Laurent biorthogonal polynomials, q-Narayana polynomials and domino tilings of the Aztec diamonds

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Abstract

A Töplitz determinant whose entries are described by a q-analogue of the Narayana polynomials is evaluated by means of Laurent biorthogonal polynomials which allow of a combinatorial interpretation in terms of Schröder paths. As an application, a new proof is given to the Aztec diamond theorem by Elkies, Kuperberg, Larsen and Propp concerning domino tilings of the Aztec diamonds. The proof is based on the correspondence with non-intersecting Schröder paths developed by Eu and Fu.

Keywords: orthogonal polynomials, Narayana polynomials, Aztec diamonds, lattice paths, Hankel determinants

1. Introduction

Laurent biorthogonal polynomials (LBPs) are orthogonal functions which play fundamental roles in the theory of two-point Padé approximants at zero and infinity [15]. In Padé approximants, LBPs appear as the denominators of the convergents of a T-fraction. (See also, e.g., [14, Chapter 7] and [12, 24].) Recently, the author exhibited a combinatorial interpretation of LBPs in terms of lattice paths called *Schröder paths* [16, 17]. In this paper, we utilize LBPs to calculate a determinant whose entries are given by a q-analogue of the Narayana polynomials [2] which have a combinatorial expression in Schröder paths. As an application, we give a new proof to the Aztec diamond theorem by Elkies, Kuperberg, Larsen and Propp [7, 8] by means of LBPs and Schröder paths.

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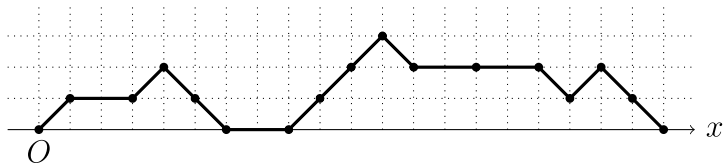

Figure 1: A Schröder path $P \in S_{10}$ such that level$(P) = 4$ and area$(P) = 20$.

A *Schröder path* P is a lattice path in the two-dimensional plane $\mathbb{Z}^2$ consisting of up steps $(1, 1)$, down steps $(1, -1)$ and level steps $(2, 0)$, and never going beneath the x-axis. See Figure 1 for example. For $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$, let S_k denote the set of Schröder paths from $(0, 0)$ to $(2k, 0)$. The number $\#S_k$ of such paths is counted by the k-th large Schröder number (A006318 in OEIS [19]). The first few of $\#S_k$ are 1, 2, 6, 22 and 90.

Enumerative or statistical properties of Schröder paths are often investigated through the *Narayana polynomials*

$$L_k(t) = \sum_{j=1}^{k} \frac{1}{k} \binom{k}{j} \binom{k}{j-1} (1+t)^j, \qquad k \in \mathbb{N}, \tag{1}$$

where $N_0(t) = 1$. (The coefficients $\frac{1}{k}\binom{k}{j}\binom{k}{j-1}$ are the *Narayana numbers*, A001263 in OEIS [19].) Bonin, Shapiro and Simion [2] interpreted the Narayana polynomials by counting the level steps in Schröder paths,

$$L_k(t) = \sum_{P \in S_k} t^{\mathrm{level}(P)} \tag{2}$$

where level(P) denotes the number of level steps in a Schröder path P. (Level steps in this paper are identified with "diagonal" steps in [2].) For more about the Narayana polynomials and related topics, see, e.g., Sulanke's paper [22] and the references therein. Besides level steps, Bonin et al. also examined the area polynomials

$$A_k(q) = \sum_{P \in S_k} q^{\mathrm{area}(P)}, \qquad k \in \mathbb{N}, \tag{3}$$

with respect to the statistic area(P) that measures the area bordered by a path P and the x-axis. (In [2], the major index is also examined, but we will not consider in this paper.) In this paper, we consider the two statistics level(P) and area(P) simultaneously in the polynomials

$$N_k(t,q) = \sum_{P \in S_k} t^{\mathrm{level}(P)} q^{\mathrm{area}(P)}, \qquad k \in \mathbb{N}. \tag{4}$$

We refer to $N_k(t,q)$ by the *q-Narayana polynomials*. Obviously, the q-Narayana polynomials satisfy that $N_k(t,1) = L_k(t)$ and $N_k(1,q) = A_k(t)$, and reduce to the large Schröder numbers, $N_k(1,1) = \#S_k$, as well as the Catalan numbers, $N_k(0,1) = \frac{1}{k+1}\binom{2k}{k}$. In Section 5, we find the LBPs of which the moments are described by the q-Narayana polynomials.

The aim of this paper is twofold: (i) to calculate a determinant whose entries are described by the q-Narayana polynomials $N_k(t,q)$; (ii) to give a new proof to the Aztec diamond theorem by Elkies, Kuperberg, Larsen and Propp [7, 8] by means of LBPs and Schröder paths.

Determinants whose entries are given by the large Schröder numbers, by the Narayana polynomials and by their q-analogues are calculated by many authors using various techniques. Ishikawa, Tagawa and Zeng [13] found a closed-form expression of Hankel determinants of a q-analogue of the large Schröder numbers in a combinatorial way based on Gessel–Viennot's lemma [11]. Petković, Barry and Rajković [20] calculated Hankel determinants described by the Narayana polynomials using an analytic method of solving a moment problem of orthogonal polynomials. In Section 6, we evaluate a Töplitz determinant described by the q-Narayana polynomials $N_k(t,q)$ by means of a combinatorial interpretation of LBPs in terms of Schröder paths.

Counting domino tilings of the Aztec diamonds is a typical problem of tilings which is exactly solvable. For $n \in \mathbb{N}$, the *Aztec diamond AD_n of order n* is the union of all unit squares which lie inside the closed region $|x| + |y| \leq n + 1$. A domino denotes a one-by-two or two-by-one rectangle. Then, a *domino tiling*, or simply a *tiling*, of AD_n is a collection of non-overlapping dominoes which exactly covers AD_n. Figure 2 shows an Aztec diamond and an example of a tiling. Let T_n denote the set of all tilings of AD_n. Elkies, Kuperberg, Larsen and Propp, in their two-parted paper [7, 8], considered the statistics $v(T)$ and $r(T)$ of a tiling T, where $v(T)$ denotes half the number of vertical dominoes in T and $r(T)$ the *rank* of T. (The definition of the rank is explained in Section 7.) They showed that the

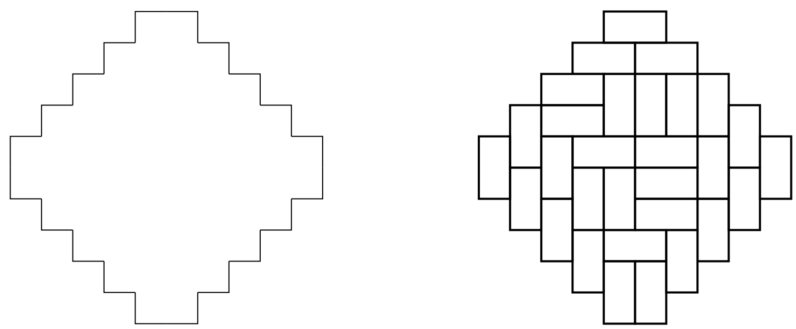

Figure 2: The Aztec diamond AD_5 (left) and a tiling of AD_5 (right).

counting polynomials

$$\mathrm{AD}_n(t,q) = \sum_{T\in T_n} t^{v(T)} q^{r(T)}, \qquad n\in\mathbb{N}, \tag{5}$$

admit the following closed-form expression.

Theorem 1 (Aztec diamond theorem [7, 8]). *For $n\in\mathbb{N}$,*

$$\mathrm{AD}_n(t,q) = \prod_{k=0}^{n-1} (1+tq^{2k+1})^{n-k}. \tag{6}$$

Especially, the number $\#T_n$ of possible tilings of AD_n equals to

$$\#T_n = \mathrm{AD}_n(1,1) = 2^{\frac{n(n+1)}{2}}. \tag{7}$$

(That is the solution to Exercise 6.49b in Stanley's book [21].) In [7, 8], a proof by means of the domino shuffling is shown for (6) as well as three different proofs for (7). Further different proofs for (7) are given by several authors [6, 18, 3, 9]. In particular, Eu and Fu [9] gives a proof of (7) by calculating Hankel determinants of the large and the small Schröder numbers. They showed a one-to-one correspondence between tilings and tuples of non-intersecting Schröder paths to apply Gessel–Viennot's lemma [11, 1] on non-intersecting paths and determinants. In this paper, we give a new proof to (6) based on the correspondence developed by Eu and Fu. Clarifying the connection between the statistics $v(T)$ and $r(T)$ of tilings T and the statistics level(P) and area(P) of Schröder paths P, we reduce the proof to the calculation of a determinant of the q-Narayana polynomials.

This paper is organized as follows. In Section 2, we recall the definitions and the fundamentals of LBPs and T-fractions focusing on the moments and a moment determinant of the Töplitz form. Sections 3–4 concern a combinatorial interpretation of LBPs in terms of Schröder paths which is applicable to general families of LBPs. In Section 3, we exhibit a combinatorial expression of the moments of LBPs (Theorem 3) with two different proofs. In Section 4, we show a combinatorial expression of the moment determinant in terms of non-intersecting Schröder paths (Theorem 6) based on Gessel–Viennot's methodology [11, 1].

Sections 5–7 concern the special case of the moments of LBPs given by the q-Narayana polynomials. In Section 5, we find the LBPs whose moments are given by the q-Narayana polynomials (Theorem 7). In Section 6, we evaluate a determinant of the q-Narayana polynomials by calculating the moment determinant of LBPs (Theorem 9). Finally, in Section 7, we give a new proof of the Aztec diamond theorem based on the discussions in the foregoing sections about LBPs, Schröder paths and the q-Narayana polynomials. Section 8 is devoted to concluding remarks.

2. Laurent biorthogonal polynomials and T-fractions

In Section 2, we recall the definition and the fundamentals of LBPs and T-fractions. See, e.g., [15, 12, 24] for more details. The formulations of LBPs may differ depending on the authors though they are essentially equivalent. In this paper, we adopt the formulation in [24].

2.1. Laurent biorthogonal polynomials

Let b_{n+1} and c_n for $n \in \mathbb{N}$ be arbitrary nonzero constants. The (monic) *Laurent biorthogonal polynomials (LBPs)* $P_n(z)$, $n \in \mathbb{N}$, is the polynomials determined from the recurrence

$$P_{n+1}(z) = (z - c_n)P_n(z) - b_n z P_{n-1}(z) \qquad \text{for } n \geq 1 \tag{8}$$

with the initial values $P_0(z) = 1$ and $P_1(z) = z - c_0$. The first few of the LBPs are

$$
\begin{aligned}
P_0(z) &= 1, && \text{(9a)}\\
P_1(z) &= z - c_0, && \text{(9b)}\\
P_2(z) &= z^2 - (b_1 + c_0 + c_1)z + c_0c_1, && \text{(9c)}\\
P_3(z) &= z^3 - (b_1 + b_2 + c_0 + c_1 + c_2)z^2 \\
&\qquad + (b_1c_2 + b_2c_0 + c_0c_1 + c_0c_2 + c_1c_2)z - c_0c_1c_2. && \text{(9d)}
\end{aligned}
$$

The LBP $P_n(z)$ is a monic polynomial in z exactly of degree n of which the constant term does not vanish. In fact,

$$P_n(0) = (-1)^n \prod_{j=0}^{n-1} c_j \neq 0. \tag{10}$$

The orthogonality of LBPs is described in the following theorem, that is sometimes referred to by *Favard type theorem*.

Theorem 2 (Favard type theorem for LBPs)**.** *There exists a linear functional $\mathcal{F}$ defined over Laurent polynomials in z with respect to which the LBPs $P_n(z)$ satisfy the orthogonality*

$$\mathcal{F}[P_n(z)z^{-k}] = h_n\delta_{n,k} \qquad \textit{for } 0 \leq k \leq n \tag{11}$$

with some constants $h_n \neq 0$, where $\delta_{n,k}$ denotes the Kronecker delta. The linear functional $\mathcal{F}$ is unique up to a constant factor.

We can prove Theorem 2 in almost the same way as Favard's theorem for orthogonal polynomials. See, e.g., Chihara's book [4, Chapter I, Theorem 4.4].

We write the *moments* of the linear functional $\mathcal{F}$,

$$f_k = \mathcal{F}[z^k], \qquad k \in \mathbb{Z}. \tag{12}$$

We fix the first moment $f_1 = \mathcal{F}[z]$ by

$$f_1 = \kappa \tag{13}$$

where κ is an arbitrary nonzero constant. We can show that the *moment determinant* of Töplitz form

$$\Delta_n^{(s)} = \det(f_{s-j+k})_{j,k=0,\ldots,n-1} = \begin{vmatrix} f_s & f_{s+1} & \cdots & f_{s+n-1} \\ f_{s-1} & f_s & \cdots & f_{s+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{s-n+1} & f_{s-n+2} & \cdots & f_s \end{vmatrix} \tag{14}$$

does not vanish for $s \in \{0,1\}$ and $n \in \mathbb{N}$. The LBPs $P_n(z)$ have the determinant expression

$$P_n(z) = \frac{1}{\Delta_n^{(0)}} \begin{vmatrix} f_0 & f_1 & \cdots & f_{n-1} & f_n \\ f_{-1} & f_0 & \cdots & f_{n-2} & f_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{-n+1} & f_{-n+2} & \cdots & f_0 & f_1 \\ 1 & z & \cdots & z^{n-1} & z^n \end{vmatrix}. \tag{15}$$

Thus, from (8) and (11), the coefficients b_n and c_n of the recurrence (8) and the constants h_n in the orthogonality (11)

$$b_n = -\frac{\Delta_{n+1}^{(1)}\Delta_{n-1}^{(0)}}{\Delta_n^{(1)}\Delta_n^{(0)}}, \qquad c_n = \frac{\Delta_{n+1}^{(1)}\Delta_n^{(0)}}{\Delta_n^{(1)}\Delta_{n+1}^{(0)}}, \qquad h_n = \frac{\Delta_{n+1}^{(0)}}{\Delta_n^{(0)}}. \tag{16}$$

The inverted polynomials

$$\tilde{P}_n(z) = \frac{z^n P_n(z^{-1})}{P_n(0)} \tag{17}$$

also make a family of LBPs which are determined by the recurrence (8) with the different coefficients

$$\tilde{b}_n = \frac{b_n}{c_{n-1}c_n}, \qquad \tilde{c}_n = \frac{1}{c_n}. \tag{18}$$

We can determine a linear functional $\tilde{\mathcal{F}}$ for $\tilde{P}_n(z)$ by the moments

$$\tilde{f}_k = \tilde{\mathcal{F}}[z^k] = f_{1-k}, \qquad k \in \mathbb{Z}. \tag{19}$$

Then,

$$\tilde{f}_1 = \tilde{\kappa} := f_0 = \frac{\kappa}{c_0}. \tag{20}$$

The equations (16) and (18) imply that

$$\Delta_n^{(1)} = (-1)^{\frac{n(n-1)}{2}} \kappa^n \prod_{k=1}^{n-1} \left(\frac{b_k}{c_{k-1}} \right)^{n-k}, \tag{21a}$$

$$\Delta_n^{(0)} = (-1)^{\frac{n(n-1)}{2}} \tilde{\kappa}^n \prod_{k=1}^{n-1} \left(\frac{\tilde{b}_k}{\tilde{c}_{k-1}} \right)^{n-k}. \tag{21b}$$

In Section 6, we make use of the formulae (21) to compute the moment determinant $\Delta_n^{(s)}$.

2.2. T-fractions

A *T-fraction* is a continued fraction

$$T(z) = \frac{\kappa}{|z - c_0} - \frac{b_1 z}{|z - c_1} - \frac{b_2 z}{|z - c_2} - \cdots. \tag{22}$$

The n-th convergent of $T(z)$

$$T_n(z) = \frac{\kappa}{|z - c_0} - \frac{b_1 z}{|z - c_1} - \cdots - \frac{b_{n-1} z}{|z - c_{n-1}} \tag{23}$$

is expressed by a ratio of polynomials

$$T_n(z) = \frac{Q_n(z)}{P_n(z)} \tag{24}$$

where $P_n(z)$ is the LBP of degree n determined by the recurrence (8), and $Q_n(z)$ is the polynomial determined by the same recurrence (8) from different initial values $Q_0(z) = 0$ and $Q_1(z) = \kappa$. Thus, we can identify the LBP $P_n(z)$ with the denominator polynomial of $T_n(z)$.

In Padé approximants, the convergent $T_n(z)$ simultaneously approximates two formal power series

$$F_+(z) = \sum_{k=1}^{\infty} f_k z^{-k} \qquad \text{and} \qquad F_-(z) = -\sum_{k=0}^{\infty} f_{-k} z^k \tag{25}$$

in the sense that

$$T_n(z) = F_+(z) + \mathrm{O}(z^{-n-1}) \qquad \text{as } z \to \infty; \tag{26a}$$

$$= F_-(z) + \mathrm{O}(z^n) \qquad \text{as } z \to 0, \tag{26b}$$

where $f_k = \mathcal{F}[z^k]$ are the moments of the LBPs $P_n(z)$. That is, expanded into series at $z = \infty$ and at $z = 0$, $T_n(z) = Q_n(z)/P_n(z)$ coincide with $F_+(z)$ and $F_-(z)$, respectively, at least in the first n terms. The approximation (26) of $F_+(z)$ and $F_-(z)$ by $T_n(z)$ is equivalent to the orthogonality (11) of LBPs.

Taking the limit $n \to \infty$ in (26), we observe that the T-fraction $T(z)$ equals to $F_+(z)$ and $F_-(z)$ as formal power series,

$$\begin{aligned} T(z) &= F_+(z) \qquad && \text{as } z \to \infty; && (27a) \\ &= F_-(z) && \text{as } z \to 0. && (27b) \end{aligned}$$

3. Moments and Schröder paths

In Section 3, we give a combinatorial interpretation to the moments of LBPs. Theorem 3 of expressing each moment in terms of Schröder paths is already shown in [16, Theorem 8]. In this paper, we review the result by providing two new simple proofs. The lattice path interpretation of LBPs is quite analogous to those in the combinatorial interpretation of orthogonal polynomials by Viennot [23]. We owe the idea of the proof in Section 3.2 by T-fractions to a combinatorial interpretation of continued fractions by Flajolet [10].

Let P be a Schröder path. We label each step in P by unity if the step is an up step, by b_n if a down step descending from the line $y = n$ and by c_n if a level step on the line $y = n$, where b_n and c_n are the coefficients of the recurrence (8) of the LBPs $P_n(z)$. We then define the *weight* $w(P)$ of P by the product of the labels of all the steps in P. For example, the path in Figure 1 weighs $w(P) = b_1^2 b_2^3 b_3 c_0 c_1 c_2^2$. In the same way, labeling each step in P using the recurrence coefficients $\tilde{b}_n$ and $\tilde{c}_n$ for $\tilde{P}_n(z)$, we define another weight $\tilde{w}(P)$. The main statement in Section 3 is the following.

Theorem 3. *The moments $f_k = \mathcal{F}[z^k]$ of LBPs admit the expressions*

$$f_{k+1} = \kappa \sum_{P \in S_k} w(P), \tag{28a}$$

$$f_{-k} = \tilde{\kappa} \sum_{P \in S_k} \tilde{w}(P) \qquad \text{for } k \in \mathbb{N}. \tag{28b}$$

For example,

$$
\begin{aligned}
f_{-2} &= \tilde{\kappa}(\tilde{b}_1\tilde{b}_2 + \tilde{b}_1^2 + \tilde{b}_1\tilde{c}_1 + 2\tilde{b}_1\tilde{c}_0 + \tilde{c}_0^2), && \text{(29a)}\\
f_{-1} &= \tilde{\kappa}(\tilde{b}_1 + \tilde{c}_0), && \text{(29b)}\\
f_0 &= \tilde{\kappa}, && \text{(29c)}\\
f_1 &= \kappa, && \text{(29d)}\\
f_2 &= \kappa(b_1 + c_0), && \text{(29e)}\\
f_3 &= \kappa(b_1b_2 + b_1^2 + b_1c_1 + 2b_1c_0 + c_0^2). && \text{(29f)}
\end{aligned}
$$

In the rest of Section 3, we show two different proofs of Theorem 3. The first proof in Section 3.1 is based on LBPs. The second proof in Section 3.2 is based on T-fractions.

3.1. Proof of Theorem 3 by LBPs

Lemma 4. *For $n \in \mathbb{N}$ and $k \in \mathbb{N}$,*

$$\mathcal{F}[P_n(z)z^{k+1}] = \kappa \sum_P w(P), \tag{30a}$$

$$\tilde{\mathcal{F}}[\tilde{P}_n(z)z^{k+1}] = \tilde{\kappa} \sum_P \tilde{w}(P) \tag{30b}$$

where both the sums range over all Schröder paths P from $(-n,-n)$ to $(2k,0)$.

Proof. Let us write $f_{n,k} = \mathcal{F}[P_n(z)z^{k+1}]$. From the recurrence (8) of $P_n(z)$, we obtain a recurrence of $f_{n,k}$

$$f_{n,k} = f_{n+1,k-1} + c_n f_{n,k-1} + b_n f_{n-1,k} \tag{31}$$

for $n \in \mathbb{N}$ and $k \in \mathbb{N}$, where the boundary values $f_{-1,k} = 0$ and $f_{n,-1} = \tilde{\kappa}\delta_{n,0}$ are induced from (11) and (20). The recurrence (31) leads us to a combinatorial expression of (30a),

$$f_{n,k} = \tilde{\kappa}c_0 \sum_P w(P) = \kappa \sum_P w(P) \tag{32}$$

where the sum ranges over all Schröder paths P from $(-n,-n)$ to $(2k,0)$. In much the same way, we can derive (30b) using Schröder paths labelled with $\tilde{b}_n$ and $\tilde{c}_n$. □

From (12) and (19), Theorem 3 is the special case of $n = 0$ in Lemma 4. That completes the proof of Theorem 3 by LBPs.

3.2. Proof of Theorem 3 by T-fractions

For a Schröder path P, we define length(P) by the sum of half the number of up and down steps and the number of level steps in P. For example, the path P in Figure 1 is as long as length$(P) = 10$.

Lemma 5. *The T-fraction $T(z)$ admits the expansions into formal power series*

$$T(z) = \kappa \sum_P w(P) z^{-\mathrm{length}(P)-1} \qquad \text{as } z \to \infty; \tag{33a}$$

$$= -\tilde{\kappa} \sum_P \tilde{w}(P) z^{\mathrm{length}(P)} \qquad \text{as } z \to 0, \tag{33b}$$

where the both (formal) sums range over all Schröder paths P from $(0,0)$ to some point on the x-axis.

Proof. Let us consider partial convergents of $T(z)$

$$T_{m,n}(z) = \frac{1}{z - c_m} - \frac{b_{m+1} z}{z - c_{m+1}} - \cdots - \frac{b_{m+n-1} z}{z - c_{m+n-1}} \tag{34}$$

for $m \in \mathbb{N}$ and $n \in \mathbb{N}$, where $T_{m,0}(z) = 0$. We first show by induction for $n \in \mathbb{N}$ that $T_{m,n}(z) = S_{m,n}(z)$ as $n \to \infty$ where $S_{m,n}(z)$ denotes the formal power series

$$S_{m,n}(z) = \sum_P w(P) z^{-\mathrm{length}(P)-1} \tag{35}$$

over all Schröder paths P from (m,m) to some point on the line $y = m$ which lie in the region bounded by $y = m$ and $y = m+n-1$. (Hence, all the points in P have the y-coordinates $\geq m$ and $\leq m+n-1$.) For $n = 0$, it is trivial that $S_{m,0}(z) = 0$ because the region in which P may live is empty. Hence, $T_{m,0}(z) = S_{m,0}(z) = 0$.

Suppose that $n \geq 1$. We classify Schröder paths P in the sum (35) into three classes: (i) the empty path only of one point at (m,m) (without steps) of weight 1; (ii) paths P_2 beginning by an up step; (iii) paths P_3 beginning by a level step. Thus,

$$S_{m,n}(z) = z^{-1} + \sum_{P_2} w(P_2) z^{-\mathrm{length}(P_2)-1} + \sum_{P_3} w(P_3) z^{-\mathrm{length}(P_3)-1}, \tag{36}$$

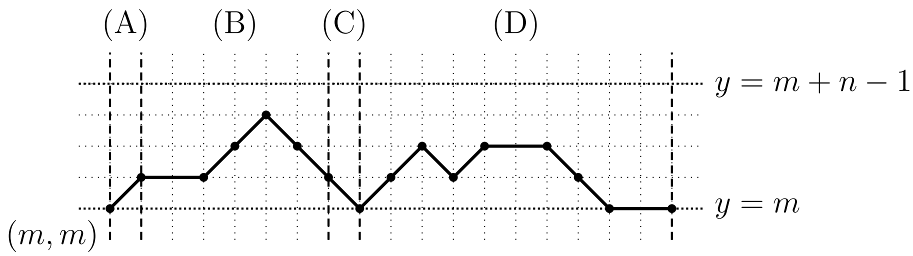

Figure 3: The decomposition of a path P_3 in the class (iii) into four parts (A), (B), (C) and (D).

where the sums with respect to P_2 and P_3 are taken over all Schröder paths in the classes (ii) and (iii), respectively. Each path P_2 in the class (ii) consists of an initial level step on $y = m$, labelled c_m, and a subpath (maybe empty) from $(m+2, m)$ to some point on $y = m$. Hence,

$$\sum_{P_2} w(P_2) z^{-\mathrm{length}(P_2)-1} = c_m z^{-1} S_{m,n}(z). \tag{37}$$

Each path P_3 in the class (iii), as shown in Figure 3, uniquely decomposed into four parts: (A) an initial up step, labelled unity; (B) a subpath (maybe empty) from $(m+1, m+1)$ to some point on $y = m+1$ never going beneath $y = m+1$; (C) the first down step descending from $y = m+1$ to $y = m$, labelled b_{m+1}; (D) a subpath (maybe empty) both of whose initial and terminal points are on $y = m$. Hence,

$$\sum_{P_3} w(P_3) z^{-\mathrm{length}(P_3)-1} = b_{m+1} S_{m+1,n-1}(z) S_{m,n}(z). \tag{38}$$

Substituting (37) and (38) into (36), we get

$$S_{m,n}(z) = \{z - c_m - b_{m+1} z S_{m+1,n-1}(z)\}^{-1}. \tag{39}$$

From the assumption of induction, we can assume that $S_{m+1,n-1}(z) = T_{m+1,n-1}(z)$ as $z \to \infty$ and hence

$$S_{m,n}(z) = \{z - c_m - b_{m+1} z T_{m+1,n-1}(z)\}^{-1} = T_{m,n}(z) \qquad \text{as } z \to \infty. \tag{40}$$

Now, let us prove Lemma 5. In taking the limit $n \to \infty$ of the identity $T_{0,n}(z) = S_{0,n}(z)$ as $z \to \infty$, the left-hand side $T_{0,n}(z)$ tends to $T(z)$ while the right-hand side $S_{0,n}$ to the right-hand side of (33a). In order to show (33b), we observe from (18) that $T_{m,n}(z)$ is equivalent to

$$T_{m,n}(z) = -\frac{\tilde{c}_m z^{-1}|}{|z^{-1} - \tilde{c}_m} - \frac{\tilde{b}_{m+1} z^{-1}|}{|z^{-1} - \tilde{c}_{m+1}} - \cdots - \frac{\tilde{b}_{m+n-1} z^{-1}|}{|z^{-1} - \tilde{c}_{m+n-1}}. \tag{41}$$

We can thereby show (33b) as a simple corollary of (33a). That completes the proof of Lemma 5. □

The expressions (28) of moments in Theorem 3 are derived just by equating (27) and (33). Indeed, every Schröder path P from $(0,0)$ to some point on the x-axis terminates at $(2k,0)$ if and only if $\mathrm{length}(P) = k$. That completes the proof of Theorem 3 by T-fractions.

4. Non-intersecting Schröder paths

In Section 4, as a consequence of Theorem 3, we examine the moment determinant $\Delta_n^{(s)}$ from a combinatorial viewpoint. We utilize *Gessel–Viennot's lemma* [11, 1] to read the determinant in terms of non-intersecting paths.

For $m \in \mathbb{N}$ and $n \in \mathbb{N}$, let $\boldsymbol{S}_{m,n}$ denote the set of n-tuples $\boldsymbol{P} = (P_0, \ldots, P_{n-1})$ of Schröder paths P_k such that (i) P_k goes from $(-k,k)$ to $(2m+k,k)$ and that (ii) every two distinct paths P_j and P_k, $j \neq k$, are *non-intersecting*, namely $P_j \cap P_k = \emptyset$. As shown in Figure 4, each n-tuple $\boldsymbol{P} \in \boldsymbol{S}_{m,n}$ can be drawn in a diagram of n non-intersecting Schröder paths which are pairwise disjoint. For simplicity, we write

$$w(\boldsymbol{P}) = \prod_{k=0}^{n-1} w(P_k), \qquad \tilde{w}(\boldsymbol{P}) = \prod_{k=0}^{n-1} \tilde{w}(P_k). \tag{42}$$

Theorem 6. *For general b_n and c_n nonzero, the moment determinant $\Delta_n^{(s)}$ admits the expressions*

$$\Delta_n^{(s)} = (-1)^{\frac{n(n-1)}{2}} \kappa^n \left(\prod_{j=1}^{n-1} b_j^{n-j} \right) \sum_{\boldsymbol{P} \in \boldsymbol{S}_{s-n,n}} w(\boldsymbol{P}) \qquad \text{if } s \geq n; \tag{43a}$$

$$= (-1)^{\frac{n(n-1)}{2}} \tilde{\kappa}^n \left(\prod_{j=1}^{n-1} \tilde{b}_j^{n-j} \right) \sum_{\boldsymbol{P} \in \boldsymbol{S}_{|s|-n+1,n}} \tilde{w}(\boldsymbol{P}) \quad \text{if } s \leq -n+1. \tag{43b}$$

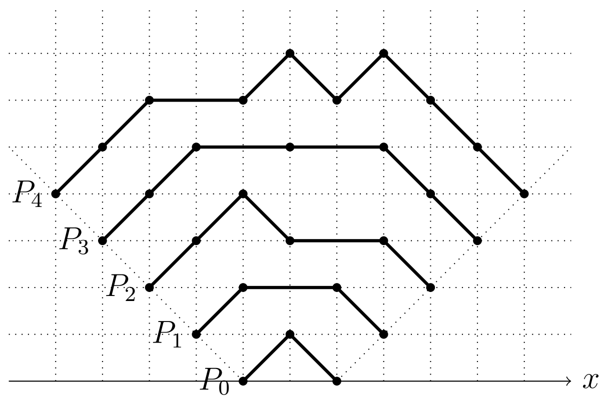

Figure 4: A quintuple $\boldsymbol{P} = (P_0, \dots, P_4) \in \boldsymbol{S}_{1,5}$ of non-intersecting Schröder paths which is drawn in a plane.

Proof. Suppose that $s \geq n \geq 0$. We rewrite $\Delta_n^{(s)}$ into Hankel form,

$$\Delta_n^{(s)} = (-1)^{\frac{n(n-1)}{2}} \det(f_{s-n+j+k+1})_{j,k=0,\dots,n-1}. \tag{44}$$

Owing to Theorem 3, the (j,k)-entry of the last Hankel determinant has the combinatorial expression

$$f_{s-n+j+k+1} = \kappa \sum_{P_{j,k}} w(P_{j,k}) \tag{45}$$

where we can assume that the sum ranges over all Schröder paths $P_{j,k}$ from $(-2j, 0)$ to $(2(s-n) + 2k, 0)$. Thus, we can apply Gessel–Viennot's lemma [11, 1] to expand the determinant (44),

$$\det(f_{s-n+j+k+1})_{j,k=0,\dots,n-1} = \kappa^n \sum_{(P_{0,0},\dots,P_{n-1.n-1})} w(P_{0,0}) \cdots w(P_{n-1,n-1}) \tag{46}$$

where the sum ranges over all n-tuples $(P_{0,0}, \dots, P_{n-1,n-1})$ of *non-intersecting* Schröder paths $P_{k,k}$ such that $P_{k,k}$ goes from $(-2k, 0)$ to $(2(s-n)+2k, 0)$ for each k. (See Figure 5 for example.) As shown in Figure 5, the first and last k steps of $P_{k,k}$ must be all up and down steps, respectively, so that the paths do not collide. Especially, $P_{k,k}$ passes the points $(-k, k)$ and $(2(s-n)+k, k)$. We thus have

$$\det(f_{s-n+j+k+1})_{j,k=0,\dots,n-1} = \kappa^n \left(\prod_{j=1}^{n-1} b_j^{n-j} \right) \sum_{\boldsymbol{P} \in \boldsymbol{S}_{s-n,n}} w(\boldsymbol{P}) \tag{47}$$

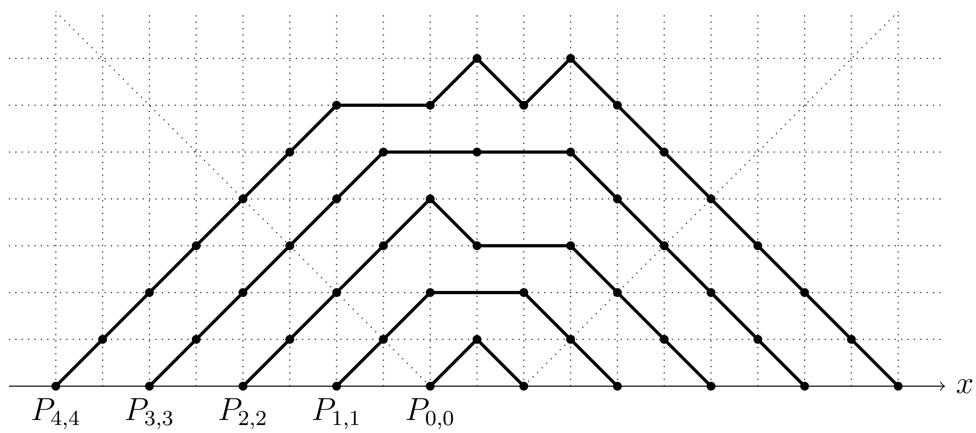

Figure 5: A quintuple $(P_{0,0}, \ldots, P_{4,4})$ of non-intersecting Schröder paths counted in the right-hand sum of (46) ($m = 1$ and $n = 5$).

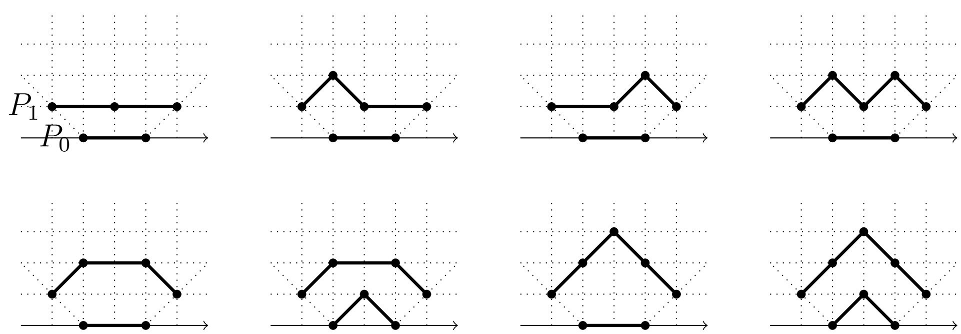

Figure 6: The eight doubles $(P_0, P_1) \in \boldsymbol{S}_{1,2}$ of non-intersecting Schröder paths.

and thereby (43a). In the same way, we can show (43b) from Theorem 3. $\square$

For example, for $m = 3$ and $n = 2$, the set $\boldsymbol{S}_{1,2}$ contains exactly eight doubles (P_0, P_1) of non-intersecting Schröder paths which are shown in Figure 6. Thus, the moment determinant $\Delta_2^{(3)}$ equals to the polynomial of eight monomials

$$\Delta_2^{(3)} = -\kappa^2 b_1(c_0c_1^2 + 2b_2c_0c_1 + b_2^2c_0 + b_2c_0c_2 + b_1b_2c_2 + b_2b_3c_0 + b_1b_2b_3) \tag{48}$$

of which each monomial corresponds to a diagram in Figure 6.

5. q-Narayana polynomials as moments

In Section 5 and the subsequent, we consider the special case of the LBPs whose moments are described by the q-Narayana polynomials. Let us recall from Section 1 the definition of the q-Narayana polynomials

$$N_k(t,q) = \sum_{P \in S_k} t^{\mathrm{level}(P)} q^{\mathrm{area}(P)} \tag{49}$$

where level(P) denotes the number of level steps in a Schröder path P, and area(P) the area bordered by P and the x-axis. For example, the first few of the q-Narayana polynomials are enumerated in

$$N_0(t,q) = 1, \tag{50a}$$
$$N_1(t,q) = t + q, \tag{50b}$$
$$N_2(t,q) = t^2 + 2tq + tq^3 + q^2 + q^4, \tag{50c}$$
$$\begin{aligned} N_3(t,q) = {} & t^3 + 3t^2q + 2t^2q^3 + t^2q^5 + 3tq^2 + 4tq^4 + 2tq^6 + tq^8 \\ & + q^3 + 2q^5 + q^7 + q^9. \end{aligned} \tag{50d}$$

In view of Theorem 3, it is easy to find the q-Narayana polynomials in the moments of LBPs.

Theorem 7. *Let us determine the LBPs $P_n(z)$ by the recurrence (8) with the coefficients*

$$b_n = q^{2n-1}, \qquad c_n = tq^{2n}. \tag{51}$$

Then, the linear functional $\mathcal{F}$ for $P_n(z)$ admits the moments $f_k = \mathcal{F}[z^k]$ described by the q-Narayana polynomials,

$$f_k = \kappa N_{k-1}(t,q) \qquad \text{for } k \geq 1; \tag{52a}$$
$$= \kappa t^{-2|k|-1} N_{|k|}(t,q^{-1}) \qquad \text{for } k \leq 0, \tag{52b}$$

where κ is an arbitrary nonzero constant.

Proof. Let $P \in S_k$. Labelled with (51), P weighs $w(P) = t^{\mathrm{level}(P)} q^{\mathrm{area}(P)}$. Hence, by virtue of Theorem 3, we have (52a) as a special case of (28a). Similarly, with

$$\tilde{b}_n = t^{-2} q^{-2n+1}, \qquad \tilde{c}_n = t^{-1} q^{-2n} \tag{53}$$

from (18), $\tilde{w}(P) = t^{-2k+\mathrm{level}(P)} q^{-\mathrm{area}(P)}$. Now $\tilde{\kappa} = \kappa t^{-1}$ from (20). Hence, we obtain (52b) from (28b). □

We remark that the q-Narayana polynomials (49) defined in a combinatorial way are already investigated by Cigler [5] who introduced the polynomials by modifying the generating function of the q-Catalan numbers. Indeed, we can deduce from (49) a recurrence

$$N_k(t,q) = tN_{k-1}(t,q) + \sum_{j=0}^{k-1} q^{2j+1} N_j(t,q) N_{k-j-1}(t,q). \tag{54}$$

We can identify (54) with the recurrence in [5, Eq. (19)].

6. Determinant of q-Narayana polynomials

In Section 6, we examine a Töplitz determinant of the q-Narayana polynomials

$$\mathcal{N}_n^{(s)}(t,q) = \det(N_{s+j-k-1}(t,q))_{j,k=0,\dots,n-1}, \tag{55}$$

where, in view of (52), we define $N_k(t,q)$ for negative k by

$$N_k(t,q) = t^{-2|k|-1} N_{|k|-1}(t,q^{-1}) \qquad \text{for } k < 0. \tag{56}$$

As the special case of the moments given by the q-Narayana polynomials, Theorem 6 allows us to read the determinant (55) in the context of non-intersecting Schröder paths. The results in this section, Theorem 9 and Corollary 10, will be applied later in Section 7 to a new proof of the Aztec diamond theorem (Theorem 1).

Theorem 7 implies that

$$\Delta_n^{(s)} = \mathcal{N}_n^{(s)}(t,q) \tag{57}$$

provided that the coefficients b_n and c_n of the recurrence (8) are given by (51) where $\kappa = 1$. Hence, we can use the formulae (21) to find the exact value of $\mathcal{N}_n^{(s)}$ for $s \in \{0,1\}$. Recall that, in using (21b), we assume $\tilde{b}_n$ and $\tilde{c}_n$ to be given by (53) and $\tilde{\kappa} = t^{-1}$.

Lemma 8. *For $s \in \{0,1\}$ and $n \in \mathbb{N}$, the exact value of the determinant $\mathcal{N}_n^{(s)}$ is given by*

$$\mathcal{N}_n^{(1)}(t,q) = (-1)^{\frac{n(n-1)}{2}} t^{-\frac{n(n-1)}{2}} q^{\frac{n(n-1)}{2}}, \tag{58a}$$

$$\mathcal{N}_n^{(0)}(t,q) = (-1)^{\frac{n(n-1)}{2}} t^{-\frac{n(n+1)}{2}} q^{-\frac{n(n-1)}{2}}. \tag{58b}$$

In order to find the value of $\mathcal{N}_n^{(s)}(t,q)$ for further $s \in \mathbb{Z}$ and $n \in \mathbb{N}$, we can use Sylvester's determinant identity:

$$X \cdot X(i,j;k,\ell) - X(i;k) \cdot X(j;\ell) + X(i;\ell) \cdot X(j;k) = 0 \tag{59}$$

where X is an arbitrary determinant and $X(i,j;k,\ell)$ denotes the minor of X obtained by deleting the i-th and the j-th rows and the k-th and the ℓ-th columns; $X(i;k)$ the minor of X with respect to the i-th row and the k-th column. Applying Sylvester's determinant identity, we get

$$\mathcal{N}_{n+1}^{(s)} \cdot \mathcal{N}_{n-1}^{(s)} - \mathcal{N}_n^{(s)} \cdot \mathcal{N}_n^{(s)} + \mathcal{N}_n^{(s+1)} \cdot \mathcal{N}_n^{(s-1)} = 0 \tag{60}$$

for $s \in \mathbb{Z}$ and $n \in \mathbb{N}$, where $\mathcal{N}_n^{(s)} = \mathcal{N}_n^{(s)}(t,q)$ except that $\mathcal{N}_{-1}^{(s)} = 0$. Using (60) as a recurrence from appropriate initial value, we can compute the value of $\mathcal{N}_n^{(s)}(t,q)$ for each $s \in \mathbb{Z}$ and $n \in \mathbb{N}$. Especially, we find a closed form of $\mathcal{N}_n^{(s)}(t,q)$ for $-n \le s \le n+1$ as follows.

Theorem 9. *For $-n \le s \le n+1$, the exact value of the determinant $\mathcal{N}_n^{(s)}(t,q)$ is given by*

$$\mathcal{N}_n^{(s)}(t,q) = \varphi_n^{(s)}(t,q) \prod_{k=1}^{s-1} (t + q^{2k-1})^{s-k} \qquad \text{for } 1 \le s \le n+1; \tag{61a}$$

$$= \varphi_n^{(s)}(t,q) \prod_{k=1}^{|s|} (t + q^{-2k+1})^{|s|-k+1} \qquad \text{for } -n \le s \le 0 \tag{61b}$$

where

$$\varphi_n^{(s)}(t,q) = (-1)^{\frac{n(n-1)}{2}} t^{-\frac{(n-s)(n-s+1)}{2}} q^{\frac{n(n-1)(2s-1)}{2}}. \tag{61c}$$

Proof. Using Sylvester's identity (60) from the initial value (58), we can easily show (61) by induction. □

Note that Cigler [5] found a closed-form expression of the Hankel determinant $\det(N_{s+j+k}(t,q))_{j,k=0,\dots,n-1}$ of the q-Narayana polynomials for $s \in \{0,1\}$ and $n \in \mathbb{N}$ by means of orthogonal polynomials. (The Hankel determinant coincides with $\mathcal{N}_n^{(s)}(t,q)$ for $s \in \{n, n+1\}$ without sign.) Theorem 9 generalizes Cigler's result [5, Eqs. (24) and (25)] for further s and n.

As a corollary of Theorem 9, equating (61a) with (43a) in Theorem 6, we obtain the following result about non-intersecting Schröder paths.

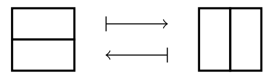

Figure 7: Rotation of a two-by-two block of two horizontal or vertical dominoes in an elementary move.

Corollary 10. *For* $m \in \{0, 1\}$ *and* $n \in \mathbb{N}$,

$$\sum_{\boldsymbol{P} \in \boldsymbol{S}_{m,n}} t^{\mathrm{level}(\boldsymbol{P})} q^{\mathrm{area}(\boldsymbol{P})} = q^{\frac{n(n-1)(3m+2n-1)}{3}} \prod_{k=1}^{m+n-1} (t + q^{2k-1})^{m+n-k} \tag{62}$$

where $\mathrm{level}(\boldsymbol{P}) = \sum_{k=0}^{n-1} \mathrm{level}(P_k)$ *and* $\mathrm{area}(\boldsymbol{P}) = \sum_{k=0}^{n-1} \mathrm{area}(P_k)$ *with* $\boldsymbol{P} = (P_0, \ldots, P_{n-1})$.

7. Proof of Aztec diamond theorem

Finally, in Section 7, we give a new proof of the Aztec diamond theorem (Theorem 1) based on the discussions in the foregoing sections. In the two-parted paper by Elkies, Kuperberg, Larsen and Propp [7, 8], the Aztec diamond theorem is proven by the technique of the domino shuffling. Whereas, the proof in this paper is based on the one-to-one correspondence between tilings of the Aztec diamonds and tuples of non-intersecting Schröder paths developed by Eu and Fu [9] who used the correspondence to prove (7).

In order to make the statement precise, as we announced in Section 1, we review from [7] the definition of the rank statistic. Let $T \in T_n$ be a tiling of the Aztec diamond AD_n. If $n \geq 1$, T certainly contains one or more two-by-two blocks of two horizontal or vertical dominoes. Thus, choosing one from such two-by-two blocks and rotating it by ninety degrees, we obtain a new tiling $T' \in T_n$. (See Figure 7.) We refer by an *elementary move* to this operation of transforming T into T' by rotating a two-by-two block. It can be shown that any tiling of AD_n can be reached from any other tiling of AD_n by a sequence of elementary moves. The rank $r(T)$ of T denotes the minimal number of elementary moves required to reach T from the "all-horizontal" tiling T^0 consisting only of horizontal dominoes, where $r(T^0) = 0$. For example, in Figure 8, the rightmost tiling T of AD_2 has the rank $r(T) = 4$ since at least four elementary moves are required to reach from the leftmost T^0.

Eu and Fu [9] developed a one-to-one correspondence between T_n and $\boldsymbol{S}_{1,n}$. We describe the bijection from T_n to $\boldsymbol{S}_{1,n}$ in a slightly different manner

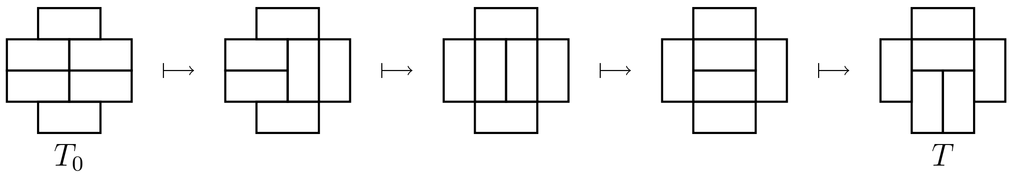

Figure 8: A sequence of elementary moves from T^0 to T of AD_2. At least four elementary moves are required to reach from T^0 only of horizontal dominoes to the rightmost T, and thereby $r(T) = 4$.

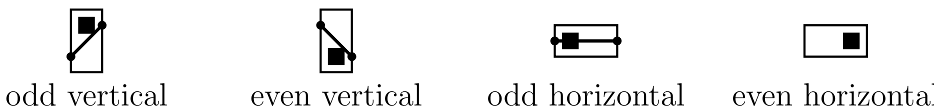

Figure 9: The rule to draw a step on a domino.

from [9]. Following [7], we color the Aztec diamond AD_n in a black-white checkerboard fashion so that all unit squares on the upper-left border of AD_n are white. We say that a horizontal domino (resp. a vertical domino) put into AD_n is *even* if the left half (resp. the upper half) of the domino covers a white unit square. Otherwise, the domino is *odd*. The bijection mapping a tiling $T \in T_n$ to an n-tuple $\boldsymbol{P} = (P_0, \dots, P_{n-1}) \in \boldsymbol{S}_{1,n}$ of non-intersecting Schröder paths is described by the following procedure: For each domino in T, as shown in Figure 9, draw an up step (resp. a down step, a level step) that goes through the center of the domino if the domino is even vertical (resp. odd vertical, odd horizontal). (For even horizontal dominoes, we do nothing.) Then, we find n non-intersecting Schröder paths $P_0, \dots, P_{n-1}$ on T of which the n-tuple $\boldsymbol{P} = (P_0, \dots, P_{n-1})$ belongs to $\boldsymbol{S}_{1,n}$. For example, see Figure 10.

The bijection connects the statistics $v(T)$ and $r(T)$ for tilings and the statistics level(P) and area(P) for Schröder paths as follows. Recall from Section 1 that $v(T)$ denotes half the number of vertical dominoes in a tiling T.

Lemma 11. *Suppose that a tiling $T \in T_n$ and an n-tuple $\boldsymbol{P} = (P_0, \dots, P_{n-1}) \in \boldsymbol{S}(1, n)$ of non-intersecting Schröder paths are in the one-to-one correspon-*

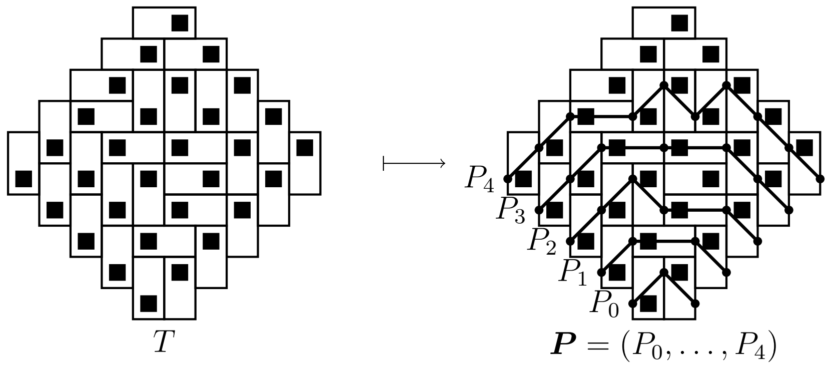

Figure 10: The bijection mapping a tiling $T \in T_5$ of AD_5 to a quintuple $\boldsymbol{P} = (P_0, \ldots, P_4) \in S_{1,5}$ of non-intersecting Schröder paths. (The Aztec diamond is colored in a checkerboard fashion.)

Figure 11: A rotation of a two-by-two block in an elementary move raises the rank of the tiling T by one (left-to-right, respectively) if and only if the corresponding deformation of a tuple $\boldsymbol{P}$ of non-intersecting Schröder paths increases area($\boldsymbol{P}$) by one.

dence by the bijection. Then,

$$v(T) = \frac{n(n+1)}{2} - \mathrm{level}(\boldsymbol{P}), \tag{63}$$

$$r(T) = \mathrm{area}(\boldsymbol{P}) - \frac{2n(n+1)(n-1)}{3}, \tag{64}$$

where $\mathrm{level}(\boldsymbol{P}) = \sum_{k=0}^{n-1} \mathrm{level}(P_k)$ *and* $\mathrm{area}(\boldsymbol{P}) = \sum_{k=0}^{n-1} \mathrm{area}(P_k)$.

Proof. The bijection implies that $v(T)$ equals to half the number of up and down steps in $\boldsymbol{P}$. The sum of half the number of up and down steps and the number of level steps in $\boldsymbol{P}$ is a constant independent of $\boldsymbol{P}$ that equals to $n(n+1)/2$. Thus, we have (63).

As shown in Figure 11, each elementary move of a tiling T raising the rank by one gives rise to a deformation of some path in $\boldsymbol{P}$ increasing the area by one. Thus, $r(T)$ and area($\boldsymbol{P}$) differ by a constant independent of T and $\boldsymbol{P}$. Since $r(T^0) = 0$ then the constant equals to $\mathrm{area}(\boldsymbol{P}^0) = 2n(n+1)(n-1)/3$, where T^0 denotes the "all-horizontal" tiling of AD_n and $\boldsymbol{P}^0 \in \boldsymbol{S}_{1,n}$ the n-tuple of non-intersecting Schröder paths only of level steps that corresponds to T^0. Thus, we have (64). $\square$

Now, we give a proof of the Aztec diamond theorem.

Proof of Theorem 1. As a consequence of Lemma 11, we can substitute (63) and (64) into (6) to obtain

$$\mathrm{AD}_n(t,q) = t^{\frac{n(n+1)}{2}} q^{-\frac{2n(n-1)(n+1)}{3}} \sum_{\boldsymbol{P} \in S_{1,n}} t^{-\mathrm{level}(\boldsymbol{P})} q^{\mathrm{area}(\boldsymbol{P})}. \tag{65}$$

From Corollary 10, the sum in the right-hand side of (65) is equated with

$$\sum_{\boldsymbol{P} \in S_{1,n}} t^{-\mathrm{level}(\boldsymbol{P})} q^{\mathrm{area}(\boldsymbol{P})} = q^{\frac{2n(n-1)(n+1)}{3}} \prod_{k=1}^{n} (t^{-1} + q^{2k-1})^{n-k+1}. \tag{66}$$

Substituting (66) into the right-hand side of (65), we have

$$\mathrm{AD}_n(t,q) = t^{\frac{n(n+1)}{2}} \prod_{k=1}^{n} (t^{-1} + q^{2k-1})^{n-k+1} = \prod_{k=1}^{n} (1 + tq^{2k-1})^{n-k+1}. \tag{67}$$

That completes the proof of Theorem 1. □

8. Concluding remarks

In this paper, we evaluated a determinant whose entries are given by the q-Narayana polynomials (Theorem 9). In order to find the value of the determinant, we utilized Laurent biorthogonal polynomials which allow of a combinatorial interpretation in terms of Schröder paths (Theorem 3 and Theorem 7). As an application, we exhibited a new proof of the Aztec diamond theorem (Theorem 1) by Elkies, Kuperberg, Larsen and Propp [7, 8] with the help of the one-to-one correspondence developed by Eu and Fu [9] between tilings of the Aztec diamonds and tuples of non-intersecting Schröder paths.

We remark that, in Theorem 9, we can evaluate the determinant $\mathcal{N}_n^{(s)}$ of the q-Narayana polynomials also for $s < -n$ and $s > n+1$ by using the formula (60) from Sylvester's identity. For example, if $s = n+2$,

$$\mathcal{N}_n^{(n+2)} = (-1)^{\frac{n(n-1)}{2}} q^{\frac{n(n-1)(2n+3)}{2}} \prod_{k=1}^{n} (t + q^{2k-1})^{n-k+1} \sum_{\ell=0}^{n} t^{n-\ell} q^{\ell^2} \binom{n+1}{\ell}_{q^2}, \tag{68}$$

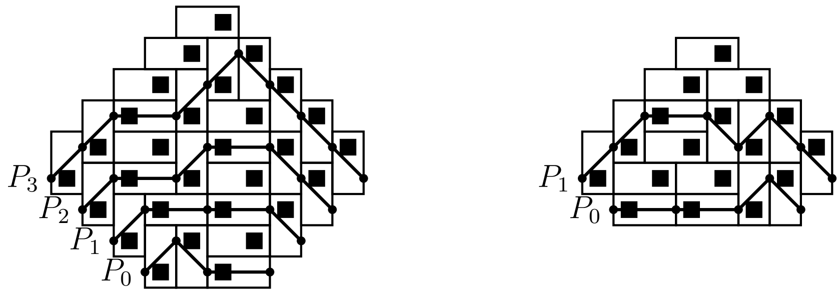

Figure 12: The one-to-one correspondence between a tiling of $AD_{m,n}$ and a n-tuple $\boldsymbol{P} = (P_0, \ldots, P_{n-1}) \in \boldsymbol{S}_{m,n}$ of non-intersecting Schröder paths. (The left figure shows an instance for $m = 2$ and $n = 4$ while the right for $m = 3$ and $n = 2$.)

where $\binom{m}{n}_q$ denotes the q-binomial coefficient

$$\binom{m}{n}_q = \prod_{k=1}^{n} \frac{1 - q^{m-k+1}}{1 - q^k}. \tag{69}$$

From Theorem 6, we can read (68) in terms of non-intersecting Schröder paths,

$$\sum_{\boldsymbol{P} \in \boldsymbol{S}_{2,n}} t^{\mathrm{level}(\boldsymbol{P})} q^{\mathrm{area}(\boldsymbol{P})} = q^{\frac{n(n-1)(2n+5)}{3}} \prod_{k=1}^{n} (t + q^{2k-1})^{n-k+1} \sum_{\ell=0}^{n} t^{n-\ell} q^{\ell^2} \binom{n+1}{\ell}_{q^2}. \tag{70}$$

We can readily observe that the bijection in Section 7 gives a one-to-one correspondence between n-tuples of non-intersecting Schröder paths in $\boldsymbol{S}_{2,n}$ and tilings of the region $AD_{2,n}$, the Aztec diamond AD_{n+1} from which two unit squares at the south corner are removed. (See Figure 12). Therefore, as a variant of (6), we have

$$\sum_{T} t^{v(T)} q^{r(T)} = \prod_{k=0}^{n-1} (1 + tq^{2k+1})^{n-k} \sum_{\ell=0}^{n} t^{\ell} q^{\ell^2} \binom{n+1}{\ell}_{q^2}, \tag{71}$$

where the sum in the left-hand side ranges over all tilings T of $AD_{2,n}$. (The rank(T) is defined in the same way as AD_n to be the minimal number of elementary moves required to reach from "all-horizontal" tilings of $AD_{2,n}$.)

Similarly, calculating the determinant $\mathcal{N}_n^{(m+n)}(t,q)$, we can obtain in principle variant formulae of (6) for tilings of the Aztec diamond AD_{m+n} from which $m(m-1)$ unit squares at the south corner are removed. However, the value of $\mathcal{N}_n^{(m+n)}(t,q)$ seems much complicated for large m, and exact formulae has not been found yet for general m and n.

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