

NUMBER OF PERMUTATIONS WITH SAME PEAK SET FOR SIGNED PERMUTATIONS

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ABSTRACT. A signed permutation $\pi = \pi_1\pi_2\dots\pi_n$ in the hyperoctahedral group B_n is a word such that each $\pi_i \in \{-n, \dots, -1, 1, \dots, n\}$ and $\{|\pi_1|, |\pi_2|, \dots, |\pi_n|\} = \{1, 2, \dots, n\}$. An index i is a peak of π if $\pi_{i-1} < \pi_i > \pi_{i+1}$ and $P_B(\pi)$ denotes the set of all peaks of π . Given any set S , we define $P_B(S, n)$ to be the set of signed permutations $\pi \in B_n$ with $P_B(\pi) = S$. In this paper we are interested in the cardinality of the set $P_B(S, n)$. In [4], Billey, Burdzy and Sagan investigated the analogous problem for permutations in the symmetric group, S_n . In this paper we extend their results to the hyperoctahedral group; in particular we show that $\#P_B(S, n) = p(n)2^{2n-|S|-1}$ where $p(n)$ is the same polynomial found in [4] which leads to the explicit computation of interesting special cases of the polynomial $p(n)$. In addition we have extended these results to the case where we add $\pi_0 = 0$ at the beginning of the permutations, which gives rise to the possibility of a peak at position 1, for both the symmetric and the hyperoctahedral groups.

1. INTRODUCTION

A permutation $\pi = \pi_1\pi_2\dots\pi_n$ in the symmetric group S_n has a peak at index i if $\pi_{i-1} < \pi_i > \pi_{i+1}$. The peak set of π is defined to be $P(\pi) = \{i \mid i \text{ is a peak of } \pi\}$, then we define

$$P(S, n) = \{\pi \in S_n \mid P(\pi) = S\}$$

to be the set of all permutations with the same peak set S . For example, the permutation $\pi = 2\ 6\ 5\ 1\ 4\ 3$ has peaks at position 2 and 5, hence $P(\pi, 6) = \{2, 5\}$.

Stembridge [9] was one of the first to study the combinatorics of peaks, in particular he gave a peak analog of Stanley's theory of poset partitions. Additional interest in the study of peaks arose when Nyman [7] showed that summing permutations according to their peak sets leads to a non-unital subalgebra of the group algebra of the symmetric group.

In a recent paper [4] Billey, Burdzy and Sagan considered the cardinalities of the sets $P(S, n)$. They discovered that $\#P(S, n) = p(n)2^{n-\#S-1}$ for some polynomial $p(n)$ depending on S ; they also computed special cases of the polynomial $p(n)$. One motivation for studying peaks of permutations lies in probability theory, in a recent paper Billey, Burdzy, Pal and Sagan [5] studied distributions on graphs that are related to random permutations with certain peak sets. Besides the applicability to probability theory, the problem of enumerating permutations and signed permutations with respect to a given statistic is an interesting

problem on its own, for example the enumeration of permutations related to peak sets has also been studied in [6, 10, 11].

It is natural that when a result related to the symmetric group (the Coxeter group of type A) is obtained one wishes to generalize it to other Coxeter groups. In this paper we generalize the results in [4] to the group of signed permutations, the hyperoctahedral group B_n (the Coxeter group of type B). A signed permutation $\pi = \pi_1\pi_2 \dots \pi_n$ is a word such that each $\pi_i \in \{-n, \dots, -1, 1, \dots, n\}$ and $\{|\pi_1|, |\pi_2|, \dots, |\pi_n|\} = \{1, 2, \dots, n\}$. A peak of a signed permutation is defined in exactly the same way as for regular permutations. We will denote by $P_B(\pi)$ the set of peaks of a signed permutation π and define

$$P_B(S, n) = \{\pi \in B_n \mid P_B(\pi) = S\}.$$

We show that $\#P_B(S, n) = 2^{2n-\#S-1}p_B(n)$, where $p_B(n)$ is the same polynomial as for the symmetric group, this generalizes Theorem 3 in [4]. We also consider special cases of the polynomials $p_B(n)$.

Peaks for signed permutations are also of interest in the construction of algebraic structures. In [3] Bergeron and Hohlweg have described peak analogues of the peak algebras for the hyperoctahedral group and Petersen [8] considered peak algebras of the hyperoctahedral group when the signed permutations are grouped by number of peaks.

The second part of our paper considers the enumeration of peak classes when we put a zero at the beginning of the permutations for both the symmetric and hyperoctahedral groups. That is, we consider permutations of the form $\pi_0\pi_1 \dots \pi_n$ where $\pi_0 = 0$. These permutations arose in the study of unital peak algebras of the symmetric group [1, 2]. In the case of the symmetric group, adding a zero at the beginning of every permutation has the effect of having the identity as the unique permutation with no peaks. We denote by $\widehat{P}(S, n)$ the set of permutations with a zero added in S_n with peak set S and $\widehat{P}_B(S, n)$ the corresponding set for the hyperoctahedral group. We generalize results obtained in [4] to $\widehat{P}(S, n)$ and $\widehat{P}_B(S, n)$. In particular, we give a method for computing $\#\widehat{P}(S, n)$ and $\#\widehat{P}_B(S, n)$ and compute these numbers for special sets S .

We now give a more detail description of the contents of this paper. In Section 2, we prove that $\#P_B(S, n) = p_B(n)2^{2n-s-1}$ where $p_B(n)$ is an integral polynomial in terms of n . In addition, we show that the polynomials $p_B(n)$ in B_n are equal to the polynomials $p(n)$ in S_n found in [4]. We also show that the values for $\#P_B(S, n)$ are symmetric for a fixed n as we vary the set S . In Section 3, we provide a method to compute $\#\widehat{P}_B(S, n)$ for any S . We find that if $S = \emptyset$, $\#\widehat{P}_B(S, n)$ can be written in terms of the Stirling numbers of the second kind. Another result in this section gives us the parity of $\#\widehat{P}_B(S, n)$. Additionally, we find that $\#P_B(S, n)$ can be written as the sum of $\#\widehat{P}_B(S, n)$ and $\#\widehat{P}_B(S \cup \{1\}, n)$ which gives us the results of the previous section when $2 \in S$. Finally, we calculate $\#\widehat{P}_B(S, n)$ for various specific sets S .

Finally, in Section 4, we focus on the symmetric group, we provide a method to compute $\#\widehat{P}(S, n)$ for any S . We also find that $\#P(S, n)$ can be written as the sum of $\#\widehat{P}(S, n)$ and $\#\widehat{P}(S \cup \{1\}, n)$ which gives us the results in [4] when

$2 \in S$. Another result in this section gives us the parity of $\#\widehat{P}(S, n)$. Finally, we calculate $\#\widehat{P}(S, n)$ for various special cases of the set S .

2. SIGNED PERMUTATIONS IN B_n

Let B_n be the hyperoctahedral group, i.e., the group of signed permutations, and let $\pi = \pi_1\pi_2 \dots \pi_n$ be a permutation in B_n . Recall that we define a position $i \in \{2, \dots, n-1\}$ as a peak if $\pi_{i-1} < \pi_i > \pi_{i+1}$, and the set $P_B(\pi)$ as the set of all peaks of π .

Define a set $S = \{i_1 < \dots < i_s\}$ to be n -admissible if $\#P_B(S, n) \neq 0$. Note that we insist the elements be listed in increasing order. Notice that S cannot contain two consecutive integers and S is a subset of $\{2, \dots, n-1\}$. The minimum possible value of n for which S is n -admissible is $i_s + 1$, and in that case S is n -admissible for all $n \geq i_s + 1$. If we make a statement about an admissible set S , we mean that S is n -admissible for some n and the statement holds for every n such that S is n -admissible. It is well-known that the number of n -admissible sets is the n -th Fibonacci number. We include a proof for completeness.

Proposition 2.1. *Let A_n be the set of n -admissible peak sets S . Then the size of A_n is given by the n -th Fibonacci number.*

Proof. Note that $A_1 = A_2 = \{\emptyset\}$, thus the result holds for $n = 1$ and $n = 2$.

Now consider $n \geq 3$. We can write A_n as a union of disjoint sets A_α and A_β where A_α is the set of n -admissible sets that do not contain $n-1$, and A_β is the set of n -admissible sets that do contain $n-1$. But note that since A_{n-1} contains all $(n-1)$ -admissible peak sets S , which cannot contain the element $n-1$, it must be equal to A_α . Also, adding $n-1$ to all the peak sets in A_{n-2} gives us A_β . Therefore $|A_n| = |A_{n-1}| + |A_{n-2}|$. □

If we fix n and the cardinality of the set S , then there exists a set T of the same cardinality as S such that $\#P_B(S, n) = \#P_B(T, n)$. We make this symmetry property more explicit in the following proposition.

Proposition 2.2. *Let $S = \{i_1, i_2, \dots, i_k\}$ and $T = \{n+1-i_k, \dots, n+1-i_1\}$. Then $\#P_B(S, n) = \#P_B(T, n)$.*

Proof. Define $f : B_n \rightarrow B_n$ such that if $\pi = \pi_1\pi_2 \dots \pi_n$, $f(\pi) = \pi_n \dots \pi_2\pi_1$. Note that f is an involution, because $f(f(\pi)) = \pi$.

Now let $\rho = \rho_1\rho_2 \dots \rho_n \in P_B(S, n)$. If j is a peak of ρ then $n+1-j$ is a peak of $f(\rho)$, hence the peak set of $f(\rho)$ is $\{n+1-i_k, \dots, n+1-i_2, n+1-i_1\}$. Thus $f(\rho) \in P_B(T, n)$. Similarly, if $f(\pi) \in P_B(T, n)$ we can show that $f(f(\pi)) \in P_B(S, n)$. Therefore $\#P_B(S, n) = \#P_B(T, n)$. □

Remark: Note that since $S_n \subseteq B_n$, this result holds in S_n as well.

We now prove the following special case which will be the base case for our induction later on. In what follows we set $[n] = \{1, 2, \dots, n\}$.

Proposition 2.3. *The number of signed permutations with no peaks is equal to 2^{2n-1} .*

Proof. Let $\pi = \pi_1\pi_2 \dots \pi_n$ be a signed permutation, and $k = \min(\pi_1, \pi_2, \dots, \pi_n)$. Since $\{|\pi_1|, |\pi_2|, \dots, |\pi_n|\} = [n]$ then there is a π_j with $|\pi_j| = 1$. Hence $k = 1$ or $k < 0$.

Thus the number of signed permutations with no peaks can be divided into those with $k = 1$ and those with $k < 0$. Let us first assume that $k = 1$. Then, we can write $\pi = k_1 1 k_2$ where k_1 denotes the portion to the left of 1 and k_2 denotes the portion to the right of 1. To have $P_B(\pi) = \emptyset$, k_1 must be decreasing and k_2 must be increasing. We notice that the size of k_1 ranges from 0 to $n - 1$ and the values of π_i in k_1 and k_2 range from 2 to n because $k = 1$. This means that the number of signed permutations with no peaks when $k = 1$ is equal to the number choices of a subset of elements from $\{2, \dots, n\}$ to be in k_1 since after the choices are made, the rest of π is determined. This implies that the number of signed permutations with no peaks and $k = 1$ equals to

$$\sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}.$$

Now let $k < 0$. Note that the numbers that can go in π are from the set $A = \{k + 1, k + 2, \dots, -1, 1, \dots, n\}$. We also notice that the size of k_1 can vary from 0 to $n - 1$ again. Although we have more than $n - 1$ options available to put in k_1 , we note that if an integer $m \in [n]$ is in the permutation, then $-m$ cannot be, and vice versa. As a result we have $n - 1$ numbers to put in k_1 and then we multiply by 2 for every element in A for which its negative is in A as well. Furthermore the number of signed permutations with no peaks and k fixed, where $k < 0$ equals $2^{n-1} \cdot 2^{k-1}$ (since we are multiplying by 2 for every element in $\{k + 1, k + 2, \dots, -1\}$). Hence the total number of signed permutations with no peaks and $k < 0$ is

$$\sum_{k=1}^n 2^{n-1} \cdot 2^{k-1} = 2^{n-1} \sum_{k=1}^n 2^{k-1} = 2^{n-1}(2^n - 1) = 2^{2n-1} - 2^{n-1}.$$

But we have to add in what we obtained in the $k = 1$ case to get the total number of signed permutations with no peaks, then the number is 2^{2n-1} . \square

We now show that the results in [4] for the symmetric group extend to the case of sign permutations.

Theorem 2.4. *Let $S = \{i_1, i_2, \dots, i_s\}$ be admissible. Then*

$$\#P_B(S, n) = p_B(n)2^{2n-s-1},$$

where $p_B(n) = p_B(S, n)$ is a polynomial depending on S such that $p_B(n)$ is an integer for all integral n . In addition, the degree of $p_B(n) = i_s - 1$ (when $S = \emptyset$, the degree of $p_B(n) = 0$).

Proof. We prove by induction on $i_1 + i_2 + \dots + i_s$, following the argument in [4]. We previously showed that $\#P_B(\emptyset, n) = 2^{2n-1}$, thus our claim is true for the base case. Our inductive hypothesis is that our claim is true for any set $\hat{S} = \{r_1, r_2, \dots, r_t\}$ where $r_1 + r_2 + \dots + r_t < i_1 + i_2 + \dots + i_s$.

We let $k = i_s - 1$. For any $n \geq i_s$, let Π be the set of all signed permutations $\pi = \pi_1\pi_2 \dots \pi_n$ such that $P_B(\pi_1\pi_2 \dots \pi_k) = S_1 = S - \{i_s\}$ and $P_B(\pi_{i_s} \dots \pi_n) =$

\emptyset . We can partition Π into blocks by the peak set of π . In addition to the peaks given by $S_1 = S - \{i_s\}$, there could be a peak at π_k , a peak at π_{i_s} , or no peak at both π_k and π_{i_s} . Note that these are all the possibilities, and that the three are disjoint. Thus, if we let $S_2 = S_1 \cup \{i_s - 1\}$, then

$$\#\Pi = \#P_B(S_2, n) + \#P_B(S, n) + \#P_B(S_1, n). \quad (1)$$

First, we find $\#\Pi$. Recall that for $\pi \in \Pi$, $P_B(\pi_1 \dots \pi_k) = S_1$ and $P_B(\pi_{i_s} \dots \pi_n)$ equals \emptyset . Therefore to construct any π , first we choose $k = i_s - 1$ elements to be in the first section. For signed permutations, if an integer $m \in [n]$ is in the permutation, then $-m$ cannot be, and vice versa. Therefore we choose k elements from a set of n elements. The number of ways to do so is $\binom{n}{k}$. Then we create a signed permutation from these n elements, arranged in a way such that their peak set is S_1 . We have denoted the number of ways to do so by $\#P_B(S_1, k)$. Finally we arrange the last $n - k$ items such that their peak set is \emptyset . The number of ways to do this is $\#P_B(\emptyset, n - k)$. Therefore, $\#\Pi = \binom{n}{k} \#P_B(S_1, k) \#P_B(\emptyset, n - k)$.

By our inductive assumption, $\#P_B(S_1, k) = p_1(k)2^{2k-(s-1)-1} = p_1(k)2^{2k-s}$, where $p_1(k)$ is a polynomial of degree $i_{s-1} - 1$, and we know from the base case that $\#P_B(\emptyset, n - k) = 2^{2(n-k)-1} = 2^{2n-2k-1}$. Thus,

$$\#\Pi = p_1(k) \binom{n}{k} 2^{2n-s-1}.$$

Similarly, we find that by our inductive assumption, $\#P_B(S_1, n) = p_1(n)2^{2n-s}$ and $\#P_B(S_2, n) = p_2(n)2^{2n-s-1}$, where $p_2(n)$ has degree $i_s - 2$. Thus,

$$\begin{aligned} \#P_B(S, n) &= \#\Pi - \#P_B(S_1, n) - \#P_B(S_2, n) \\ &= p_1(k) \binom{n}{k} 2^{2n-s-1} - p_1(n)2^{2n-s} - p_2(n)2^{2n-s-1} \\ &= \left(p_1(k) \binom{n}{k} - 2p_1(n) - p_2(n) \right) 2^{2n-s-1}. \end{aligned}$$

Note that $p_1(k) \binom{n}{k}$ has degree k . We know $2p_1(n)$ has degree $i_{s-1} - 1 < k$, and $p_2(n)$ has degree $i_s - 2 < k$, then the coefficient of 2^{2n-s-1} is a polynomial of degree $k = i_s - 1$. Since $\binom{n}{k}$, $2p_1(n)$, and $p_2(n)$ all have integral values at integer n , the difference also has integral values at integer n . \square

The proof of Theorem 2.4 immediately yields the following recursive formula for the polynomial $p_B(S, n)$.

Corollary 2.5. *If $S \neq \emptyset$ is admissible and $m = \max S$ then*

$$p_B(S, n) = p_1(m-1) \binom{n}{m-1} - 2p_1(n) - p_2(n)$$

where $S_1 = S - \{m\}$, $S_2 = S_1 \cup \{m-1\}$, and $p_i(n) = p_B(S_i, n)$ for $i = 1, 2$.

Proof. Looking at the proof of Theorem 2.4, we can see that the greatest element in S is i_s which means that $m = i_s$. We also let $k = i_s - 1 = m - 1$, then by

Theorem 2.4 we have

$$p_B(S, n) = p_1(k) \binom{n}{k} - 2p_1(n) - p_2(n) = p_1(m-1) \binom{n}{m-1} - 2p_1(n) - p_2(n).$$

□

The recursive formula for the polynomial $p_B(n) = p_B(S, n)$ in terms of $p_B(S_1, k)$, $p_B(S_1, n)$, and $p_B(S_2, n)$, given in Corollary 2.5, is identical to the recursive formula for the polynomial when working in S_n , where Theorem 3 in [4] shows that $\#P(S, n) = p(n)2^{n-|S|-1}$. In fact, we can show in general that $p_B(n) = p(n)$.

Lemma 2.6. *For any non-empty $S = \{i_1, i_2, \dots, i_s\}$, let $S_2 = \{i_1, i_2, \dots, i_{s-1}, i_s - 1\}$. Then $p_B(S, n) + p_B(S_2, n) = p(S, n) + p(S_2, n)$ implies $p_B(S, n) = p(S, n)$ for any n .*

Proof. We assume that $p_B(S, n) + p_B(S_2, n) = p(S, n) + p(S_2, n)$ and consider two cases $s = 1$ and $s > 1$.

Case $s = 1$: Since S has only one element, we can write $S = \{m\}$ and induct on m . The base case is $m = 1$, since we cannot have a peak at position 1, $p_B(S, n) = p(S, n) = 0$. Now, we assume our claim is true for $S = \{m\}$, where $m \geq 1$. Consider $S = \{m+1\}$, our assumption gives us that $p_B(\{m+1\}, n) + p_B(\{m\}, n) = p(\{m+1\}, n) + p(\{m\}, n)$, and our inductive hypothesis gives us our result.

Case $s > 1$: In this case, we notice that if we write $i_s = i_{s-1} + z$, then $S = \{i_1, i_2, \dots, i_{s-1}, i_{s-1} + z\}$ and $S_2 = \{i_1, i_2, \dots, i_{s-1}, i_{s-1} + (z-1)\}$. Therefore, by induction on z we can easily see that, assuming $p_B(S, n) + p_B(S_2, n) = p(S, n) + p(S_2, n)$ implies $p_B(S, n) = p(S, n)$. □

Theorem 2.7. *For any $S = \{i_1, i_2, \dots, i_s\}$ and for any n , if $\#P(S, n) = p(n)2^{n-s-1}$ and $\#P_B(S, n) = p_B(n)2^{2n-s-1}$, then $p(n) = p_B(n)$ is the same polynomial in n depending on S for both S_n and B_n .*

Proof. First, let $S_1 = S - \{i_s\}$ and $S_2 = S_1 \cup \{i_s - 1\}$. By Corollary 2.5 we can write

$$p_B(S, n) + p_B(S_2, n) = \binom{n}{i_s - 1} p_B(S_1, i_s - 1) - 2p_B(S_1, n). \quad (2)$$

By [4], we can also write

$$p(S, n) + p(S_2, n) = \binom{n}{i_s - 1} p(S_1, i_s - 1) - 2p(S_1, n). \quad (3)$$

By Lemma 2.6 it is enough to show that the recursive formulas for the polynomials for S of size s are equal, i.e. $p_B(S, n) + p_B(S_2, n) = p(S, n) + p(S_2, n)$. We show this by inducting on s .

The base case is $s = 0$. Here, $S_1 = \emptyset$, then Equation (2) and Equation (3) are in terms of the polynomials for $S = \emptyset$. By Proposition 2.3, we have that $p_B(\emptyset, n) = 1$, and the same is true for $p(\emptyset, n)$ [4]. Therefore $p_B(S, n) + p_B(S_2, n) = p(S, n) + p(S_2, n)$ in the base case.

Now, we assume that our claim holds for $s-1 \geq 0$. Then if we let $S_3 = (S_1 - \{i_{s-1}\}) \cup \{i_{s-1} - 1\}$, our inductive hypothesis gives us that $p_B(S_1, n) +$

$p_B(S_3, n) = p(S_1, n) + p(S_3, n)$. By Lemma 2.6 this hypothesis implies that $p_B(S_1, n) = p(S_1, n)$. By the same argument, since our inductive hypothesis applies for any n , $p_B(S_1, i_s - 1) = p(S_1, i_s - 1)$. Therefore Equation (2) and Equation (3) will give us our claim for any s . \square

In [4], the polynomials $p(n)$ have been computed for several special cases of S using Corollary 2.5. Hence using Theorem 2.7 we obtain the same polynomials for B_n given in the following corollaries.

Corollary 2.8. *If $S = \{m\}$ is admissible then*

$$p_B(S, n) = \binom{n-1}{m-1} - 1.$$

Corollary 2.9. *If $S = \{2, m\}$ is admissible then*

$$p_B(S, n) = (m-3) \binom{n-2}{m-1} + (m-2) \binom{n-2}{m-2} - \binom{n-2}{1}.$$

Corollary 2.10. *If $S = \{2, m, m+2\}$ is admissible then*

$$p_B(S, n) = m(m-3) \binom{n}{m+1} - 2(m-3) \binom{n-2}{m-1} - 2(m-2) \binom{n-2}{m-2} + 2 \binom{n-2}{1}.$$

3. PERMUTATIONS IN B_n WITH $\pi_0 = 0$

Recall that a peak is defined such that the permutation $\pi \in B_n$ has a peak at position i if $\pi_{i-1} < \pi_i > \pi_{i+1}$. Therefore if we introduce the assumption that $\pi_0 = 0$ for all $\pi \in B_n$, then a permutation π can have a peak at position 1 if $0 < \pi_1 > \pi_2$, that is if π_1 is positive and π has a descent at 1 (i.e. $\pi_1 > \pi_2$). We define $\widehat{P}_B(S, n)$ to be the set of all permutations in B_n with peak set S with the assumption that $\pi_0 = 0$. The number of n -admissible sets is also given by the Fibonacci sequence.

Proposition 3.1. *Let A_n be the set of n -admissible peak sets S . Then the size of A_n is given by the $(n+1)$ th Fibonacci number.*

Proof. The proof is exactly the same as for Proposition 2.1 with initial values $A_1 = \{\emptyset\}$ and $A_2 = \{\emptyset, \{1\}\}$. \square

The next theorem shows that similar recursion holds for computing values for $\#\widehat{P}_B(S, n)$. We then proceed to compute special cases for various n -admissible peak sets S .

Theorem 3.2. *Let $S = \{i_1, i_2, \dots, i_s\}$ be an admissible set then,*

$$\#\widehat{P}_B(S, n) = \binom{n}{i_s-1} \#\widehat{P}_B(S_1, i_s-1) 2^{2(n-i_s)+1} - \#\widehat{P}_B(S_1, n) - \#\widehat{P}_B(S_2, n).$$

where $S_1 = S - \{i_s\}$ and $S_2 = S_1 \cup \{i_s - 1\}$.

Proof. This equation is based on the construction given in the proof of Theorem 2.4. We omit the details as they are exactly the same as in that proof. \square

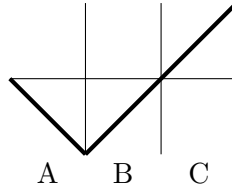


FIGURE 1. General shape for $\pi \in \widehat{P}_B(\emptyset, n)$.

In the following theorem we show that $\#\widehat{P}(\emptyset, n)$ is given by sequence A007051 in the OEIS. The first few values of this sequence are 1, 2, 5, 14, 41, 122, 365, \dots

Theorem 3.3. *Let $S = \emptyset$. Then*

$$\#\widehat{P}_B(S, n) = \frac{3^n + 1}{2}.$$

Proof. Let $\pi \in \widehat{P}_B(\emptyset, n)$. A general shape for π is given by Figure 1, where the section labeled A is negative and decreasing, the section labeled B is negative and increasing, and the section labeled C is positive and increasing. According to these sections, we can partition π into sections $\pi = \pi_0\pi_A\pi_B\pi_C$. In general, up to two of these sections can be empty. We also assume that π_B contains the entire section of the permutation that is negative and ascending, including the minimum of π . For example, if $\pi = 0 - 4 - 5 - 6 - 2 - 1 \ 3$, then $\pi_A = -4 - 5$, $\pi_B = -6 - 2 - 1$ and $\pi_C = 3$.

Now, define a function f from B_n to the set of partitions of $[n + 1]$ into at most 3 blocks. Let $\pi \in B_n$. If $\pi = \pi_0\pi_A\pi_B\pi_C$, then we let A , B , and C be the subsets of $[n]$ that correspond to the absolute values of the sections π_A , π_B , and π_C , respectively. Then f maps π to the partition of $[n + 1]$ into at most 3 blocks, given by $\{A, B, C \cup \{n + 1\}\}$, where if a section is empty it is not represented in the partition. Then $f(\pi)$ is in the set of partitions of $[n + 1]$ into at most 3 blocks.

Next, we define the inverse of f from the set of partitions of $[n + 1]$ into at most 3 blocks to B_n . Let P be such a partition. We write P as a set of three blocks, where we allow some of the blocks to be empty. I.e., if $P = \{P_1, P_2\}$, we write $P = \{P_1, P_2, \emptyset\}$. If P_j is the block containing $n + 1$, then we let $C = P_j - \{n + 1\}$. If P_i is the block containing the maximum value of the remaining two blocks, then we let $B = P_i$, and we let A be the remaining block. Hence, P maps to the signed permutation $\pi = \pi_0\pi_A\pi_B\pi_C$, such that π_A is given by negating the elements of A and ordering them so they are decreasing, π_B is given by negating the elements of B and ordering them so they are increasing, and π_C is given by ordering the elements of C so they are increasing.

It is known that the size of the set of partitions of $[n + 1]$ into at most 3 blocks is given by the first three Stirling numbers of the second kind, $S(n + 1, 1) + S(n + 1, 2) + S(n + 1, 3) = \frac{3^{n+1}}{2}$. Therefore, the size of $\widehat{P}_B(S, n)$ is $(3^n + 1)/2$. \square

3.1. Parity of $\widehat{P}_B(S, n)$. In the previous section we showed that $P_B(S, n)$ was always a multiple of a power of 2, and hence always even. This is no longer the case for $\widehat{P}_B(S, n)$ as we show in the next theorem.

Theorem 3.4. *Let $S = \{i_1, i_2, \dots, i_s\}$. Then $\#\widehat{P}_B(S, n)$ is even if S contains some even number or if n is odd, and is odd otherwise.*

Proof. We induct on $i_1 + i_2 + \dots + i_s$.

Our base case is $i_1 + i_2 + \dots + i_s = 0$, where, $S = \emptyset$. Clearly, S contains no even elements. Note that $\#\widehat{P}_B(\emptyset, n) = (3^n + 1)/2$ is even if n is odd and odd if n is even, thus our claim holds.

Recall Theorem 3.2, which states that if $S_1 = S - \{i_s\}$ and $S_2 = S_1 \cup \{i_s - 1\}$, then

$$\#\widehat{P}_B(S, n) = \binom{n}{i_s - 1} \#\widehat{P}_B(S_1, i_s - 1) 2^{2(n-i_s)+1} - \#\widehat{P}_B(S_1, n) - \#\widehat{P}_B(S_2, n).$$

Note that the first term will always be even, since it is multiplied by 2 with some positive exponent. Therefore $\#\widehat{P}_B(S, n)$ is even if and only if $\#\widehat{P}_B(S_1, n) + \#\widehat{P}_B(S_2, n)$ is even. If n is odd, then by our inductive assumption, $\#\widehat{P}_B(S_1, n)$ and $\#\widehat{P}_B(S_2, n)$ are both even, then their sum is even.

Now, consider the case where n is even. If S has at least one even element, let i_j be the first even element in S . Either $i_j \in S_1$ or $i_j = i_s$. In the first case, our inductive hypothesis implies that $\#\widehat{P}_B(S_1, n)$ and $\#\widehat{P}_B(S_2, n)$ are both even, then their sum is even. In the second case, S_1 has no even elements, thus by our inductive hypothesis, $\#\widehat{P}_B(S_1, n)$ is odd. Note that if i_s is even, then $i_s - 1$ is odd and S_2 has no even elements. Therefore $\#\widehat{P}_B(S_2, n)$ is also odd, thus their sum is even.

Now consider the case where S contains no even elements and n is still even. Since S_1 contains no even elements, by our inductive hypothesis $\#\widehat{P}_B(S_1, n)$ is odd. But since i_s is odd, $i_s - 1$ must be even. Therefore by our inductive hypothesis $\#\widehat{P}_B(S_2, n)$ is even, hence their sum is odd. \square

3.2. Relationship between $\#P_B(S, n)$ and $\#\widehat{P}_B(S, n)$. The following relation between $\#P_B(S, n)$ and $\#\widehat{P}_B(S, n)$ allows us to extrapolate some results from Section 2.

Proposition 3.5. *If S is admissible, then*

$$\#P_B(S, n) = \#\widehat{P}_B(S, n) + \#\widehat{P}_B(S \cup \{1\}, n).$$

Proof. For any $\pi \in P_B(S, n)$, either π has a descent at position 1 (i.e. $\pi_1 < \pi_2$), or it does not. Therefore we can write $P_B(S, n)$ as a union of disjoint sets $P_B(S, n) = P_\alpha(S, n) \cup P_\beta(S, n)$ where $\pi \in P_\alpha(S, n)$ has a descent at position 1 and $\pi \in P_\beta(S, n)$ does not. Note that $\pi \in P_\alpha(S, n)$ correspond to an element in $\widehat{P}_B(S \cup \{1\}, n)$ by adding a zero at the beginning of π . Hence, $\#P_\alpha(S, n) = \#\widehat{P}_B(S \cup \{1\}, n)$. Similarly, any $\pi \in P_\beta(S, n)$ corresponds to an element in $\widehat{P}_B(S, n)$ and thus $\#P_\beta(S, n) = \#\widehat{P}_B(S, n)$. Therefore $\#P_B(S, n) = \#\widehat{P}_B(S, n) + \#\widehat{P}_B(S \cup \{1\}, n)$. \square

Proposition 3.5 implies the following corollary.

Corollary 3.6. *If S is admissible and $2 \in S$, then*

$$\#P_B(S, n) = \#\widehat{P}_B(S, n).$$

Proof. If $2 \in S$, then that means that the peak set $S \cup \{1\}$ has two consecutive peaks which is clearly not possible. This means that $\#\widehat{P}_B(S \cup \{1\}, n) = 0$, implying that $\#P_B(S, n) = \#\widehat{P}_B(S, n)$ using Proposition 3.5. \square

Using this relation, we are able to find a formula for $\#\widehat{P}_B(S, n)$ in the case where the permutation has one peak.

Proposition 3.7. *Let $S = \{m\}$ be admissible, then*

$$\#\widehat{P}_B(\{m\}, n) = 4^{n-m-1} \sum_{i=1}^m \binom{n}{m-i} (3^{m-i} + 1) 4^i (-1)^{i+1} - (m \bmod 2) \left\lfloor \frac{3^n + 1}{2} \right\rfloor.$$

Proof. We will induct on m . First let $m = 1$ then using Theorem 3.3 and Proposition 3.5,

$$\begin{aligned} \#\widehat{P}_B(\{1\}, n) &= \#P_B(\emptyset, n) - \#\widehat{P}_B(\emptyset, n) \\ &= 2^{2n-1} - \left\lfloor \frac{3^n + 1}{2} \right\rfloor \\ &= 4^{n-2}(2)(4) - \left\lfloor \frac{3^n + 1}{2} \right\rfloor \\ &= 4^{n-1-1} \sum_{i=1}^1 \binom{n}{1-i} (3^{1-i} + 1) 4^i (-1)^{i+1} - (1 \bmod 2) \left\lfloor \frac{3^n + 1}{2} \right\rfloor. \end{aligned}$$

We assume our claim is true for m and we consider $m + 1$. Apply Theorem 3.2 for the peak set $S = \{m + 1\}$ to obtain the following

$$\#\widehat{P}_B(\{m+1\}, n) = \binom{n}{m} \#\widehat{P}_B(\emptyset, m) \#P_B(\emptyset, n-m) - \#\widehat{P}_B(\emptyset, n) - \#\widehat{P}_B(\{m\}, n).$$

Apply Proposition 2.3, Theorem 3.3 and the inductive hypothesis, then

$$\begin{aligned} \#\widehat{P}_B(\{m+1\}, n) &= \binom{n}{m} \left(\frac{3^m + 1}{2} \right) 2^{2(n-m)-1} - \left(\frac{3^n + 1}{2} \right) - \\ &\quad 4^{n-m-1} \sum_{i=1}^m \left[\binom{n}{m-i} (3^{m-i} + 1) 4^i (-1)^{i+1} \right] + (m \bmod 2) \left(\frac{3^n + 1}{2} \right) \\ &= -4^{n-m-1} \left(\binom{n}{m-0} (3^{m-0} + 1) (4)^0 (-1)^{0+1} \right. \\ &\quad \left. + \sum_{i=1}^m \binom{n}{m-i} (3^{m-i} + 1) 4^i (-1)^{i+1} \right) - (m+1 \bmod 2) \left(\frac{3^n + 1}{2} \right) \\ &= -4^{n-m-1} \sum_{i=0}^m \binom{n}{m-i} (3^{m-i} + 1) 4^i (-1)^{i+1} - (m+1 \bmod 2) \left(\frac{3^n + 1}{2} \right) \\ &= -4^{n-(m+1)-1} \sum_{i=1}^{m+1} \binom{n}{(m+1)-i} (3^{(m+1)-i} + 1) 4^i (-1)^{i+1} \\ &\quad - (m+1 \bmod 2) \left(\frac{3^n + 1}{2} \right). \end{aligned} \quad \square$$

Again, applying the relation to the case $S = \{m\}$ gives us a result for $S = \{1, m\}$.

Corollary 3.8. *Let $S = \{1, m\}$ be admissible, then*

$$\begin{aligned} \#\widehat{P}_B(S, n) &= 2^{2n-2} \left[\binom{n-1}{m-1} - 1 \right] - 4^{n-m-1} \sum_{i=1}^m \binom{n}{m-i} (3^{m-i} + 1)(4)^i (-1)^{i+1} \\ &\quad + (m \bmod 2) \left\lfloor \frac{3^n + 1}{2} \right\rfloor. \end{aligned}$$

Proof. Apply Proposition 3.5 to obtain the following

$$\#P_B(\{m\}, n) = \#\widehat{P}_B(\{m\}, n) + \#\widehat{P}_B(\{m\} \cup \{1\}, n).$$

We then use Theorem 2.4 together with Corollary 2.8 for $\#P_B(\{m\}, n)$ and Proposition 3.7 for $\#\widehat{P}_B(\{m\}, n)$. The rest follows. \square

The following result is a general result for a two-element peak set S .

Proposition 3.9. *Let $S = \{m, m+z\}$ be admissible, then $\#\widehat{P}_B(S, n)$ equals*

$$\begin{aligned} &\sum_{i=0}^{z-2} (-1)^i 2^{2(n-m-z+i+1)-1} \binom{n}{m+z-1-i} \left[4^{z-i-2} \sum_{j=1}^m \left((3^{m-j} + 1) 4^j (-1)^{j+1} \right. \right. \\ &\quad \left. \left. \binom{m+z-i-1}{m-j} \right) - (m \bmod 2) \left(\frac{3^{m+z-i-1} + 1}{2} \right) \right] - (z-1 \bmod 2) \left[4^{n-m-1} \right. \\ &\quad \left. \sum_{i=1}^m \left((3^{m-i} + 1) 4^i (-1)^{i+1} \binom{n}{m-i} \right) - (m \bmod 2) \left(\frac{3^n + 1}{2} \right) \right]. \end{aligned}$$

Proof. Let $S = \{m, m+z\}$ be admissible, let $S_1 = \{m\}$ and $S_2 = \{m, m+z-1\}$. Apply Theorem 3.2 to obtain the following recursive formula,

$$\begin{aligned} \#\widehat{P}_B(S, n) &= \binom{n}{m+z-1} \#\widehat{P}_B(S_1, m+z-1) \#P_B(\emptyset, n - (m+z) + 1) \\ &\quad - \#\widehat{P}_B(S_1, n) - \#\widehat{P}_B(S_2, n). \end{aligned}$$

Then apply the recursion $\#\widehat{P}_B(S_2, n)$ until $m+z-1$ approaches $m+1$, then if $a = m+z-1$ we arrive at the following formula for $\#\widehat{P}_B(S, n)$

$$\begin{aligned} &\sum_{i=0}^{z-2} \left[(-1)^i \binom{n}{a-i} \#\widehat{P}_B(\{m\}, m+z-1-i) \#P_B(\emptyset, n - (m+z) + 1 - i) \right] \\ &\quad - (z-1 \bmod 2) \#\widehat{P}_B(\{m\}, n). \end{aligned}$$

Using Proposition 3.7 and Proposition 2.3 we obtain the result. \square

We have the following special case when S is a three element set.

Proposition 3.10. *Let $S = \{1, m, m + 2\}$ be admissible, then*

$$\begin{aligned} \#\widehat{P}_B(S, n) &= 4^{n-m-1} \sum_{i=1}^m (3^{m-i} + 1)(4)^i (-1)^{i+1} \left[\binom{n}{m-i} - \frac{1}{2} \binom{n}{m+1} \binom{m+1}{m-i} \right] \\ &\quad + 4^{n-1} \left[\frac{m-1}{2} \binom{n}{m+1} + 1 - \binom{n-1}{m-1} \right] \\ &\quad + (m \bmod 2) \left[\binom{n}{m+1} \left(\frac{3^{m+1} + 1}{2} \right) 2^{2(n-m-1)-1} - \frac{3^n + 1}{2} \right]. \end{aligned}$$

Proof. Let $S_1 = \{1, m\}$ and let $S_2 = \{1, m, m + 1\}$ and apply Theorem 3.2. Note that $\#\widehat{P}_B(S_2, n) = 0$, then

$$\#\widehat{P}_B(S, n) = \binom{n}{m+1} \#\widehat{P}_B(S_1, m+1) \#P_B(\emptyset, n - (m+2) + 1) - \#\widehat{P}_B(S_1, n).$$

The result follows from Corollary 3.8 and Proposition 2.3. \square

4. PERMUTATIONS IN S_n WITH $\pi_0 = 0$

Let S_n be the set of all permutations $\pi = \pi_1 \pi_2 \dots \pi_n$ of $[n]$. Recall that we define the set $P(\pi)$ as the set of all peaks of π . Now, we introduce the condition $\pi_0 = 0$, which will allow our peak set to contain $i = 1$. Define $\widehat{P}(\pi)$ as the set of all peaks of π with $\pi_0 = 0$, and $\widehat{P}(S, n)$ as the set of all permutations of S_n having $\pi_0 = 0$ and peak set S .

We first give the recursive method that will allow us to compute formulas for the special peak sets $S = \{\{m\}, \{1, m\}, \{1, m, m + 2\}, \{1, m, n - 1\}\}$. This recursive formula is based on Theorem 2.4.

Theorem 4.1. *Let $S = \{i_1, i_2, \dots, i_s\}$ be an admissible set then,*

$$\#\widehat{P}(S, n) = \binom{n}{i_s - 1} \#\widehat{P}(S_1, i_s - 1) 2^{n-i_s} - \#\widehat{P}(S_1, n) - \#\widehat{P}(S_2, n).$$

Proof. The method to obtain this expression is based on the construction of Π in Theorem 2.4. For this we let the $\max(S) = i_s$ and $k = i_s - 1$, and the rest follows. \square

We need the next result in order to prove future cases.

Proposition 4.2. *The number of permutations with peak set $\widehat{P}(\pi) = \emptyset$ is equal to one.*

Proof. Let $\pi = \pi_0 \pi_1 \dots \pi_n$ be a permutation in $\widehat{P}(\emptyset, n)$, and $\pi_0 = 0$. Now suppose $\pi_1 > 1$, then there is an integer $m \in \{2, 3, \dots, n\}$ such that $\pi_m = 1$, hence there is a peak at a position $i \in \{1, \dots, m - 1\}$. Therefore, in order to have no peaks, π_1 must equal 1. Now suppose $\pi_2 > 2$, then there is an integer $m \in \{3, 4, \dots, n\}$ such that $\pi_m = 2$, hence there is a peak at a position $i \in \{2, \dots, m - 1\}$. Therefore, in order to have no peaks, π_2 must equal 2. Apply the same procedure $n - 1$ times. Thus in order to have a permutation of the form $\pi = \pi_0 \pi_1 \dots \pi_n$ with $\pi_0 = 0$ and peak set $\widehat{P}(\pi) = \emptyset$, the permutation must satisfy $\pi_i = i$, for $i \in \{0, 1, \dots, n\}$. \square

We need the next result as a base case for the special case $S = \{m\}$.

Proposition 4.3. *Let $S = \{1\}$. Then*

$$\widehat{P}(S, n) = 2^{n-1} - 1.$$

Proof. Let $\pi \in P(\emptyset, n)$. Then either $\pi \in \widehat{P}(\{1\}, n)$ or by Proposition 4.2, π is the identity in S_n . Therefore by [4] Proposition 2 we have $\#\widehat{P}(\{1\}, n) = \#P(\emptyset, n) - 1 = 2^{n-1} - 1$. \square

Now we make use of the recursive formula in Theorem 4.1 and the result in Proposition 4.2 to obtain a recursive formula for the case when $S = \{m\}$. This will lead us to have a closed formula for this case.

Lemma 4.4. *Let $S = \{m\}$ be admissible. Then we can find the number of permutations with peak set S recursively by*

$$\#\widehat{P}(S, n) = 2^{n-m} \binom{n}{m-1} - \#\widehat{P}(\{m-1\}, n) - 1.$$

Proof. Let Π be the set of all $\pi \in S_n$ such that if $\pi = \pi_1\pi_2 \dots \pi_{m-1}\pi_m \dots \pi_n$, then $\widehat{P}(\pi_1\pi_2 \dots \pi_{m-1}) = \emptyset$ and $P(\pi_m \dots \pi_n) = \emptyset$.

Note that since π could have a peak at position $m-1$, position m , or neither, then Π is a union of disjoint sets $\Pi = \widehat{P}(\{m-1\}, n) \cup \widehat{P}(\{m\}, n) \cup \widehat{P}(\emptyset, n)$. Thus,

$$\#\widehat{P}(\{m\}, n) = \#\Pi - \#\widehat{P}(\emptyset, n) - \#\widehat{P}(\{m-1\}, n). \quad (4)$$

We can construct Π by first choosing the $m-1$ first elements, arranging them so their peak set is the empty set, and arranging the $n-m+1$ other elements so their peak set is the empty set. Therefore by Proposition 4.2 and by [4] Proposition 2 we have

$$\begin{aligned} \#\Pi &= \binom{n}{m-1} \#\widehat{P}(\emptyset, m-1) \#P(\emptyset, n-m+1) \\ &= \binom{n}{m-1} 2^{n-m}. \end{aligned}$$

Also, by Proposition 4.2, $\#\widehat{P}(\emptyset, n) = 1$. Therefore by (4) the result follows. \square

Proposition 4.5. *Let $S = \{m\}$ be admissible. Then*

$$\#\widehat{P}(S, n) = \sum_{i=1}^m 2^{n-i} \binom{n}{i-1} (-1)^{m-i} - (m \pmod{2}).$$

Proof. We induct on m . Our base case is $m = 1$. Then by Proposition 4.3, our claim is true. Now, by our inductive assumption,

$$\widehat{P}(\{m-1\}, n) = \sum_{i=1}^{m-1} 2^{n-i} \binom{n}{i-1} (-1)^{m-1-i} - (m-1 \pmod{2}).$$

Using this value for $\widehat{P}(\{m-1\}, n)$ in the recursive formula given in Lemma 4.4, we find

$$\begin{aligned} \#\widehat{P}(S, n) &= 2^{n-m} \binom{n}{m-1} - 1 - \left(\sum_{i=1}^{m-1} 2^{n-i} \binom{n}{i-1} (-1)^{m-1-i} - (m-1 \pmod{2}) \right) \\ &= 2^{n-m} \binom{n}{m-1} (-1)^0 + \sum_{i=1}^{m-1} 2^{n-i} \binom{n}{i-1} (-1)^{m-i} - (m \pmod{2}) \\ &= \sum_{i=1}^m 2^{n-i} \binom{n}{i-1} (-1)^{m-i} - (m \pmod{2}). \quad \square \end{aligned}$$

From Proposition 4.5 we notice that we can factor a power of two out of the summation, in this way we obtain a new formula for the case $S = \{m\}$.

Proposition 4.6. *Let $S = \{m\}$ be admissible, then*

$$\#\widehat{P}(\{m\}, n) = \frac{p_{m-1}(n)2^{n-m}}{(m-1)!} - (m \pmod{2}) \quad (5)$$

where $p_{m-1}(n) = p(S, n)$ is a polynomial depending on S such that $p(n)$ is an integer for all integral n . Also, $\deg(p_{m-1}(n)) = m-1$.

Proof. We will prove this by induction on m . The case where $m = 1$ is true since we already found that $\#\widehat{P}(\{1\}, n) = 2^{n-1} - 1$ where $p(n) = 1$ and is of degree $1-1 = 0$. We will assume that the proposition is true for m and we will prove it for the $m+1$ case. We will first use the recurrence relation in Lemma 4.4 with our inductive assumption to get

$$\begin{aligned} \#\widehat{P}(\{m+1\}, n) &= \binom{n}{m} 2^{n-(m+1)} - 1 - \left(\frac{p_m(n)2^{n-m}}{(m-1)!} - (m \pmod{2}) \right) \\ &= \frac{2^{n-(m+1)}}{m!} \left(\frac{n!}{(n-m)!} - 2mp_m(n) \right) - (m+1 \pmod{2}). \end{aligned}$$

We now have to prove that

$$\frac{n!}{(n-m)!} - 2mp_m(n)$$

is a polynomial in terms of n with degree m . Because m is fixed, we can see that this expression is a polynomial in terms of n and we can also clearly see that $n!/(n-m)!$ has degree m and because of the inductive hypothesis $2m \cdot p_m(n)$ has degree $m-1$ which means that the whole expression has degree m which completes the induction. \square

Corollary 4.7. *Additionally,*

$$p_m(n) = m! \sum_{i=0}^m 2^i (-1)^i \binom{n}{m-i}. \quad (6)$$

Proof. From Proposition 4.6 we have a recursive formula for $p_m(n)$,

$$p_m(n) = \frac{n!}{(n-m)!} - 2mp_{m-1}(n).$$

We will prove this by induction on m . The case where $m = 0$ is obviously true since using the formula we get that $p_0(n) = 1$ which agrees with $\#\widehat{P}(\{1\}, n) = 2^{n-1} \cdot 1 - 1$. We will assume that the proposition is true for m and we will prove it for the $m + 1$ case. Using the recursive formula and the inductive hypothesis we get

$$\begin{aligned} p_{m+1}(n) &= \frac{n!}{(n-(m+1))!} - 2(m+1) \cdot p_m(n) \\ &= \frac{n!}{(n-(m+1))!} - 2(m+1) \left(\frac{n!}{(n-m)!} + m! \sum_{i=1}^m 2^i (-1)^i \binom{n}{m-i} \right) \\ &= \frac{n!}{(n-(m+1))!} - 2(m+1) \left(m! \binom{n}{m} + m! \sum_{i=1}^m 2^i (-1)^i \binom{n}{m-i} \right) \\ &= \frac{n!}{(n-(m+1))!} + (m+1)! \sum_{i=1}^{m+1} 2^i (-1)^i \binom{n}{m+1-i} \\ &= (m+1)! \sum_{i=0}^{m+1} 2^i (-1)^i \binom{n}{m+1-i}. \end{aligned}$$

which is what we wanted thus completing the induction. \square

In the following proposition we compute $\#\widehat{P}(\{m\}, n)$ using a different approach. The new formula we obtain will help us to compute other special cases such when $S = \{1, n-1\}$ in a simpler way.

Proposition 4.8. *Let $S = \{n-m\}$ be admissible. Then*

$$\#\widehat{P}(S, n) = \sum_{i=0}^{m-1} 2^i \binom{n-(m-i)}{i+1}.$$

Proof. Let $\pi = \pi_1 \pi_2 \dots \pi_n$ be a permutation in S_n . We will prove this proposition by induction on m . We will first prove the base case, when $m = 1$. Letting $m = 1$ means that we will have a peak only in the $(n-1)$ -th position. Note that $\pi_i \in [n]$ which means that the number on the $n-1$ position has to be n because otherwise there would either be no peaks or more than one peaks. We know that the numbers before the $n-1$ position must be in increasing order, thus the permutation is completely determined by the element in the n th position. There are $\binom{n-1}{1}$ ways to choose the last element.

Now assume the proposition is true for $m \geq 1$, we will prove that it is true for $m+1$. This means that we have a peak at the $n-(m+1)$ -th position, then using reasoning similar to the one used in the inductive hypothesis, we know that n is either in position $n-(m+1)$ or in the last position. If n is in the position of the peak, the number of permutations that satisfy this condition is equal to the number of ways to choose the last $m+1$ numbers in the permutation times

the number of ways to arrange these $m + 1$ numbers so that they do not form a peak. This number is equal $2^m \binom{n-1}{m+1}$.

If n is in the n -th position of the permutation, then we can reduce the computation to the m -th case of the induction. Thus,

$$2^m \binom{n-1}{m+1} + \sum_{i=0}^{m-1} 2^i \binom{n-(m-i)}{i+1} = \sum_{i=0}^m 2^i \binom{n-(m+1-i)}{i+1}$$

which was what we were looking for. \square

Note that doing a change of variable in the previous result will lead us to obtain better results for the case $S = \{m\}$.

Remark 4.9. Let $S = \{m\}$ be admissible. Notice from the Proposition 4.8 that we can write $\#\hat{P}(S, n)$, as

$$\#\hat{P}(S, n) = \sum_{i=0}^{n-(m+1)} 2^i \binom{m+i}{i+1}.$$

Proposition 4.10. Let $S = \{1, m\}$ be admissible, then

$$\begin{aligned} \#\hat{P}(\{1, m\}, n) &= \sum_{i=1}^{m-2} \binom{n}{m-i} (2^{m-i-1} - 1)(2^{n-(m-i+1)})(-1)^{i+1} \\ &\quad - (m \bmod 2)(2^{n-1} - 1). \end{aligned}$$

Proof. Let $S = \{1, m\}$ be admissible and let $S_1 = \{1\}$ and $S_2 = \{1, m-1\}$. Recall Theorem 4.1 provides the following recursive fomula

$$\begin{aligned} \#\hat{P}(S, n) &= \binom{n}{m-1} \#\hat{P}(\{1\}, m-1) \#P(\emptyset, n-m+1) - \#\hat{P}(\{1\}, n) \\ &\quad - \#\hat{P}(\{1, m-1\}, n) \\ &= \binom{n}{m-1} (2^{m-2} - 1)(2^{n-m}) - (2^{n-1} - 1) - \#\hat{P}(\{1, m-1\}, n). \end{aligned}$$

To obtain the terms $(2^{m-2} - 1)$ and $(2^{n-1} - 1)$ apply Proposition 4.3 and the term $(2^{n-m+1-1})$ follows from Proposition 2 in [4]. The result follows by induction. \square

Proposition 4.11. Let $S = \{1, m, m+2\}$ be admissible, then $\#\hat{P}(S, n)$ equals

$$\begin{aligned} &\sum_{i=1}^{m-2} (2^{m-i-1} - 1)(-1)^{i+1} (2^{n-m+i-1}) \left[\binom{n}{m+1} \binom{m+1}{m-i} \frac{1}{2} - \binom{n}{m-i} \right] \\ &+ (m \bmod 2) \left(2^{n-1} - 1 - \binom{n}{m+1} 2^{n-m-2} (2^m - 1) \right). \end{aligned}$$

Proof. We apply the same method as in Proposition 4.10. For this we let $S_1 = \{1, m\}$ and $S_2 = \{1, m, m+1\}$. Note that $\#\hat{P}(\{1, m, m+1\}, n) = 0$ since we can not have consecutive peaks. Then we construct Π based on the

number of ways to arrange the permutations in S_1 and S_2 and use Theorem 4.1 to obtain the recursive formula.

$$\begin{aligned} \#\widehat{P}(\{1, m, m+2\}, n) &= \Pi - \#\widehat{P}(\{1, m\}, n) \\ &= \binom{n}{m+1} \#\widehat{P}(\{1, m\}, m+1) \#P(\emptyset, n - (m+2) + 1) - \#\widehat{P}(\{1, m\}, n) \end{aligned}$$

Now for the terms $\#\widehat{P}(\{1, m\}, m+1)$ and $\#\widehat{P}(\{1, m\}, n)$ apply Proposition 4.10, and for $\#P(\emptyset, n - (m+2) + 1)$ apply Proposition 2 in [4]. The result follows. \square

Proposition 4.12. *Let $S = \{1, m, n-1\}$ be admissible, then we have the following recursive formula for $\#\widehat{P}(S, n)$*

$$\begin{aligned} &\sum_{i=1}^{m-2} (2^{m-i-1} - 1)(2^{n-m+i-1})(-1)^{i+1} \left(\frac{1}{2} \binom{n}{n-2} \binom{n-2}{m-i} - \binom{n}{m-i} \right) \\ &- (m \bmod 2) \left((2^{n-2} - 2) \binom{n}{2} - 2^{n-1} + 1 \right) - \#\widehat{P}(\{1, m, n-2\}, n). \end{aligned}$$

Proof. let $S_1 = \{1, m\}$ and $S_2 = \{1, m, n-2\}$. Apply Theorem 4.1 to obtain the recursive formula.

$$\begin{aligned} \#\widehat{P}(\{1, m, n-1\}, n) &= 2 \binom{n}{2} \#\widehat{P}(\{1, m\}, n-2) \#P(\emptyset, n - (n-1) + 1) \\ &\quad - \#\widehat{P}(\{1, m\}, n) - \#\widehat{P}(\{1, m, n-2\}, n). \end{aligned}$$

Now for $\#\widehat{P}(\{1, m\}, n-2)$ and $\#\widehat{P}(\{1, m\}, n)$ apply Proposition 4.10. The result follows. \square

4.1. Relationship between $\#P(S, n)$ and $\#\widehat{P}(S, n)$. In this section we use the relationship between $\#P(S, n)$ and $\#\widehat{P}(S, n)$ to find new formulas for special cases. We begin by giving this relationship.

Proposition 4.13. *If S is admissible then*

$$\#P(S, n) = \#\widehat{P}(S, n) + \#\widehat{P}(S \cup \{1\}, n).$$

We omit the proof of this proposition since it is identical at the proof of Proposition 3.5.

Corollary 4.14. *Let $S = \{1, m\}$ be an admissible set, then*

$$\#\widehat{P}(\{1, m\}, n) = \left(\binom{n-1}{m-1} - 1 \right) 2^{n-2} - \#\widehat{P}(\{m\}, n).$$

Proof. We apply Proposition 4.13 to the case $S = \{m\}$, using the fact that $\#P(\{m\}, n) = \left(\binom{n-1}{m-1} - 1 \right) 2^{n-2}$ from Theorem 6 in [4]. \square

Corollary 4.15. *Let $S = \{1, n-1\}$ be an admissible set then,*

$$\#\widehat{P}(S, n) = 2^{n-2}(n-2) - (n-1).$$

Proof. Note that $S = \{1, n-1\}$ is a special case of $S = \{1, m\}$ with $m = n-1$. Now apply Corollary 4.14 and Remark 4.9. Then we have,

$$\begin{aligned} \#\widehat{P}(\{1, n-1\}, n) &= \left(\binom{n-1}{(n-1)-1} - 1 \right) 2^{n-2} - \sum_{i=0}^{n-((n-1)+1)} 2^i \binom{(n-1)+i}{i+1} \\ &= 2^{n-2}(n-2) - (n-1). \quad \square \end{aligned}$$

We propose now a result that also comes as a result of Proposition 4.13.

Corollary 4.16. *Let $S = \{i_1, i_2, \dots, i_s\}$ where $i_1 = 2$. Then*

$$\#\widehat{P}(S, n) = p(n)2^{n-s-1}$$

where $p(n) = p(S, n)$ is an polynomial depending on S with degree $i_s - 1$, such that $p(n)$ is an integer for all integral n .

Furthermore, if we let $m = \max(S)$, $S_1 = S - \{m\}$, and $S_2 = S_1 \cup \{m-1\}$, then

$$p(S, n) = p(S_1, m-1) \binom{n}{m-1} - 2p(S_1, n) - p(S_2, n).$$

Proof. We apply Proposition 4.13, and note that $\#\widehat{P}(S \cup \{1\}, n) = 0$ if $2 \in S$. Thus,

$$\#\widehat{P}(S, n) = \#P(S, n).$$

Hence, we can apply Theorem 3 in [4] for $\#P(S, n)$ to this special case of $\#\widehat{P}(S, n)$. \square

4.2. Parity of $\#\widehat{P}(S, n)$. Notice from the previous results how the number of permutations varies according to the parity of some integer related to the peaks. This lead us to establish a relation between the parity of the numbers of permutations with a given peak set and the parity of the numbers in the peak set.

Theorem 4.17. *Let $S = \{i_1, i_2, \dots, i_s\}$ be admissible. Then $\#\widehat{P}(S, n)$ is even if and only if there exists $i_j \in S$ such that i_j is even.*

Proof. We induct on $i_1 + i_2 + \dots + i_s$. Our base case is $i_1 + i_2 + \dots + i_s = 0$, where, $S = \emptyset$. Clearly, S contains no even elements. By Proposition 4.2, $\#P(\emptyset, n) = 1$ for all n , thus our claim holds.

Now, by Theorem 4.1 if we let $S_1 = S - \{i_s\}$ and $S_2 = S_1 \cup \{i_s - 1\}$, then

$$\#\widehat{P}(S, n) = \binom{n}{i_s-1} 2^{n-i_s} \#\widehat{P}(S_1, i_s-1) - \#\widehat{P}(S_1, n) - \#\widehat{P}(S_2, n).$$

S admissible implies $i_s < n$. Since $n - i_s > 0$, then $\binom{n}{i_s-1} 2^{n-i_s} \#\widehat{P}(S_1, i_s-1)$ is even in all cases.

Assume that $\#\widehat{P}(S, n)$ is even. Then either $\#\widehat{P}(S_1, n)$ and $\#\widehat{P}(S_2, n)$ are both even, or they are both odd. If $\#\widehat{P}(S_1, n)$ is even, then, by the inductive hypothesis, S_1 contains some even element. Since $S_1 \subset S$, S then contains some even element. If $\#\widehat{P}(S_2, n)$ is odd, then, by the inductive hypothesis, all its elements are odd, including $i_s - 1$. Therefore i_s is even, hence S contains some even element.

Assume that S contains some even element. Then either S_1 contains some even element and i_s is even, or S_1 contains some even element and i_s is odd, or S_1 does not contain any even elements and i_s is even. In the first case and the second case, by inductive hypothesis $\#\widehat{P}(S_1, n)$ and $\#\widehat{P}(S_2, n)$ are both even, then $\#\widehat{P}(S, n)$ is even. In the third case, $i_s - 1$ is odd, thus by the inductive hypothesis $\#\widehat{P}(S_1, n)$ and $\#\widehat{P}(S_2, n)$ are both odd, hence $\#\widehat{P}(S, n)$ is even. \square

5. ACKNOWLEDGEMENTS

We thank Ivelisse Rubio, and MSRI for their help and support. This work was conducted at MSRI-UP and supported by the National Security Agency (NSA) grant H-98230-13-1-0262 and the National Science Foundation (NSF) grant 1156499.

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