# A note about combinatorial sequences and Incomplete Gamma function 

H. Bergeron ${ }^{\text {a }}$, E.M.F. Curado ${ }^{\text {b,c }}$,<br>J. P. Gazeau ${ }^{\mathrm{b}, \mathrm{d}}$, Ligia M.C.S. Rodrigues ${ }^{\mathrm{b}}{ }^{(*)}$<br>a Univ Paris-Sud, ISMO, UMR 8214, 91405 Orsay, France<br>b Centro Brasileiro de Pesquisas Fisicas<br>${ }^{\text {c }}$ Instituto Nacional de Ciência e Tecnologia - Sistemas Complexos Rua Xavier Sigaud 150, 22290-180 - Rio de Janeiro, RJ, Brazil<br>d $A P C, U M R$ 7164,<br>Univ Paris Diderot, Sorbonne Paris Cité, 75205 Paris, France

September 19, 2018


#### Abstract

In this short note we present a set of interesting and useful properties of a one-parameter family of sequences including factorial and subfactorial, and their relations to the Gamma function and the incomplete Gamma function.


## Contents

1 Introduction 1
2 The function $T_{n}(x, y) \quad 2$
3 The function $\mathcal{T}_{n}(z) \quad 2$
4 The relation to the incomplete Gamma function 4
5 Comment 5

## 1 Introduction

The purpose of this note is to show a set of properties of a one-parameter family of sequences $\mathcal{T}_{n}(z), z$ is a complex variable, which are apparently not known in the literature. Among these sequences we have factorials and

[^0]subfactorials and they are seen to be related to the incomplete Gamma function. This family was introduced in a recent paper about generalized binomial distribution [1].

## 2 The function $T_{n}(x, y)$

Let us consider the following two-variable polynomial:

$$
\begin{equation*}
T_{n}(x, y):=\sum_{m=0}^{n}\binom{n}{m}(m+x)^{m}(n-m+y)^{n-m} \tag{1}
\end{equation*}
$$

which is clearly symmetrical in $x, y$. Actually, we have the stronger result:
Proposition 2.1. The polynomial $T_{n}(x, y)$ depends on the sum $z=x+y$ only:

$$
\begin{equation*}
T_{n}(x, z-x)=\sum_{m=0}^{n}\binom{n}{m}(m+x)^{m}(n-m+z-x)^{n-m} \equiv \mathcal{T}_{n}(z) \tag{2}
\end{equation*}
$$

where $x$ can be arbitrarily chosen with the convention that $0^{0} \equiv 1$ in the cases $x=0$ or $x=z$.
Proof. Let us prove by recurrence on $n$ that $T_{n}(x, z-x)$ does not depend explicitly on $x$ for all $z$. For that, we take the partial derivative of $T_{n}(x, z-$ $x)$ with respect to $x$. We have successively, by appropriately shift the summation variable,

$$
\begin{aligned}
\frac{\partial}{\partial x} T_{n}(x, z-x)= & \sum_{m=0}^{n}\binom{n}{m}\left[m(m+x)^{m-1}(n-m+z-x)^{n-m}\right. \\
& \left.-(n-m)(m+x)^{m}(n-m+z-x)^{n-m-1}\right] \\
= & n \sum_{m=0}^{n-1}\binom{n-1}{m}(m+1+x)^{m}(n-1-m+z-x)^{n-1-m} \\
& -n \sum_{m=0}^{n-1}\binom{n-1}{m}(m+x)^{m}(n-1-m+z+1-x)^{n-1-m} \\
= & n\left[T_{n-1}(1+x, z-x)-T_{n-1}(x, z+1-x)\right] .
\end{aligned}
$$

Now, we note that $T_{0}(x, z-x)=1$ and so does not depend explicitly on $x$. Suppose that the property " $T_{k}(x, z-x)$ does not depend explicitly on $x$ for all $z$ " holds true for all integer $1 \leq k \leq n-1$. Then $T_{n-1}(1+x, z-$ $x)=T_{n-1}(x, z+1-x)$ and this implies that $\partial / \partial x T_{n}(x, z-x)=0$, i.e., $T_{n}(x, z-x)$ does not depend explicitly on $x$ for all $z$.

## 3 The function $\mathcal{T}_{n}(z)$

Thanks to Proposition 2.1 it is now possible to find a convenient alternate expression of $T_{n}(x, z-x) \equiv \mathcal{T}_{n}(z)$. First, let us establish a recurrence formula.

Proposition 3.1. The polynomial $\mathcal{T}_{n}(z)$ satisfies the following recurrence formula:

$$
\begin{equation*}
\mathcal{T}_{n}(z)=(z+n)^{n}+n \mathcal{T}_{n-1}(z+1), \quad \mathcal{T}_{0}(z)=1 \tag{3}
\end{equation*}
$$

Proof. Taking $x=0$ in (2), let us write that expression as

$$
\begin{aligned}
\mathcal{T}_{n}(z) & =\sum_{m=0}^{n}\binom{n}{m}(m)^{m}(n-m+z)^{n-m} \\
& =(n+z)^{n}+n \sum_{m=1}^{n}\binom{n-1}{m-1} \times \\
& \times(m-1+1)^{m-1}(n-1-(m-1)+z+1-1)^{n-1-(m-1)} \\
& =(n+z)^{n}+n T_{n-1}(1, z+1-1)=(n+z)^{n}+n \mathcal{T}_{n-1}(z+1),
\end{aligned}
$$

where we have applied Proposition 2.1 with $x=1$.
It is then straightforward to deduce from this formula the following result.
Proposition 3.2. The polynomial $\mathcal{T}_{n}(z)$ admits the following expansion in powers of $(z+n)$ :

$$
\begin{equation*}
\mathcal{T}_{n}(z)=\sum_{k=0}^{n} \frac{n!}{k!}(z+n)^{k} . \tag{4}
\end{equation*}
$$

We note in particular the interesting corollary proving that the family $\left(\mathcal{T}_{n}(z)\right)_{z \in \mathbb{Z}}$ of integer sequences includes the factorial:

$$
\begin{equation*}
\mathcal{T}_{n}(-n)=n! \tag{5}
\end{equation*}
$$

and consequently also the Gamma function as

$$
\begin{equation*}
\mathcal{T}_{n}(-n)=\Gamma(n+1) \tag{6}
\end{equation*}
$$

The expression (4) allows to easily derive two asymptotic behaviors of $\mathcal{T}_{n}(z):$

$$
\begin{array}{ll}
\text { At large } z & \mathcal{T}_{n}(z) \sim z^{n} \\
\text { At large } n & \mathcal{T}_{n}(z) \sim n^{n} \tag{8}
\end{array}
$$

Indeed, in the latter the dominant term of the sum is for $k=n$ (the term $k=0$ is not dominant due to the decreasing exponential factor in the Stirling's formula $n!\sim n^{n} \sqrt{2 \pi n} e^{-n}$ ). This indicates that the dominant term of the sum in (4) is $(z+n)^{n}$.

A second expression of $\mathcal{T}_{n}(z)$ is also of interest.
Proposition 3.3. The polynomial $\mathcal{T}_{n}(z)$ admits the following expansion in powers of $(z+n+1)$ :

$$
\begin{equation*}
\mathcal{T}_{n}(z)=\sum_{k=0}^{n} a_{k}(z+n+1)^{k}, \tag{9}
\end{equation*}
$$

where the coefficients $a_{k}$ are given by

$$
\begin{equation*}
a_{k}=\binom{n}{k} \mathcal{T}_{n-k}(k-n-1) \equiv\binom{n}{k} d_{n-k} \tag{10}
\end{equation*}
$$

Proof. Applying formula

$$
a_{k}=\left.\frac{1}{k!} \frac{d^{k}}{d z^{k}} \mathcal{T}_{n}(z)\right|_{z=-n-1}
$$

to the original expression (2) where we put $x=0$, we easily derive

$$
\begin{aligned}
\left.\frac{1}{k!} \frac{d^{k}}{d z^{k}} \mathcal{T}_{n}(z)\right|_{z=-n-1} & =\sum_{m=0}^{n-k}(-1)^{n-m-k}\binom{n}{m}\binom{n-m}{k} m^{m}(1+m)^{n-m-k} \\
& =\binom{n}{k} \sum_{m=0}^{n-k}(-1)^{n-m-k}\binom{n-k}{m} m^{m}(1+m)^{n-m-k} \\
& =\binom{n}{k} \mathcal{T}_{n-k}(k-n-1)
\end{aligned}
$$

The sequence of numbers $\left(d_{n} \equiv \mathcal{T}_{n}(-n-1)\right)_{n \in \mathbb{N}}$, for which (4) gives

$$
\begin{equation*}
d_{n}=\sum_{k=0}^{n} \frac{n!}{k!}(-1)^{k} \tag{11}
\end{equation*}
$$

and whose first terms are $1,0,1,2,9,44, \ldots$, is well known for more than 3 centuries 4, 5, 6, 7, 8. Its OEIS name is A000166. Its numbers are named subfactorial (and then denoted as $!n$ ) or rencontres numbers, or derangements, since $d_{n}$ is the number of permutations of $n$ elements with no fixed points. They obey the recurrence relations (Euler) $d_{n}=$ $(n-1) d_{n-1}+(-1)^{n}$ and $d_{n}=n\left(d_{n-1}+d_{n-2}\right)$. Their generating function is

$$
D(x)=\sum_{n=0}^{\infty} d_{n} \frac{x^{n}}{n!}=\frac{e^{-x}}{1-x},
$$

and their asymptotic behavior at large $n$ is

$$
\lim _{n \rightarrow \infty} \frac{d_{n}}{n!}=\frac{1}{e}
$$

## 4 The relation to the incomplete Gamma function

Let us finally establish the relation between the polynomial $\mathcal{T}_{n}(z)$ and the incomplete Gamma function [3].
Proposition 4.1. The polynomial $\mathcal{T}_{n}(z)$ is related to the incomplete Gamma function

$$
\begin{equation*}
\Gamma(a, x)=\int_{x}^{\infty} t^{a-1} e^{-t} d t, \quad \operatorname{Re}(a)>0 \tag{12}
\end{equation*}
$$

as follows:

$$
\begin{equation*}
\mathcal{T}_{n}(z)=e^{z+n} \Gamma(n+1, z+n) \tag{13}
\end{equation*}
$$

Note that in the particular case that $z=-n$, from (13) we reobtain (6).

Proof. It is straightforward to prove, by integration by part, the recurrence formula obeyed by the incomplete Gamma function:

$$
\Gamma(a, x)=e^{-x} x^{a-1}+(a-1) \Gamma(a-1, x) \Leftrightarrow e^{x} \Gamma(a, x)=x^{a-1}+(a-1) e^{x} \Gamma(a-1, x) .
$$

Applied to polynomial $\mathcal{T}_{n}(z)$ this formula gives

$$
\begin{aligned}
\mathcal{T}_{n}(z) & =e^{z+n} \Gamma(n+1, z+n)=(z+n)^{n}+n e^{z+n} \Gamma(n, z+1+n-1) \\
& =(z+n)^{n}+n \mathcal{T}_{n-1}(z+1)
\end{aligned}
$$

and this is precisely (3) with the same initial condition $\mathcal{T}_{0}(z)=e^{z} \Gamma(1, z)=$ 1.

## 5 Comment

It is probable that $\mathcal{T}_{n}(z)$ has a combinatorial interpretation (and alternate simpler expression) for each $z \in \mathbb{Z}$. For instance the elements of the sequence $\left(\mathcal{T}_{n}(1)\right)_{n \in \mathbb{N}}$, whose first terms are $1,3,17,142,1569, \ldots$ are the numbers of connected functions on $n$ labeled nodes, and we also have (OEIS number: A001865, see 8 for references)

$$
\mathcal{T}_{n}(1)=\sum_{k=0}^{n} \frac{n!}{k!}(n+1)^{k}=e^{n+1} \int_{n+1}^{\infty} x^{n} e^{-x} d x .
$$

We can present other examples. For $z=2$ : the elements of the sequence $\left(\mathcal{T}_{n}(2)\right)_{n \in \mathbb{N}}$, whose first terms are $1,4,26,236,2760, \ldots$, are the numbers of normalized total height of rooted trees with $n$ nodes (OEIS number: A001863, see [9]). The sequence $\left(\mathcal{T}_{n}(3)\right)$, whose first terms are $1,5,37,366,4553, \ldots$, is A129137: $\mathcal{T}_{n}(3)$ is the number of trees on $1,2,3, \ldots, n \equiv[n]$, rooted at 1 , in which 2 is a descendant of 3 . And so on.

## References

[1] H. Bergeron, E.M.F. Curado, J.P. Gazeau, L. M. C. S. Rodrigues, Symmetric generalized binomial distributions, submitted (2013); arXiv:1308.4863 [math-phys]
[2] W. Magnus, F. Oberhettinger and R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, SpringerVerlag, Berlin, 3rd Edition, 1996.
[3] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, Applied Mathematical Series55, 10th Edition, Washington, 1972.
[4] P. R. de Montmort, On the Game of Thirteen (1713), reprinted in Annotated Readings in the History of Statistics, ed. H. A. David and A. W. F. Edwards, Springer-Verlag, 2001, pp. 25-29.
[5] L. Euler, Solution quaestionis curiosae ex doctrina combinationum, Mémoires Académie sciences St. Pétersburg 3 (1809/1810), 57-64; also E738 in his Collected Works, series I, volume 7, pp. 435-440.
[6] L. Comtet, Advanced Combinatorics, Reidel, 1974, p. 182.
[7] J. Desarmenien, Une autre interpretation du nombre de derangements Séminaire Lotharingien de Combinatoire, B08b, 1982, 6 pp. [Formerly : Publ. I.R.M.A. Strasbourg, 1984, 229/S-08 Actes 8e Séminaire Lotharingien, p. 11-16].
[8] The On-Line Encyclopedia of Integer Sequences!, https://oeis.org
[9] J. Riordan and N. J. A. Sloane, Enumeration of rooted trees by total height, J. Austral. Math. Soc., 10 278-282 (1969)


[^0]:    *e-mail: herve.bergeron@u-psud.fr, evaldo@cbpf.br, gazeau@apc.univ-paris7.fr, ligia@cbpf.br

