon_2\right)}{\Gamma\left(1+l'-\varepsilon_2\right)}\frac{\Gamma\left(1+3k-2l-\frac{3\varepsilon_1}{2}+2\varepsilon_2\right)}{\Gamma\left(-3k'+2l'+\frac{3\varepsilon_1}{2}-2\varepsilon_2\right)}\right] \times
$$
$$
\times \left[\frac{\Gamma\left(-k-\frac{\varepsilon_1}{2}\right)}{\Gamma\left(1+k'+\frac{\varepsilon_1}{2}\right)}\frac{\Gamma\left(1+2k-l-\varepsilon_1+\varepsilon_2\right)}{\Gamma\left(-2k'+l'+\varepsilon_1-\varepsilon_2\right)}\right]^4, \qquad\qquad (\text{B.39})
$$

with $k' = k - m_1$, $l' = l - 2m_1 - m_2$. The second pole gives

$$
Z_{S^2}^{(2b)} = \frac{1}{4}\sum_{\substack{k,l,k',l'=0 \\ k(k')+l(l')=even}}^{\infty}\oint\frac{d\varepsilon_1 d\varepsilon_2}{(2\pi i)^2}\left[-\frac{1}{16}\left(2 + 6k - 2l - 3\varepsilon_1 + 4\varepsilon_2\right)\left(2 + 6k' - 2l' - 3\varepsilon_1 + 4\varepsilon_2\right)\right] \times
$$
$$
\times e^{-8qr}e^{4\pi r\frac{1}{4}(-2-k-k'-l-l'+\frac{\varepsilon_1}{4}+\varepsilon_2)}e^{-i\theta\frac{1}{2}(k-k'+l-l')}e^{-i\pi(-k+k'+3l-3l')} \times
$$
$$
\times \left[\frac{\Gamma\left(\frac{1}{2}+\frac{k}{2}+\frac{l}{2}-\frac{\varepsilon_1}{4}-\varepsilon_2\right)}{\Gamma\left(\frac{1}{2}-\frac{k'}{2}-\frac{l'}{2}+\frac{\varepsilon_1}{4}+\varepsilon_2\right)}\right]^7\left[\frac{\Gamma\left(-l+2\varepsilon_2\right)}{\Gamma\left(1+l'-2\varepsilon_2\right)}\frac{\Gamma\left(-\frac{1}{2}-\frac{3k}{2}-\frac{l}{2}+\frac{3\varepsilon_1}{4}+\varepsilon_2\right)}{\Gamma\left(\frac{3}{2}+\frac{3k'}{2}+\frac{l'}{2}-\frac{3\varepsilon_1}{4}-\varepsilon_2\right)}\right] \times
$$
$$
\times \left[\frac{\Gamma\left(\frac{1}{2}+\frac{k}{2}-\frac{l}{2}-\frac{\varepsilon_1}{4}+\varepsilon_2\right)}{\Gamma\left(\frac{1}{2}-\frac{k'}{2}+\frac{l'}{2}+\frac{\varepsilon_1}{4}-\varepsilon_2\right)}\frac{\Gamma\left(-k+\frac{\varepsilon_1}{2}\right)}{\Gamma\left(1+k'-\frac{\varepsilon_1}{4}\right)}\right]^4, \qquad\qquad (\text{B.40})
$$

with $k' = k - m_1$, $l' = l - m_1 - 2m_2$. Looking at the five terms, there are transformations such that

$$k \leftrightarrow l, \; \varepsilon_1 \to 2\varepsilon_2, \; \varepsilon_2 \to -\frac{\varepsilon_1}{2}: \qquad Z_{S^2}^{(1b)} \to -Z_{S^2}^{(2a)} \qquad (\text{B.41})$$

$$k \leftrightarrow l, \; \varepsilon_1 \to 4\varepsilon_2, \; \varepsilon_2 \to \frac{\varepsilon_1}{4}: \qquad Z_{S^2}^{(1c)} \to Z_{S^2}^{(2b)}. \qquad (\text{B.42})$$

Therefore the complete sphere partition function simplifies to

$$Z_{S^2} = Z_{S^2}^{(1a)} + 2Z_{S^2}^{(1c)}. \qquad (\text{B.43})$$

With the same tools as for the previous examples and the definition $z = -e^{-2\pi r + i\theta}$ the three terms can be further manipulated to give the following.

$$
Z_{S^2}^{(1a)} = -\frac{(z\bar{z})^{2q}}{16}\oint\frac{d\varepsilon_1 d\varepsilon_2}{(2\pi i)^2}\sum_{\substack{k,k',l,l'=0 \\ \text{mod}(-2+k(k')+l(l'),3)\neq 0}}^{\infty}\pi^3(z\bar{z})^{\frac{1}{3}-\frac{\varepsilon_1}{6}-\varepsilon_2}(-1)^{k+l}(-1)^{k-k'-l+l'} \times
$$

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$$\times \frac{\left[\sin \pi \left(\frac{1}{3}(2 - k' - l') + \frac{\varepsilon_1}{6} + \varepsilon_2\right)\right]^7}{\sin \pi \frac{\varepsilon_1}{2} \sin 3\pi \varepsilon_2 \left[\sin \pi \left(\frac{2}{3}(2 - k - l) + \frac{\varepsilon_1}{3} - \varepsilon_2\right)\right]^4 \left[\sin \pi \left(\frac{2}{3}(2 - k - l) + \frac{\varepsilon_1}{6} - 2\varepsilon_2\right)\right]^4} \times$$

$$\times \left\{ z^{\frac{k+l}{3}} (2(k - l) - \varepsilon_1 + 6\varepsilon_2) \frac{\left[\Gamma\left(\frac{1}{3}(1 + k + l) - \frac{\varepsilon_1}{6} - \varepsilon_2\right)\right]^7}{\Gamma\left(1 + k - \frac{\varepsilon_1}{2}\right)\Gamma(1 + l - 3\varepsilon_2)} \times \right.$$

$$\left. \times \frac{1}{\left[\Gamma\left(\frac{1}{3}(2 + 2k - l) - \frac{\varepsilon_1}{3} + \varepsilon_2\right)\right]^4 \left[\Gamma\left(\frac{1}{3}(2 - k + 2l) + \frac{\varepsilon_1}{6} - 2\varepsilon_2\right)\right]^4} \right\} \{k', l', \bar{z}\} \qquad \text{(B.44)}$$

The expression $\{k', l', \bar{z}\}$ means to multiply with the same factor and the indicated replacements. Note that $Z_{S^2}^{(1a)}$ does not factorize, which is unexpected. Computing the residue gives a non-zero expression which does not factorize either. However, when one explicitly evaluates the sums one actually finds that $Z_{S^2}^{(1a)} = 0$! To see that $Z_{S^2}^{(1a)} = 0$ we note that after evaluating the residue the expression has the following form:

$$Z_{S^2}^{(1a)} = \frac{2\pi}{3}(z\bar{z})^{\frac{1}{3}+2q} \sum_{\substack{k,k',l,l'=0 \\ \mathrm{mod}(-2+k(k')+l(l'),3)\neq 0}}^{\infty} \frac{\left[\sin \pi \left(\frac{1}{3}(-2 + k' + l')\right)\right]^7}{\left[\sin \pi \left(\frac{2}{3}(-2 + k + l)\right)\right]^8}(-1)^{k+l}(-1)^{-k'+l'}\pi \times$$

$$\times \left\{ (k - l)z^{\frac{k+l}{3}} \frac{\left[\Gamma\left(\frac{1}{3}(1 + k + l)\right)\right]^7}{\left[\Gamma\left(\frac{1}{3}(2 - k + 2l)\right)\right]^4 \left[\Gamma\left(\frac{1}{3}(2 + 2k - l)\right)\right]^4 \Gamma(1 + k)\Gamma(1 + l)} \right\} \{k', l', \bar{z}\}$$

$$\text{(B.45)}$$

This immediately shows that for fixed (k, l) every summand with a particular (k', l') cancels against the summand where $k' \leftrightarrow l'$, which is the same up the an overall sign. Therefore the sum is 0, even though the individual summands are not.

The only thing that is left is $Z_{S^2}^{(1c)}$ which indeed factorizes:

$$Z_{S^2}^{(1c)} = \frac{(z\bar{z})^q}{64} \oint \frac{d\varepsilon_1 d\varepsilon_2}{(2\pi i)^2} \pi^3 (z\bar{z})^{\frac{1}{2} - \frac{\varepsilon_1}{4} - \varepsilon_2} \frac{\left[\cos \pi \left(\frac{\varepsilon_1}{4} + \varepsilon_2\right)\right]^7}{\sin \pi \frac{\varepsilon_1}{2} \cos \pi \left(\frac{\varepsilon_1}{4} + 3\varepsilon_2\right) \left[\cos \pi \left(\frac{\varepsilon_1}{4} - \varepsilon_2\right)\right]^4 \left[\sin 2\pi \varepsilon_2\right]^4} \times$$

$$\times \left| \sum_{\substack{k,l=0 \\ k+l=even}}^{\infty} (2 - 2k + 6l + \varepsilon_1 - 12\varepsilon_2)(-z)^{\frac{k+l}{2}} \times \right.$$

$$\left. \times \frac{\left[\Gamma\left(\frac{1}{2}(1 + k + l) - \frac{\varepsilon_1}{4} - \varepsilon_2\right)\right]^7}{\Gamma\left(1 + k - \frac{\varepsilon_1}{2}\right)\Gamma\left(\frac{3}{2} + \frac{k}{2} + \frac{3l}{2} - \frac{\varepsilon_1}{4} - 3\varepsilon_2\right)\left[\Gamma\left(\frac{1}{2} + \frac{k}{2} - \frac{l}{2} - \frac{\varepsilon_1}{4} + \varepsilon_2\right)\right]^4 \left[\Gamma(1 + l - 2\varepsilon_2)\right]^4} \right|^2$$

$$\text{(B.46)}$$

B.4 X_7

The field content of this model only differs very slightly from X_{13}, yet it exhibits completely different behavior in the $r \gg 0$-region. The gauge group is $U(2)$ and the

weights and R-charges of the matter fields are

$$
\begin{array}{c|cccc}
 & p^{1,\ldots,5} & p^{6,7} & x_{1,2} & x_{3,\ldots,5} \\
\hline
\chi(\sigma) & -\sigma_1 - \sigma_2 & -2\sigma_1 - 2\sigma_2 & \begin{pmatrix} 2\sigma_1 + \sigma_2 \\ \sigma_1 + 2\sigma_2 \end{pmatrix} & \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \\
U(1)_V & -4q & -8q & 1 + 6q & 1 + 2q
\end{array}
\qquad (B.47)
$$

$(-\frac{1}{6} < q < 0$ is assumed.) In terms of the variables $\tau_{1,2} = q - \sigma_{1,2}$ the sphere partition function looks like the one of X_{13} except for $Z_{p^{6,7}}$ and different multiplicities.

$$
\begin{aligned}
Z_{S^2} =\ & \frac{1}{2} \sum_{m_1,m_2=-\infty}^{\infty} \int_{q-i\infty}^{q+i\infty} \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{(2\pi i)^2} \left[\frac{(m_1 - m_2)^2}{4} - (\tau_1 - \tau_2)^2 \right] \times \\
& \times e^{4\pi r(\tau_1 + \tau_2)} e^{-8\pi r q} e^{-i\theta(m_1 + m_2)} e^{-i\pi(m_1 - m_2)} \times \\
& \times Z_{p^{1,\ldots,5}} Z_{p^{6,7}} Z_{x_{1,2}} Z_{x_{3,\ldots,5}}
\end{aligned}
\qquad (B.48)
$$

where

$$
Z_{p^{1,\ldots,5}} = \left[\frac{\Gamma\left(-\tau_1 - \tau_2 + \frac{m_1 + m_2}{2}\right)}{\Gamma\left(1 + \tau_1 + \tau_2 + \frac{m_1 + m_2}{2}\right)} \right]^5
\qquad (B.49)
$$

$$
Z_{p^{6,7}} = \left[\frac{\Gamma\left(-2\tau_1 - 2\tau_2 + m_1 + m_2\right)}{\Gamma\left(1 + 2\tau_1 + 2\tau_2 + m_1 + m_2\right)} \right]^2
\qquad (B.50)
$$

$$
Z_{x_{1,2}} = \left[\frac{\Gamma\left(\frac{1}{2} + 2\tau_1 + \tau_2 - \frac{2m_1 + m_2}{2}\right) \Gamma\left(\frac{1}{2} + \tau_1 + 2\tau_2 - \frac{m_1 + 2m_2}{2}\right)}{\Gamma\left(\frac{1}{2} - 2\tau_1 - \tau_2 - \frac{2m_1 + m_2}{2}\right) \Gamma\left(\frac{1}{2} - \tau_1 - 2\tau_2 - \frac{m_1 + 2m_2}{2}\right)} \right]^2
\qquad (B.51)
$$

$$
Z_{x_{3,\ldots,5}} = \left[\frac{\Gamma\left(\frac{1}{2} + \tau_1 - \frac{m_1}{2}\right) \Gamma\left(\frac{1}{2} + \tau_2 - \frac{m_2}{2}\right)}{\Gamma\left(\frac{1}{2} - \tau_1 - \frac{m_1}{2}\right) \Gamma\left(\frac{1}{2} - \tau_2 - \frac{m_2}{2}\right)} \right]^3 .
\qquad (B.52)
$$

Since we are interested in the $r \gg 0$ region the poles of $Z_{p^{1,\ldots,5}}$ and $Z_{p^{6,7}}$ will not contribute to the residue integrals. The only difference to X_{13} is that for the calculation of $Z_{S^2}^{(1a)}$ there is an additional condition $\mathrm{mod}(-1 + 2k + 2l, 3) \neq 0$ due to extra pole cancellations coming from the denominator of $Z_{p^{6,7}}$. The partition function is $Z_{S^2} = Z_{S^2}^{(1a)} + 2 Z_{S^2}^{(1c)}$ with

$$
\begin{aligned}
Z_{S^2}^{(1a)} =\ & -\frac{(z\bar{z})^{2q}}{16} \oint \frac{\mathrm{d}\varepsilon_1 \mathrm{d}\varepsilon_2}{(2\pi i)^2} \sum_{\substack{k,k',l,l'=0 \\ \mathrm{cond.}}}^{\infty} \pi^3 (z\bar{z})^{\frac{1}{3} - \frac{\varepsilon_1}{6} - \varepsilon_2} (-1)^{k+l} (-1)^{k-k'-l+l'} \times \\
& \times \frac{\left[\sin\pi\left(\frac{1}{3}(2 - k' - l') + \frac{\varepsilon_1}{6} + \varepsilon_2\right)\right]^5 \left[\sin\pi\left(\frac{1}{3}(1 - 2k' - 2l') + \frac{\varepsilon_1}{3} + 2\varepsilon_2\right)\right]^2}{\left[\sin\pi\frac{\varepsilon_1}{2}\right]^2 \left[\sin 3\pi\varepsilon_2\right]^2 \left[\sin\pi\left(\frac{2}{3}(2 - k - l) + \frac{\varepsilon_1}{3} - \varepsilon_2\right)\right]^3 \left[\sin\pi\left(\frac{2}{3}(2 - k - l) + \frac{\varepsilon_1}{6} - 2\varepsilon_2\right)\right]^3} \times \\
& \times \left\{ z^{\frac{k+l}{3}} (2(k - l) - \varepsilon_1 + 6\varepsilon_2) \frac{\left[\Gamma\left(\frac{1}{3}(1 + k + l) - \frac{\varepsilon_1}{6} - \varepsilon_2\right)\right]^5 \left[\Gamma\left(\frac{2}{3}(1 + k + l) + \frac{\varepsilon_1}{3} + 2\varepsilon_2\right)\right]^2}{\left[\Gamma\left(1 + k - \frac{\varepsilon_1}{2}\right)\right]^2 \left[\Gamma\left(1 + l - 3\varepsilon_2\right)\right]^2} \times \right. \\
& \left. \times \frac{1}{\left[\Gamma\left(\frac{1}{3}(2 + 2k - l) - \frac{\varepsilon_1}{3} + \varepsilon_2\right)\right]^3 \left[\Gamma\left(\frac{1}{3}(2 - k + 2l) + \frac{\varepsilon_1}{6} - 2\varepsilon_2\right)\right]^3} \right\} \{k', l', \bar{z}\},
\end{aligned}
\qquad (B.53)
$$

where *cond.* stands for $\mod(-2+k+l, 3) \neq 0$ and $\mod(-1+2k+2l, 3) \neq 0$. Furthermore

$$
\begin{aligned}
Z_{S^2}^{(1c)} &= -\frac{(z\bar{z})^q}{64} \oint \frac{d\varepsilon_1 d\varepsilon_2}{(2\pi i)^2} \pi^3 (z\bar{z})^{\frac{1}{2}-\frac{\varepsilon_1}{4}-\varepsilon_2} \times \\
&\quad \times \frac{\left[\cos\pi\left(\frac{\varepsilon_1}{4}+\varepsilon_2\right)\right]^5 \left[\sin\pi\left(\frac{\varepsilon_1}{2}+2\varepsilon_2\right)\right]^2}{\left[\sin\pi\frac{\varepsilon_1}{2}\right]^2 \left[\cos\pi\left(\frac{\varepsilon_1}{4}+3\varepsilon_2\right)\right]^2 \left[\cos\pi\left(\frac{\varepsilon_1}{4}-\varepsilon_2\right)\right]^3 \left[\sin 2\pi\varepsilon_2\right]^3} \times \\
&\quad \times \left| \sum_{\substack{k,l=0 \\ k+l=even}}^{\infty} (2-2k+6l+\varepsilon_1-12\varepsilon_2)(-z)^{\frac{k+l}{2}} \times \right. \\
&\quad \times \frac{\left[\Gamma\left(\frac{1}{2}(1+k+l)-\frac{\varepsilon_1}{4}-\varepsilon_2\right)\right]^5}{\left[\Gamma\left(1+k-\frac{\varepsilon_1}{2}\right)\right]^2 \left[\Gamma\left(\frac{3}{2}+\frac{k}{2}+\frac{3l}{2}-\frac{\varepsilon_1}{4}-3\varepsilon_2\right)\right]^2} \times \\
&\quad \left. \times \frac{\left[\Gamma\left(1+k+l-\frac{\varepsilon_1}{2}+2\varepsilon_2\right)\right]^2}{\left[\Gamma\left(\frac{1}{2}+\frac{k}{2}-\frac{l}{2}-\frac{\varepsilon_1}{4}+\varepsilon_2\right)\right]^3 \left[\Gamma(1+l-2\varepsilon_2)\right]^3} \right|^2
\end{aligned}
\tag{B.54}
$$

The leading term in the $z \to 0$ limit comes from $Z_{S^2}^{(1a)}$. However, since it is

$$
Z_{S^2}^{(1a)} = \frac{2}{\sqrt{3}\pi} \frac{\Gamma\left(\frac{1}{3}\right)^{10}}{\Gamma\left(\frac{2}{3}\right)^8} (z\bar{z})^{2q+\frac{1}{3}}, \qquad -\frac{1}{6} < q < 0
\tag{B.55}
$$

the contribution to $g_{z\bar{z}}$ is zero. Therefore we have to take into account the next to leading order contributed by $Z_{S^2}^{(1c)}$. With that we get,

$$
e^{-K} = \frac{2}{\sqrt{3}\pi} \frac{\Gamma\left(\frac{1}{3}\right)^{10}}{\Gamma\left(\frac{2}{3}\right)^8} (z\bar{z})^{2q+\frac{1}{3}} - 4(z\bar{z})^{2q+\frac{1}{2}} (36 + 8\log 4 - 3\log z\bar{z}) + \dots
\tag{B.56}
$$

The leading term of the Kähler metric is

$$
g_{z\bar{z}} = -\frac{\pi\Gamma\left(\frac{2}{3}\right)^8 \log^3\left(\frac{z\bar{z}}{2^{\frac{16}{3}}}\right)}{6\sqrt{3}\Gamma\left(\frac{1}{3}\right)^{10} (z\bar{z})^{\frac{5}{6}}}.
\tag{B.57}
$$

B.5 $(\mathbf{S}^{2,+}_{(-1)^2,(-2)^3,1^4})$

The gauge group is $U(1) \times O(2)$ and the weights and the R-charges of the matter fields are

$$
\begin{array}{c|ccc}
 & p^{1,2} & p^{3,\dots,5} & x_{1,\dots,4} \\
\hline
\chi(\sigma) & -\sigma_0 & -2\sigma_0 & \begin{pmatrix} \sigma_0 + \sigma_1 \\ \sigma_0 - \sigma_1 \end{pmatrix} \\
U(1)_V & 1-2q & 2-4q & 2q
\end{array}.
\tag{B.58}
$$

Since $O(2)$ has no root, we have
$$Z_{gauge} = 1. \tag{B.59}$$

To compute Z_{class} we note that only the $U(1)$-factor contributes an FI-parameter r. The one from $O(2)$ is zero. However, since $N - k = 4 - 2 = 2$ is even, we need the discrete θ-angle of π for the regularity. Therefore, the one gets
$$Z_{class} = e^{-4\pi r\sigma_0 - i\theta m_0 - i\pi m_1}. \tag{B.60}$$

Defining $\tau_0 = q - i\sigma_0$ and $\tau_1 = -i\sigma_1$, the contribution from the chiral fields is
$$Z_{p^{1,2}} = \left[\frac{\Gamma\left(\frac{1}{2} - \tau_0 + \frac{m_0}{2}\right)}{\Gamma\left(\frac{1}{2} + \tau_0 + \frac{m_0}{2}\right)}\right]^2 \qquad Z_{p^{3,\ldots,5}} = \left[\frac{\Gamma\left(1 - 2\tau_0 + m_0\right)}{\Gamma\left(2\tau_0 + m_0\right)}\right]^3, \tag{B.61}$$

$$Z_{x_{1,\ldots,4}} = \left[\frac{\Gamma\left(\tau_0 + \tau_1 - \frac{(m_0 + m_1)}{2}\right)}{\Gamma\left(1 - \tau_0 - \tau_1 - \frac{(m_0 + m_1)}{2}\right)} \frac{\Gamma\left(\tau_0 - \tau_1 - \frac{m_0 - m_1}{2}\right)}{\Gamma\left(1 - \tau_0 + \tau_1 - \frac{m_0 - m_1}{2}\right)}\right]^4. \tag{B.62}$$

Therefore the sphere partition function is
$$Z_{S^2} = \sum_{m_0, m_1 = -\infty}^{\infty} \int_{q-i\infty}^{q+i\infty} \int_{-i\infty}^{i\infty} \frac{d\tau_0 d\tau_1}{(2\pi i)^2} e^{-4\pi rq - i\theta m_0} (-1)^{m_1} e^{4\pi r\tau_0} Z_{p^{1,2}} Z_{p^{3,\ldots,5}} Z_{x_{1,\ldots,4}} \tag{B.63}$$

The calculation proceeds exactly like in the previous examples. It is recommended to do the τ_1 integration first. While the closing of the contour can be chosen freely, we always choose to close it in the same direction as for τ_0. Since the $r \ll 0$-phase is also new in this case, we start by computing the partition function for this case. For the τ_1-integration where we close the contour on the right half, the only contribution comes from the pole at
$$\tau_1 = \tau_0 + k - \frac{(m_0 - m_1)}{2} \qquad m_0 - m_1 \leq k \tag{B.64}$$

For the τ_0-integration, there are two possible poles coming from the singularities of Z_p:
$$\tau_0 = \frac{1}{2} + l + \frac{m_0}{2}$$
$$\tau_0 = \frac{1}{2} + \frac{l}{2} + \frac{m_0}{2} \tag{B.65}$$

Both types of poles get canceled by the pole $\tau_0 = \frac{1}{2}(1 + l_1 - k - m_1)$ in the denominator of $Z_{x_{1,\ldots,4}}$. However, this is only a fourth order pole, wheres the combined pole order of Z_p if five. So there will always be something left. We only have to take into account the first type of pole in (B.65) because for the second type terms with odd l have no poles

in $Z_{p^{1,2}}$ and l thus has to be even, whereupon the second type of pole becomes the first. After the usual manipulations, one gets

$$Z_{S^2}^{r\ll0} = \frac{(z\bar{z})^{q-\frac{1}{2}}}{2}\lim_{\delta\to0}\oint\frac{d\varepsilon_1 d\varepsilon_2}{(2\pi i)^2}(z\bar{z})^{q-\frac{1}{2}-\varepsilon_2}\pi^5\frac{[\sin\pi(\varepsilon_1-2\varepsilon_2)]^4}{[\sin\pi(\varepsilon_2)]^2[\sin 2\pi\varepsilon_2]^3[\sin\pi\varepsilon_1]^4}\times$$
$$\times\left|\sum_{k,l=0}^{\infty}(-e^{-\delta})^k(-z)^{-l}\frac{[\Gamma(1+k+2l+\varepsilon_1-2\varepsilon_2)]^4}{[\Gamma(1+l-\varepsilon_2)]^2[\Gamma(1+2l-2\varepsilon_2)]^3[\Gamma(1+k+\varepsilon_1)]^4}\right|^2,$$
(B.66)

For the $r\gg0$-phase we close the contours of both τ_0 and τ_1 in the negative half plane. The calculation is identical to the $U(1)\times SU(2)$-examples and the result is

$$Z_{S^2}^{r\gg0} = \frac{(z\bar{z})^q}{2}\oint\frac{d\varepsilon_1 d\varepsilon_2}{(2\pi i)^2}(z\bar{z})^{-\varepsilon_1-2\varepsilon_2}\pi^3\frac{\left[\cos\pi\left(\frac{\varepsilon_1}{2}+\varepsilon_2\right)\right]^2[\sin\pi(\varepsilon_1+2\varepsilon_2)]^3}{[\sin\pi\varepsilon_1]^4[\sin 2\pi\varepsilon_2]^4}\times$$
$$\times\left|\sum_{\substack{k,l=0\\k+l=even}}^{\infty}(-z)^{\frac{k+l}{2}}\frac{\left[\Gamma\left(\frac{1}{2}+\frac{k}{2}+\frac{l}{2}-\frac{\varepsilon_1}{2}-\varepsilon_2\right)\right]^2[\Gamma(1+k+l-\varepsilon_1-2\varepsilon_2)]^3}{[\Gamma(1+k-\varepsilon_1)]^4[\Gamma(1+l-2\varepsilon_1)]^4}\right|^2$$
(B.67)

C   Elliptic genus of $(\mathbf{S}_{(-1)^2,(-2)^3,1^4}^{2,+})$

We compute the $q\to0$ limit of the elliptic genus of the model $(\mathbf{S}_{(-1)^2,(-2)^3,1^4}^{2,+})$ applying the recently developed technique from [11, 12]. Recall that the model has gauge group $U(1)\times O(2)$. It is initially expressed as an integration over the moduli space of flat $U(1)\times O(2)$ connections on the torus, which can further be simplified into a contour integral of a meromorphic form on the moduli space along a certain middle dimensional cycle in the complement of the poles.

The moduli space of flat $U(1)\times O(2)$ connections on the torus is the direct product $\mathfrak{M}^{U(1)}\times\mathfrak{M}^{O(2)}$ of the moduli spaces for the groups $U(1)$ and $O(2)$. The first factor is parametrized by the $U(1)$ holonomies along the two directions. With a natural complex structure, it can be identified as the torus itself, $\mathfrak{M}^{U(1)}\cong\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, i.e. $u_0\equiv u_0+1\equiv u_0+\tau$ where τ is the complex structure of the torus. The second factor $\mathfrak{M}^{O(2)}$ consists of seven components:[1] one is the space of $u_1\in\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ modulo $u_1\equiv-u_1$ and the others are six points represented by the commuting pairs of holonomies

---

[1]Here we regard $O(2)$ as the semi-direct product $U(1)\rtimes\mathbb{Z}_2$, where the generator $\gamma\in\mathbb{Z}_2$ acts on $U(1)$ as the inversion.

$((1, \gamma), (\pm 1, 1))$, $((\pm 1, 1), (1, \gamma))$, $((\pm 1, \gamma), (1, \gamma))$. Accordingly, the moduli space of flat $U(1) \times O(2)$ connections consists of seven components — one two-dimensional component $\mathfrak{M}_{\mathrm{cont}} \cong (\mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}))^2$ and six one-dimensional components $\mathfrak{M}_a \cong \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$, where a parametrizes the above six discrete $O(2)$ holonomy pairs.

Under $O(2) \cong U(1) \rtimes \mathbb{Z}_2$, an $O(2)$ doublet corresponds to a multiplet of two fields ϕ_1, ϕ_2 of $U(1)$ charge $1, -1$ which are exchanged by the element $(1, \gamma) \in U(1) \rtimes \mathbb{Z}_2$. For the six discrete $O(2)$ holonomies, it is convenient to take the combinations $\phi_\pm = \phi_1 \pm \phi_2$. Both of them are odd under the element $(-1, 1)$ while ϕ_+ (*resp.* ϕ_-) is even (*resp.* odd) under $(1, \gamma)$. The $U(1)$ gauge multiplet is even under $(-1, 1)$ and odd under $(1, \gamma)$.

The elliptic genus for the theory with gauge group $U(1) \times O_\epsilon(2)$ with the discrete theta angle θ_D ($\epsilon = \pm$, $\theta_D = 0, \pi$) is given by

$$Z_{T^2}^{\epsilon, \theta_D}(\tau, z) = \frac{1}{2} \sum_{u_* \in \mathfrak{M}_{\mathrm{cont}}^*} \oint_{u_*} du_1 \, du_0 \, Z_{\mathrm{cont}}(\tau, u_0, u_1)$$

$$+ \frac{1}{4} \sum_a \epsilon_a \, e^{i\theta_a} \sum_{u_{0*} \in \mathfrak{M}_a^*} \oint_{u_{0*}} du_0 \, Z_{\mathrm{disc}}(\tau, u_0, a). \tag{C.1}$$

$\mathfrak{M}_{\mathrm{cont}}^*$ is a part of the intersection points of the singular hyperplanes where the matter scalars have zero modes, selected by a choice of a generic weight vector $\eta = (\eta^0, \eta^1)$. For η near the positive η^0 axis, the intersection points of the singular hyperplanes, $u_0 + u_1 \equiv 0$ and $u_0 - u_1 \equiv 0$, for the $O(2)$ doublets are selected,

$$\mathfrak{M}_{\mathrm{cont}}^* = \left\{ (0, 0), \, (\tfrac{1}{2}, \tfrac{1}{2}), \, (\tfrac{\tau}{2}, \tfrac{\tau}{2}), \, (\tfrac{1+\tau}{2}, \tfrac{1+\tau}{2}) \right\}. \tag{C.2}$$

These are all non-degenerate and the contour around each $u_* \in \mathfrak{M}_{\mathrm{cont}}^*$ is uniquely specified. $\mathfrak{M}_a^*$ is the set of u_0's where the $O(2)$ doublets have zero modes,

$$\mathfrak{M}_a^* = \{-a_+, -a_-\}. \tag{C.3}$$

See below for the definition of $a_\pm$. The integrands are given by the one-loop determinants

$$Z_{\mathrm{cont}}(\tau, u_0, u_1) = \left(\frac{-i\eta(\tau)^3}{\vartheta(-z)}\right)^2 \left(\frac{\vartheta(-\frac{z}{2} - u_0)}{\vartheta(\frac{z}{2} - u_0)}\right)^2 \left(\frac{\vartheta(-2u_0)}{\vartheta(z - 2u_0)}\right)^3 \times$$

$$\left(\frac{\vartheta(-z + u_0 + u_1)}{\vartheta(u_0 + u_1)}\right)^4 \left(\frac{\vartheta(-z + u_0 - u_1)}{\vartheta(u_0 - u_1)}\right)^4, \tag{C.4}$$

$$Z_{\mathrm{disc}}(\tau, u_0, a) = \frac{-i\eta(\tau)^3}{\vartheta(-z)} \frac{\vartheta(a_{\mathrm{v}})}{\vartheta(-z + a_{\mathrm{v}})} \left(\frac{\vartheta(-\frac{z}{2} - u_0)}{\vartheta(\frac{z}{2} - u_0)}\right)^2 \left(\frac{\vartheta(-2u_0)}{\vartheta(z - 2u_0)}\right)^3 \times$$

$$\left(\frac{\vartheta(-z + u_0 + a_+)}{\vartheta(u_0 + a_+)}\right)^4 \left(\frac{\vartheta(-z + u_0 + a_-)}{\vartheta(u_0 + a_-)}\right)^4, \tag{C.5}$$

where
$$\vartheta(z) := \theta_1(\tau|z) = -iq^{1/8}y^{1/2}\prod_{k=1}^{\infty}(1-q^k)(1-yq^k)(1-y^{-1}q^{k-1}) \tag{C.6}$$

with $q = e^{2\pi i\tau}$ and $y = e^{2\pi iz}$. Finally, the data ϵ_a, θ_a, a_{v} and $a_{\pm}$ are given by

a	$((1,\gamma),(1,1))$	$((1,\gamma),(-1,1))$	$((1,1),(1,\gamma))$	$((-1,1),(1,\gamma))$	$((1,\gamma),(1,\gamma))$	$((-1,\gamma),(1,\gamma))$
$(a_{\mathrm{v}}, a_+, a_-)$	$(\frac{1}{2},0,\frac{1}{2})$	$(\frac{1}{2},-\frac{\tau}{2},\frac{1+\tau}{2})$	$(\frac{\tau}{2},0,\frac{\tau}{2})$	$(\frac{\tau}{2},-\frac{1}{2},\frac{1+\tau}{2})$	$(\frac{1+\tau}{2},0,\frac{1+\tau}{2})$	$(\frac{1+\tau}{2},-\frac{1}{2},\frac{\tau}{2})$
(ϵ_a, θ_a)	$(1,1)$	$(1,\theta_D)$	$(\epsilon,1)$	(ϵ,θ_D)	$(1,1)$	$(1,\theta_D)$

$$\tag{C.7}$$

In the model $(\mathrm{S}^{2,+}_{(-1)^2,(-2)^3,1^4})$ we take $\epsilon = +$ and $\theta_D = \pi$.

We evaluate the integrals in the $q \to 0$ limit. The contributions from the four points of $\mathfrak{M}^*_{\mathrm{cont}}$ are

$$
\begin{aligned}
(0,0): \quad & -\frac{1}{4}(1+y^{\frac{1}{2}})^2 y^{-\frac{3}{2}}(1+7y^{\frac{1}{2}}+28y+7y^{\frac{3}{2}}+y^2)+O(q)\\[4pt]
(\tfrac{1}{2},\tfrac{1}{2}): \quad & -\frac{1}{4}(1-y^{\frac{1}{2}})^2 y^{-\frac{3}{2}}(1-7y^{\frac{1}{2}}+28y-7y^{\frac{3}{2}}+y^2)+O(q)\\[4pt]
(\tfrac{\tau}{2},\tfrac{\tau}{2}): \quad & \frac{1}{4}(y-1)^2 y^{-\frac{3}{2}}(1+y)+O(q)\\[4pt]
(\tfrac{\tau+1}{2},\tfrac{\tau+1}{2}): \quad & \frac{1}{4}(y-1)^2 y^{-\frac{3}{2}}(1+y)+O(q),
\end{aligned}
$$

which sum up to

$$-22(y^{-\frac{1}{2}}+y^{\frac{1}{2}})+O(q). \tag{C.8}$$

The contributions from the six components $\mathfrak{M}_a$ sum up to zero for the choice $\theta_D = \pi$: The contributions of the first and second a's, the third and the fourth a's, and the fifth and the sixth a's cancel against each other because of $e^{i\theta_D} = -1$. Therefore, (C.8) is the total.

D Topology of $\widetilde{Y}_S$

We now explain how to obtain the topological data (4.12) for the Calabi-Yau manifold $\widetilde{Y}_S$ that appears in the strongly coupled phase of the model $(\mathrm{S}^{2,+}_{(-1)^2,(-2)^3,1^4})$.[1] Recall that $\widetilde{Y}_S$ is a double cover of the symmetric determinantal variety $Y_S = \{\mathrm{rank}S(p) \leq 3\}$ inside the weighted projective space $\mathbb{P}^4_{11222}$. It is a simultaneous unfolding of the A_1 singularity of Y_S along two disjoint curves, C_S and Σ_S. C_S is the locus where rank $S(p) = 2$ and Σ_S is the intersection of Y_S and the orbifold locus $\mathfrak{S}$ of $\mathbb{P}^4_{11222}$.

---

[1] We would like to thank Alexander Kuznetsov and Hiromichi Takagi from whom we learned most of what is described here. We also thank Alexey Bondal for much help.

First, let us put $(y_0, \ldots, y_5) = ((p^1)^2, p^1 p^2, (p^2)^2, p^3, p^4, p^5)$. Then, the matrix $S(p)$ can be written as a linear expression $S(y)$. We also find that $\mathbb{P}^4_{11222}$ can be regarded as the degree 2 hypersurface $y_0 y_2 = y_1^2$ of $\mathbb{P}^5$. Therefore, Y_S can be regarded as the intersection of the degree 2 and degree 4 hypersurfaces, $y_0 y_2 = y_1^2$ and $\det S(y) = 0$. The two hypersurfaces have singularities which are resolved respectively by inserting $(z_1, z_2) \in \mathbb{P}^1$ and $(x_1, \ldots, x_4) \in \mathbb{P}^3$ obeying

$$y_0 z_1 = y_1 z_2, \quad y_1 z_1 = y_2 z_2, \tag{D.1}$$

$$\sum_{j=1}^{4} S(y)^{ij} x_j = 0, \quad i = 1, 2, 3, 4. \tag{D.2}$$

Therefore, the singularity of $Y_S \subset \mathbb{P}^5$ is resolved by $Z_S \subset \mathbb{P}^5 \times \mathbb{P}^1 \times \mathbb{P}^3$ defined by (D.1) and (D.2): The curves Σ_S and C_S of A_1 singularity in Y_S become divisors $\Delta_S = \{(y, z, x(y))\}$ and $D_S = \{(y, z(y), x)\}$ in Z_S, which are $\mathbb{P}^1$ fibrations over the curves. We introduce $\widetilde{Z}_S$ as the fiber product:

$$
\begin{array}{ccccc}
\widetilde{Z}_S & \longrightarrow & Z_S & \subset & \mathbb{P}^5 \times \mathbb{P}^1 \times \mathbb{P}^3 \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{Y}_S & \longrightarrow & Y_S & \subset & \mathbb{P}^5
\end{array}
\tag{D.3}
$$

The arrow $\widetilde{Z}_S \to \widetilde{Y}_S$ is the blow up of $\widetilde{Y}_S$ at C_S and Σ_S. (We denote the preimagaes of C_S and Σ_S under $\widetilde{Y}_S \to Y_S$ by the same symbols; we shall do the same for D_S and Δ_S.) Therefore, the Hodge numbers of $\widetilde{Z}_S$ and $\widetilde{Y}_S$ are related by

$$h^{p,q}(\widetilde{Z}_S) = h^{p,q}(\widetilde{Y}_S) + h^{p-1,q-1}(C_S) + h^{p-1,q-1}(\Sigma_S). \tag{D.4}$$

Since the map $\widetilde{Z}_S \to Z_S$ is one to one on $D_S \cup \Delta_S$ and two to one on the complement, we have the relation between the Euler numbers,

$$
\begin{aligned}
\chi(\widetilde{Z}_S) &= \chi(\widetilde{Z}_S \setminus (D_S \cup \Delta_S)) + \chi(D_S \cup \Delta_S) \\
&= 2\chi(Z_S \setminus (D_S \cup \Delta_S)) + \chi(D_S \cup \Delta_S) \\
&= 2\chi(Z_S) - \chi(D_S \cup \Delta_S).
\end{aligned}
\tag{D.5}
$$

Let h_1, h_2 and h_3 be the hyperplane classes of $\mathbb{P}^5$, $\mathbb{P}^1$ and $\mathbb{P}^3$ respectively. The classes of the divisors Δ_S and D_S are

$$[\Delta_S] = (h_1 - 2h_2)|_{Z_S}, \qquad [D_S] = (3h_1 - 2h_3)|_{Z_S}. \tag{D.6}$$

To show the first, we consider the sequence of embeddings $\Delta \subset \widehat{\mathbb{P}}^4_{11222} \subset \mathbb{P}^5 \times \mathbb{P}^1$ where $\widehat{\mathbb{P}}^4_{11222}$ (which we write $\widehat{\mathbb{P}}^4$ for simplicity) is the resolution (D.1) of $\mathbb{P}^4_{11222}$ and Δ is the

preimage $y_0 = y_1 = y_2$ of the singularity $\mathfrak{S}$. This induces an exact sequence of the normal bundles

$$0 \to N_{\Delta,\widehat{\mathbb{P}}^4} \longrightarrow N_{\Delta,\mathbb{P}^5 \times \mathbb{P}^1} \longrightarrow N_{\widehat{\mathbb{P}}^4,\mathbb{P}^5 \times \mathbb{P}^1} \to 0. \tag{D.7}$$

Since Δ and $\widehat{\mathbb{P}}^4$ are zeroes of the vector bundles $\mathcal{O}(1,0)^{\oplus 3}$ and $\mathcal{O}(1,1)^{\oplus 2}$ over $\mathbb{P}^5 \times \mathbb{P}^1$, the normal bundles on the middle and on the right have first Chen-classes $3h_1$ and $2(h_1 + h_2)$ respectively. Therefore, we find $c_1(N_{\Delta,\widehat{\mathbb{P}}^4}) = 3h_1 - 2(h_1 + h_2) = h_1 - 2h_2$. Including the $\mathbb{P}^3$ factor and restricting to Z_S we find the first equality. To show the second equality, we consider the following diagrams of sheaves over $\mathbb{P}^5 \times \mathbb{P}^1 \times \mathbb{P}^3$,

$$\begin{array}{ccccccc}
0 & \to & \mathcal{O}(-h_3) & \longrightarrow & V \otimes \mathcal{O} & \longrightarrow & E & \to & 0 \\
& & & & \downarrow S(y) & & & & \\
0 & \to & E^*(h_1) & \longrightarrow & V^* \otimes \mathcal{O}(h_1) & \longrightarrow & \mathcal{O}(h_1 + h_3) & \to & 0
\end{array} \tag{D.8}$$

where $V \cong \mathbb{C}^4$. The upper line is the tautological exact sequence on $\mathbb{P}^3$ and the lower line is its dual tensored with $\mathcal{O}(h_1)$. Note that $\det E = \mathcal{O}(h_3)$. Z_S is the locus where the map from $\mathcal{O}(-h_3)$ to $V^* \otimes \mathcal{O}(h_1)$ vanishes or, by the symmetry of $S(y)$, the locus where the map from $V \otimes \mathcal{O}$ to $\mathcal{O}(h_1 + h_3)$ vanishes. On this locus, there is a map from E to $E^*(h_1)$ and D_S is where this degenerates. That is, it is the zero of a section of $\det E^*(h_1) \otimes \det^{-1} E = \mathcal{O}(3h_1) \otimes \det^{-2} E = \mathcal{O}(3h_1 - 2h_3)$. This proves the second equality.

Since $Z_S \subset \mathbb{P}^5 \times \mathbb{P}^1 \times \mathbb{P}^3$ is the zero of a section of $F = \mathcal{O}(h_1 + h_2)^{\oplus 2} \oplus \mathcal{O}(h_1 + h_3)^{\oplus 4}$, it is a Calabi-Yau manifold and its Euler number is

$$\begin{aligned}
\chi(Z_S) &= \int_{\mathbb{P}^5 \times \mathbb{P}^1 \times \mathbb{P}^3} c_{\text{top}}(F) \cdot \frac{c(\mathbb{P}^5 \times \mathbb{P}^1 \times \mathbb{P}^3)}{c(F)} \\
&= \int_{\mathbb{P}^5 \times \mathbb{P}^1 \times \mathbb{P}^3} [(h_1 + h_2)^2 (h_1 + h_3)^4] \cdot \frac{(1 + h_1)^6 (1 + h_2)^2 (1 + h_3)^4}{(1 + h_1 + h_2)^2 (1 + h_1 + h_3)^4} = -88.
\end{aligned} \tag{D.9}$$

The equations (D.6) show that D_S and Δ_S, as submanifolds of $\mathbb{P}^5 \times \mathbb{P}^1 \times \mathbb{P}^3$, are zeroes of sections of $F \oplus \mathcal{O}(h_1 - 2h_2)$ and $F \oplus \mathcal{O}(3h_1 - 2h_3)$. By the similar computation as above, we find

$$\chi(D_S) = -80, \qquad \chi(\Delta_S) = -8. \tag{D.10}$$

Since D_S and Δ_S are $\mathbb{P}^1$ bundles over C_S and Σ_S, this means

$$\chi(C_S) = -40, \qquad \chi(\Sigma_S) = -4. \tag{D.11}$$

Therefore, C_S and Σ_S have genus 21 and 3 respectively, meaning that $h^{1,0}(C_S) = 21$ and $h^{1,0}(\Sigma_S) = 3$.

By (D.5), we find

$$\chi(\widetilde{Z}_S) = -2 \times 88 - (-80 - 8) = -88, \tag{D.12}$$

and by (D.4), we find

$$h^{2,1}(\widetilde{Z}_S) = h^{2,1}(\widetilde{Y}_S) + 21 + 3, \qquad h^{1,1}(\widetilde{Z}_S) = h^{1,1}(\widetilde{Y}_S) + 2. \tag{D.13}$$

We also know from a separate account that $h^{1,0}(\widetilde{Z}_S) = h^{2,0}(\widetilde{Z}_S) = 0$ and $h^{3,0}(\widetilde{Z}_S) = 1$ which implies via (D.4) that $h^{1,0}(\widetilde{Y}_S) = h^{2,0}(\widetilde{Y}_S) = 0$ and $h^{3,0}(\widetilde{Y}_S) = 1$. Therefore, we find

$$h^{2,1}(\widetilde{Z}_S) - h^{1,1}(\widetilde{Z}_S) = 44, \tag{D.14}$$

and

$$h^{2,1}(\widetilde{Y}_S) - h^{1,1}(\widetilde{Y}_S) = 44 - 22 = 22. \tag{D.15}$$

Thus, if we can show either $h^{1,1}(\widetilde{Y}_S) = 1$ or $h^{2,1}(\widetilde{Y}_S) = 23$, we are done about the Hodge numbers of $\widetilde{Y}_S$. We can show the latter by following the argument used in [17] for the similar purpose. It uses the fact from the deformation theory [40] that, by $h^{2,0}(\widetilde{Y}_S) = 0$, any deformation of $\widetilde{Y}_S$ comes from a deformation of Y_S. It is also known that the number of deformations of Y_S is equal to the naïve count (4.5), i.e., 23, if the Picard number of Y_S is one. This shows that $h^{2,1}(\widetilde{Y}_S) \leq 23$. By (D.15) and $h^{1,1}(\widetilde{Y}_S) \geq 1$, we find $h^{1,1}(\widetilde{Y}_S) = 1$ and $h^{2,1}(\widetilde{Y}_S) = 23$.

Finally, we would like to compute the intersection numbers. Let M be the pull back of $M_{Y_S} = \mathcal{O}_{\mathbb{P}^5}(1)|_{Y_S}$ to $\widetilde{Y}_S$. We have

$$M^3 = 2M_{Y_S}^3 = 2 \int_{\mathbb{P}^5} (2h_1) \cdot (4h_1) \cdot h_1^3 = 16. \tag{D.16}$$

Since M is ample, by Kodaira's vanishing theorem, we have $\chi(\mathcal{O}_{\widetilde{Y}_S}(M)) = h^0(\mathcal{O}_{\widetilde{Y}_S}(M))$. One can also show that this is equal to $h^0(\mathcal{O}_{Y_S}(M_{Y_S})) = h^0(\mathcal{O}_{\mathbb{P}^5}(1)) = 6$. Using Riemann-Roch formula $\chi(\mathcal{O}_{\widetilde{Y}_S}(M)) = \frac{1}{6}M^3 + \frac{1}{12}c_2(\widetilde{Y}_S) \cdot M$ and (D.16), we find

$$c_2(\widetilde{Y}_S) \cdot M = 40. \tag{D.17}$$

Let H_{Y_S} be the Weil divisor of Y_S defined by $p^1 = 0$. Then, there is a relation $M_{Y_S} = 2H_{Y_S}$ in Y_S which pulls back to the relation

$$M = 2H \quad \text{in} \quad \widetilde{Y}_S, \tag{D.18}$$

where now H is a Cartier divisor of $\widetilde{Y}_S$ which is smooth. This shows what we wanted: $H^3 = 2$ and $c_1(\widetilde{Y}_S) \cdot H = 20$.

E   Mirror symmetry checks

We have used the sphere partition function to check whether the hybrid point in the $r \gg 0$ phase is at finite distance in the moduli space. In all the examples we have discussed in detail the Picard-Fuchs operator is known. In [41] (see also [42]) it was shown how to extract the symplectic pairing of the periods from the Picard-Fuchs, or rather, the associated Gauss-Manin system. We will demonstrate this procedure for X_5. It works the same way for the other examples, except for X_7. The reason is that this approach requires the knowledge of the flat coordinate in the given phase. Due to its exceptional behavior it is not even clear what the flat coordinate is in the pseudo-hybrid phase.

In order to compute the metric $g_{z\bar{z}} = \partial_z \partial_{\bar{z}} K(z, \bar{z})$ we have to compute the Kähler potential. On the mirror it is given by $e^{-K} = \Pi^\dagger Q \Pi$, where Q is the symplectic form and Π is the period vector. We need to determine Q for a given set of periods which is given by the solutions of the Picard-Fuchs equations. For this purpose, one defines an antisymmetric bilinear differential operator which acts on the set solutions of the Picard-Fuchs equations

$$D_1 \wedge D_2(f_1, f_2) = \frac{1}{2} \left(D_1 f_1 D_2 f_2 - D_2 f_1 D_1 f_2 \right). \tag{E.1}$$

Using this, we make an ansatz for Q as an asymmetric bidifferential operator, specializing to the one-parameter case:

$$Q(z) = \sum_{k,l} Q_{k,l}(z) D_k(\theta) \wedge D_l(\theta), \tag{E.2}$$

where z denotes the complex structure modulus on the mirror. Since the symplectic form must be constant over the moduli space, one has to impose the conditions

$$\theta Q(z) = 0. \tag{E.3}$$

This determines the coefficients Q_{kl}, which are in general z-dependent. The Picard-Fuchs equation enters into the above condition.

Let us demonstrate this procedure for X_5. The Picard-Fuchs operator in the Pfaffian phase is [18]

$$
\begin{aligned}
\mathcal{L}^{r \ll 0} =\ & \theta^4 + z(2000\theta^4 + 3904\theta^3 + 2708\theta^2 + 756\theta + 76) \\
& + z^2(63488\theta^4 + 63488\theta^3 - 21376\theta^2 - 18624\theta - 2832) \\
& + z^3(512000\theta^4 + 24576\theta^3 - 37888\theta^2 + 6144\theta + 3072) \\
& + z^4(4096(2\theta + 1)^4).
\end{aligned}
\tag{E.4}
$$

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To get the Picard-Fuchs operator in the $r \gg 0$ phase we make the transformation[1] $\tilde{z} = \frac{1}{\sqrt{z}}$. The Picard-Fuchs equation is $\mathcal{L}\Pi = \int(\Omega, \theta\Omega, \theta^2\Omega, \theta^2\Omega) = 0$, where Ω is the holomorphic threeform and the vector denotes elements of H^3 of the mirror Calabi-Yau. It is convenient to rescale the holomorphic three form $\Omega(\tilde{z}) \to \tilde{z}\Omega(\tilde{z})$. Setting $\tilde{z} \equiv z$, the Picard-Fuchs operator in the $r \gg 0$ phase becomes

$$
\begin{aligned}
\mathcal{L}^{r\gg0} &= 4096\theta^4 + z^2(19456 + 96768\theta + 173312\theta^2 + 124928\theta^3 + 32000\theta^4) \\
&\quad + z^4(-2832 - 9312\theta - 5344\theta^2 + 7936\theta^3 + 3968\theta^4) \\
&\quad + z^6(12 + 12\theta - 37\theta^2 + 12\theta^3 + 125\theta^4) \\
&\quad + z^8\left(\frac{1}{16} + \frac{\theta}{4} + \frac{3\theta^2}{8} + \frac{\theta^3}{4} + \frac{\theta^4}{16}\right).
\end{aligned}
\tag{E.5}
$$

We make the following ansatz for $Q(z)$ [29]:

$$
Q(z) = c_1(z)1 \wedge \theta^3 + c_2(z)\theta \wedge \theta^2 + c_3(z)1 \wedge \theta^2 + c_4(z)1 \wedge \theta
\tag{E.6}
$$

The condition $\theta Q(z) = 0$ leads to a system of differential equations for the coefficients $c_i(z)$. Furthermore all θ^4-terms can be eliminated using the Picard-Fuchs equation. The result is

$$
c_1 = -c_2 = \frac{256 + 1968z^2 + z^4}{16 + z^2} \quad c_3 = \frac{2z^2(31232 + 32z^2 + z^4)}{(16 + z^2)^2} \quad c_4 = \frac{2z^2(24192 - 408z^2 + z^4)}{(16 + z^2)^2}
\tag{E.7}
$$

With that we determine the symplectic pairing for a basis of solution of the Picard-Fuchs equation. To see the behavior of the solutions we rewrite the Picard-Fuchs equation as a matrix differential equation, the Gauss-Manin system: $\theta\Pi = M(z)\Pi$ where the period vector Π is as above. The leading behavior of the solutions can be read off from the Jordan normal form of the connection matrix M evaluated at $z = 0$. If all the eigenvalues are zero, it is also the logarithm of the monodromy matrix $\mathcal{T} = e^{2\pi i M(0)}$ (see for instance [43]). With our choice of coordinates and normalization of the holomorphic threeform, we get

$$
M(0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\tag{E.8}
$$

This is exactly what one would expect at a large radius point. Therefore the leading behavior of the period vector is $\pi \sim (1, \frac{1}{2\pi i}\log z, \frac{1}{(2\pi i)^2}\log^2 z, \frac{1}{(2\pi i)^3}\log^3 z)$. The factors $(2\pi i)$

---

[1]Using this transformation instead of $\tilde{z} = \frac{1}{z}$ reveals that this model has maximally unipotent monodromy.

are required for the correct monodromy behavior. The solutions can be obtained from a power series ansatz and we refrain from giving them here. Inserting π into the expression for $Q(z)$, one finds the following symplectic matrix for the given basis of solutions:

$$Q = \frac{1}{\pi^3} \begin{pmatrix} 0 & 0 & 0 & 6i \\ 0 & 0 & -2i & 0 \\ 0 & 2i & 0 & 0 \\ -6i & 0 & 0 & 0 \end{pmatrix} \tag{E.9}$$

Using this and the period matrix to compute e^{-K}, the Kähler metric at $r \gg 0$ is

$$g_{z\bar{z}} = \frac{3}{z\bar{z}(\log|z|^2)^2}, \tag{E.10}$$

which is exactly the result we got from the sphere partition function. The same procedure can be repeated for X_{10}, X_{13} and the two phases of the $U(1) \times O(2)$ example.

F   Mirror symmetry calculations for $d < 3$

F.1   Mirror of $(\mathbf{A}^2_{5,5})$

For this example we can use mirror symmetry to check the result for the fundamental period we obtained from the sphere partition function. In the $\ll 0$ phase the mirror of the Pfaffian has been constructed in [37] using tropical geometry. The mirror is given by the rank 2 locus of the following skew-symmetric matrix

$$\tilde{A}(\tilde{p}) = \begin{pmatrix} 0 & \psi\tilde{p}^5 & \tilde{p}^3 & \tilde{p}^4 & t\tilde{p}^1 \\ & 0 & \psi\tilde{p}^4 & \tilde{p}^1 & \tilde{p}^2 \\ & & 0 & \psi\tilde{p}^2 & \tilde{p}^5 \\ & & & 0 & \psi\tilde{p}^3 \\ & & & & 0 \end{pmatrix}, \tag{F.1}$$

where we denote by $\tilde{p}^i$ the dual coordinates of the p^i and ψ is the complex structure parameter of the mirror Pfaffian. The remaining entries of $\tilde{A}(\tilde{p})$ are determined by the skew-symmetry. Following the prescription in [15] one can calculate the fundamental period by direct integration. For a Pfaffian Calabi-Yau there is a globally defined holomorphic threeform

$$\Omega = \frac{(-1)^\mu (2\pi i)^3 \text{Pf}_{\mu_1\mu_2\mu_3}\Omega_0}{d\text{Pf}_{\mu_1} \wedge d\text{Pf}_{\mu_2} \wedge d\text{Pf}_{\mu_3}} = \text{Res} \frac{(-1)^\mu \text{Pf}_{\mu_1\mu_2\mu_3}\Omega_0}{\text{Pf}_{\mu_1}\text{Pf}_{\mu_2}\text{Pf}_{\mu_3}}, \tag{F.2}$$

where

$$\Omega_0 = \frac{(\tilde{p}^1)^5}{(2\pi i)^4} \bigwedge_{i=2}^{5} d\left(\frac{\tilde{p}^i}{\tilde{p}^1}\right). \tag{F.3}$$

The subscripts of the Pf indicate which rows and columns are deleted from $\tilde{A}(\tilde{p})$ and μ is an integer. The formula above holds for any three $\mu_i \in \{1,\ldots,5\}$. The fundamental period is than defined by

$$X^0(\psi) = \int_{\gamma_0(\psi)} \Omega(\psi) = \int_{\Gamma} \Psi(\psi). \tag{F.4}$$

In the last equality we used the fact that the period integral can be rewritten as an integral of a six-form $\Psi(t)$ over a six-cycle Γ around zero. Ψ is defined such that its residue is Ω:

$$\Psi = \Omega \wedge \frac{d\mathrm{Pf}_{\mu_0}}{2\pi i \mathrm{Pf}_{\mu_0}} \wedge \frac{d\mathrm{Pf}_{\mu_1}}{2\pi i \mathrm{Pf}_{\mu_1}} \wedge \frac{d\mathrm{Pf}_{\mu_2}}{2\pi i \mathrm{Pf}_{\mu_2}} = \frac{(-1)^\mu \mathrm{Pf}_{\mu_1\mu_2\mu_3}}{(2\pi i)^4 \mathrm{Pf}_{\mu_1}\mathrm{Pf}_{\mu_2}\mathrm{Pf}_{\mu_3}}(\tilde{p}^1)^5 \bigwedge_{i=2}^{5} d\left(\frac{\tilde{p}^i}{\tilde{p}^1}\right) \tag{F.5}$$

In order to calculate this integral explicitly we choose $\mu_1, \mu_2, \mu_3 = 2, 4, 5$ and repeat the calculation in [15] step by step.

$$
\begin{aligned}
\frac{(-1)^\mu \mathrm{Pf}_{245}}{\mathrm{Pf}_2\mathrm{Pf}_4\mathrm{Pf}_5} &\overset{\tilde{p}^1=1}{=} -\frac{\tilde{p}^3}{(-\tilde{p}^1\tilde{p}^3 + \psi(\tilde{p}^4)^2 + \psi^2\tilde{p}^2\tilde{p}^5)(\psi^2\tilde{p}^1\tilde{p}^2 + \psi(\tilde{p}^3)^2 - \tilde{p}^4\tilde{p}^5)(-\tilde{p}^2\tilde{p}^3 + \psi^2\tilde{p}^1\tilde{p}^4\psi(\tilde{p}^5)^2)} \\
& \frac{1}{(\tilde{p}^2\tilde{p}^3\tilde{p}^4\tilde{p}^5)\left(1 - \psi\frac{(\tilde{p}^4)^2}{\tilde{p}^3} - \psi^2\frac{\tilde{p}^2\tilde{p}^5}{\tilde{p}^3}\right)\left(1 - \psi\frac{(\tilde{p}^3)^2}{\tilde{p}^4\tilde{p}^5} - \psi^2\frac{\tilde{p}^2}{\tilde{p}^4\tilde{p}^5}\right)\left(1 - \psi\frac{(\tilde{p}^5)^2}{\tilde{p}^2\tilde{p}^3} - \psi^2\frac{\tilde{p}^4}{\tilde{p}^2\tilde{p}^3}\right)} \\
&:= \frac{1}{(\tilde{p}^2\tilde{p}^3\tilde{p}^4\tilde{p}^5)(1 - v_{11} - v_{12})(1 - v_{21} - v_{22})(1 - v_{31} - v_{32})} \\
&= \frac{1}{\tilde{p}^2\tilde{p}^3\tilde{p}^4\tilde{p}^5} \sum_{n=1}^{\infty} (v_{11} + v_{12})^n \sum_{m=1}^{\infty} (v_{21} + v_{22})^m \sum_{k=1}^{\infty} (v_{31} + v_{32})^k \\
&= \frac{1}{\tilde{p}^2\tilde{p}^3\tilde{p}^4\tilde{p}^5} \sum_{n=1}^{\infty} \sum_{a=0}^{n} \binom{n}{a} v_{11}^a v_{12}^{n-a} \sum_{m=1}^{\infty} \sum_{b=0}^{m} \binom{m}{b} v_{21}^b v_{22}^{m-b} \sum_{k=1}^{\infty} \sum_{c=0}^{k} \binom{k}{c} v_{31}^c v_{32}^{k-c}
\end{aligned}
\tag{F.6}
$$

There will only be a contribution to the residue integral if v_{ij}-terms are independent of x_i. There are the following relations:

$$
\begin{aligned}
v_{11}v_{21}v_{31}v_{22} &= \psi^5 \equiv z_1 \\
v_{12}v_{32}v_{21} &= \psi^5 \equiv z_2
\end{aligned}
\tag{F.7}
$$

Thus, only the following monomials will contribute to the residue:

$$z_1^\alpha z_2^\beta = v_{11}^\alpha v_{12}^\beta v_{21}^{\alpha+\beta} v_{22}^\alpha v_{31}^\alpha v_{32}^\beta \quad \alpha + \beta = n \tag{F.8}$$

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Comparing with the expression above, there are non-zero contributions if $a = \alpha, n = \alpha + \beta, b = \alpha, m = 2\alpha + \beta, c = \alpha, k = \alpha + \beta$. Defining $z = \psi^5$ we find the following expression for the fundamental period:

$$
\begin{aligned}
\varpi_0(z) &= \sum_{n=0}^{\infty} \sum_{\alpha+\beta=n} \binom{\alpha+\beta}{\alpha} \binom{2\alpha+\beta}{\alpha} \binom{\alpha+\beta}{\alpha} z^n \\
&= \sum_{n=0}^{\infty} \sum_{\alpha=0}^{n} \binom{n}{\alpha}^2 \binom{n+\alpha}{\alpha} z^n.
\end{aligned}
\tag{F.9}
$$

This is precisely the result we got from the sphere partition function in both phases.

F.2   Mirror of $(\mathbf{S}_{3,3}^{2,+})$

For this example we can check the localization calculation in the $r \gg 0$ phase where the elliptic curve is a free $\mathbb{Z}_2$ quotient of a complete intersection $\tilde{X}$ of three quadric hypersurfaces $Q_i(u_i, v_i) = 0$ of co-dimension 3 in $\mathbb{P}^2 \times \mathbb{P}^2$. Modding out by the involution $\sigma = (u_i, v_i) \leftrightarrow (v_i, u_i)$ we obtain the Calabi-Yau $X \simeq \tilde{X}/\langle\sigma\rangle$. To obtain the mirror we use the Batyrev-Borisov construction for $\tilde{X}$ [44]. This is the exactly same procedure that was used to construct the mirror of the Reye congruence in [17] which we repeat now for two dimensions less. One of the $\mathbb{P}^2$ is denoted by the lattice polytope Δ whose vertices provide a basis of sections $H^0(\mathbb{P}^2, \mathcal{O}(-K_{\mathbb{P}^2}))$. Its dual polytope Δ^* is given by Conv.$(e_1, e_2, -e_1 - e_2)$ where $\{e_i\}$ is the canonical basis of $\mathbb{R}^2$. $\mathbb{P}^2 \times \mathbb{P}^2$ is then given by $\Delta \times \Delta$. The dual polytope $(\Delta \times \Delta)^*$ has six vertices $\{(e_1, 0), \ldots, (0, -e_1 - e_2)\}$ which we denote by $\nu_1, \ldots, \nu_6$. They are in one-to-one correspondence with toric divisors $D_i = D_{\nu_i}$. In order to specify a complete intersection we also have to give a nef-partition of the anti-canonical class $-K_{\mathbb{P}_{\Delta \times \Delta}}$. This is given by $\{D_i, D_{i+3}\}_{i=1,\ldots,3}$. Associated to this nef-partition we define subpolytopes ∇_i such that their points are the sections $H^0(P_{\Delta \times \Delta}, \mathcal{O}_{D_i + D_{i+3}})$. The complete intersection $\tilde{X}$ is then given by the polynomials $f_{\nabla_i} = 0$. The dual of ∇_i is given by $\Delta_i^* = \text{Conv}(\{0, \nu_i, \nu_{i+3}\})$. The mirror $\tilde{X}^\vee$ is then given by $f_{\Delta_i^*} = 0$. These polynomials have the following form:

$$
f_{\Delta_i^*} = a_i + b_i U_i + c_i V_i \qquad i = 1, \ldots, 3
\tag{F.10}
$$

for $U_{1,\ldots,2}, V_{1,\ldots,2} \in (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^2$ and $U_1 U_2 U_3 = V_1 V_2 V_3 = 1$. Furthermore a_i, b_i, c_i are parameters which, redundantly, parametrize the complex structure of $\tilde{X}^\vee$. The dual $X^\vee$ should be invariant under the involution $\sigma^\vee : U_i \leftrightarrow V_i$. This is realized when we set $b_i = c_i$. This reduces the number of complex structure moduli from two to one.

We are going to compute the fundamental period by direct integration. The period integral for the complete intersection is:

$$
\begin{aligned}
\tilde{X}^0 &= \frac{1}{(2\pi i)^4} \int_\gamma \frac{a_1 a_2 a_3}{f_{\Delta_1^*} f_{\Delta_2^*} f_{\Delta_3^*}} \prod_{i=1}^2 \frac{dU_i}{U_i} \frac{dV_i}{V_i} \\
&= \frac{1}{(2\pi i)^4} \int_\gamma \prod_{i=1}^3 \frac{dU_i}{U_i} \frac{dV_i}{V_i} \frac{a_1 a_2 a_3}{\prod_{i=1}^2 (a_i + b_i U_i + c_i V_i)(a_3 + \frac{b_3}{U_1 U_2} + \frac{c_3}{V_1 V_2})} \\
&= \frac{1}{(2\pi i)^4} \int_\gamma \prod_{i=1}^2 \frac{dU_i}{U_i} \frac{dV_i}{V_i} \sum_{n=0}^\infty (-1)^n \left(\frac{b_1}{a_1} U_1 + \frac{c_1}{a_1} V_1 \right)^n \sum_{m=0}^\infty (-1)^m \left(\frac{b_2}{a_2} U_2 + \frac{c_2}{a_2} V_2 \right)^m \times \\
&\quad \sum_{k=0}^\infty (-1)^k \left(\frac{b_3}{a_3} \frac{1}{U_1 U_2} + \frac{c_3}{a_3} \frac{1}{V_1 V_2} \right)^k
\end{aligned}
\tag{F.11}
$$

There is only a contribution to the residue if the product of the four infinite sums is independent of U_i, V_i. Defining,

$$
x = \frac{b_1 b_2 b_3}{a_1 a_2 a_3} \qquad y = \frac{c_1 c_2 c_3}{a_1 a_2 a_3}
\tag{F.12}
$$

we get:

$$
\tilde{\Pi}_0 = \sum_{a,b=0}^\infty \binom{a+b}{a}^3 x^a y^b
\tag{F.13}
$$

In order to make the transition from $\tilde{X}^\vee$ to $X^\vee$ we set $x = y \equiv z$. Then the fundamental period of $X^\vee$ is,

$$
X^0 = \sum_{n=0}^\infty \sum_{k=0}^n \binom{n}{k}^3 z^n,
\tag{F.14}
$$

which again coincides with the result from the sphere partition function.

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