A note about combinatorial sequences and Incomplete Gamma function

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Abstract

In this short note we present a set of interesting and useful properties of a one-parameter family of sequences including factorial and subfactorial, and their relations to the Gamma function and the incomplete Gamma function.

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1 Introduction

The purpose of this note is to show a set of properties of a one-parameter family of sequences $\mathcal{T}_n(z)$, z is a complex variable, which are apparently not known in the literature. Among these sequences we have factorials and

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subfactorials and they are seen to be related to the incomplete Gamma function. This family was introduced in a recent paper about generalized binomial distribution [1].

2 The function $T_n(x, y)$

Let us consider the following two-variable polynomial:

$$T_n(x,y) := \sum_{m=0}^n \binom{n}{m} (m+x)^m (n-m+y)^{n-m}, \qquad (1)$$

which is clearly symmetrical in x, y. Actually, we have the stronger result:

Proposition 2.1. The polynomial $T_n(x, y)$ depends on the sum z = x + y only:

$$T_n(x, z - x) = \sum_{m=0}^n \binom{n}{m} (m+x)^m (n-m+z-x)^{n-m} \equiv \mathcal{T}_n(z), \quad (2)$$

where x can be arbitrarily chosen with the convention that $0^0 \equiv 1$ in the cases x = 0 or x = z.

Proof. Let us prove by recurrence on n that $T_n(x, z-x)$ does not depend explicitly on x for all z. For that, we take the partial derivative of $T_n(x, z-x)$ with respect to x. We have successively, by appropriately shift the summation variable,

$$\frac{\partial}{\partial x}T_n(x,z-x) = \sum_{m=0}^n \binom{n}{m} \left[m(m+x)^{m-1}\left(n-m+z-x\right)^{n-m} -(n-m)(m+x)^m\left(n-m+z-x\right)^{n-m-1}\right]$$
$$= n\sum_{m=0}^{n-1} \binom{n-1}{m}(m+1+x)^m\left(n-1-m+z-x\right)^{n-1-m}$$
$$-n\sum_{m=0}^{n-1} \binom{n-1}{m}(m+x)^m\left(n-1-m+z+1-x\right)^{n-1-m}$$
$$= n\left[T_{n-1}(1+x,z-x) - T_{n-1}(x,z+1-x)\right].$$

Now, we note that $T_0(x, z - x) = 1$ and so does not depend explicitly on x. Suppose that the property " $T_k(x, z - x)$ does not depend explicitly on x for all z" holds true for all integer $1 \le k \le n - 1$. Then $T_{n-1}(1 + x, z - x) = T_{n-1}(x, z + 1 - x)$ and this implies that $\partial/\partial x T_n(x, z - x) = 0$, i.e., $T_n(x, z - x)$ does not depend explicitly on x for all z.

3 The function $\mathcal{T}_n(z)$

Thanks to Proposition 2.1, it is now possible to find a convenient alternate expression of $T_n(x, z - x) \equiv \mathcal{T}_n(z)$. First, let us establish a recurrence formula.

Proposition 3.1. The polynomial $\mathcal{T}_n(z)$ satisfies the following recurrence formula:

$$\mathcal{T}_n(z) = (z+n)^n + n \,\mathcal{T}_{n-1}(z+1), \quad \mathcal{T}_0(z) = 1.$$
 (3)

Proof. Taking x = 0 in (2), let us write that expression as

$$\begin{aligned} \mathcal{T}_n(z) &= \sum_{m=0}^n \binom{n}{m} (m)^m \left(n - m + z\right)^{n-m} \\ &= (n+z)^n + n \sum_{m=1}^n \binom{n-1}{m-1} \times \\ &\times (m-1+1)^{m-1} (n-1-(m-1)+z+1-1)^{n-1-(m-1)} \\ &= (n+z)^n + n T_{n-1} (1, z+1-1) = (n+z)^n + n \mathcal{T}_{n-1} (z+1) \end{aligned}$$

where we have applied Proposition 2.1 with x = 1.

It is then straightforward to deduce from this formula the following result.

Proposition 3.2. The polynomial $\mathcal{T}_n(z)$ admits the following expansion in powers of (z + n):

$$\mathcal{T}_{n}(z) = \sum_{k=0}^{n} \frac{n!}{k!} (z+n)^{k} \,. \tag{4}$$

We note in particular the interesting corollary proving that the family $(\mathcal{T}_n(z))_{z \in \mathbb{Z}}$ of integer sequences includes the factorial:

$$\mathcal{T}_n(-n) = n!\,,\tag{5}$$

and consequently also the Gamma function as

$$\mathcal{T}_n(-n) = \Gamma(n+1). \tag{6}$$

The expression (4) allows to easily derive two asymptotic behaviors of $\mathcal{T}_n(z)$:

At large
$$z \quad \mathcal{T}_n(z) \sim z^n$$
, (7)

At large
$$n \quad \mathcal{T}_n(z) \sim n^n$$
. (8)

Indeed, in the latter the dominant term of the sum is for k = n (the term k = 0 is not dominant due to the decreasing exponential factor in the Stirling's formula $n! \sim n^n \sqrt{2\pi n} e^{-n}$). This indicates that the dominant term of the sum in (4) is $(z + n)^n$.

A second expression of $\mathcal{T}_n(z)$ is also of interest.

Proposition 3.3. The polynomial $\mathcal{T}_n(z)$ admits the following expansion in powers of (z + n + 1):

$$\mathcal{T}_{n}(z) = \sum_{k=0}^{n} a_{k} (z+n+1)^{k} , \qquad (9)$$

where the coefficients a_k are given by

$$a_{k} = \binom{n}{k} \mathcal{T}_{n-k}(k-n-1) \equiv \binom{n}{k} d_{n-k}.$$
 (10)

Proof. Applying formula

$$a_k = \frac{1}{k!} \frac{d^k}{dz^k} \left. \mathcal{T}_n(z) \right|_{z=-n-1}$$

to the original expression (2) where we put x = 0, we easily derive

$$\frac{1}{k!} \frac{d^k}{dz^k} \left. \mathcal{T}_n(z) \right|_{z=-n-1} = \sum_{m=0}^{n-k} (-1)^{n-m-k} \binom{n}{m} \binom{n-m}{k} m^m (1+m)^{n-m-k}$$
$$= \binom{n}{k} \sum_{m=0}^{n-k} (-1)^{n-m-k} \binom{n-k}{m} m^m (1+m)^{n-m-k}$$
$$= \binom{n}{k} \mathcal{T}_{n-k}(k-n-1) \,.$$

The sequence of numbers $(d_n \equiv \mathcal{T}_n(-n-1))_{n \in \mathbb{N}}$, for which (4) gives

$$d_n = \sum_{k=0}^n \frac{n!}{k!} (-1)^k , \qquad (11)$$

and whose first terms are 1, 0, 1, 2, 9, 44, ..., is well known for more than 3 centuries [4, 5, 6, 7, 8]. Its OEIS name is A000166. Its numbers are named *subfactorial* (and then denoted as !n) or *rencontres numbers*, or *derangements*, since d_n is the number of permutations of n elements with no fixed points. They obey the recurrence relations (Euler) $d_n = (n-1)d_{n-1} + (-1)^n$ and $d_n = n(d_{n-1} + d_{n-2})$. Their generating function is

$$D(x) = \sum_{n=0}^{\infty} d_n \, \frac{x^n}{n!} = \frac{e^{-x}}{1-x} \,,$$

and their asymptotic behavior at large n is

$$\lim_{n \to \infty} \frac{d_n}{n!} = \frac{1}{e}.$$

4 The relation to the incomplete Gamma function

Let us finally establish the relation between the polynomial $\mathcal{T}_n(z)$ and the incomplete Gamma function [3].

Proposition 4.1. The polynomial $\mathcal{T}_n(z)$ is related to the incomplete Gamma function

$$\Gamma(a,x) = \int_{x}^{\infty} t^{a-1} e^{-t} dt \,, \quad \text{Re}(a) > 0 \,, \tag{12}$$

as follows:

$$\mathcal{T}_n(z) = e^{z+n} \Gamma(n+1, z+n) \,. \tag{13}$$

Note that in the particular case that z = -n, from (13) we reobtain (6).

Proof. It is straightforward to prove, by integration by part, the recurrence formula obeyed by the incomplete Gamma function:

$$\Gamma(a,x) = e^{-x} x^{a-1} + (a-1)\Gamma(a-1,x) \Leftrightarrow e^x \Gamma(a,x) = x^{a-1} + (a-1)e^x \Gamma(a-1,x) = x^{a-1} +$$

Applied to polynomial $\mathcal{T}_n(z)$ this formula gives

$$\mathcal{T}_n(z) = e^{z+n} \Gamma(n+1, z+n) = (z+n)^n + ne^{z+n} \Gamma(n, z+1+n-1)$$

= $(z+n)^n + n\mathcal{T}_{n-1}(z+1)$,

and this is precisely (3) with the same initial condition $\mathcal{T}_0(z) = e^z \Gamma(1, z) = 1$.

5 Comment

It is probable that $\mathcal{T}_n(z)$ has a combinatorial interpretation (and alternate simpler expression) for each $z \in \mathbb{Z}$. For instance the elements of the sequence $(\mathcal{T}_n(1))_{n \in \mathbb{N}}$, whose first terms are 1, 3, 17, 142, 1569,... are the numbers of connected functions on n labeled nodes, and we also have (OEIS number: A001865, see [8] for references)

$$\mathcal{T}_n(1) = \sum_{k=0}^n \frac{n!}{k!} (n+1)^k = e^{n+1} \int_{n+1}^\infty x^n e^{-x} \, dx \,.$$

We can present other examples. For z = 2: the elements of the sequence $(\mathcal{T}_n(2))_{n \in \mathbb{N}}$, whose first terms are 1, 4, 26, 236, 2760,..., are the numbers of normalized total height of rooted trees with *n* nodes (OEIS number: A001863, see [9]). The sequence $(\mathcal{T}_n(3))$, whose first terms are 1, 5, 37, 366, 4553,..., is A129137: $\mathcal{T}_n(3)$ is the number of trees on $1, 2, 3, \ldots, n \equiv [n]$, rooted at 1, in which 2 is a descendant of 3. And so on.

References

- H. Bergeron, E.M.F. Curado, J.P. Gazeau, L. M. C. S. Rodrigues, Symmetric generalized binomial distributions, submitted (2013); arXiv:1308.4863 [math-phys]
- [2] W. Magnus, F. Oberhettinger and R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag, Berlin, 3rd Edition, 1996.
- [3] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, Applied Mathematical Series-55, 10th Edition, Washington, 1972.
- [4] P. R. de Montmort, On the Game of Thirteen (1713), reprinted in Annotated Readings in the History of Statistics, ed. H. A. David and A. W. F. Edwards, Springer-Verlag, 2001, pp. 25-29.
- [5] L. Euler, Solution quaestionis curiosae ex doctrina combinationum, Mémoires Académie sciences St. Pétersburg 3 (1809/1810), 57-64; also E738 in his Collected Works, series I, volume 7, pp. 435-440.

- [6] L. Comtet, Advanced Combinatorics, Reidel, 1974, p. 182.
- [7] J. Desarmenien, Une autre interpretation du nombre de derangements Séminaire Lotharingien de Combinatoire, B08b, 1982, 6 pp.
 [Formerly : Publ. I.R.M.A. Strasbourg, 1984, 229/S-08 Actes 8e Séminaire Lotharingien, p. 11-16].
- [8] The On-Line Encyclopedia of Integer Sequences!, https://oeis.org
- [9] J. Riordan and N. J. A. Sloane, Enumeration of rooted trees by total height, J. Austral. Math. Soc., 10 278-282 (1969)