

Monotonicity and log-behavior of some functions related to the Euler Gamma function *

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Abstract

The aim of this paper is to develop analytic techniques to deal with certain monotonicity of combinatorial sequences. On the one hand, a criterion for the monotonicity of the function $\sqrt[x]{f(x)}$ is given, which is a continuous analog for one result of Wang and Zhu. On the other hand, the log-behavior of the functions $\theta(x) = \sqrt[x]{2\zeta(x)\Gamma(x+1)}$ and $F(x) = \sqrt[x]{\frac{\Gamma(ax+b+1)}{\Gamma(cx+d+1)\Gamma(ex+f+1)}}$ is considered, where $\zeta(x)$ and $\Gamma(x)$ are the Riemann zeta function and the Euler Gamma function, respectively. As consequences, the strict log-concavities of the function $\theta(x)$ (a conjecture of Chen *et al.*) and $\{\sqrt[n]{z_n}\}$ for some combinatorial sequences (including the Bernoulli numbers, the Tangent numbers, the Catalan numbers, the Fuss-Catalan numbers, the Binomial coefficients $\binom{2n}{n}$, $\binom{3n}{n}$, $\binom{4n}{n}$, $\binom{5n}{n}$, $\binom{5n}{2n}$) are demonstrated. In particular, this contains some results of Chen *et al.*, Luca and Stănică.

Finally, by researching logarithmically complete monotonicity of some functions, the infinite log-monotonicity of the sequence $\{\frac{(n_0+ia)!}{(k_0+ib)!(k_0+ib)!}\}_{i \geq 0}$ is proved. This generalizes two results of Chen *et al.* that both the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$ and central binomial coefficients $\binom{2n}{n}$ are infinitely log-monotonic and strengthens one result of Su and Wang that $\binom{dn}{\delta n}$ is log-convex in n for positive integers $d > \delta$. In addition, the asymptotically infinite log-monotonicity of derangement numbers is showed. In order to research the stronger properties of the above functions $\theta(x)$ and $F(x)$, the logarithmically complete monotonicity of functions $1/\sqrt[x]{a\zeta(x+b)\Gamma(x+c)}$ and $\sqrt[x]{\rho \prod_{i=1}^n \frac{\Gamma(x+a_i)}{\Gamma(x+b_i)}}$ is also obtained, which generalizes the results of Lee and Tepedelenlioglu, Qi and Li.

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1 Introduction

Let $\{z_n\}_{n \geq 0}$ be a sequence of positive numbers. It is called *log-concave* (resp. *log-convex*) if $z_{n-1}z_{n+1} \leq z_n^2$ (resp. $z_{n-1}z_{n+1} \geq z_n^2$) for all $n \geq 1$. Clearly, the sequence $\{z_n\}_{n \geq 0}$ is log-concave (resp. log-convex) if and only if the sequence $\{z_{n+1}/z_n\}_{n \geq 0}$ is decreasing (resp. increasing). Generally speaking, a sequence will have good behavior (e.g., distribution properties, bounds by inequalities) if it is log-concave or log-convex. In addition, sequences with log-behaviour arise often in combinatorics, algebra, geometry, analysis, probability and statistics and have been extensively investigated (see [3, 10, 15, 20, 23] for instance).

Motivated by a series of conjectures of Sun [17] about monotonicity of sequences of the forms $\{\sqrt[n]{z_n}\}$ and $\{\sqrt[n+1]{z_{n+1}}/\sqrt[n]{z_n}\}$, where $\{z_n\}_{n \geq 0}$ is a familiar number-theoretic or combinatorial sequence, e.g., the Bernoulli numbers, the Fibonacci numbers, the derangements numbers, the Tangent numbers, the Euler numbers, the Schröder numbers, the Motzkin numbers, the Domb numbers, and so on. These conjectures have recently been investigated by some researchers (see Chen *et al.* [4, 5], Hou *et al.* [8], Luca and Stănică [11], Wang and Zhu [21]). The main aim of this paper is to develop some analytic techniques to deal with monotonicity of $\{\sqrt[n]{z_n}\}$ and $\{\sqrt[n+1]{z_{n+1}}/\sqrt[n]{z_n}\}$ (Note that the monotonicity of $\{\sqrt[n+1]{z_{n+1}}/\sqrt[n]{z_n}\}$ equals to the log-behavior of $\{\sqrt[n]{z_n}\}$).

Recently, Wang and Zhu [21] observed sufficient conditions that the log-behaviour of $\{z_n\}_{n \geq 0}$ implies the monotonicity that of $\{\sqrt[n]{z_n}\}_{n \geq 1}$. For example, for a positive log-convex sequence $\{z_n\}_{n \geq 0}$, if $z_0 \leq 1$, then the sequence $\{\sqrt[n]{z_n}\}_{n \geq 1}$ is increasing. Using the analytic approach of Chen *et al.* [4], the following continuous analog can be proved, whose proof is arranged in Section 2.

Theorem 1.1. *Let N be a positive number. If $f(x)$ is a positive increasing log-convex function for $x \geq N$ and $f(N) \leq 1$, then $\sqrt[x]{f(x)}$ is strictly increasing on (N, ∞) .*

Remark 1.2. Theorem 1.1 can be applied to the monotonicity of $\{\sqrt[n]{z_n}\}_{n \geq 1}$ for some combinatorial sequences $\{z_n\}_{n \geq 0}$. Some further examples and applications related to Theorem 1.1 can be found in [4].

Thus, one may ask whether there are some analytic techniques to deal with the log-behavior of $\{\sqrt[n]{z_n}\}_{n \geq 1}$. This is another motivation of this paper. In particular, the following conjecture of Chen *et al.* [4] is still open.

Example 1.3. Recall that the classical Bernoulli numbers are defined by

$$B_0 = 1, \quad \sum_{k=0}^n \binom{n+1}{k} B_k = 0, \quad n = 1, 2, \dots$$

It is well known that $B_{2n+1} = 0$, $(-1)^{n-1}B_{2n} > 0$ for $n \geq 1$ and

$$(-1)^{n-1}B_{2n} = \frac{2(2n)!\zeta(2n)}{(2\pi)^{2n}},$$

see [6, (6.89)] for instance. In order to show that $\{\sqrt[n]{(-1)^{n-1}B_{2n}}\}$ is increasing, Chen et al [4] introduced the function $\theta(x) = \sqrt[x]{2\zeta(x)\Gamma(x+1)}$, where

$$\zeta(x) = \sum_{n \geq 1} \frac{1}{n^x}$$

is the Riemann zeta function and $\Gamma(x)$ is the Euler Gamma function. Thus

$$\sqrt[n]{(-1)^{n-1}B_{2n}} = \theta^2(2n)/4\pi^2.$$

They proved that $\theta(x)$ is increasing on $(6, \infty)$. In addition, in order to get the log-concavity of $\{\sqrt[n]{(-1)^{n-1}B_{2n}}\}_{n \geq 1}$, they further conjectured.

Conjecture 1.4. [4] *The function $\theta(x) = \sqrt[x]{2\zeta(x)\Gamma(x+1)}$ is log-concave on $(6, \infty)$.*

Using some inequalities of the Riemann zeta function and the Euler Gamma function, in Section 3, this conjecture will almost be confirmed, see Theorem 3.3. As applications, the results of Luca and Stănică [11] on strict log-concavities of $\{\sqrt[n]{(-1)^{n-1}B_{2n}}\}_{n \geq 1}$ and $\{\sqrt[n]{T(n)}\}_{n \geq 1}$ can be verified, where $T(n)$ are the Tangent numbers.

In addition, motivated by the strict log-concavities of $\sqrt[n]{\binom{2n}{n}}$ and $\sqrt[n]{\frac{1}{2n+1}\binom{2n}{n}}$ (Chen et al. [5]), the log-behavior of the function

$$F(x) = \sqrt[x]{\frac{\Gamma(ax+b+1)}{\Gamma(cx+d+1)\Gamma(ex+f+1)}}$$

is considered (see Theorem 3.6). As consequences, for any positive integers $p \geq 2$ and $a > c$, the strict log-concavities of $\{\sqrt[n]{\frac{1}{(p-1)n+1}\binom{pn}{n}}\}_{n \geq 2}$ and $\{\sqrt[n]{\binom{an}{cn}}\}_{n \geq 30}$ are obtained, see Corollary 3.7. For more examples, the sequences $\{\sqrt[n]{\frac{1}{2n+1}\binom{2n}{n}}\}_{n \geq 1}$, $\{\sqrt[n]{\binom{2n}{n}}\}_{n \geq 1}$, $\{\sqrt[n]{\binom{3n}{n}}\}_{n \geq 1}$, $\{\sqrt[n]{\binom{4n}{n}}\}_{n \geq 1}$, $\{\sqrt[n]{\binom{5n}{n}}\}_{n \geq 1}$ and $\{\sqrt[n]{\binom{5n}{2n}}\}_{n \geq 1}$ are strictly log-concave, respectively.

To study the conjectures of Sun on the monotonicity of $\{\sqrt[n+1]{z_{n+1}}/\sqrt[n]{z_n}\}$, Chen et al. [5] found a connection between the log-behavior of $\{\sqrt[n]{z_n}\}_{n \geq 1}$ and that of $\{z_{n+1}/z_n\}_{n \geq 0}$. Moreover, they introduced a stronger concept as follows: define an operator R on a sequence $\{z_n\}_{n \geq 0}$ by

$$R\{z_n\}_{n \geq 0} = \{x_n\}_{n \geq 0},$$

where $x_n = z_{n+1}/z_n$. The sequence $\{z_n\}_{n \geq 0}$ is called *infinitely log-monotonic* if the sequence $R^r\{z_n\}_{n \geq 0}$ is log-concave for all positive odd r and is log-convex for all nonnegative even r . In fact, the infinite log-monotonicity is related to the logarithmically completely monotonic function.

Recall that a function $f(x)$ is said to be *completely monotonic* on an interval I if $f(x)$ has derivatives of all orders on I which alternate successively in sign, that is,

$$(-1)^n f^{(n)}(x) \geq 0$$

for all $x \in I$ and for all $n \geq 0$. If inequality is strict for all $x \in I$ and for all $n \geq 0$, then $f(x)$ is said to be strictly completely monotonic. A positive function $f(x)$ is said to be *logarithmically completely monotonic* on an interval I if $\log f(x)$ satisfies

$$(-1)^n [\log f(x)]^n \geq 0$$

for all $x \in I$ and for all $n \geq 1$. A logarithmically completely monotonic function is completely monotonic, but not vice versa, see Berg [2]. The reader can refer to [22] for the properties of completely monotonic functions and [13] for a survey of logarithmically completely monotonic functions. In [5], Chen *et al.* found the link between logarithmically completely monotonic functions and infinite log-monotonicity of combinatorial sequences. Thus, in Section 4, the logarithmically complete monotonicity of some functions related to the combinatorial sequences will be considered. As applications, for nonnegative integers n_0, k_0, \bar{k}_0 and positive integers a, b, \bar{b} , if $a \geq b + \bar{b}$ and $-1 \leq k_0 - (n_0 + 1)b/a \leq 0$, then the sequence

$$\left\{ \frac{(n_0 + ia)!}{(k_0 + ib)!(\bar{k}_0 + i\bar{b})!} \right\}_{i \geq 0}$$

is infinitely log-monotonic. This generalizes two results of Chen *et al.* [5] that both the Catalan numbers $\frac{1}{n+1} \binom{2n}{n}$ and central binomial coefficients $\binom{2n}{n}$ are infinitely log-monotonic and strengthens one result of Su and Wang [16] that $\binom{dn}{\delta n}$ is log-convex in n for positive integers $d > \delta$. In addition, the asymptotically infinite log-monotonicity of derangement numbers is also demonstrated.

In order to research the stronger properties of the above functions $\theta(x)$ and $F(x)$, the logarithmically complete monotonicity of functions $1/\sqrt{x} a \zeta(x+b) \Gamma(x+c)$ and $\sqrt{x} \rho \prod_{i=1}^n \frac{\Gamma(x+a_i)}{\Gamma(x+b_i)}$ is also given, which generalizes one result of Lee and Tepedelenlioğlu about the logarithmically complete monotonicity of $\sqrt{x} \frac{2\sqrt{\pi} \Gamma(x+1)}{\Gamma(x+1/2)}$, and one result of Qi and Li about the logarithmically complete monotonicity of $\sqrt{x} \frac{a \Gamma(x+b)}{\Gamma(x+c)}$.

2 Analytic results for the monotonicity of the sequence $\sqrt[n]{z_n}$

This section is to give the proof of the analytic result Theorem 1.1.

Proof. Let $y = \sqrt[x]{f(x)}$. Then one can get

$$y' = \frac{y}{x} \left(\frac{f'(x)}{f(x)} - \frac{\log f(x)}{x} \right).$$

In order to show that $\sqrt[x]{f(x)}$ is strictly increasing, it suffices to prove

$$\frac{f'(x)}{f(x)} - \frac{\log f(x)}{x} > 0 \quad (2.1)$$

for $x \geq N$. Since $f(N) \leq 1$ and $f(x)$ is increasing, one can derive that

$$\frac{\log f(x)}{x} \leq \frac{\log f(x) - \log f(N)}{x} < \frac{\log f(x) - \log f(N)}{x - N} \quad (2.2)$$

for $x \geq N$.

By the mean value theorem, one can obtain

$$\frac{\log f(x) - \log f(N)}{x - N} = \frac{f'(\xi)}{f(\xi)}, \quad (2.3)$$

where $N \leq \xi \leq x$. On the other hand, it follows from log-convexity of the function $f(x)$ that

$$(\log f(x))'' = \left(\frac{f'(x)}{f(x)} \right)' = \frac{f''(x)f(x) - f'(x)^2}{f^2(x)} \geq 0, \quad (2.4)$$

which implies that $\frac{f'(x)}{f(x)}$ is increasing. Thus, it follows that

$$\frac{f'(\xi)}{f(\xi)} \leq \frac{f'(x)}{f(x)} \quad (2.5)$$

for $x \geq \xi$. Combining (2.2), (2.3) and (2.5), one can obtain (2.1). So $\sqrt[x]{f(x)}$ is increasing. \square

3 Analytic results for the log-behavior of the sequence

$$\sqrt[n]{z_n}$$

In order to deal with the log-behavior of the sequence $\sqrt[n]{z_n}$, some analytic methods will be developed in this section. There are two main results in this section, one being the proof of Conjecture 1.4 and the other being the log-behavior of the function $F(x)$.

In the proofs, the following some known facts are needed. It follows from [1, Theorem 8] that the function

$$G_0(x) = -\log \Gamma(x) + (x - 1/2) \log x - x + \log \sqrt{2\pi} + \frac{1}{12x}$$

is strictly completely monotonic on $(0, \infty)$. This implies that

$$\log \Gamma(x) < (x - 1/2) \log x - x + \log \sqrt{2\pi} + \frac{1}{12x}, \quad (3.1)$$

$$(\log \Gamma(x))' > \log x - \frac{1}{2x} - \frac{1}{12x^2}, \quad (3.2)$$

$$(\log \Gamma(x))'' < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}. \quad (3.3)$$

On the other hand, [1, Theorem 8] also says that the function

$$F_0(x) = \log \Gamma(x) - (x - 1/2) \log x + x - \log \sqrt{2\pi}$$

is strictly completely monotonic on $(0, \infty)$. So

$$\log \Gamma(x) > (x - 1/2) \log x - x + \log \sqrt{2\pi}, \quad (3.4)$$

$$(\log \Gamma(x))' < \log x - \frac{1}{2x}, \quad (3.5)$$

$$(\log \Gamma(x))'' > \frac{1}{x} + \frac{1}{2x^2}. \quad (3.6)$$

Thus, by combining these inequalities, one can get the next result, which will be used repeatedly in the proofs.

Lemma 3.1. *Let $a > 0$. Assume that $h(x) = \log \Gamma(x)$. If $b \geq -1$ and $ax + b \geq 0$, then*

$$x^3 \left(\frac{h(ax + b + 1)}{x} \right)'' \leq -ax + (2b + 1) \log(ax + b + 1) - 3b - \frac{3}{2} + \log 2\pi + \frac{b^2 + b + 1/2}{ax + b + 1};$$

$$x^3 \left(\frac{h(ax + b + 1)}{x} \right)'' \geq -ax + (2b + 1) \log(ax + b + 1) - 3b - 3 + \log 2\pi.$$

Proof. By $h(x) = \log \Gamma(x)$, it is not hard to deduce that

$$\left(\frac{h(ax + b + 1)}{x} \right)'' = \frac{a^2 x^2 h''(ax + b + 1) - 2ax h'(ax + b + 1) + 2h(ax + b + 1)}{x^3}.$$

By (3.1), (3.2) and (3.3), it follows that

$$\begin{aligned} & a^2 x^2 h''(ax + b + 1) - 2ax h'(ax + b + 1) + 2h(ax + b + 1) \\ \leq & -ax + (2b + 1) \log(ax + b + 1) - 3b - \frac{3}{2} + \log 2\pi + \frac{b^2 + b + 1/2}{ax + b + 1}. \end{aligned}$$

In addition, by (3.4), (3.5) and (3.6), one can also obtain that

$$\begin{aligned} & a^2 x^2 h''(ax + b + 1) - 2ax h'(ax + b + 1) + 2h(ax + b + 1) \\ \geq & -ax + (2b + 1) \log(ax + b + 1) - 3b - 3 + \log 2\pi. \end{aligned}$$

This completes the proof. □

In order to prove Conjecture 1.4, the next result will be used.

Lemma 3.2. *Let $\zeta(x) = \sum_{n \geq 1} \frac{1}{n^x}$ the Riemann zeta function. Define a function $\eta(x) = \zeta(x) - 1$. The bound $\eta(x) \leq 3/2^x$ holds for all $x \geq 4$.*

Proof. Since

$$\begin{aligned}\eta(x) &= \frac{1}{2^x} \left(1 + \frac{1}{1.5^x} + \frac{1}{2^x} + \cdots \right) \\ &\leq \frac{1}{2^x} \left(1 + \frac{1}{1.5^x} + 2(\zeta(x) - 1) \right) \\ &\leq \frac{1}{2^x} \left(1 + \frac{1}{2} + 2\eta(x) \right)\end{aligned}$$

for $x \geq 4$, one can get $\eta(x) \leq \frac{3}{2^x}$. □

Now a result for Conjecture 1.4 can be stated as follows.

Theorem 3.3. *The function*

$$\theta(x) = \sqrt[x]{2\zeta(x)\Gamma(x+1)}$$

is log-concave on $(7.1, \infty)$.

Proof. In order to show that $\theta(x)$ is log-concave on $(7.1, \infty)$, it suffices to prove

$$\begin{aligned}(\log \theta(x))'' &= \left(\frac{\log 2}{x} \right)'' + \left(\frac{\log \zeta(x)}{x} \right)'' + \left(\frac{\log \Gamma(x+1)}{x} \right)'' \\ &= \frac{2 \log 2}{x^3} + \left(\frac{\log \zeta(x)}{x} \right)'' + \left(\frac{\log \Gamma(x+1)}{x} \right)'' \\ &< 0.\end{aligned}\tag{3.7}$$

Noting that $\log x < \sqrt{x}$ for $x \geq 2$, one has $\zeta''(x) < \eta(x-1)$ and $|\zeta'(x)| < \eta(x-0.5)$. In addition, it follows from $\log(x+1) \leq x$ for $x > 0$ that $\log(1 + \eta(x)) \leq \eta(x) \leq \frac{3}{2^x}$ by Lemma 3.2. Thus, for $x \geq 7.1$, it follows that

$$\begin{aligned}x^3 \left(\frac{\log \zeta(x)}{x} \right)'' &= x^2 \left(\frac{\zeta(x)\zeta''(x) - \zeta'(x)^2}{\zeta(x)^2} \right) - 2x \frac{\zeta'(x)}{\zeta(x)} + 2 \log \zeta(x) \\ &< \frac{x^2 \zeta''(x)}{\zeta(x)} - \frac{2x \zeta'(x)}{\zeta(x)} + 2 \log \zeta(x) \\ &< 2.67,\end{aligned}\tag{3.8}$$

where the final inequality can be obtained by considering the monotonicity of the right function.

On the other hand, by Lemma 3.1, one can get

$$\begin{aligned}x^3 \left(\frac{\log \Gamma(x+1)}{x} \right)'' &\leq -x + \log(x+1) - 1 + \log 2\pi + \frac{1}{2(x+1)} \\ &< -4.1\end{aligned}\tag{3.9}$$

for $x \geq 7.1$.

Thus, combining (3.7), (3.8) and (3.9), one can conclude

$$\begin{aligned} (\log \theta(x))'' &= \frac{2 \log 2}{x^3} + \left(\frac{\log \zeta(x)}{x} \right)'' + \left(\frac{\log \Gamma(x+1)}{x} \right)'' \\ &< 0, \end{aligned}$$

as desired. This completes the proof. \square

Notice that

$$\sqrt[n]{(-1)^{n-1} B_{2n}} = \frac{\theta^2(2n)}{4\pi^2}.$$

Thus, it follows from the strict log-concavity of $\{\theta(2n)\}_{n \geq 4}$ that $\{\sqrt[n]{(-1)^{n-1} B_{2n}}\}_{n \geq 4}$ is strictly log-concave. In addition, it is easy to check that $\{\sqrt[n]{(-1)^{n-1} B_{2n}}\}_{n \geq 0}$ is strictly log-concave for $1 \leq n \leq 4$. Thus, the following result is immediate, which was conjectured by Sun [17, Conjecture 2.15] and has been verified by Luca and Stănică [11] and Chen *et al.* [5] by different methods, respectively.

Corollary 3.4. *The sequence $\{\sqrt[n]{(-1)^{n-1} B_{2n}}\}_{n \geq 1}$ is strictly log-concave.*

Now consider the tangent numbers

$$\{T(n)\}_{n \geq 0} = \{1, 2, 16, 272, 7936, 353792, \dots\}, \quad [14, A000182]$$

which are defined by

$$\tan x = \sum_{n \geq 1} T(n) \frac{x^{2n-1}}{(2n-1)!}$$

and are closely related to the Bernoulli numbers:

$$T(n) = (-1)^{n-1} B_{2n} \frac{(4^n - 1)}{2n} 4^n,$$

see [6, (6.93)] for instance. So

$$\sqrt[n]{T(n)} = 4 \sqrt[n]{(-1)^{n-1} B_{2n}} \sqrt[n]{4^n - 1} \sqrt[n]{\frac{1}{2n}}.$$

It is not difficult to verify that both $\sqrt[n]{4^n - 1}$ and $\sqrt[n]{\frac{1}{2n}}$ are log-concave in n (we leave the details to the reader). The product of log-concave sequences is still log-concave. So the next result is immediate, which was conjectured by Sun [17, Conjecture 3.5] and was verified by Luca and Stănică [11] by a discrete method.

Corollary 3.5. *The sequence $\{\sqrt[n]{T(n)}\}_{n \geq 1}$ is strictly log-concave.*

In order to develop analytic techniques to deal with the log-behavior of $\{\sqrt[n]{z_n}\}$. In the following, the log-behavior of a function $F(x)$ related to the Euler Gamma function will be considered, which can be applied to some interesting binomial coefficients.

Theorem 3.6. *Given real numbers b, d, f and nonnegative real numbers a, c, e , define the function*

$$F(x) = \sqrt[x]{\frac{\Gamma(ax + b + 1)}{\Gamma(cx + d + 1)\Gamma(ex + f + 1)}}.$$

- (i) *If $a > c + e$, then $F(x)$ is an asymptotically log-concave function.*
- (ii) *Assume $a = c + e$. If $c \geq e > 0$ and $b < d + f + 1/2$, then $F(x)$ is an asymptotically log-concave function. In particular, if $c \geq 1$ and $b = d = f = 0$, then we have $F(x)$ is a log-concave function for $x \geq 30$; if $c \geq 1$, $b = d = 0$ and $f \geq 1$, then $F(x)$ is a log-concave function for $x \geq 2$.*
- (iii) *Assume $a = c + e$. If $c > e = 0$ and $b < d$, then $F(x)$ is an asymptotically log-concave function.*
- (iv) *If $a < c + e$, then $F(x)$ is an asymptotically log-convex function.*

Proof. Let $h(x) = \log \Gamma(x)$. By Lemma 3.1, one has

$$\begin{aligned} (\log F(x))'' &= \left(\frac{h(ax + b + 1)}{x} \right)'' - \left(\frac{h(cx + d + 1)}{x} \right)'' - \left(\frac{h(ex + f + 1)}{x} \right)'' \\ &= (c + e - a)x + \log \frac{(ax + b + 1)^{(2b+1)}}{(cx + d + 1)^{(2d+1)}(ex + f + 1)^{(2f+1)}} + 3(d + f - b) \\ &\quad + \frac{9}{2} - \log 2\pi + \frac{b^2 + b + 1/2}{ax + b + 1}. \end{aligned} \tag{3.10}$$

It is easy to prove for $a > c + e$ that

$$\lim_{x \rightarrow +\infty} (c + e - a)x + \log \frac{(ax + b + 1)^{(2b+1)}}{(cx + d + 1)^{(2d+1)}(ex + f + 1)^{(2f+1)}} = -\infty,$$

and for $a = c + e$ that

$$\lim_{x \rightarrow +\infty} \log \frac{(ax + b + 1)^{(2b+1)}}{(cx + d + 1)^{(2d+1)}(ex + f + 1)^{(2f+1)}} = -\infty$$

if $c \geq e > 0$ and $b < d + f + 1/2$ or $c > e = 0$ and $b < d$. Thus, under conditions of (i), (ii) and (iii), respectively, by (3.10) one can get

$$\lim_{x \rightarrow +\infty} (\log F(x))'' = -\infty,$$

implying that $F(x)$ is an asymptotically log-concave function.

Assume that $a = c + e$ and $c \geq e \geq 1$. If $b = d = f = 0$, then, by (3.10),

$$\begin{aligned} (\log F(x))'' &< \log \frac{(ax+1)}{(cx+1)(ex+1)} + \frac{9}{2} - \log 2\pi + \frac{1}{2(ax+1)} \\ &< -0.04 \end{aligned}$$

for $x \geq 30$. If $b = d = 0$ and $f \geq 1$, then, by (3.10),

$$\begin{aligned} (\log F(x))'' &< \log \frac{(ax+1)}{(cx+1)(ex+2)^3} + \frac{9}{2} - \log 2\pi + \frac{1}{2(ax+1)} \\ &< -0.37 \end{aligned}$$

for $x \geq 2$.

Finally, since the proof of (iv) is similar to that of (i), which is omitted for brevity. This completes the proof. \square

By Theorem 3.6, the next result is immediate.

Proposition 3.7. *Let integers a, b, c, d, f satisfy $a > c > 0$ and $b < d + f + 1/2$. Then the sequence*

$$\left\{ \sqrt[n]{\frac{\Gamma(an+b+1)}{\Gamma(cn+d+1)\Gamma((a-c)n+f+1)}} \right\}_{n \geq 1}$$

is asymptotically log-concave. In particular, $\{\sqrt[n]{\binom{an}{cn}}\}_{n \geq 30}$ and

$$\left\{ \sqrt[n]{\frac{\Gamma(an+1)}{\Gamma(cn+1)\Gamma((a-c)n+f+1)}} \right\}_{n \geq 2}$$

is strictly log-concave for $f \geq 1$.

For integer $p \geq 2$, Fuss-Catalan numbers [7] are given by the formula

$$C_p(n) = \frac{1}{(p-1)n+1} \binom{pn}{n} = \frac{\Gamma(pn+1)}{\Gamma(n+1)\Gamma((p-1)n+2)}.$$

It is well known that the Fuss-Catalan numbers count the number of paths in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ (with directed vertices from (i, j) to either $(i, j+1)$ or $(i+1, j)$) from the origin $(0, 0)$ to $(n, (p-1)n)$ which never go above the diagonal $(p-1)x = y$. Su and Wang [16] showed that $\{\binom{an}{bn}\}_{n \geq 0}$ is log-convex for positive integers $a > b$. Thus it is easy to see that $\{C_p(n)\}_{n \geq 0}$ is log-convex. Chen *et al.* [5] proved that $\sqrt[n]{\frac{1}{2n+1}\binom{2n}{n}}$ and $\sqrt[n]{\binom{2n}{n}}$ are strictly log-concave, respectively. By verifying the first few terms, one can get the following corollary by Corollary 3.7.

Corollary 3.8. *The sequences $\{\sqrt[n]{\frac{1}{2n+1}\binom{2n}{n}}\}_{n \geq 1}$, $\{\sqrt[n]{\binom{2n}{n}}\}_{n \geq 1}$, $\{\sqrt[n]{\binom{3n}{n}}\}_{n \geq 1}$, $\{\sqrt[n]{\binom{4n}{n}}\}_{n \geq 1}$, $\{\sqrt[n]{\binom{5n}{n}}\}_{n \geq 1}$, $\{\sqrt[n]{\binom{5n}{2n}}\}_{n \geq 1}$ and $\{\sqrt[n]{C_p(n)}\}_{n \geq 2}$ are strictly log-concave for any positive integer $p \geq 2$, respectively.*

4 Logarithmically completely monotonic functions

Since logarithmically completely monotonic functions have many applications, it is important to know which function has such property. In particular, Chen *et al.* [5] found the connection between logarithmically completely monotonic functions and infinite log-monotonicity of combinatorial sequences as follows.

Theorem 4.1. [5] *Assume that a function $f(x)$ such that $[\log f(x)]''$ is completely monotonic for $x \geq 1$ and $a_n = f(n)$ for $n \geq 1$. Then the sequence $\{a_n\}_{n \geq 1}$ is infinitely log-monotonic.*

Thus it is very interesting to research logarithmically complete monotonicity of some functions related to combinatorial sequences, which is the aim of this section.

Many sequences of binomial coefficients share various log-behavior properties, see Tanny and Zuker [18, 19], Su and Wang [16] for instance. In particular, Su and Wang proved that $\binom{dn}{\delta n}$ is log-convex in n for positive integers $d > \delta$. Recently, Chen *et al.* [5] proved that both the Catalan numbers $\frac{1}{n+1} \binom{2n}{n}$ and central binomial coefficients $\binom{2n}{n}$ are infinitely log-monotonic. Motivated by these results, a generalization can be stated as follows.

Theorem 4.2. *Let n_0, k_0, \bar{k}_0 be nonnegative integers and a, b, \bar{b} be positive integers. Define the function*

$$G(x) = \frac{\Gamma(n_0 + ax + 1)}{\Gamma(k_0 + bx + 1)\Gamma(\bar{k}_0 + x\bar{b} + 1)}.$$

If $a \geq b + \bar{b}$ and $-1 \leq k_0 - (n_0 + 1)b/a \leq 0$, then $(\log G(x))''$ is a completely monotonic function for $x \geq 0$. In particular,

$$\left\{ \frac{(n_0 + ia)!}{(k_0 + ib)!(\bar{k}_0 + i\bar{b})!} \right\}_{i \geq 0}$$

is infinitely log-monotonic.

Proof. By Theorem 4.1, it suffices to show that $(\log G(x))''$ is a completely monotonic function for $x \geq 0$. Let $g(x) = \log G(x)$. So

$$\begin{aligned} & [g(x)]^{(n)} \\ &= [\log \Gamma(n_0 + ax + 1)]^{(n)} - [\log \Gamma(k_0 + bx + 1)]^{(n)} - [\log \Gamma(\bar{k}_0 + x\bar{b} + 1)]^{(n)} \\ &= (-1)^n \int_0^\infty \frac{t^{n-1}}{1 - e^{-t}} \left[a^n e^{-t(n_0 + ax + 1)} - b^n e^{-t(k_0 + bx + 1)} - \bar{b}^n e^{-t(\bar{k}_0 + x\bar{b} + 1)} \right] dt \quad (4.1) \\ &= (-1)^n \int_0^\infty a^n t^{n-1} e^{-tax} \left[\frac{e^{-(n_0+1)t}}{1 - e^{-t}} - \frac{e^{-ta(k_0+1)/b}}{1 - e^{-at/b}} - \frac{e^{-ta(\bar{k}_0+1)/\bar{b}}}{1 - e^{-at/\bar{b}}} \right] dt \end{aligned}$$

since

$$[\log \Gamma(x)]^{(n)} = (-1)^n \int_0^\infty \frac{t^{n-1} e^{-tx}}{1 - e^{-t}} dt$$

for $x > 0$ and $n \geq 2$, see [12, p.16] for instance.

It follows from $a > b > 0$ that for further simplification denote $u = k_0 - (n_0 + 1)b/a$, $p = a/b$, and $q = a/\bar{b}$. Clearly, $\frac{1}{p} + \frac{1}{q} \leq 1$. So one can deduce that

$$(-1)^n [g(x)]^{(n)} = \int_0^\infty a^n t^{n-1} e^{-t(n_0+ax+1)} h(t, u) dt, \quad (4.2)$$

where

$$h(t, u) = \frac{1}{1 - e^{-t}} - \frac{e^{-tp(u+1)}}{1 - e^{-pt}} - \frac{e^{uqt}}{1 - e^{-qt}}.$$

Furthermore, one can obtain the next claim for $-1 \leq k_0 - (n_0 + 1)b/a \leq 0$.

Claim 1. *If $-1 \leq u \leq 0$, then $h(t, u) > 0$.*

Proof of Claim: It is obvious that $h(t, u)$ is concave in u . Thus it suffices to show $h(t, u) > 0$ for $u = -1$ and $u = 0$. Setting $u = 0$ since the case $u = -1$ can be obtained by switching the roles of p and q , one has

$$h(t, 0) = \frac{e^{-t}}{1 - e^{-t}} - \frac{e^{-tp}}{1 - e^{-pt}} - \frac{e^{-qt}}{1 - e^{-qt}}.$$

Noting for $s > 0$ that function

$$f(s) = \frac{se^{-s}}{1 - e^{-s}}$$

strictly decreases in s and $\frac{1}{p} + \frac{1}{q} \leq 1$, one gets that

$$\begin{aligned} h(t, 0) &\geq \left(\frac{1}{p} + \frac{1}{q}\right) \frac{e^{-t}}{1 - e^{-t}} - \frac{e^{-tp}}{1 - e^{-pt}} - \frac{e^{-qt}}{1 - e^{-qt}} \\ &= \frac{f(t) - f(tp)}{tp} + \frac{f(t) - f(tq)}{tq} \\ &\geq 0. \end{aligned}$$

This completes the proof of this Claim.

Thus, by (4.2) and this Claim, one has $(-1)^n [g(x)]^{(n)} > 0$, which implies that $(\log G(x))''$ is a completely monotonic function. This completes the proof. \square

By Theorem 4.2, the following two corollaries are immediate.

Corollary 4.3. *Let n_0, k_0, d, δ be four nonnegative integers. Define the sequence*

$$C_i = \binom{n_0 + id}{k_0 + i\delta}, \quad i = 0, 1, 2, \dots$$

If $d > \delta > 0$ and $-1 \leq k_0 - (n_0 + 1)\delta/d \leq 0$, then the sequence $\{C_n\}_{n \geq 0}$ infinitely log-monotonic.

Corollary 4.4. *The Fuss-Catalan sequence $\{C_p(n)\}_{n \geq 0}$ is infinitely log-monotonic, where $p \geq 2$ and $C_p(n) = \frac{1}{(p-1)n+1} \binom{pn}{n}$.*

The derangements number d_n is a classical combinatorial number. It is log-convex and ratio log-concave, see [10] and [4] respectively. Noted that $\{\Gamma(n)\}_{n \geq 1}$ is strictly infinitely log-monotonic (see Chen *et al.* [5]) and

$$\left|d_n - \frac{n!}{e}\right| \leq \frac{1}{2} \quad (4.3)$$

for $n \geq 3$ (see [8]), the following interesting result can be demonstrated.

Theorem 4.5. *The sequence of the derangements numbers $\{d_n\}_{n \geq 3}$ is asymptotically infinitely log-monotonic.*

Proof. From (4.3), one can deduce

$$\frac{n!}{e} - \frac{1}{2} \leq d_n \leq \frac{n!}{e} + \frac{1}{2},$$

which implies

$$\Gamma(n+1) - \frac{3}{2} \leq ed_n \leq \Gamma(n+1) + \frac{3}{2}.$$

Thus

$$\begin{aligned} & e^2(d_{n+1}d_{n-1} - d_n^2) \\ & \geq [\Gamma(n+2) - 1.5][\Gamma(n) - 1.5] - [\Gamma(n+1) + 1.5]^2 \\ & > 0 \end{aligned}$$

for $n \geq 4$, which implies that $\{d_n\}_{n \geq 4}$ is log-convex. Note that

$$\begin{aligned} & e^4(d_{n+1}^3d_{n-1} - d_n^3d_{n+2}) \\ & \geq [\Gamma(n+2) - 1.5]^3[\Gamma(n) - 1.5] - [\Gamma(n+1) + 1.5]^3[\Gamma(n+3) + 1.5] \\ & > 0 \end{aligned}$$

for $n \geq 8$, which implies that $R\{d_n\}_{n \geq 8}$ is log-concave. Because $\{\Gamma(n)\}_{n \geq 1}$ is strictly infinitely log-monotonic, similarly, it can be proceeded to the higher order log-monotonicity. Thus, for any positive integer k , by the sign-preserving property of limits, one can obtain that there exists a positive N such that the sequence $R^r\{d_n\}_{n \geq N}$ is log-concave for positive odd r and is log-convex for positive even r . Thus, the sequence of the derangements numbers $\{d_n\}_{n \geq 3}$ is asymptotically infinitely log-monotonic. \square

In the following, we will continuous to give two kinds of logarithmically completely monotonic functions. In order to consider a stronger result for Theorem 3.3, given $a, b, c > 0$, define the function

$$\theta_{a,b,c}(x) = \sqrt[x]{a\zeta(x+b)\Gamma(x+c)}.$$

It is known that the Riemann zeta function $\zeta(x)$ is logarithmically completely monotonic on $(1, +\infty)$ and the function $[\log \Gamma(x)]''$ is completely monotonic on $(0, +\infty)$, see Chen *et al.* [5]. Basing on these results, one can demonstrate the next.

Theorem 4.6. *Let a, b, c be positive real numbers, where $b \geq 1$. If $a\zeta(b)\Gamma(c) \leq 1$, then the reciprocal of the function $\theta_{a,b,c}(x)$ is logarithmically completely monotonic on $(1, \infty)$.*

Proof. Since

$$\log \theta_{a,b,c}^{-1}(x) = -\frac{\log(a\zeta(x+b)\Gamma(x+c))}{x} = -\frac{\log a + \log \zeta(x+b) + \log \Gamma(x+c)}{x},$$

in order to show that $\theta_{a,b,c}^{-1}(x)$ is logarithmically completely monotonic on $(1, \infty)$, it suffices to prove

$$(-1)^n \log^{(n)} \theta_{a,b,c}^{-1}(x) \geq 0$$

for all $n \geq 1$. Note that a known formula as follows:

$$\left(\frac{g(x)}{x}\right)^{(n)} = \frac{(-1)^n g(0)n!}{x^{n+1}} + x^{-n-1} \int_0^x t^n g^{(n+1)}(x) dt, \quad (4.4)$$

which can be easily proved by induction. Thus, one can deduce for $n \geq 1$ and $x > 1$ that

$$\begin{aligned} & (-1)^n \log^{(n)} \theta_{a,b,c}^{-1}(x) \\ &= \frac{-n! \log a\zeta(b)\Gamma(c)}{x^{n+1}} + x^{-n-1} \int_0^x t^n (-1)^{n+1} \left[(\log \zeta(x+b))^{(n+1)} + (\log \Gamma(x+c))^{(n+1)} \right] dt \\ &\geq 0 \end{aligned}$$

since $\log a\zeta(b)\Gamma(c) \leq 0$, $(-1)^{n+1} (\log \zeta(x+b))^{(n+1)} \geq 0$ and $(-1)^{n+1} (\log \Gamma(x+c))^{(n+1)} \geq 0$. This completes the proof. \square

In [1], the next result was proved by Alzer.

Theorem 4.7. [1] *Let nonnegative sequences $0 \leq a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n$ and $0 \leq b_1 \leq b_2 \leq b_3 \leq \dots \leq b_n$. If $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for $k = 1, 2, \dots, n$, then the function*

$$\prod_{i=1}^n \frac{\Gamma(x+a_i)}{\Gamma(x+b_i)}$$

is completely monotonic on $(0, \infty)$.

On the other hand, in [9], Lee and Tepedelenlioğlu proved the function $\sqrt[x]{\frac{2\sqrt{\pi}\Gamma(x+1)}{\Gamma(x+1/2)}}$ originating from the coding gain is logarithmically completely monotonic on $(0, \infty)$. In addition, Qi and Li [13] considered the logarithmically completely monotonicity of $\sqrt[x]{\frac{a\Gamma(x+b)}{\Gamma(x+c)}}$. In what follows a general result for a kind of logarithmically completely monotonic functions is obtained.

Theorem 4.8. Let $0 \leq a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n$, $0 \leq b_1 \leq b_2 \leq b_3 \leq \dots \leq b_n$ and $\rho > 0$, define the function

$$\chi(x) = \sqrt[x]{\rho \prod_{i=1}^n \frac{\Gamma(x + a_i)}{\Gamma(x + b_i)}}.$$

- (i) If $\rho \prod_{i=1}^n \frac{\Gamma(a_i)}{\Gamma(b_i)} \geq 1$ and $\sum_{i=1}^k a_i \geq \sum_{i=1}^k b_i$ for $k = 1, 2, \dots, n$, then the function $\chi(x)$ is logarithmically completely monotonic on $(0, \infty)$.
- (ii) If $\rho \prod_{i=1}^n \frac{\Gamma(a_i)}{\Gamma(b_i)} \leq 1$ and $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for $k = 1, 2, \dots, n$, then the reciprocal of the function $\chi(x)$ is logarithmically completely monotonic on $(0, \infty)$.

Proof. Because (ii) can be obtained in the similar way, it only needs to prove (i). Define the function $h(x) = \sum_{i=1}^n \log \Gamma(x + a_i) - \log \Gamma(x + b_i)$. Then

$$\log \sqrt[x]{\rho \prod_{i=1}^n \frac{\Gamma(x + a_i)}{\Gamma(x + b_i)}} = \frac{\log \rho \prod_{i=1}^n \frac{\Gamma(x + a_i)}{\Gamma(x + b_i)}}{x} = \frac{\log \rho + h(x)}{x}.$$

So it is not hard to get

$$(-1)^k [\log \chi(x)]^{(k)} = \frac{k!(\log \rho + h(0))}{x^{k+1}} + x^{-k-1} \int_0^x t^k (-1)^k h^{(k+1)}(x) dt. \quad (4.5)$$

If $\rho \prod_{i=1}^n \frac{\Gamma(a_i)}{\Gamma(b_i)} \geq 1$, then it is clear that

$$\log \rho + h(0) \geq 0.$$

In addition, Alzer [1] proved that $(-1)^k h^{(k+1)}(x) \geq 0$ for $k \geq 0$ and $x \geq 0$. Thus,

$$(-1)^k [\log \chi(x)]^{(k)} \geq 0,$$

that is, $\chi(x)$ is logarithmically completely monotonic on $(0, \infty)$. This completes the proof. \square

Remark 4.9. If $\rho = 2\sqrt{\pi}$, $a_1 = 1$ and $b_1 = \frac{1}{2}$, then $\frac{2\sqrt{\pi}\Gamma(1)}{\Gamma(1/2)} = 2 > 1$. So the function $\sqrt[x]{\frac{2\sqrt{\pi}\Gamma(x+1)}{\Gamma(x+1/2)}}$ is logarithmically completely monotonic on $(0, \infty)$, see [9]. In addition, if $n = 1$ in Theorem 4.8, then it was proved by Qi and Li [13]. Thus, the result in Theorem 4.8 is a generalization.

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