A Lower Bound for $\tau(n)$ of k-Multiperfect Number

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Abstract

A natural number n is said to be k- multiperfect number if $\sigma(n) = k{\cdot}n$ for some integer $k > 2$. In this paper, I will provide a lower bound for $\tau(n)$ of any k- multiperfect numbers. The lower bound for $\tau(n)$ will help in distinguishing if the number is k-multiperfect or not.

1 Preliminary Concepts

The sum-of-positive divisor function $\sigma_m(n)$ is defined as

$$\sigma_m(n) = \sum_{d|n} d^m$$

where d is a factor of n for natural numbers n and complex numbers m. In this definition, we concentrate only for $m = 0$ and $m = 1$ and denote them as $\tau(n)$ and $\sigma(n)$ respectively. It is easy to see then that $\tau(n)$ counts the number of divisors of n and $\sigma(n)$ gives the sum of the divisors of n. It is a known theorem that for any natural number $n = \prod_{i=1}^{n} p_i^{\alpha_1}$,

$$\tau(n) = \prod_{i=1}^{n} (\alpha_i + 1)$$

For $\sigma(n)$, n is said to be perfect if $\sigma(n) = 2n$.But if $\sigma(n) > 2n$ and $\sigma(n) < 2n$, it is said to be abundant and deficient numbers respectively. In addition,a natural number n is said to be k-multiperfect number if $\sigma(n) = k \cdot n$ for some integer $k > 2$. It should be noted that for integer $k > 3$ of k- multiperfect numbers, all these k- multiperfect numbers are abundant.

In studying perfect numbers, the abundancy index is helpful and defined as

$$I(n) = \frac{\sigma(n)}{n}$$

. If n is k-multiperfect, then $\sigma(n) = k \cdot n$ and that implies $I(n) = k$.

It is easy to see that

$$I(n)=\sum_{d|n}\frac{1}{d}=k$$

.

On the other hand, we know that the nth harmonic number denoted by H_n is defined as

$$H_n=\sum_{i=1}^{n}\frac{1}{i}$$

. Clearly, $I(n)\leq H_n$ for all natural numbers n.

2 Some Results

Let us first consider some lemmas.

Lemma 1. *For $n\in\mathbb{N}$, the inequality*

$$\Big(1+\frac{1}{k(k+2)}\Big)^k\leq 1+\frac{1}{k+1}\leq\Big(1+\frac{1}{k(k+1)}\Big)^k$$

holds.

Proof. Consider first the inequality

$$1+\frac{1}{k+1}\leq\Big(1+\frac{1}{k(k+1)}\Big)^k$$

By binomial expansion on the RHS of the inequality, we have

$$\Big(1+\frac{1}{k(k+1)}\Big)^k=\sum_{i=0}^{k}\binom{k}{i}1^{k-i}\Big(\frac{1}{k(k+1)}\Big)^i=1+\frac{1}{k+1}+\sum_{i=2}^{k}\binom{k}{i}1^{k-i}\Big(\frac{1}{k(k+1)}\Big)^i.$$

Clearly,

$$0\leq\sum_{i=2}^{k}\binom{k}{i}1^{k-i}\Big(\frac{1}{k(k+1)}\Big)^i.$$

Adding both sides by $1+\frac{1}{k+1}$, we arrive on the desired inequality. On the other hand, consider the inequality

$$\Big(1+\frac{1}{k(k+2)}\Big)^k\leq 1+\frac{1}{k+1}$$

Raising both sides by $k+2$, we get

$$\Big(1+\frac{1}{k(k+2)}\Big)^{k(k+2)}\leq\Big(1+\frac{1}{k+1}\Big)^{k+2}\Leftrightarrow\Big(1+\frac{1}{x}\Big)^x\leq\Big(1+\frac{1}{y}\Big)^{y+1}.$$

Since the

$$\Big(\lim_{x\to+\infty}\Big(1+\frac{1}{x}\Big)^x=e\Big)\wedge\Big(\lim_{y\to+\infty}\Big(1+\frac{1}{y}\Big)^{y+1}=e\Big)$$

from below and from above respectively, then that proves the inequality. □

Lemma 2. *The inequality*

$$\sum_{i=2}^{n} \frac{1}{i} < \int_{1}^{n} \frac{1}{x}\, dx < \sum_{i=1}^{n} \frac{1}{i}$$

holds.

Proof. By lemma 1,

$$\left(1 + \frac{1}{k(k+2)}\right)^{k} \leq 1 + \frac{1}{k+1}$$

By some manipulations,

$$\left(\frac{(k+1)(k+1)}{k(k+2)}\right)^{k} \leq \frac{k+2}{k+1} \Rightarrow \left(\frac{k+1}{k}\right)^{k} \leq \left(\frac{k+2}{k+1}\right)\left(\frac{k+2}{k+1}\right)^{k} = \left(\frac{k+2}{k+1}\right)^{k+1}.$$

Thus, we get

$$\left(1 + \frac{1}{k}\right)^{k} \leq \left(1 + \frac{1}{k+1}\right)^{k+1} < e$$

Now, we consider the inequality

$$\left(1 + \frac{1}{k}\right)^{k} < e \Rightarrow e^{\ln\left(1+\frac{1}{k}\right)} < e^{\frac{1}{k}} \Rightarrow \ln\left(\frac{k+1}{k}\right) < \frac{1}{k}.$$

Therefore,

$$\begin{aligned}
\ln\left(\frac{k+1}{k}\right) &< \frac{1}{k} \\
\ln(k+1) - \ln(k) &< \\
\int_{k}^{k+1} \frac{1}{x}\, dx &< \\
\sum_{k=1}^{n} \int_{k}^{k+1} \frac{1}{x}\, dx &< \sum_{k=1}^{n} \frac{1}{k} \\
\int_{1}^{n} \frac{1}{x}\, dx &< \sum_{k=1}^{n} \frac{1}{k}
\end{aligned}$$

The other inequality can be solved in similar fashion. □

The previous lemma can be written as

$$H_n - 1 < H_n - (\gamma + \epsilon) < H_n$$

where γ is the Euler- Mascheroni constant and ϵ, a positive number that can be expressed as

$$\sum_{m=2}^{\infty} = \frac{\zeta(n, m+1)}{m}$$

and where $\zeta(n, m+1)$ is said to be the Hurwitz zeta function. From this inequality, we can have a bound for γ.

$$-\epsilon < \gamma < 1 - \epsilon$$

As $n \to +\infty$, $\epsilon \to 0$ and that will give us $0 < \gamma < 1$. In fact, $\gamma = 0.57721...$ (see Sloane's A001620 at OEIS.org)

3 Main Results

We can now rewrite H_n as

$$H_n = \ln(n) + \gamma + \epsilon$$

Since we know that $\epsilon < 1 - \gamma < 0.5$, then the margin of error ϵ becomes minimal and can be "ignored".Before we proceed to the main result, let us have some necessary results.

Theorem 1. *For nonnegative integers k_i,*

$$\sum_{i=1}^{n} \frac{1}{k_i} \leq \sum_{i=1}^{n} \frac{1}{i}$$

where for every k_i and k_j, $k_i \neq k_j$ and for all k_i and k_{i+1} , $k_i < k_{i+1}$.

Proof. It should be noted that equality holds if $k_i = i$. Now suppose that there exists $k_i \neq i$. This would mean that in the set $S = \{1, 2, 3, ..., n\}$, there is $k_i \notin S$. Thus, $k_i > n$. Now, we have k_i's such that

$$\frac{1}{k_i} < \frac{1}{n} < \frac{1}{j}$$

for all $j \in S$ such that $j \neq k_i$. Adding all unit fractions $\frac{1}{j}$ for $j \neq k_i$ and $j = k_i$, we get

$$\sum_{j \neq k_i} \frac{1}{k_i} + \sum_{j = k_i} \frac{1}{k_i} \leq \sum_{j \neq k_i} \frac{1}{j} + \sum_{j = k_i} \frac{1}{j}$$

and thus,

$$\sum_{i=1}^{n} \frac{1}{k_i} \leq \sum_{i=1}^{n} \frac{1}{i}$$

□

Suppose that k_i's are not just any random natural numbers but rather all $k_i | n$ and the n in the $\sum_{i=1}^{n} \frac{1}{k_i}$ will be replaced with $\tau(n)$. From this, we can rewrite the above inequality as

$$k = I(n) = \sum_{d|n} \frac{1}{d} = \sum_{i=1; d_i|n}^{\tau(n)} \frac{1}{d_i} \leq H_{\tau(n)}$$

Theorem 2 (A Lower bound of $\tau(n)$)**.** *For any natural n, the natural number n can be a k-multiperfect if the property*

$$e^{k-\gamma} < \tau(n)$$

is satisfied.

Proof. It was already established that

$$k < H_{\tau(n)} = \ln(\tau(n)) + \gamma + \epsilon$$

From here, we eliminate can eliminate ϵ and we have

$$k - \gamma < \ln(\tau(n)) \Rightarrow e^{k-\gamma} < \tau(n)$$

□

4 Illustration of the Theorem

It is necessary to verify for some small natural numbers due to the effect if ϵ is not included. The table below will provide numerical information up to $k = 26$, that is the least $\tau(n)$ for every k-multiperfect numbers.

k	$e^{k-\gamma}$	$min(\tau(n))$ for $H_{\tau(n)} > k$
1	1.526205112	1
2	4.148655621	4
3	11.27721519	11
4	30.65464912	31
5	83.32797566	83
6	226.5089221	227
7	615.7150868	616
8	1673.687132	1674
9	4549.553317	4550
10	12366.96811	12367
11	33616.90469	33617
12	91380.22114	91380
13	248397.1946	248397
14	675213.5803	675214
15	1835420.806	1835421
16	4989191.024	4989191
17	13562027.30	13562027
18	36865412.36	36865412
19	100210580.5	100210581
20	272400600.1	272400600
21	740461601.2	740461601
22	02012783315	2012783315
23	05471312310	5471312310
24	14872568831	14872568831
25	40427833596	40427833596

The table illustrates that suppose $\tau(n) = 2000000$, then n can never be 16-multiperfect. This helps us distinguish of a particular n can be k- multiperfect based on its $\tau(n)$. Although the lower bound is not that *tight* for every k- multiperfect number, at the very least, it does provide some information about it.

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