A Criterion for Deficient Numbers Using the Abundancy Index and Deficiency Functions

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Abstract: We show that n is almost perfect if and only if $I(n) - 1 < D(n) \leq I(n)$, where $I(n)$ is the abundancy index of n and $D(n)$ is the deficiency of n. This criterion is then extended to the case of integers m satisfying $D(m) > 1$.
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1 Introduction

If n is a positive integer, then we write $\sigma(n)$ for the sum of the divisors of n. A number n is *almost perfect* if $\sigma(n) = 2n - 1$. It is currently unknown whether there are any other almost perfect numbers apart from those of the form 2^k, where $k \geq 0$.

We denote the abundancy index I of the positive integer w as $I(w) = \dfrac{\sigma(w)}{w}$. We also denote the deficiency D of the positive integer x as $D(x) = 2x - \sigma(x)$ [4].

2 Preliminary Lemmata

We begin with some preliminary results.

Note that $I(y) = D(y)$ if and only if

$$\sigma(y) = 2y^2 - y\sigma(y)$$

which corresponds to

$$(y+1)\sigma(y) = 2y^2 \Longleftrightarrow \sigma(y) = \frac{2y^2}{y+1} = \frac{2y^2 - 2}{y+1} + \frac{2}{y+1} = 2(y-1) + \frac{2}{y+1}.$$

Since $\sigma(y)$ and $2(y-1)$ are both integers, this implies that $(y+1) \mid 2$, from which it follows that $y + 1 \leq 2$. Hence, $y \leq 1$. Together with $1 \leq y$, this means that $y = 1$.

We state this result as our initial lemma.

Lemma 2.1. *$I(n) = D(n)$ if and only if $n = 1$.*

Next, we show conditions that are sufficient and necessary for n to be almost perfect.

Lemma 2.2. *If n is a positive integer which satisfies the inequality*

$$\frac{2n}{n+1} \leq I(n) < 2,$$

then n is almost perfect.

Proof. Let n be a positive integer, and suppose that

$$\frac{2n}{n+1} \leq I(n) < 2.$$

Then we have

$$\frac{2n^2}{n+1} \leq \sigma(n) < 2n.$$

But

$$\frac{2n^2}{n+1} = 2n - 2 + \frac{2}{n+1} \leq \sigma(n) < 2n.$$

Since $\sigma(n)$ is an integer and $n \geq 1$, this last chain of inequalities forces

$$\sigma(n) = 2n - 1,$$

and we are done. □

Lemma 2.3. *If n is almost perfect, then n satisfies the inequality*

$$\frac{2n}{n+1} \leq I(n) < 2.$$

Proof. Let n be a positive integer, and suppose that $\sigma(n) = 2n - 1$.

It follows that

$$I(n) = \frac{\sigma(n)}{n} = 2 - \frac{1}{n} < 2.$$

Now we want to show that

$$\frac{2n}{n+1} \leq I(n).$$

Assume to the contrary that $I(n) < \frac{2n}{n+1}$. (Note that this forces $n > 1$.) Mimicking the proof in Lemma 2.2, we have

$$\sigma(n) < \frac{2n^2}{n+1},$$

from which it follows that

$$\sigma(n) < 2n - 2 + \frac{2}{n+1} < (2n-2) + 1 = 2n - 1.$$

This contradicts our assumption that n is almost perfect. Hence the reverse inequality

$$\frac{2n}{n+1} \leq I(n)$$

holds. □

Remark 2.1. *By their definition, all almost perfect numbers are automatically deficient. But of course, not all deficient numbers are almost perfect.*

By Remark 2.1, it seems natural to try to establish an upper bound for the abundancy index of an almost perfect number n, that is *strictly* less than 2, and which (perhaps) can be expressed as a rational function of n (similar to the form of the lower bound given in Lemma 2.2 and Lemma 2.3).

Lemma 2.4. *If n is a positive integer which satisfies the inequality*

$$\frac{2n}{n+1} \leq I(n) < \frac{2n+1}{n+1},$$

then n is almost perfect.

Proof. Let n be a positive integer, and suppose that

$$\frac{2n}{n+1} \leq I(n) < \frac{2n+1}{n+1}.$$

Again, mimicking the proof in Lemma 2.2, we have

$$\frac{2n^2}{n+1} \leq \sigma(n) < \frac{2n^2+n}{n+1},$$

from which it follows that

$$2n - 2 + \frac{2}{n+1} \leq \sigma(n) < 2n - 1 + \frac{1}{n+1}.$$

Since $n \geq 1$, this last chain of inequalities forces the equation

$$\sigma(n) = 2n - 1$$

to be true. Consequently, n is almost perfect, and we are done. □

We now show that the (nontrivial) upper bound obtained for the abundancy index of n in Lemma 2.4 is also necessary for n to be almost perfect.

Lemma 2.5. *If n is almost perfect, then n satisfies the inequality*

$$\frac{2n}{n+1} \leq I(n) < \frac{2n+1}{n+1}.$$

Proof. It suffices to prove that if n is almost perfect, then the inequality

$$I(n) < \frac{2n+1}{n+1}$$

holds. To this end, assume to the contrary that

$$\frac{2n+1}{n+1} \leq I(n).$$

Mimicking the proof in Lemma 2.4, we obtain

$$2n - 1 + \frac{1}{n+1} = \frac{2n^2+n}{n+1} \leq \sigma(n),$$

from which it follows that

$$2n - 1 < \sigma(n).$$

This contradicts our assumption that n is almost perfect, and we are done. □

3 Main Results

Collecting all the results from Lemma 2.2, Lemma 2.3, Lemma 2.4 and Lemma 2.5, we now have the following theorem.

Theorem 3.1. *Let n be a positive integer. Then n is almost perfect if and only if the following chain of inequalities hold:*

$$\frac{2n}{n+1} \leq I(n) < \frac{2n+1}{n+1}.$$

Remark 3.1. *Note that equality holds in*

$$\frac{2n}{n+1} \leq I(n)$$

if and only if $n = 1$.

Also, it is trivial to verify that the known almost perfect numbers $n = 2^k$ (for integers $k \geq 0$) satisfy the inequalities in Theorem 3.1. In fact, Theorem 3.1 can be used to rule out particular families of integers from being almost perfect numbers. (For example, $n_1 = p^r$ and $n_2 = p^r q^s$, where p, q are odd primes and r, s are even, are easily shown not to satisfy $\sigma(n_i) = 2n_i - 1$ ($i = 1, 2$) using the criterion in Theorem 3.1.)

This can, of course, also be attempted for the case $M = 2^r b^2$ with $\sigma(M) = 2M - 1$, where $r \geq 1$ and b is an odd composite indivisible by 3, although a complete proof appears to be difficult [1].

We now give a proof for the following result (which was originally conjectured in `http://arxiv.org/pdf/1308.6767v4.pdf`).

Theorem 3.2. *The bounds in Theorem 3.1 are best-possible.*

Proof. The bounds

$$\frac{2n}{n+1} \leq I(n) = \frac{\sigma(n)}{n} < \frac{2n+1}{n+1}$$

are easily seen to be equivalent to

$$2n \cdot n \leq (n+1) \cdot \sigma(n) < 2n \cdot n + n$$

which further implies that

$$n\,(2n - \sigma(n)) \leq \sigma(n)$$

and

$$\sigma(n) - n < (2n - \sigma(n))\,n$$

so that we obtain

$$I(n) - 1 < 2n - \sigma(n) = D(n) \leq I(n).$$

Since

$$D(n) = 1 \leq I(n) = \frac{\sigma(n)}{n} < 2 = D(n) + 1$$

when n is almost perfect, we obtain the claimed result. □

Remark 3.2. *Following the proof of Theorem 3.2, and by using Lemma 2.1, we have the following additional observations when* $m > 1$*:*

1. *If* $D(m) < I(m) < 2$*, then* m *is almost perfect, by Lemma 2.2.*

2. *The case* $I(m) < D(m) < 2$ *cannot occur, as it implies that* $2m - 2 < \sigma(m) < 2m$*, so that* $D(m) = 2m - \sigma(m) = 1$*, contradicting* $1 \leq I(m) < D(m) = 1$.

3. *If* $I(m) < 2 \leq D(m)$*, then* m *is not almost perfect, by Lemma 2.2.*

Next, we attempt to extend the results in Theorem 3.1 to the case of positive integers m satisfying $D(m) > 1$. To begin with, notice that, if we write $1 = D(n)$, then the bounds in Theorem 3.1 take the following form:

$$\frac{2n}{n + D(n)} \leq I(n) < \frac{2n + D(n)}{n + D(n)}.$$

By Remark 3.1, equality holds in

$$\frac{2n}{n + D(n)} \leq I(n)$$

if and only if $n = 1$, which is true if and only if $I(n) = D(n)$ by Lemma 2.1.

Now, assume that m is a positive integer with $D(m) > 1$. We want to show that the following theorem holds.

Theorem 3.3. *Let* m *be a positive integer, and suppose that* $D(m) > 1$*. Then we have the following bounds for the abundancy index of* m*, in terms of the deficiency of* m*:*

$$\frac{2m}{m + D(m)} < I(m) < \frac{2m + D(m)}{m + D(m)}.$$

Proof. Assume that m is a positive integer satisfying $D(m) > 1$. (In particular, note that $m > 1$ (by Lemma 2.1), and therefore that $I(m) > 1$ (since $I(m) = 1$ if and only if $m = 1$).)

Suppose to the contrary that

$$I(m) \leq \frac{2m}{m + D(m)} = \frac{2m}{3m - \sigma(m)}.$$

Then we have (noting that $3m - \sigma(m) > 2m - \sigma(m) > 1$)

$$3m\sigma(m) - (\sigma(m))^2 \leq 2m^2.$$

Dividing through by m^2, we get

$$(I(m))^2 - 3I(m) + 2 \geq 0$$

which implies that

$$(I(m) - 2)(I(m) - 1) \geq 0.$$

This is a contradiction, as we know that $1 < I(m) < 2$.

Now, assume that

$$I(m) \geq \frac{2m + D(m)}{m + D(m)} = \frac{4m - \sigma(m)}{3m - \sigma(m)}.$$

Then we obtain (noting that $4m - \sigma(m) > 3m - \sigma(m) > 2m - \sigma(m) > 1$)

$$3m\sigma(m) - (\sigma(m))^2 \geq 4m^2 - m\sigma(m).$$

Again, dividing through by m^2, we get

$$(I(m))^2 - 4I(m) + 4 \leq 0$$

which implies that

$$(I(m) - 2)^2 \leq 0.$$

This contradicts $1 < I(m) < 2$.

Consequently, we have the bounds

$$\frac{2m}{m + D(m)} < I(m) < \frac{2m + D(m)}{m + D(m)}$$

if $D(m) > 1$, and we are done. □

We end this section with the following result, which closely parallels that of Theorem 3.2.

Theorem 3.4. *The bounds in Theorem 3.3 are best-possible.*

Proof. The bounds

$$\frac{2m}{m + D(m)} < I(m) < \frac{2m + D(m)}{m + D(m)}$$

are easily seen to be equivalent to

$$2m < \sigma(m) + D(m)I(m) < 2m + D(m)$$

which further implies that

$$D(m) = 2m - \sigma(m) < D(m)I(m) < (2m - \sigma(m)) + D(m) = 2D(m)$$

so that we obtain

$$1 < I(m) < 2$$

since $D(m) > 1$. Since $1 < I(m) < 2$ also holds for deficient integers m satisfying $D(m) > 1$, the claimed result follows. □

4 Conclusion

The results in this article originated from the author's attempts to show that d is almost perfect, if $e^f d^2$ is an odd perfect number with Euler prime e satisfying $e \equiv f \equiv 1 \pmod 4$. (That is, since

$$I(d) < I(d^2) = \frac{2}{I(e^f)} \leq \frac{2e}{e+1},$$

the author was hoping to show $d < e$, and thereby prove the Descartes-Frenicle-Sorli conjecture (i.e., $f = 1$) as an immediate consequence.)

It is now known that, in fact, one has

$$I(d^2) < \frac{2d^2}{d^2+1}$$

and

$$I(d) < \frac{2d}{d+1},$$

so that d^2 (and therefore, d) are not almost perfect.

Additionally, Brown [3] has recently announced a proof for $e < d$, and a partial proof that $e^f < d$ holds "in many cases".

Nonetheless, work is in progress in [1] to try to rule out even almost perfect numbers other than the powers of two.

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References

[1] J. R. M. Antalan and J. A. B. Dris, Some new results on even almost perfect numbers which are not powers of two, preprint (2016), `http://arxiv.org/abs/1602.04248`.

[2] J. R. M. Antalan and R. P. Tagle, Revisiting forms of almost perfect numbers, preprint (2014).

[3] P. A. Brown, A partial proof of a conjecture of Dris, preprint (2016), `http://arxiv.org/abs/1602.01591`.

[4] N. J. A. Sloane, OEIS sequence A033879 - Deficiency of n, or $2n - \sigma(n)$, `http://oeis.org/A033879`.