

THE COGROWTH SERIES FOR $BS(N, N)$ IS D-FINITE

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ABSTRACT. We compute the cogrowth series for Baumslag-Solitar groups $BS(N, N) = \langle a, b | a^N b = b a^N \rangle$, which we show to be D-finite. It follows that their cogrowth rates are algebraic numbers.

1. INTRODUCTION

The function $c : \mathbb{N} \rightarrow \mathbb{N}$ where $c(n)$ is the number of words of length n in the generators and inverses of generators of a finitely generated group that represent the identity element is called the *cogrowth function* and the corresponding generating function is called the *cogrowth series*. The rate of exponential growth of the cogrowth function $\limsup c(n)^{1/n}$ is the *cogrowth* of the group (with respect to a chosen finite generating set). Note that cogrowth can also be defined by counting only *freely reduced* trivial words — see Remark 1.1 at the end of this section.

In this article we study the cogrowth of the groups $BS(N, M)$ with presentation

$$BS(N, M) = \langle a, b \mid a^N b = b a^M \rangle$$

for positive integers N, M . We prove in Theorem 4.1 that for groups $BS(N, N)$ the cogrowth series is D-finite, that is, satisfies a linear differential equation with polynomial coefficients.

The class of D-finite (or holonomic) functions includes rational and algebraic functions, and many of the most famous functions in mathematics and physics. See [23, 24] for background on D-finite generating functions. If $\{a_n\}$ is a sequence and $A(z) = \sum_n a_n z^n$ is its corresponding generating function then $A(z)$ is D-finite if and only if $\{a_n\}$ is P-recursive (satisfies a linear recurrence with polynomial coefficients). D-finite functions are closed under addition and multiplication, and the composition of D-finite function with an algebraic function is D-finite [24]. Further, if a generating function is D-finite and the differential equation is known, then the coefficients of corresponding sequence can be computed quickly and their asymptotics are readily computed (see for example [26]).

The class of group presentations for which the cogrowth series has been computed explicitly (in terms of a closed-form expression, or system of simple recurrences, for example) is limited. Kouksov proved that the cogrowth series is a rational function if and only if the group is finite [15], and computed closed-form expressions for some free products of finite groups and free groups [16] which

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are algebraic functions. Humphries gave recursions and closed-form functions for various abelian groups [11, 12].

We note that Dykema and Redelmeier have also tried to compute cogrowth for general Baumslag-Solitar groups [3], and the problem appears to be a difficult one.

Grigorchuk and independently Cohen [2, 8] proved that a finitely generated group is amenable if and only if its cogrowth rate is twice the number of generators. For more background on amenability and cogrowth see [19, 25]. The free group on two (or more) letters is known to be non-amenable, and subgroups of amenable groups are also amenable. It follows that if a group contains a subgroup isomorphic to the free group on two generators, then it cannot be amenable. $\mathbb{Z}^2 \cong \text{BS}(1, 1)$ is amenable, while for $N > 1$ the subgroup of $\text{BS}(N, N)$ generated by at and at^{-1} is free, so these groups are non-amenable. We compute cogrowth rates for $\text{BS}(N, N)$ for $N \leq 10$ (see Table 1), and observe that the rate appears to converge to that of the free group of rank 2. This is in line with the result of Guyot and Stalder [9] that the limit (in the space of *marked groups*) of $\text{BS}(N, M)$ as $N, M \rightarrow \infty$ is the free group of rank 2.

In [5] a numerical method was used to find bounds for the cogrowth of groups, and a lower bound for the cogrowth rate of $\text{BS}(N, M)$ for small values of N, M were computed. A significant improvement of this numerical work has been undertaken by the authors, which can be found in [4].

The article is organised as follows. In Section 2 we briefly explain the well-known result that the cogrowth for $\text{BS}(1, 1) \cong \mathbb{Z}^2$ is D-finite and not algebraic. In Section 3 we introduce some two variable generating functions that count various words in Baumslag-Solitar groups $\text{BS}(N, M)$, and give a system of equations that they satisfy in Proposition 3.6. In Section 4 we restrict these equations to the case when $N = M$, and prove that the cogrowth series for $\text{BS}(N, N)$ is D-finite. In Section 5 we compute precise numerical values for the cogrowth rates, and in Section 6 we discuss computational work to find explicit formulae for the differential equations proved above, and conjecture an asymptotic form for the cogrowth series.

Remark 1.1. We note that the cogrowth of a group is often defined in terms of freely reduced words. Let $d : \mathbb{N} \rightarrow \mathbb{N}$ where $d(n)$ is the number of freely reduced words of length n . The associated generating functions of $c(n)$ and $d(n)$ are related via the following algebraic substitutions.

Lemma 1.2 (Lemma 1 of [27]; [15]). *Let $C(z) = \sum c(n)z^n$ and $D(z) = \sum d(n)z^n$ be the generating functions associated to $c(n)$ and $d(n)$, and assume the group in question is generated by p elements and their inverses. Then*

$$(1.1) \quad D(z) = \frac{1 - z^2}{1 + (2p - 1)z^2} C\left(\frac{z}{1 + (2p - 1)z^2}\right)$$

$$(1.2) \quad C(z) = \frac{1 - p + p\sqrt{1 - 4(2p - 1)z^2}}{1 - 4p^2z^2} D\left(\frac{1 - \sqrt{1 - 4(2p - 1)z^2}}{2(2p - 1)z}\right)$$

In this work we show that $C(z)$ is D-finite for $\text{BS}(N, N)$. The closure properties of D-finite functions then imply that $D(z)$ is also D-finite. More precisely, D-finite functions are closed under multiplication and the result of composing a D-finite

function with an algebraic function is also D-finite (see Theorems 2.3 and 2.7 in [24]).

2. $BS(1, 1)$

The contents of this section are well known, see for example sequence A002894 in [22] or page 90 of [6]. The group $BS(1, 1) = \langle a, b \mid ab = ba \rangle$ is simply the free abelian group \mathbb{Z}^2 of rank 2. The *Cayley graph* is just the square grid, and trivial words correspond to closed paths of even length starting and ending at the origin.

Now rotate the grid 45° and rescale by $\sqrt{2}$ — see Figure 1. Each step in a closed path changes the x -ordinate by ± 1 . At the same time, each step changes the y -ordinate by ± 1 , and these two processes are independent. In a path of $2n$ -steps, n steps must increase the x -ordinate and n must decrease it and so giving $\binom{2n}{n}$ possibilities. The same occurs independently for the y -ordinates and so we get another factor of $\binom{2n}{n}$. Hence the total number of possible trivial words of length $2n$ is $\binom{2n}{n}^2$.

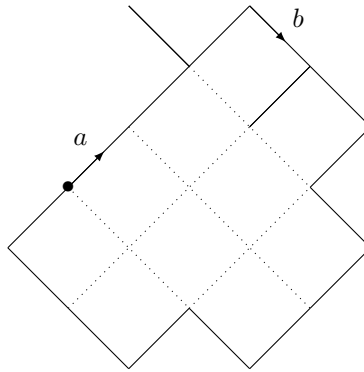


FIGURE 1. A trivial word $aab^{-1}baba^{-1}aba^{-1}ba^{-1}a^{-1}b^{-1}a^{-1}b^{-1}b^{-1}a$ in the Cayley graph of $\langle a, b \mid ab = ba \rangle$.

Let $c_{2n} = \binom{2n}{n}^2$ and $c_{2n+1} = 0$, and notice that $(n+1)^2 c_{2n+2} = 4(2n+1)^2 c_{2n}$. Then the sequence $\{c_n\}$ satisfies the polynomial recurrence

$$(n/2 + 1)^2 c_{n+2} = 4(n+1)^2 c_n$$

and is therefore *P-recursive*, which implies that the corresponding cogrowth series satisfies a linear differential equation with polynomial coefficients (Theorem 1.5 [24]), that is, the cogrowth series is *D-finite*.

One can show that the cogrowth series is not algebraic by considering its asymptotics. In particular, the coefficients of an algebraic function must grow as $A\mu^n n^\gamma$ where μ is an algebraic number, and γ belongs to the set $\mathbb{Q} \setminus \{-1, -2, -3, \dots\}$ (see Theorem D from [7]). An application of Stirling's formula shows that

$$(2.1) \quad c_n \sim 4^n \cdot \frac{2}{\pi n},$$

and so the factor of n^{-1} implies the corresponding generating function is not algebraic.

3. SERIES FOR GENERAL BAUMSLAG-SOLITAR GROUPS

Let us fix the following notation. If words u, v are identical as strings we write $u \equiv v$, and if they represent the same group element we write $u = v$. If a word w represents an element in a subgroup U , we write $w \in U$.

Consider the Baumslag-Solitar group

$$\text{BS}(N, M) = \langle a, b \mid a^N b = b a^M \rangle$$

with N, M positive integers.

Any word in $\{a^{\pm 1}, b^{\pm 1}\}^*$ can be transformed into a *normal form* ([18] p.181) for the corresponding group element by “pushing” each a and a^{-1} in the word as far to the right as possible using the identities

$$\begin{aligned} a^{\pm 1} a^{\mp 1} &= 1, & b^{\pm 1} b^{\mp 1} &= 1, \\ a^{\pm N} b &= b a^{\pm M} & a^{\pm M} b^{-1} &= b^{-1} a^{\pm N}, \\ a^{-i} b &= a^{N-i} b a^{-M} & a^{-j} b^{-1} &= a^{M-j} b^{-1} a^{-N} \end{aligned}$$

where $0 < i < N$ and $0 < j < M$, so that only positive powers of a appear before a $b^{\pm 1}$ letter. The resulting normal form can be written as Pa^k , where P is a freely reduced word in the alphabet $\{b, ab, \dots, a^{N-1}b, b^{-1}, ab^{-1}, \dots, a^{M-1}b^{-1}\}$. Call P the *prefix* and k the *a-exponent* of the normal form word Pa^k .

Recall Britton’s lemma ([18] p.181) which in the case of $\text{BS}(N, M)$ states that if a trivial word $w \in \{a^{\pm 1}, b^{\pm 1}\}^*$ is freely reduced and contains a $b^{\pm 1}$ letter, then w must contain a subword of the form $ba^{lN}b^{-1}$ or $b^{-1}a^{lM}b$ for some $l \in \mathbb{Z}$.

Let \mathcal{H} be the set of words in $\{a^{\pm 1}, b^{\pm 1}\}^*$ that represent elements in $\langle a \rangle$. Define $g_{n,k}$ to be the number of words $w \in \mathcal{H}$ of length n having normal form with a -exponent k .

The cogrowth function for $\text{BS}(N, M)$ can be obtained from $g_{n,k}$ by setting $k = 0$, as the next lemma shows. The reason for considering a more general function is that we found the corresponding two-variable generating function much easier to work with than the cogrowth series directly.

Lemma 3.1. $g_{n,0} = c(n)$ where $c : \mathbb{N} \rightarrow \mathbb{N}$ is the cogrowth function for $\text{BS}(N, M)$.

Proof. $g_{n,0}$ counts words $u \in \langle a \rangle$ with normal form Pa^0 , so $Pa^0 u^{-1} = Pa^k = 1$ for some k so by Britton’s Lemma P must be the empty string. \square

Lemma 3.2. $g_{n,k} = g_{n,-k}$.

Proof. Exchange each a by a^{-1} and vice versa in all words of length n equal to a^k to obtain all words of length n equal to a^{-k} . \square

Define two subsets of \mathcal{H} as follows.

- Let \mathcal{L} be the set of words $w \in \mathcal{H}$ such that for any prefix u of w , u does not have normal form $b^{-1}a^j$ for any integer j .
- Let \mathcal{K} be the set of words $w \in \mathcal{H}$ such that for any prefix u of w , u does not have normal form ba^j for any integer j .

Define $l_{n,j}, k_{n,j}$ to be the number of words of length n and a -exponent j in \mathcal{L}, \mathcal{K} respectively.

Lemma 3.3. *Let $w \in \mathcal{H}$. Then $w \in \mathcal{L}$ if and only if $w \neq xb^{-1}y$ with $x \in \langle a^N \rangle$. Similarly $w \in \mathcal{K}$ if and only if $w \neq xby$ with $x \in \langle a^M \rangle$.*

Proof. If $w \equiv xb^{-1}y$ with $x \in \langle a^N \rangle$, then $w = a^{iN}b^{-1}y = b^{-1}a^{iM}y$ so $w \notin \mathcal{L}$. Conversely if $w \notin \mathcal{L}$ then $w \equiv uv$ with $u = b^{-1}a^j$ for some j . Let Pa^k be the normal form for v . Since $w \in \mathcal{H}$ then $w = a^i$ for some i , and so

$$1 = wa^{-i} = uva^{-i} = b^{-1}a^jPa^{k-i}.$$

Since P contains no subwords of the form $ba^{lN}b^{-1}$ or $b^{-1}a^{lM}b$, Britton's lemma implies that $j = lM$ and P starts with a b , so $w \equiv xy$ with $x = b^{-1}a^{lM}b$. The result for \mathcal{K} follows by a similar argument. \square

Define the following two-variable generating functions

$$G(z; q) = \sum_{n,j} g_{n,j} z^n q^j,$$

$$L(z; q) = \sum_{n,j} l_{n,j} z^n q^j, \quad K(z; q) = \sum_{n,j} k_{n,j} z^n q^j.$$

These are all formal power series in z with coefficients that are Laurent polynomials in q . The functions L and K are related as follows.

Lemma 3.4. *In any group $BS(N, M)$ we have $L(z; 1) = K(z; 1)$.*

Proof. $L(z; 1) = \sum_n (\sum_k l_{n,k}) z^n$, where the inner sum is the number of words of length n in \mathcal{L} . If $w \in \mathcal{H} \setminus \mathcal{L}$ then $w \equiv xb^{-1}y$ with $x \in \langle a^N \rangle$. Assume x is chosen to be of minimal length – that is, if $x \equiv ub^{-1}v$ then u is not in $\langle a^N \rangle$. Since $w \in \mathcal{H}$ then $xb^{-1}yw^{-1} = a^{Ni}b^{-1}ya^i = 1$ so by Britton's lemma we must have $y = zbt$ with $z \in \langle a^M \rangle$. Again choose z to be minimal. Then $zbx b^{-1}t \in \mathcal{H} \setminus \mathcal{K}$ has the same length as w . Since x, z are uniquely determined, this gives a bijection between $\mathcal{H} \setminus \mathcal{L}$ and $\mathcal{H} \setminus \mathcal{K}$, and therefore between \mathcal{L} and \mathcal{K} . \square

In the case that $N = M$ we can say even more.

Lemma 3.5. *In $BS(N, N)$ we have $L(z; q) = K(z; q)$.*

Proof. If $w \in \mathcal{L}$ has length n then replacing b by b^{-1} and vice versa, we obtain a word which has a prefix equal to ba^j and of length n , and represents the same power of a as w , so is in \mathcal{K} . \square

Note that if $N \neq M$ then the above proof breaks down — replacing b by b^{-1} in $u = ba^N b^{-1} \in \mathcal{L}$ gives a word not in $\langle a \rangle$, so the resulting word is not in \mathcal{K} .

Next for $d, e \in \mathbb{N}$ define an operator $\Phi_{d,e}$ which acts on Laurent polynomials in q by

$$\Phi_{d,e} \circ (Aq^k) = \begin{cases} Aq^{ej} & \text{if } k = dj \\ 0 & \text{otherwise.} \end{cases}$$

For example, $\Phi_{2,3}(4zq + 5zq^4) = 5zq^6$. We then extend this operator to power series in z with coefficients that are Laurent polynomials in q , in the obvious way

$$\Phi_{d,e} \circ \left(\sum_n z^n \sum_k c_{n,k} q^k \right) = \sum_n z^n \sum_j c_{n,dj} q^{ej}.$$

Note that $\Phi_{N,N}$ simply deletes all terms except those of the form $cz^i q^{jN}$.

With these definitions we can write down a set of equations satisfied by the functions G, L and K .

Proposition 3.6. *The generating functions $G \equiv G(z; q), L \equiv L(z; q)$ and $K \equiv K(z; q)$ satisfy the following system of equations.*

$$\begin{aligned} L &= 1 + z(q + q^{-1})L + z^2 L \cdot [\Phi_{N,M} \circ L + \Phi_{M,N} \circ K] - z^2 [\Phi_{M,N} \circ K] \cdot [\Phi_{N,N} \circ L], \\ K &= 1 + z(q + q^{-1})K + z^2 K \cdot [\Phi_{M,N} \circ K + \Phi_{N,M} \circ L] - z^2 [\Phi_{N,M} \circ L] \cdot [\Phi_{M,M} \circ K], \end{aligned}$$

and

$$G = 1 + z(q + q^{-1})G + z^2 [\Phi_{N,M} \circ L + \Phi_{M,N} \circ K] G.$$

Proof. We first establish the equation

$$G = 1 + zqG + zq^{-1}G + z^2 [\Phi_{N,M} \circ L] G + z^2 [\Phi_{M,N} \circ K] G$$

Factor words in \mathcal{H} recursively by considering the first letter in any word $w \in \mathcal{H}$. This gives five cases which will correspond to the five terms on the RHS of the above equation:

- w is the empty word. This is counted by the 1 in the expression for G .
- The first letter is a . Then $w \equiv av$ for some $v \in \mathcal{H}$. If w has length n and a -exponent k , then v has length $n - 1$ and a -exponent $k - 1$. Summing over the contribution of all possible such w gives $zqG(z; q)$ at the level of generating functions.
- The first letter is a^{-1} . Then $w \equiv a^{-1}v$ for some $v \in \mathcal{H}$. At the level of generating functions this gives $zq^{-1}G(z; q)$, since the number of words counted by $g_{n,k}$ of this form is the number of words counted in \mathcal{H} of length $n - 1$ and a -exponent $k + 1$.
- The first letter is b . Write $w \equiv uv$ where u is the shortest prefix of w so that $u \in \langle a \rangle$. Thus, $u \equiv bu'b^{-1}$ for some $u' \in \langle a^N \rangle$ and so $u \in \langle a^M \rangle$. The minimality of u ensures $u' \in \mathcal{L}$.

It follows that words $w \equiv bu'b^{-1}v$ of length n and a -exponent k are counted by

$$\sum_{i=0}^{n-2} \sum_j (l_{i,jN}) (g_{n-i-2,k-jM})$$

where $l_{i,jN}$ counts the words u' of length i and a -exponent jN , and $g_{n-i-2,k-jM}$ counts the words v of length $n - i - 2$ and a -exponent $k - jM$.

So the term

$$\left(\sum_{i=0}^{n-2} \sum_j (l_{i,jN}) (g_{n-i-2,k-jM}) \right) z^n q^k = z^2 \left(\sum_{i=0}^{n-2} \sum_j (l_{i,jN}) (g_{n-i-2,k-jM}) \right) z^{n-2} q^k$$

is the contribution to $g_{n,k}z^nq^k$ from words starting with b , and summing over all such words gives their contribution to $G(z; q)$.

We claim that

$$\sum_{n,k} \left(\sum_{i=0}^{n-2} \sum_j (l_{i,jN}) (g_{n-i-2,k-jM}) \right) z^{n-2}q^k = [\Phi_{N,M} \circ L(z; q)] G(z; q)$$

Let us start by computing the action of the Φ operator

$$\begin{aligned} \Phi_{N,M} \circ L(z; q) &= \Phi_{N,M} \circ \left(\sum_r z^r \sum_j l_{r,i} q^i \right) \\ &= \sum_r z^r \cdot \Phi_{N,M} \circ \left(\sum_i l_{r,i} q^i \right) = \sum_r z^r \cdot \Phi_{N,M} \circ \left(\sum_j \sum_d l_{r,Nj+d} q^{Nj+d} \right) \\ &= \sum_r z^r \sum_j l_{r,Mj} q^{Nj} \end{aligned}$$

Now let us expand the right hand side.

$$\begin{aligned} [\Phi_{N,M} \circ L(z; q)] G(z; q) &= \left(\sum_r z^r \sum_j l_{r,jN} q^{jM} \right) \cdot \left(\sum_i z^i \sum_e g_{i,e} q^e \right) \\ &= \sum_{r,i} z^{r+i} \left(\sum_j q^{jM} \cdot l_{r,jN} \right) \cdot \left(\sum_e q^e \cdot g_{i,e} \right) \\ &= \sum_{r,i} z^{r+i} \sum_{j,e} q^{jM+e} \cdot l_{r,jN} \cdot g_{i,e} \end{aligned}$$

Set $r = \alpha$, $i = n - 2 - \alpha$ and $e = k - jN$

$$= \sum_{n,\alpha} z^{n-2} \sum_{j,k} q^k \cdot l_{\alpha,jN} \cdot g_{n-2-\alpha,k-jN}$$

which is the required form.

- The first letter is b^{-1} . Factor $w \equiv uv$ where u is the shortest word so that $u \in \langle a \rangle$. As per the previous case, $u \equiv b^{-1}u'b$ for some $u' \in \langle a^M \rangle$ with $u' \in \mathcal{K}$, and so $u \in \langle a^N \rangle$. It follows (by similar reasoning) that the contribution of these words to $G(z; q)$ is $z^2 (\Phi_{M,N} \circ K(z; q)) \cdot G(z, q)$.

We now prove the equation satisfied by $L(z, q)$:

$$L = 1 + z(q + q^{-1})L + z^2L \cdot [\Phi_{N,M} \circ L] + z^2(L - [\Phi_{N,N} \circ L]) \cdot [\Phi_{M,N} \circ K]$$

Consider an element $w \in \mathcal{L}$, and we note that \mathcal{L} (and \mathcal{K}) is closed under appending the generators a and a^{-1} , but not prepending. For this reason we will factor words in \mathcal{L} recursively by considering the last letter of w , which again gives use five cases.

- w is the empty word, accounting for the 1 in the expression.
- The last letter is a or a^{-1} . Then $w \equiv va$ or $w \equiv va^{-1}$ for some $v \in \mathcal{L}$, increasing the length by 1 and altering the a -exponent by ± 1 . This yields the term $z(q + q^{-1})L(z; q)$.

- The last letter is b^{-1} . Factor $w \equiv uv$ where u is the longest subword such that $u \in \langle a \rangle$ and v is non-empty. This forces $v \equiv bv'b^{-1}$ with the restriction that $v' \in \mathcal{L}$. Since both $v, v' \in \mathcal{L}$ we must have $v' \in \langle a^N \rangle$ and $v \in \langle a^M \rangle$. By similar arguments to those used above for G , the contribution of all such words is $z^2 L(z; q) \cdot \Phi_{N, M} \circ L(z; q)$.
- The last letter is b . Factor $w \equiv uv$ where u is the longest subword such that $u \in \langle a \rangle$ and v is non-empty. This forces $v \equiv b^{-1}v'b$ with the restriction that $v' \in \mathcal{K}$. Lemma 3.3 implies the subword $u \notin \langle a^N \rangle$.

The generating function for $\{u \in \mathcal{L} \mid u \notin \langle a^N \rangle\}$ is given by $(L - \Phi_{N, N} \circ L)$, and so this last case gives $z^2(L(z; q) - \Phi_{N, N} \circ L(z; q)) \cdot \Phi_{M, N} \circ K(z; q)$.

Again the details are similar to those used in the above argument for G .

Putting all of these cases together and rearranging gives the result. The equation for \mathcal{K} follows a similar argument. \square

4. SOLUTION FOR $\text{BS}(N, N)$

Let $G(z; q)$ be the function defined above for the group $\text{BS}(N, N)$. Recall that our main interest is the coefficient of q^0 in $G(z; q)$ rather than the full generating function itself, so let $[q^0]G(z; q) = \sum_n g_{n,0} z^n$ which by Lemma 3.1 is the cogrowth series for $\text{BS}(N, N)$.

Theorem 4.1. *The generating function $G(z, q)$ is algebraic. Consequently the cogrowth series $[q^0]G(z; q)$ is D-finite, and the cogrowth rate is an algebraic number.*

Proof. We claim that the generating function $G(z; q)$ is algebraic, satisfying a polynomial equation (of degree $N + 1$). We prove this claim below and first show that the remainder of the theorem follows from this claim.

Assume $G(z; q)$ is algebraic; since all algebraic functions are D-finite [24], it is also D-finite. The series $G(zq; q^{-1})$ is also algebraic and D-finite. The cogrowth series can then be expressed as the diagonal of this series, where the diagonal of a power series in z and q is defined to be

$$I_{z,q} \circ \sum_{n,k} f_{n,k} z^n q^k = \sum_n f_{n,n} z^n.$$

In particular

$$I_{z,q} \circ G(zq; q^{-1}) = I_{z,q} \circ \sum_{n,k} z^n q^{n-k} g_{n,k} = \sum_n z^n g_{n,0}.$$

A result of Lipshitz [17] states that the diagonal of a D-finite series is itself D-finite. Thus $I_{z,q} G(zq; q^{-1}) = [q^0]G(z; q)$ is D-finite.

By definition, any D-finite generating function, $f(z)$, satisfies a linear differential equation with polynomial coefficients which can be written as

$$f^{(n)}(z) + \sum_{j=0}^{n-1} \frac{p_j(z)}{p_n(z)} f^{(j)}(z) = 0$$

where the $p_i(z)$ are polynomial.

The singularities of $f(z)$ correspond to the singularities of the coefficients of the DE (see Chapter 1 of [20]). Thus the singularities of the solution and its

radius of convergence are all algebraic numbers. The exponential growth rate of the coefficients of the expansion of $f(z)$ about zero is equal to the reciprocal of the radius of convergence (see, for example, Theorem IV.7 in [6]). Thus the cogrowth rate is an algebraic number.

So to establish the theorem we need to show that $G(z; q)$ satisfies an polynomial equation.

Since $N = M$ we have $K(z; q) = L(z; q)$ by Lemma 3.5, and so the equations in Proposition 3.6 simplify considerably to

$$\begin{aligned} L &= 1 + z(q + q^{-1})L + 2z^2L \cdot [\Phi_{N,N} \circ L] - z^2 [\Phi_{N,N} \circ L]^2, \\ G &= 1 + z(q + q^{-1})G + 2z^2G \cdot [\Phi_{N,N} \circ L]. \end{aligned}$$

To simplify notation in what follows, write

$$L_0(z; q) = \Phi_{N,N} \circ L(z; q) = \sum_{n,d} l_{n,dN} z^n q^{dN}.$$

So the equations for L and G may be written as

$$\begin{aligned} L(z; q) &= \frac{1 - z^2 L_0(z; q)^2}{1 - z(q + q^{-1}) - 2z^2 L_0(z; q)}, \\ G(z; q) &= \frac{1}{1 - z(q + q^{-1}) - 2z^2 L_0(z; q)}. \end{aligned}$$

To complete the proof it suffices to show that $L_0(z; q)$ satisfies an algebraic equation of degree $(N + 1)$.

Let $\omega = e^{2\pi i/N}$ be an N^{th} root of unity. We claim that

$$\begin{aligned} L_0(z; \omega^j q) &= L_0(z; q) && \text{and} \\ NL_0(z; q) &= \sum_{j=0}^{N-1} L(z; \omega^j q). \end{aligned}$$

To see the first, by definition $L_0(z; \omega^j q) = \sum_{n,d} l_{n,dN} z^n (\omega^j q)^{dN}$ and $\omega^N = 1$. To derive the second consider the sum $\sum_{j=0}^{N-1} \omega^{kj}$, with k in \mathbb{Z} . If $k = dN$, $d \in \mathbb{Z}$ then

$$\sum_{j=0}^{N-1} \omega^k = \sum_{j=0}^{N-1} \omega^{Nd} = \sum_{j=0}^{N-1} 1 = N.$$

On the other hand, if $k = Nd + l$ with $0 < l < N$:

$$\begin{aligned} \sum_{j=0}^{N-1} \omega^{kj} &= \sum_{j=0}^{N-1} (\omega^N)^{dj} \cdot \omega^{jl} = \sum_{j=0}^{N-1} \omega^{jl} \\ &= 1 + \omega^l + \dots + \omega^{(N-1)l} = \frac{1 - \omega^{Nl}}{1 - \omega^l} = 0 \end{aligned}$$

since $\omega^l \neq 1$ and $\omega^{Nl} = 1$. Returning to the second claim

$$\begin{aligned} \sum_{j=0}^{N-1} L(z; \omega^j q) &= \sum_{j=0}^{N-1} \sum_{n,k} g_{n,k} z^n q^k \omega^{kj} \\ &= \sum_{n,k} g_{n,k} z^n q^k \sum_{j=0}^{N-1} \omega^{kj} \\ &= \sum_{n,d} g_{n,dN} z^n q^{dN} \cdot N = NL_0(z; q) \end{aligned}$$

as required.

Combining the above expression for $L(z; q)$ in terms of $L_0(z; q)$ with that expressing $L_0(z; q)$ in terms of $L(z; q\omega^j)$ we obtain

$$(4.1) \quad NL_0(z; q) = \sum_{j=0}^{N-1} \frac{1 - z^2 L_0(z; q)^2}{1 - z(q\omega^j + q^{-1}\omega^{-j}) - 2z^2 L_0(z; q)}$$

where we have used the fact that $L_0(z; q\omega^j) = L_0(z; q)$. Clearing the denominator of this expression we have

$$\begin{aligned} NL_0(z; q) \cdot \prod_{j=0}^{N-1} (1 - z(q\omega^j + q^{-1}\omega^{-j}) - 2z^2 L_0(z; q)) \\ = \sum_{j=0}^{N-1} (1 - z^2 L_0(z; q)^2) \cdot \prod_{\substack{0 \leq l \leq N \\ l \neq j}} (1 - z(q\omega^j + q^{-1}\omega^{-j}) - 2z^2 L_0(z; q)). \end{aligned}$$

Thus $L_0(z; q)$ satisfies an algebraic equation of degree at most $N + 1$.

Now rewrite $L_0(z; q)$ in terms of $G(z; q)$:

$$(4.2) \quad L_0(z; q) = \frac{1}{2z^2} - \frac{q + q^{-1}}{2z} - \frac{1}{2z^2 G(z; q)}.$$

Substituting this into the above equation for $L_0(z; q)$ and clearing denominators yields an equation for $G(z; q)$ of degree at most $N + 1$. \square

We remark that for small values of N , the equation satisfied for G is relatively easy to write down (with the aid of a computer algebra system). Set $Q = q + q^{-1}$, then for $N = 2, 3, 4, 5$ the generating function $G(z; q)$ satisfies the following $(N+1)$ -degree polynomial equations

$$(4.3) \quad 1 + 3zQG - (1 - 4z^2 - z^2Q^2)G^2 - zQ(1 - zQ - 2z)(1 - zQ + 2z)G^3 = 0,$$

$$(4.4) \quad 1 + 4zQG + (6Q^2z^2 - z^2 - 1)G^2 + 2z(Qz + 1)(Q^2z - Q + 2z)G^3 \\ + z^2(1 - Q)(1 + Q)(Qz + 2z - 1)(Qz - 2z - 1)G^4 = 0,$$

$$(4.5) \quad 1 + 5GQz + (10Q^2z^2 - 2z^2 - 1)G^2 + z(10Q^3z^2 - 6Qz^2 - 3Q + 4z)G^3 \\ + z^2(3Q^4z^2 + 2Q^2z^2 - 3Q^2 + 8Qz - 8z^2 + 2)G^4 \\ - z^3Q(Q^2 - 2)(Qz + 2z - 1)(Qz - 2z - 1)G^5 = 0$$

and

$$(4.6) \quad 1 + 6QzG + (15Q^2z^2 - 3z^2 - 1)G^2 \\ + 4(5Q^3z^2 - 3Qz^2 - Q + z)zG^3 + 3(5Q^4z^2 - 6Q^2z^2 - 2Q^2 + 4Qz - z^2 + 1)z^2G^4 \\ + 2(2Q^5z^2 - Q^3z^2 - 2Q^3 + 6Q^2z - 8Qz^2 + 3Q - 4z)z^3G^5 \\ - (Q^2 + Q - 1)(Qz + 2z - 1)(Qz - 2z - 1)(Q^2 - Q - 1)z^4G^6 = 0$$

respectively.

5. COGROWTH RATES

Theorem 4.1 proves that the cogrowth rates are algebraic numbers but does not give an explicit method for computing them. Since for $N \geq 2$ the groups $BS(N, N)$ are non-amenable, we know by Grigorchuk and Cohen [8, 2] these numbers are strictly less than 4. The following theorem enables us to compute the cogrowth rates explicitly.

Recall that the reciprocal of the radius of convergence of a power series $\sum_n a_n z^n$ is $\limsup a_n^{1/n}$.

Theorem 5.1. *For $BS(N, N)$, the generating functions $G(z; 1)$ and $[q^0]G(z; q)$ have the same radius of convergence. Hence the cogrowth rate is the reciprocal of the radius of convergence of $G(z, 1)$.*

The proof of the above theorem hinges on the following observation

Lemma 5.2. *For $BS(N, N)$, $g_{n,k} = 0$ for $|k| > n$.*

Proof. Let w be a word of length n and a -exponent k in \mathcal{H} . If w contains a canceling pair of generators then removing them does not increase length, and replacing $a^{\pm N}b^{\pm 1}$ with $b^{\pm 1}a^{\pm N}$ similarly does not increase length, so applying these moves to w we obtain a freely reduced word of length at most n of the form $a^{\eta_1}b^{\epsilon_1} \dots a^{\eta_s}b^{\epsilon_s}a^r$ where $\eta_i, \epsilon_i, r, s \in \mathbb{Z}$, $|\eta_i| < N$, $\epsilon_i = \pm 1$ and $s \geq 0$. Since $w \in H$ we have $w = a^p$ for some $p \in \mathbb{Z}$, so $a^{\eta_1}b^{\epsilon_1} \dots a^{\eta_s}b^{\epsilon_s}a^r a^{-p} = 1$, and Britton's lemma implies that $s = 0$ and $p = k \leq n$. \square

Proof of Theorem 5.1. Let g_n denote the number of words in \mathcal{H} of length n , that is, $g_n = \sum_k g_{n,k}$, and $G(z; 1) = \sum_{n=0}^{\infty} g_n z^n$. Write $\limsup g_n^{1/n} = \mu_{all}$ and $\limsup g_{n,0}^{1/n} = \mu_0$. Since $g_{n,k} \geq 0$ we have that $g_n \geq g_{n,0}$ and so $\mu_{all} \geq \mu_0$.

To prove the reverse inequality we proceed as follows. For a fixed n , by Lemma 5.2, $g_{n,k}$ is positive for at most $2n + 1$ values of k . Since $g_{n,k}$ is positive for only finitely many k , there is some integer k^* (depending on n) so that $g_{n,k^*} \geq g_{n,k}$ for all k . Then we have

$$(5.1) \quad g_{n,k^*} \leq g_n \leq (2n + 1)g_{n,k^*}$$

and hence $\limsup g_{n,k^*}^{1/n} = \mu_{all}$.

Keeping n fixed, consider a word that contributes to g_{n,k^*} and another that contributes to $g_{n,-k^*}$. Concatenating them together gives a word that contributes to $g_{2n,0}$. It follows that

$$(5.2) \quad g_{n,k^*} g_{n,-k^*} \leq g_{2n,0}.$$

Since $g_{n,k^*} = g_{n,-k^*}$ by Lemma 3.2 this gives

$$(5.3) \quad g_{n,k^*}^2 \leq g_{2n,0}$$

Raising both sides to the power $1/2n$ and taking $\limsup_{n \rightarrow \infty}$ gives

$$(5.4) \quad \limsup_{n \rightarrow \infty} g_{n,k^*}^{1/n} \leq \limsup_{n \rightarrow \infty} (g_{2n,0})^{1/2n}$$

and so $\mu_{all} \leq \mu_0$. □

The preceding proof is known as a “most popular” argument in statistical mechanics, where it is commonly used to prove equalities of free-energies (see [10] for example). In our case a most popular value is k^* , since this is a value that maximises $g_{n,k}$. Numerical experiments show that $k^* = 0$, with a distribution tightly peaked around 0. This tells us experimentally that the most numerous words evaluating to an element in $\langle a \rangle$ are those with zero a -exponent.

For fixed N , we can use a computer algebra system to combine equations (4.1) and (4.2) in the proof of Theorem 4.1 to obtain explicit polynomial equations satisfied by $G(z, q)$. We listed these for $N = 2, 3, 4, 5$ at the end of the previous section. Substituting $q = 1$ (and $Q = q + q^{-1} = 2$) gives an explicit polynomial equation for $G(z, 1)$. One can then find its radius of convergence by solving the discriminant of the polynomial — see for example [14]. Alternatively one can convert the algebraic equation satisfied by $G(z, 1)$ to a differential equation and then a linear recurrence satisfied by g_n (using say, the `gfun` package [21] for the `Maple` computer algebra system). The asymptotics of g_n can then be determined using the methods described in [26].

These computations become extremely slow for even modest values of N . We give the results for $1 \leq N \leq 10$ in Table 1.

Some simple numerical analysis of these numbers suggests that the cogrowth rate approaches $\sqrt{12}$ exponentially with increasing N . This finding agrees with work of Guyot and Stalder [9], who examined the limit of the *marked* groups $BS(M, N)$ as $M, N \rightarrow \infty$, and found that the groups tend towards the free group on two letters, which has an asymptotic cogrowth rate of $\sqrt{12}$.

Remark 5.3. For $BS(N, M)$ with $N \neq M$, $g_{n,k}$ can be nonzero for $|k| > n$ (for example in $BS(2, 4)$ we have $t^i a^2 t^{-i} = a^{2^{i+1}}$). So while the left-hand inequality in equation (5.1) still holds, the factor of $(2n + 1)$ in the right-hand inequality would be replaced with a term that grows exponentially with n , which means the proof of Theorem 5.1 breaks down. We computed $g_{n,k}$ for small n for several values of N, M and while the number of k -values for which $g_{n,k} > 0$ grows exponentially with n , we observed that the distribution of $g_{n,k}$ is again very tightly peaked around $k = 0$. This suggests that $g_{n,0}$ is the dominant contribution to g_n and thus it may still be the case that $G(z, 1)$ and $[q^0]G(z, q)$ have the same radii of convergence.

N	μ	λ
1	4	3
2	3.792765039	2.668565568
3	3.647639445	2.395062561
4	3.569497357	2.215245886
5	3.525816111	2.091305394
6	3.500607636	2.002421757
7	3.485775158	1.936941986
8	3.476962757	1.887871818
9	3.471710431	1.850717434
10	3.468586539	1.822458708

TABLE 1. The cogrowth rate μ in $BS(N, N)$ and the corresponding growth rate of freely reduced trivial words λ . Note that μ and λ are related by $\mu = \lambda + 3/\lambda$ (see Lemma 1.2), and that the cogrowth rate in the free group on 2 generators is $\sqrt{12} = 3.464101615$.

6. DISCUSSION OF DIFFERENTIAL EQUATIONS SATISFIED BY COGROWTH SERIES

Our proof of Theorem 4.1 guarantees the existence of differential equations satisfied by the cogrowth series. It does, however, give a recipe for producing them. In this section we use a recently developed algorithm due to Chen, Kauers and Singer [1] to obtain explicit differential equations for small values of N .

Applying the algorithms described in [1] to the generating function $G(z; q)$ for $BS(2, 2)$ we found a 6th order linear differential equation satisfied by $[q^0]G(z; q)$. The polynomial coefficients of this equation have degrees up to 47 and so we have not listed it here¹. Again applying the same methods, we found an ODE of order 8 with coefficients of degree up to 105 for $BS(3, 3)$ and for $BS(4, 4)$ it is order 10 with coefficients of degree up to 154. We thank Manuel Kauers for his assistance with these computations.

By studying these differential equations we can determine the asymptotic behaviour of $g_{n,0}$ in more detail and demonstrate that the cogrowth series is not algebraic.

Proposition 6.1. *The coefficients of the cogrowth series grow as*

$$g_{n,0} = \binom{n}{n/2}^2 \sim \frac{2}{\pi n} \cdot 4^n \quad N = 1$$

$$g_{n,0} \sim \frac{A_N}{n^2} \cdot \mu^n \quad N = 2, 3, 4$$

for even n where A_N is some real number. As a consequence the cogrowth series for $BS(N, N)$ is not algebraic for $N = 1, 2, 3, 4$.

¹The differential equation for the cogrowth of $BS(N, N)$ for $1 \leq N \leq 10$ can be found at http://www.math.ubc.ca/~andrewr/pub_list.html

Proof. We can compute the differential equations satisfied by the cogrowth series using the techniques in [1]. From these equations we then use the methods developed in [26] to determine the asymptotics of $g_{n,0}$. The presence of the factors of n^{-1} and n^{-2} in the asymptotic forms proves (see Theorem D from [7]) that the associated generating function cannot be algebraic. \square

While there is no theoretical barrier at $N = 4$, the derivation of differential equations by this method slows quickly with increasing N and we were unable to compute the differential equations for $N \geq 5$ rigorously. However, using clever guessing techniques (see [13] for a description) Manuel Kauers also found differential equations for $N = 5, \dots, 10$. For BS(5, 5) the DE is order 12 with coefficients of degree up to 301, while that of BS(10, 10) is 22nd order with coefficients of degree up to 1153 — the computation for $N = 10$ took about 50 days of computer time, and when written in text file is over 6 Mb!

Using these differential equations we also determine that for $N \leq 10$ we have

$$g_{n,0} \sim \frac{A_N}{n^2} \mu_N^n.$$

In light of this evidence we make the following conjecture:

Conjecture 1. *The number of trivial words in BS(N, N) grows as*

$$g_{n,0} \sim \frac{A_N}{n^2} \cdot \mu_N^n \quad \text{for even } n$$

for $N \geq 2$ and consequently the cogrowth series is not algebraic.

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