

On the frequencies of patterns of rises and falls

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Abstract

We investigate the probability of observing a given pattern of n rises and falls in a random stationary data series. The data are modelled as a sequence of $n + 1$ independent and identically distributed random numbers. This probabilistic approach has a combinatorial equivalent, where the data are modelled by a random permutation on $n + 1$ objects. The probability of observing a long pattern of rises and falls decays exponentially with its length n in general. The associated decay rate α is interpreted as the embedding entropy of the pattern. This rate is evaluated exactly for all periodic patterns. In the most general case, it is expressed in terms of a determinant of generalized hyperbolic or trigonometric functions. Alternating patterns have the smallest rate $\alpha_{\min} = \ln(\pi/2) = 0.451582\dots$, while other examples lead to arbitrarily large rates. The probabilities of observing uniformly chosen random patterns are demonstrated to obey multifractal statistics. The typical value $\alpha_0 = 0.806361\dots$ of the rate plays the role of a Lyapunov exponent. A wide range of examples of patterns, either deterministic or random, is also investigated.

Keywords: Data series, Patterns, Rises and falls, Entropy, Multifractals, Combinatorics, Permutations

1. Introduction

Consider a data series, such as e.g. the daily temperature at a given weather station over one year. The most obvious features of such a data series are its rises and falls. Physics and other branches of science provide plenty of examples of datasets where the statistics of geometrical features, such as maxima and minima, or rises and falls, is of central interest. One example from statistical physics is provided by energy landscapes, which are ubiquitously present in theoretical studies of systems ranging from glasses to proteins [1].

In this work we investigate the probability of observing a given pattern of n rises and falls in a random stationary data series. This question has hardly

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been addressed so far in the physics literature, in strong contrast with the statistics of extreme values, which has recently attracted a lot of attention in many areas, including random walks, disordered systems, growth processes and random matrices [2, 3, 4, 5, 6, 7].

The setting of the present work is meant to provide a null model, to which real data could be compared. A first attempt has been made recently in this direction, with the analysis of microarray time series data in genetics [8]. We model the data series as a sequence of $n + 1$ i.i.d. (independent and identically distributed) random numbers x_i drawn from a continuous distribution. As these random numbers will only occur in inequalities, their distribution can be chosen to be uniform on the unit interval. This probabilistic approach is exposed in Section 3. An equivalent combinatorial approach (see Section 4) is obtained by coarse-graining the random numbers according to the permutation which brings them to an increasing order. We are thus led to model the data as a uniformly chosen random permutation on $n + 1$ objects. This line of thought dates back to the pioneering study of alternating permutations by André [9, 10], and it has since then been addressed regularly in the mathematical literature [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28] (this list of references is not meant to be exhaustive). To close, let us mention that the combinatorial approach to our problem pertains to the more general topic of patterns in permutations, which has been for long an active area of discrete mathematics [29, 30, 31].

2. Summary of results

Our goal is to provide a comprehensive and self-contained exposition of the calculation of the frequencies of patterns of rises and falls in a random stationary data series. We aim at using a language accessible to a broad readership in statistical physics. Let us give the detailed setup of this paper and summarize our findings.

The probabilistic and combinatorial approaches, respectively exposed in Sections 3 and 4, provide two equivalent definitions of the probability $P_n(\varepsilon_1 \dots \varepsilon_n)$ of observing a given pattern $\varepsilon_1 \dots \varepsilon_n$ of n rises and falls. The equivalence between both approaches has already been underlined in several works [8, 21, 23, 24, 25, 27, 28]. It will become clear in the following that each approach has its advantages: the probabilistic one is more suitable for analytical investigations, while the combinatorial one results in a simple recursive structure, lending itself to exact numerical calculations.

In Section 5 we show explicit results for small patterns (up to $n = 4$). We then present a heuristic analysis demonstrating that the probability of observing a pattern is essentially determined by its excursion, as long as its length is modest.

In the remainder of the paper, the emphasis is on asymptotic properties in the regime of most interest, at least from the viewpoint of statistical physics, i.e., where the length n of the pattern is large. In this regime the probability

$P_n(\varepsilon_1 \dots \varepsilon_n)$ typically falls off exponentially as

$$P_n \sim e^{-\alpha n}. \quad (1)$$

The decay rate α will be our central object of interest. This quantity can be viewed as the *embedding entropy* of the binary pattern $\varepsilon_1 \dots \varepsilon_n$, i.e., the entropic cost per unit length for embedding this pattern into a sequence of i.i.d. random numbers. It is worth noticing that the above definition is entirely parameter-free. If all the 2^n patterns of length n had equal probabilities $P_n = 2^{-n}$, the rate would be constant and equal to $\alpha = \ln 2$. The observed wide range of possible values of the rate α , from $\alpha_{\min} = \ln(\pi/2) = 0.451582\dots$ to infinity, testifies the richness of the problem.

Periodic patterns are investigated in sections 6, 7, and 8. As already mentioned, the subject is an old classic of discrete mathematics. Our comprehensive approach allows us to recover many known results by more elementary means, and often to express them in simpler terms. Section 6 is a self-contained presentation of some of the beauties of the historical example of alternating patterns, for which the rate α assumes its minimal value α_{\min} . Section 7 deals with the family of p -alternating periodic patterns, whose motif (unit cell) consists of $p-1$ rises followed by a fall. The rate reads $\alpha = \ln z_0$, where z_0 is the smallest real positive zero of a generalized trigonometric function. In Section 8 we show how the rate α can be evaluated exactly for an arbitrary periodic pattern, with any period $p \geq 2$: z_0 is now the smallest zero of a determinant of generalized hyperbolic or trigonometric functions, whose size is at most $p/2$. Many examples are treated explicitly.

The rest of the paper covers entirely novel areas. Sections 9 and 10 serve as an intermezzo. In Section 9 we deal with examples of aperiodic patterns which are built from three classical self-similar sequences: Fibonacci, Thue-Morse, and Rudin-Shapiro. The probabilities P_n exhibit an exponential decay, characterized by a well-defined rate α , modulated by a fractal amplitude which reflects the self-similarity of the underlying sequence. Section 10 is devoted to chirping patterns, consisting mostly of rises, whereas falls are more and more scarce (or vice versa). In this case the probabilities P_n are found to decay super-exponentially. Their asymptotic form is predicted more precisely in the situation of most interest where the density of falls follows a power law.

Section 11 is devoted to the heart of the problem, namely the statistics of the probabilities P_n if patterns are chosen in various ensembles of random patterns of fixed length n . The uniform ensemble, where all patterns are considered with equal weights, is studied thoroughly. The probabilities P_n of generic patterns have the typical rate $\alpha_0 = 0.806361\dots$. The latter number can be interpreted as a Lyapunov exponent. The whole set of probabilities P_n is shown to obey multifractal statistics, with a non-trivial spectrum of multifractal dimensions $f(\alpha)$, increasing from $f(\alpha_{\min}) = 0$ to $f(\alpha_0) = 1$. Other ensembles of random patterns of fixed length n , namely the ensemble at fixed concentration c of rises and a symmetric Markovian ensemble defined by a persistence probability r , are also investigated. The probabilities P_n now generically decay according to

effective typical rates $\beta(c)$ and $\gamma(r)$, which depend continuously on the ensemble parameters.

Two appendices are respectively devoted to the explicit correspondence between the combinatorial and probabilistic approaches (Appendix A) and to generalized hyperbolic and trigonometric functions (Appendix B).

3. Probabilistic approach

The probabilistic approach goes as follows. The data series is modeled by a sequence of $n+1$ i.i.d. random numbers x_i ($i = 0, \dots, n$), drawn from the uniform distribution on the unit interval $[0, 1]$. This sequence of random numbers yields a pattern $\varepsilon_1 \dots \varepsilon_n$ of n rises and falls defined as follows. For $i = 1, \dots, n$:

$$\begin{aligned} \text{If } x_i > x_{i-1}, \text{ there is a rise at the } i\text{th place, and } \varepsilon_i = +, \\ \text{If } x_i < x_{i-1}, \text{ there is a fall at the } i\text{th place, and } \varepsilon_i = -. \end{aligned} \quad (2)$$

Let us start with the example shown in Figure 1. This configuration obeys the inequalities $x_0 < x_1 < x_2 > x_3$, and therefore yields the pattern $++-$. The probability of observing this pattern reads

$$\begin{aligned} P_3(++-) &= \int_0^1 dx_0 \int_{x_0}^1 dx_1 \int_{x_1}^1 dx_2 \int_0^{x_2} dx_3 \\ &= \int_0^1 dx_3 \int_{x_3}^1 dx_2 \int_0^{x_2} dx_1 \int_0^{x_1} dx_0 \\ &= \frac{1}{8}. \end{aligned} \quad (3)$$

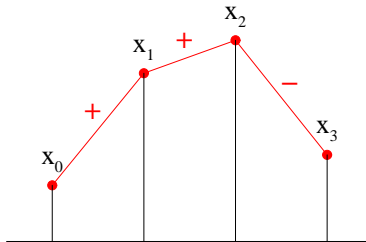


Figure 1: An example of a configuration of 4 random numbers.

Let us now turn to the general case. The probability of an arbitrary pattern $\varepsilon_1 \dots \varepsilon_n$ can be calculated recursively by conditioning on x_n . Let $f_n(x) dx$ be the probability that the sequence x_0, \dots, x_n yields the pattern $\varepsilon_1 \dots \varepsilon_n$ and that $x < x_n < x + dx$. The conditional densities $f_n(x)$ obey the recursion relations:

$$\begin{aligned} \text{If } \varepsilon_n = +, \text{ then } f_n(x) &= \int_0^x f_{n-1}(y) dy, \\ \text{If } \varepsilon_n = -, \text{ then } f_n(x) &= \int_x^1 f_{n-1}(y) dy, \end{aligned} \quad (4)$$

with the initial condition $f_0(x) = 1$. It follows that $f_n(x)$ is a polynomial in x with degree n . We have

$$P_n(\varepsilon_1 \dots \varepsilon_n) = \int_0^1 f_n(x) dx. \quad (5)$$

This recursive scheme has already been described in [23, 24, 28]. For the example of Figure 1, we get

$$\begin{aligned} f_1(x) &= \int_0^x dy = x, \\ f_2(x) &= \int_0^x f_1(y) dy = \frac{x^2}{2}, \\ f_3(x) &= \int_x^1 f_2(y) dy = \frac{1-x^3}{6}, \\ P_3(++-) &= \int_0^1 f_3(x) dx = \frac{1}{8}. \end{aligned} \quad (6)$$

The second of the nested integral expressions (3) is thus recovered.

4. Combinatorial approach

In the combinatorial approach the data series is modeled by a permutation σ , chosen uniformly among the $(n+1)!$ permutations on $n+1$ objects labelled $i = 0, \dots, n$. Such a permutation yields a pattern $\varepsilon_1 \dots \varepsilon_n$, defined as follows. For $i = 1, \dots, n$:

$$\begin{aligned} \text{If } \sigma_i > \sigma_{i-1}, \text{ there is a rise at the } i\text{th place, and } \varepsilon_i &= +, \\ \text{If } \sigma_i < \sigma_{i-1}, \text{ there is a fall at the } i\text{th place, and } \varepsilon_i &= -. \end{aligned} \quad (7)$$

The pattern $\varepsilon_1 \dots \varepsilon_n$ is said to be the *up-down signature* of the permutation σ .

Let us again consider the example shown in Figure 1. We have

$$x_0 < x_3 < x_1 < x_2. \quad (8)$$

Let us coarse-grain this configuration of random numbers by representing it as a permutation σ . The rule is that the inverse permutation σ^{-1} gives the order of indices in the inequalities (8). We thus have $\sigma^{-1} = (0312)$,¹ and so $\sigma = (0231)$, which indeed yields the pattern $++-$.

How many permutations on 4 objects yield the pattern $++-$? This simple question can be solved by inspection. There are 3 such permutations: (0132), (0231) and (1230). We thus recover the result (see (3))

$$P_3(++-) = \frac{3}{4!} = \frac{1}{8}. \quad (9)$$

¹We use the one-line notation $\sigma = (\sigma_0 \sigma_1 \dots \sigma_n)$.

It is indeed clear that all the $4!$ permutations on 4 objects are equally probable. This observation demonstrates (on an example) the equivalence between the probabilistic approach of Section 3 and the present combinatorial one (see [8, 21, 23, 24, 25, 27, 28]).

Let us now turn to the general case. The number $A_n(\varepsilon_1 \dots \varepsilon_n)$ of permutations on $n+1$ objects yielding a given pattern $\varepsilon_1 \dots \varepsilon_n$ of n rises and falls can be calculated by an efficient recursive scheme, which seems to have been discovered several times independently [14, 19, 20]. The gist of the method is similar to that of the probabilistic approach. It consists in conditioning the permutation σ on its last entry σ_n , and to relate the patterns formed by $n+1$ and n objects. Let $a_{n,j}$ be the number of permutations which yield the pattern $\varepsilon_1 \dots \varepsilon_n$ and have $\sigma_n = j$. These numbers can be shown to obey the recursion relations

$$\begin{aligned} \text{If } \varepsilon_n = +, \text{ then } & \begin{cases} a_{n,0} = 0, \\ a_{n,j} = a_{n,j-1} + a_{n-1,j-1} \quad (j = \overrightarrow{1, \dots, n}), \end{cases} \\ \text{If } \varepsilon_n = -, \text{ then } & \begin{cases} a_{n,n} = 0, \\ a_{n,j} = a_{n,j+1} + a_{n-1,j} \quad (j = \overleftarrow{0, \dots, n-1}), \end{cases} \end{aligned} \quad (10)$$

with the initial condition $a_{0,0} = 1$. Arrows indicate the order in which the recursion relations have to be used. We thus build a triangular array of integers:

$$\begin{array}{c} a_{0,0} \\ a_{1,0} \leftrightarrow a_{1,1} \\ a_{2,0} \leftrightarrow a_{2,1} \leftrightarrow a_{2,2} \\ a_{3,0} \leftrightarrow a_{3,1} \leftrightarrow a_{3,2} \leftrightarrow a_{3,3} \\ a_{4,0} \leftrightarrow a_{4,1} \leftrightarrow a_{4,2} \leftrightarrow a_{4,3} \leftrightarrow a_{4,4} \end{array} \quad (11)$$

At the n th row, all arrows go from left to right if $\varepsilon_n = +$, and from right to left if $\varepsilon_n = -$ (see (26) for an example).

The requested numbers of permutations

$$A_n(\varepsilon_1 \dots \varepsilon_n) = \sum_{j=0}^n a_{n,j} \quad (12)$$

can also be read off from the array $a_{n,j}$. We have indeed

$$\begin{aligned} \text{If } \varepsilon_{n+1} = +, \text{ then } & A_n = a_{n+1,n+1}, \\ \text{If } \varepsilon_{n+1} = -, \text{ then } & A_n = a_{n+1,0}. \end{aligned} \quad (13)$$

Finally, the probability that a random permutation on $n+1$ objects yields the pattern $\varepsilon_1 \dots \varepsilon_n$ reads

$$P_n(\varepsilon_1 \dots \varepsilon_n) = \frac{A_n(\varepsilon_1 \dots \varepsilon_n)}{(n+1)!}. \quad (14)$$

To close, let us write down explicitly the correspondence between the probabilistic and combinatorial approaches. For a given pattern $\varepsilon_1 \dots \varepsilon_n$, the probabilistic approach involves the n th degree polynomial $f_n(x)$, which has $n+1$

coefficients, while the combinatorial one involves the $n + 1$ integers $a_{n,j}$. These two sets of numbers carry the same information. The precise correspondence, to be established in Appendix A, is summarized by the formula

$$f_n(x) = \sum_{j=0}^n a_{n,j} \frac{x^j(1-x)^{n-j}}{j!(n-j)!}. \quad (15)$$

5. Explicit results for small patterns

Table 1 presents explicit results for patterns of length up to $n = 4$. For each length n , the 2^n patterns are listed in lexicographical order. For each pattern $\varepsilon_1 \dots \varepsilon_n$, the Table gives the number $A_n(\varepsilon_1 \dots \varepsilon_n)$ of permutations yielding that pattern and the corresponding probability $P_n(\varepsilon_1 \dots \varepsilon_n)$, obtained by means of (10), (12), and (14). These numbers exhibit the expected up-down and left-right symmetries. They also manifest a scatter which increases rapidly with n .

$n = 1$					
ε_1	A_1	P_1	ε_1	A_1	P_1
+	1	1/2	-	1	1/2

$n = 2$					
$\varepsilon_1\varepsilon_2$	A_2	P_2	$\varepsilon_1\varepsilon_2$	A_2	P_2
++	1	1/6	-+	2	1/3
+-	2	1/3	--	1	1/6

$n = 3$					
$\varepsilon_1\varepsilon_2\varepsilon_3$	A_3	P_3	$\varepsilon_1\varepsilon_2\varepsilon_3$	A_3	P_3
+++	1	1/24	-++	3	1/8
++-	3	1/8	-+-	5	5/24
+ - +	5	5/24	--+	3	1/8
+ - -	3	1/8	---	1	1/24

$n = 4$					
$\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$	A_4	P_4	$\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$	A_4	P_4
++++	1	1/120	-++++	4	1/30
+++ -	4	1/30	-+ + -	11	11/120
++ - +	9	3/40	-+ - +	16	2/15
++ - -	6	1/20	-+ - -	9	3/40
+ - ++	9	3/40	-- + +	6	1/20
+ - + -	16	2/15	-- + -	9	3/40
+ - - +	11	11/120	-- - +	4	1/30
+ - - -	4	1/30	----	1	1/120

Table 1: Explicit results for all patterns of rises and falls of length up to $n = 4$: numbers $A_n(\varepsilon_1 \dots \varepsilon_n)$ of permutations and probabilities $P_n(\varepsilon_1 \dots \varepsilon_n)$.

The data shown in Table 1 can be analyzed in the following heuristic way. The probability $P_n(\varepsilon_1 \dots \varepsilon_n)$ can be expected to be strongly penalized for patterns which make large excursions in the vertical direction. This notion can be formalized as follows. To the pattern $\varepsilon_1 \dots \varepsilon_n$ we associate a random walk with steps ε_i , i.e.,

$$h_i = h_{i-1} + \varepsilon_i, \quad (16)$$

with $h_0 = 0$, and we define the *excursion* of the pattern as the variance of the position of that walk:

$$\begin{aligned} \Delta_n^2(\varepsilon_1 \dots \varepsilon_n) &= \frac{1}{n+1} \sum_{i=0}^n h_i^2 - \left(\frac{1}{n+1} \sum_{i=0}^n h_i \right)^2 \\ &= \frac{n(n+2)}{6(n+1)} + \frac{2}{(n+1)^2} \sum_{1 \leq i < j \leq n} i(n+1-j) \varepsilon_i \varepsilon_j. \end{aligned} \quad (17)$$

For a fixed length n , the patterns with the largest excursion are the two straight ones, consisting only of rises, or of falls, for which we have

$$\Delta_n^2 = \frac{n(n+2)}{12}. \quad (18)$$

These patterns also have the smallest probabilities. Consider for definiteness the rising pattern ($\varepsilon_1 \dots \varepsilon_n = + \dots +$). The probabilistic approach yields

$$f_n(x) = \frac{x^n}{n!}, \quad (19)$$

while the combinatorial approach yields

$$a_{n,j} = \delta_{n,j}, \quad A_n = 1, \quad (20)$$

where the Kronecker symbol $\delta_{n,j}$ equals 1 if $j = n$ and 0 else. There is indeed one single permutation yielding the rising pattern, namely the identity ($\sigma_j = j$ for all j). Both approaches consistently give

$$P_n = \frac{1}{(n+1)!}. \quad (21)$$

This factorial decay corresponds to a logarithmically divergent rate

$$\alpha(n) \approx \ln n - 1, \quad (22)$$

and so $\alpha_{\max} = \infty$.

The patterns with the smallest excursion are the two alternating ones, $+ - + - \dots$ and $- + - + \dots$, for which we have

$$\Delta_n^2 = \frac{1}{4} \quad (n \text{ odd}), \quad \Delta_n^2 = \frac{n(n+2)}{4(n+1)^2} \quad (n \text{ even}). \quad (23)$$

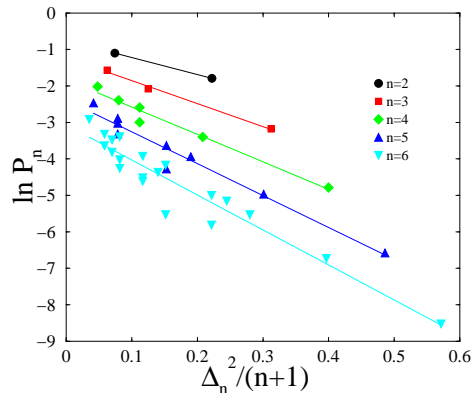


Figure 2: Logarithmic plot of probability P_n against reduced excursion $\Delta_n^2/(n+1)$ for all patterns of length $n = 2$ to 6 . Symbols: data for individual patterns. Straight lines: least-square fits.

These patterns, which will be investigated thoroughly in Section 6, are also known to have the largest probabilities [13, 19]. The corresponding rate α takes its minimum value $\alpha_{\min} = \ln(\pi/2)$ (see (41)).

More generally, we observe (see Figure 2) that the probability P_n and the excursion Δ_n^2 are strongly anti-correlated.

The observed negative correlation however progressively fades away as the length of patterns is increased. Figure 3 shows a plot of exact numerical data² for the absolute correlation coefficient $|c_n|$ between Δ_n^2 and $\ln P_n$ for lengths up to $n = 30$. The fit to the data suggests that this coefficient falls off asymptotically as $n^{-1/2}$.

A statistical analysis of the probabilities P_n adapted to their behavior for large lengths n , based on the multifractal formalism, will be presented in Section 11.1.

6. Alternating patterns

This Section is devoted to the historical example of alternating patterns $+ - + - + - \dots$, studied long ago by André in the framework of alternating permutations [9, 10].

²Throughout the following, the expression *exact numerical data* refers to results obtained by iterating numerically the combinatorial recursion (10).

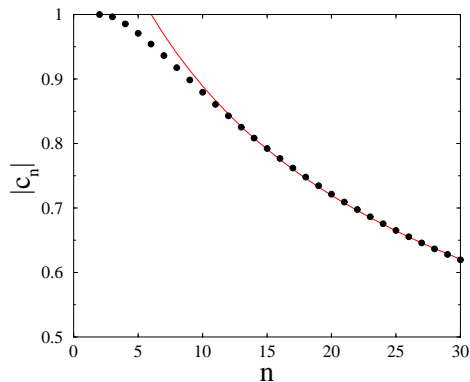


Figure 3: Absolute correlation coefficient $|c_n|$ between Δ_n^2 and $\ln P_n$ against length n . Black symbols: exact numerical data for lengths up to $n = 30$. Full red line: fit of the data for $n \geq 10$ to the form $c_n = a(n + n_0)^{-1/2}$.

6.1. Probabilistic approach

We have $\varepsilon_{2k} = -$ and $\varepsilon_{2k+1} = +$, and so the recursion (4) for the polynomials $f_n(x)$ takes the form

$$f_{2k+1}(x) = \int_0^x f_{2k}(y) dy, \quad f_{2k}(x) = \int_x^1 f_{2k-1}(y) dy. \quad (24)$$

We thus obtain

$$\begin{aligned} f_0(x) &= 1, & f_1(x) &= x, & f_2(x) &= \frac{1}{2}(1 - x^2), \\ f_3(x) &= \frac{x}{6}(3 - x^2), & f_4(x) &= \frac{1}{24}(1 - x^2)(5 - x^2), & \dots \end{aligned} \quad (25)$$

Integrating over x , using (5) and (14), we recover the first few entries of the sequence (27).

6.2. Combinatorial approach

The construction (11) of the array of integers $a_{n,j}$ takes the following form:

$$\begin{array}{cccccccc} & & & & 0 & & & & \\ & & & & 0 \rightarrow 1 & & & & \\ & & & & 1 \leftarrow 1 \leftarrow 0 & & & & \\ & & & & 0 \rightarrow 1 \rightarrow 2 \rightarrow 2 & & & & (26) \\ & & & & 5 \leftarrow 5 \leftarrow 4 \leftarrow 2 \leftarrow 0 & & & & \\ & & & & 0 \rightarrow 5 \rightarrow 10 \rightarrow 14 \rightarrow 16 \rightarrow 16 & & & & \\ & & & & 61 \leftarrow 61 \leftarrow 56 \leftarrow 46 \leftarrow 32 \leftarrow 16 \leftarrow 0 & & & & \end{array}$$

The above boustrophedon construction [32] seems to date back to Seidel [33]. The word *boustrophedon*, from two words of ancient Greek meaning *ox* and

turn, applies to writing systems where the direction of drawing is alternatively changed line after line from right to left and left to right, somewhat like the motion of a plough in a field. The resulting sequence for the numbers of alternating permutations:

$$\begin{aligned} A_0 = 1, \quad A_1 = 1, \quad A_2 = 2, \quad A_3 = 5, \quad A_4 = 16, \quad A_5 = 61, \\ A_6 = 272, \quad A_7 = 1385, \quad A_8 = 7936, \quad A_9 = 50521, \quad \dots \end{aligned} \quad (27)$$

to be investigated below, is referred to as sequence A000111 in the OEIS [34], where many further references and links are given. The integers A_n have been referred to as the up-down Euler-Bernoulli numbers or the Entringer numbers, and they have received many combinatorial interpretations.

6.3. Generating-series method

The probabilistic recursion (4) can be solved analytically by means of generating series, for any periodic pattern. We explain the method in detail in the present case of alternating patterns, and apply it to other patterns in Sections 7 and 8. Let us mention that such generating series can also be derived by purely combinatorial means. That route however requires an advanced knowledge of the theory of symmetric functions [26].

Introduce the generating series

$$F(z, x) = \sum_{n \geq 0} f_n(x) z^n \quad (28)$$

and

$$\Pi(z) = \sum_{n \geq 0} P_n z^n = \sum_{n \geq 0} A_n \frac{z^n}{(n+1)!} = \int_0^1 F(z, x) dx. \quad (29)$$

In the present case, it is advisable to set

$$F(z, x) = F_0(z, x) + F_1(z, x), \quad (30)$$

where

$$F_0(z, x) = \sum_{k \geq 0} f_{2k}(x) z^{2k}, \quad F_1(z, x) = \sum_{k \geq 0} f_{2k+1}(x) z^{2k+1}. \quad (31)$$

The recursion (4) translates to the coupled integral equations

$$F_0(z, x) = 1 + z \int_x^1 F_1(z, y) dy, \quad F_1(z, x) = z \int_0^x F_0(z, y) dy, \quad (32)$$

or, equivalently, to the coupled differential equations

$$\frac{\partial F_0}{\partial x} = -z F_1, \quad \frac{\partial F_1}{\partial x} = z F_0, \quad (33)$$

with boundary conditions $F_0(z, 1) = 1$, $F_1(z, 0) = 0$, and, finally, to the single differential equation

$$\frac{\partial^2 F_0}{\partial x^2} = -z^2 F_0, \quad (34)$$

with boundary conditions $F_0(z, 1) = 1$, $\partial F_0(z, 0)/\partial x = 0$. The solution to the latter equation reads

$$F_0(z, x) = \frac{\cos zx}{\cos z}, \quad F_1(z, x) = \frac{\sin zx}{\cos z}. \quad (35)$$

Using (32), we have

$$\Pi(z) = \frac{1}{z} (F_1(z, 1) + F_0(z, 0) - 1). \quad (36)$$

We thus obtain the explicit expression³

$$\Pi(z) = \frac{\sin z + 1 - \cos z}{z \cos z} = \frac{1}{z} (\tan z + \sec z - 1). \quad (37)$$

Splitting the above result, we get

$$\begin{aligned} \tan z &= \sum_{k \geq 0} P_{2k} z^{2k+1} = \sum_{k \geq 0} A_{2k} \frac{z^{2k+1}}{(2k+1)!}, \\ \sec z &= 1 + \sum_{k \geq 0} P_{2k+1} z^{2k+2} = 1 + \sum_{k \geq 0} A_{2k+1} \frac{z^{2k+2}}{(2k+2)!}. \end{aligned} \quad (38)$$

This is the main result obtained by André [9, 10].

To be complete, let us mention that the above expressions imply the following relationships between the probabilities P_n of alternating patterns, or equivalently the Euler-Bernoulli or Entringer numbers A_n of alternating permutations, and the Bernoulli numbers B_n , the Euler numbers E_n , and the value of Riemann's zeta function at even integers:

$$\begin{aligned} P_{2k} &= (-)^k \frac{2^{2k+2} (2^{2k+2} - 1)}{(2k+2)!} B_{2k+2} = \frac{2(2^{2k+2} - 1)}{\pi^{2k+2}} \zeta(2k+2), \\ P_{2k+1} &= (-)^{k+1} \frac{E_{2k+2}}{(2k+2)!}. \end{aligned} \quad (39)$$

The asymptotic behavior of the probabilities P_n is governed by the first pole at $\pi/2$ of the generating series $\Pi(z)$. We thus obtain the exponential decay

$$P_n \approx \frac{8}{\pi^2} \left(\frac{2}{\pi} \right)^n. \quad (40)$$

³ $\tan z = \sin z / \cos z$, $\sec z = 1 / \cos z$.

The rate α assumes its minimal value

$$\alpha_{\min} = \ln(\pi/2) = 0.451582\dots \quad (41)$$

This result already appears in the physics literature, albeit in some disguised form, in an investigation of the one-dimensional Ising spin glass at zero temperature by Derrida and Gardner [35].

7. p -alternating patterns

We pursue our investigation with the periodic patterns with any period $p \geq 2$ whose motif (unit cell) consists of $p-1$ rises followed by one single fall. In other words, we have $\varepsilon_{kp+q} = +$ for $q = 1, \dots, p-1$ and $\varepsilon_{kp} = -$. These p -alternating patterns have been investigated in [15, 16, 24, 26, 28].

7.1. General solution

The generating-series method of Section 6.3 extends to p -alternating patterns as follows. Setting

$$F(z, x) = \sum_{q=0}^{p-1} F_q(z, x), \quad F_q(z, x) = \sum_{k \geq 0} f_{kp+q}(x) z^{kp+q}, \quad (42)$$

the recursion (4) translates to the coupled integral equations

$$\begin{aligned} F_0(z, x) &= 1 + z \int_x^1 F_1(z, y) dy, \\ F_q(z, x) &= z \int_0^x F_{q-1}(z, y) dy \quad (q = 1, \dots, p-1), \end{aligned} \quad (43)$$

or, equivalently, to the differential equation

$$\frac{\partial^p F_0}{\partial x^p} = -z^p F_0, \quad (44)$$

with boundary conditions $F_0(z, 1) = 1$ and $\partial^q F_0(z, 0)/\partial x^q = 0$ ($q = 1, \dots, p-1$). The solution reads

$$F_q(z, x) = \frac{T_{p,q}(zx)}{T_{p,0}(z)} \quad (q = 0, \dots, p-1), \quad (45)$$

where the $T_{p,q}$ are the generalized trigonometric functions introduced in Appendix B. The generating series $\Pi(z)$ of the probabilities P_n then reads

$$\Pi(z) = \frac{1}{z T_{p,0}(z)} \left(\sum_{q=1}^{p-1} T_{p,q}(z) + 1 - T_{p,0}(z) \right). \quad (46)$$

This result has been obtained by purely combinatorial means by Mendes and Remmel [26]. The corresponding numbers A_n of p -alternating partitions have been called generalized Euler numbers and investigated in [36, 37].

For $p \geq 3$, the expression (46) has two main novel features with respect to its analogue (37) for alternating permutations ($p = 2$). These properties also apply to arbitrary periodic patterns, to be investigated in Section 8.

- (i) The denominator $T_{p,0}(z)$ is an entire function of z^p . It therefore has p smallest zeros sitting at the vertices of a regular p -gon, i.e., $z_q = z_0 \omega^q$ ($q = 0, \dots, p-1$), where z_0 is real positive, and $\omega = e^{2\pi i/p}$ (see (B.3)). We thus obtain an asymptotic decay of the probabilities P_n of the form

$$P_n \approx C_n e^{-\alpha n}, \quad (47)$$

where the rate α is given by

$$\alpha = \ln z_0, \quad (48)$$

while the other zeros z_q are responsible for the occurrence of a periodic amplitude C_n of n with period p (to be illustrated in Figure 4 below). This periodic modulation does not occur in the case of alternating patterns ($p = 2$). The expression (37) indeed has no pole at $z = -z_0 = -\pi/2$, as its numerator also vanishes there.

- (ii) If periodic patterns are deduced one from the other by a cyclic permutation, such as those defined by the motifs $(- + +)$, $(+ - +)$ and $(+ + -)$, they share the same rate α , but different periodic amplitudes. This phenomenon too is absent for $p = 2$, because of up-down symmetry.

7.2. The case $p = 3$ $(+ + -)$

The generating series $\Pi(z)$ reads (see (B.18))

$$\Pi(z) = \frac{3 - e^{-z} - 2e^{z/2} \cos \frac{z\sqrt{3}}{2} + 2\sqrt{3}e^{z/2} \sin \frac{z\sqrt{3}}{2}}{z \left(e^{-z} + 2e^{z/2} \cos \frac{z\sqrt{3}}{2} \right)}. \quad (49)$$

We have

$$z_0 = 1.849812\dots, \quad \alpha = 0.615084\dots \quad (50)$$

The resulting sequence for the numbers of 3-alternating permutations:

$$\begin{aligned} A_0 = 1, \quad A_1 = 1, \quad A_2 = 1, \quad A_3 = 3, \quad A_4 = 9, \quad A_5 = 19, \\ A_6 = 99, \quad A_7 = 477, \quad A_8 = 1513, \quad A_9 = 11259, \quad \dots \end{aligned} \quad (51)$$

is referred to as sequence A178963 in the OEIS [34], where an expression equivalent to (49) is also given.

7.3. *The case $p = 4$ (+ + + -)*

The generating series $\Pi(z)$ reads (see (B.19))

$$\Pi(z) = \frac{\sinh \zeta \sin \zeta + \sqrt{2} \cosh \zeta \sin \zeta - \cosh \zeta \cos \zeta + 1}{z \cosh \zeta \cos \zeta}, \quad (52)$$

with $\zeta = z/\sqrt{2}$. We have

$$z_0 = \frac{\pi}{\sqrt{2}} = 2.221441 \dots, \quad \alpha = 0.798156 \dots \quad (53)$$

The resulting sequence for the numbers of 4-alternating permutations:

$$\begin{aligned} A_0 = 1, \quad A_1 = 1, \quad A_2 = 1, \quad A_3 = 1, \quad A_4 = 4, \quad A_5 = 14, \\ A_6 = 34, \quad A_7 = 69, \quad A_8 = 496, \quad A_9 = 2896, \quad \dots \end{aligned} \quad (54)$$

is referred to as sequence A178964 in the OEIS [34], where an expression equivalent to (52) is also given.

Figure 4 illustrates the modulation of the exponential decay of the probabilities P_n by a periodic amplitude \mathcal{C}_n of the pattern length n with period p (see (47)).

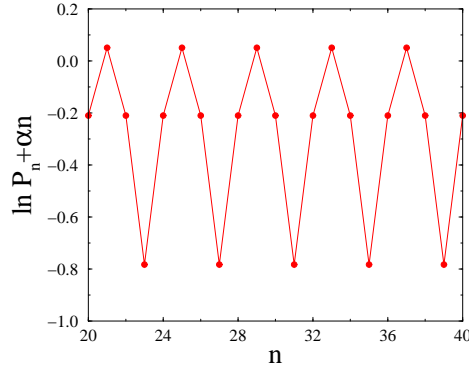


Figure 4: Plot of $\ln P_n + \alpha n \approx \ln \mathcal{C}_n$ for 4-alternating patterns against the pattern length n in some range, illustrating the modulation of the exponential decay of the probabilities P_n by the periodic amplitude \mathcal{C}_n (see (47)).

7.4. *Behavior of the rate as a function of p*

It is of interest to investigate the behavior of the rate α as a function of the period p of the p -alternating patterns.

In the regime where p is large, it is legitimate to approximate the full generalized trigonometric function $T_{p,0}$ by the first two terms of its series expansion (B.15), i.e., $T_{p,0}(z) \approx 1 - z^p/p!$. We thus obtain the estimates

$$z_0 \approx (p!)^{1/p}, \quad \alpha \approx \frac{\ln p!}{p}, \quad (55)$$

implying the asymptotic logarithmic growth

$$\alpha \approx \ln p - 1. \tag{56}$$

Figure 5 shows a plot of the rate α against p . The large- p approximation is observed to be extremely accurate, except for $p = 2$ and 3. Corrections to (55), due to higher-order terms in the expansion (B.15), are indeed exponentially small in p .

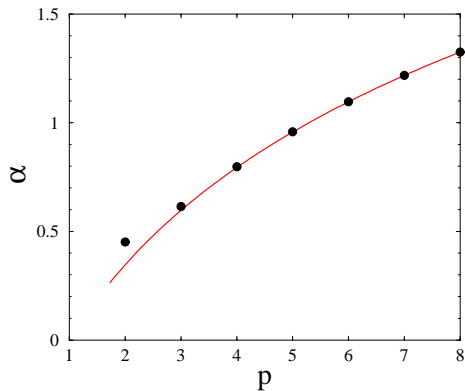


Figure 5: Rate α for p -alternating patterns against their period p . Black symbols: exact values. Full red line: result (55) of the large- p approximation.

8. Arbitrary periodic patterns

We now turn to the analysis of an arbitrary periodic pattern with period $p \geq 2$, whose motif consists of $p - \nu$ rises and ν falls. If $\nu = 0$ or $\nu = p$, the pattern is one of the straight ones discussed in Section 5. Without loss of generality, we can assume $1 \leq \nu \leq p/2$ (otherwise exchange the roles of rises and falls). Finally, as we are chiefly interested in the rate α , we can use cyclic invariance (see point (ii) below (48)) to ensure that the motif ends with a fall.

8.1. General form of the solution

The generating-series method of Section 6.3 extends to the general case as follows. Using again the splitting (42) of the generating series $F(z, x)$, the recursion (4) translates to the differential equation

$$\frac{\partial^p F_0}{\partial x^p} = (-1)^\nu z^p F_0, \tag{57}$$

with $F_0(z, 1) = 1$, while the $p - 1$ other boundary conditions ($q = 1, \dots, p - 1$) read:

$$\begin{aligned} \text{If } \varepsilon_{p-q} = +, \text{ then } \partial^q F_0 / \partial x^q(z, 0) &= 0, \\ \text{If } \varepsilon_{p-q} = -, \text{ then } \partial^q F_0 / \partial x^q(z, 1) &= 0. \end{aligned} \tag{58}$$

The differential equation (57) appears in [24] and [28]. Let us show that its solution can be simply expressed in terms of the generalized hyperbolic and trigonometric functions introduced in Appendix B. Consider for definiteness the case where ν is even. We are led to look for $F_0(z, x)$ as a linear combination of generalized hyperbolic functions:

$$F_0(z, x) = \sum_q C_q(z) H_{p,q}(zx), \quad (59)$$

where the sum runs over the ν indices q such that $\varepsilon_{p-q} = -$. Imposing the boundary conditions at $x = 1$ yields a system of ν linear equations for the coefficients $C_q(z)$. The key object is the denominator $\Delta(z)$ of the solution to the latter system. It is a $\nu \times \nu$ determinant of generalized hyperbolic functions $H_{p,q}(z)$. If ν is odd, the same holds with generalized trigonometric functions. In both cases, $\Delta(z)$ is an entire function of z^p . As a consequence, the asymptotic decay of the probabilities P_n is given by the formulas (47), (48) in full generality, where z_0 is the smallest real positive zero of $\Delta(z)$.

8.2. An explicit example: the motif (+ + --)

Let us illustrate the above formalism on the example of the motif (+ + --). We have $p = 4$ and $\nu = 2$. The expansion (59) reads

$$F_0(z, x) = C_0(z) H_{4,0}(zx) + C_1(z) H_{4,1}(zx). \quad (60)$$

The boundary conditions at $x = 1$ are

$$\begin{aligned} C_0(z) H_{4,0}(z) + C_1(z) H_{4,1}(z) &= 1, \\ C_0(z) H_{4,3}(z) + C_1(z) H_{4,0}(z) &= 0. \end{aligned} \quad (61)$$

We thus obtain

$$C_0(z) = \frac{H_{4,0}(z)}{\Delta(z)}, \quad C_1(z) = -\frac{H_{4,3}(z)}{\Delta(z)}, \quad (62)$$

where $\Delta(z)$ is the determinant of the 2×2 system (61), i.e.,

$$\Delta(z) = H_{4,0}(z)^2 - H_{4,1}(z) H_{4,3}(z) = \frac{1}{2}(\cosh z \cos z + 1). \quad (63)$$

We obtain after some algebra

$$\begin{aligned} 2z\Delta(z)\Pi(z) &= (\sin z - \cos z + 1)(\cosh z - 1) \\ &+ (\sin z + \cos z + 1) \sinh z. \end{aligned} \quad (64)$$

We have

$$z_0 = 1.875104\dots, \quad \alpha = 0.628664\dots \quad (65)$$

The resulting sequence of numbers of permutations

$$\begin{aligned} A_0 &= 1, & A_1 &= 1, & A_2 &= 1, & A_3 &= 3, & A_4 &= 6, & A_5 &= 26, \\ A_6 &= 71, & A_7 &= 413, & A_8 &= 1456, & A_9 &= 10576, & \dots & & & \end{aligned} \quad (66)$$

is referred to as sequence A058258 in the OEIS [34], where an expression equivalent to (64) is also given.

The four examples (27), (51), (54), and (66) seem to exhaust the list of sequences of numbers of permutations with prescribed up-down signatures which are given in the OEIS [34], together with the corresponding exponential generating series. The sequence of numbers of 5-alternating permutations is also given there as sequence A181936, but without the corresponding generating series. The latter is given by (46) for $p = 5$. This result is however not very useful from a merely computational viewpoint, as the expressions of generalized hyperbolic and trigonometric functions for periods $p > 4$ are rather cumbersome.

8.3. Two falls per motif

Let us consider a general pattern with period p whose motif contains two falls separated by distances a and b . We have therefore $\varepsilon_q = +$ except $\varepsilon_a = \varepsilon_{a+b} = -$, and so $p = a + b$ and $\nu = 2$.

The formalism of Section 8.1 leads to

$$\Delta(z) = \begin{vmatrix} H_{p,0}(z) & H_{p,a}(z) \\ H_{p,b}(z) & H_{p,0}(z) \end{vmatrix} = H_{p,0}(z)^2 - H_{p,a}(z)H_{p,b}(z). \quad (67)$$

It is again worth investigating the behavior of the rate α in the regime where a and b are large. Following the lines which led to (55), we thus obtain the estimate

$$z_0 \approx (a!b!)^{1/p}. \quad (68)$$

We have thus again a logarithmic growth law, of the form

$$\alpha \approx \ln p_{\text{eff}} - 1, \quad p_{\text{eff}} = (a^a b^b)^{1/p}. \quad (69)$$

The effective period p_{eff} interpolates between $p_{\text{eff}} \approx p$ if the falls are close to each other (i.e., $a \ll p$ or $b \ll p$) and $p_{\text{eff}} \approx p/2$ if a and b are nearly equal (i.e., $a \approx b \approx p/2$).

8.4. Three falls per motif

Consider a pattern with period p whose motif contains three falls separated by distances a , b and c . We have therefore $p = a + b + c$ and $\nu = 3$. The formalism of Section 8.1 leads to

$$\Delta(z) = \begin{vmatrix} T_{p,0}(z) & T_{p,c}(z) & T_{p,b+c}(z) \\ -T_{p,a+b}(z) & T_{p,0}(z) & T_{p,b}(z) \\ -T_{p,a}(z) & -T_{p,a+c}(z) & T_{p,0}(z) \end{vmatrix}. \quad (70)$$

Notice that all the minus signs are located in the lower triangle.

8.5. Four falls per motif

Consider a pattern with period p whose motif contains four falls separated by distances a , b , c and d . We have $p = a + b + c + d$ and $\nu = 4$. The formalism of Section 8.1 leads to

$$\Delta(z) = \begin{vmatrix} H_{p,0}(z) & H_{p,d}(z) & H_{p,c+d}(z) & H_{p,b+c+d}(z) \\ H_{p,a+b+c}(z) & H_{p,0}(z) & H_{p,c}(z) & H_{p,b+c}(z) \\ H_{p,a+b}(z) & H_{p,a+b+d}(z) & H_{p,0}(z) & H_{p,b}(z) \\ H_{p,a}(z) & H_{p,a+d}(z) & H_{p,a+c+d}(z) & H_{p,0}(z) \end{vmatrix}. \quad (71)$$

The general structure of our prediction for the rate of arbitrary periodic patterns clearly emerges from the expressions (67), (70) and (71).

8.6. Summary

Let us summarize our findings. For an arbitrary periodic pattern, with any period $p \geq 2$, the probabilities P_n have an asymptotic exponential decay, modulated by a periodic function of the pattern length n with period p (see (47)). The rate α is given by (48) in terms of the smallest real positive zero z_0 of a $\nu \times \nu$ determinant $\Delta(z)$ of generalized hyperbolic or trigonometric functions, whose size ν is at most $p/2$. The rate is invariant under cyclic permutations of the motif, under the exchange of rises and falls (up-down symmetry) and under reversal (left-right symmetry). Finally, any two periodic patterns are related by one of the above symmetries, they share the same rate, but different periodic modulations in general.

Table 2 gives a list of irreducible motifs up to period $p = 7$, with the exact numerical values of the corresponding rates. For each period, all symmetries have been used in order to identify a minimal set of patterns. The motifs thus obtained have been ordered according to increasing values of α . For all motifs with periods up to $p = 4$, where analytical results have been derived, the Table also lists the number of the equation giving the generating series $\Pi(z)$ and the OEIS reference [34] of the sequence A_n of numbers of permutations.

9. Deterministic aperiodic patterns

We have seen that the probabilities P_n of periodic patterns have an exponential decay, modulated by an oscillatory function of the pattern length n . The same property extends to a much wider class of patterns.

This holds in particular for patterns which are built from aperiodic sequences, such as e.g. the Fibonacci sequence. An interesting class of deterministic aperiodic sequences are the self-similar sequences generated by substitutions on a finite alphabet [38]. These sequences exhibit an intermediate degree of order between periodic and random. Investigations of the properties of various physical models defined on such sequences are reviewed in [39].

We shall successively consider three classic examples of such sequences: Fibonacci, Thue-Morse, and Rudin-Shapiro (see [38] for historical references and

$p = 2$

motif	α	equation	OEIS
(+-)	0.451582	(37)	A000111

$p = 3$

motif	α	equation	OEIS
(++-)	0.615084	(49)	A178963

$p = 4$

motif	α	equation	OEIS
(++--)	0.628664	(64)	A058258
(+++)	0.798156	(52)	A178964

$p = 5$

motif	α	motif	α
(++-+-)	0.542722	(++++-)	0.958296
(++++)	0.740839		

$p = 6$

motif	α	motif	α
(++-+-)	0.581879	(++++-)	0.866884
(++++)	0.669441	(+++++)	1.096722
(++++)	0.799654		

$p = 7$

motif	α	motif	α
(++-+-)	0.516159	(++++-)	0.797400
(++++)	0.619535	(++++-)	0.889929
(++++)	0.674316	(+++++)	0.988659
(++++)	0.718458	(+++++)	1.217921

Table 2: List of irreducible motifs of periodic patterns up to period $p = 7$, with exact numerical values of the rate α characterizing the asymptotic exponential decay of the probabilities P_n . The equation number of the corresponding generating series $\Pi(z)$ and the OEIS reference [34] of the sequence A_n of numbers of permutations are also given when applicable ($p \leq 4$).

details). In each case, we demonstrate by means of exact numerical calculations that the probabilities P_n exhibit an exponential decay, with a well-defined rate α , modulated by an aperiodic amplitude, either bounded or very slowly increasing, whose fractal structure reflects the self-similarity of the underlying sequence (see Figures 6, 7, and 8).

Fibonacci sequence.

It is generated by the substitution

$$S_{\text{Fib}} : \begin{cases} A \rightarrow AB \\ B \rightarrow A \end{cases} \quad (72)$$

on two letters. Starting with A and iterating the above rules, we obtain the Fibonacci sequence $ABAABABAABAAB \dots$. This sequence is quasiperiodic, and provides a one-dimensional analogue of the icosahedral quasicrystals discovered in 1984 [40].

Interpreting every letter A as a rise ($\varepsilon = +$) and every letter B as a fall ($\varepsilon = -$), we have thus constructed a family of patterns of any length n . Figure 6 shows a plot of exact numerical data for the quantity $\ln P_n + \alpha_{\text{Fib}}n$ for patterns with length up to 1000. The following accurate value of the rate has been obtained by fitting the data:

$$\alpha_{\text{Fib}} = 0.562168\dots \quad (73)$$

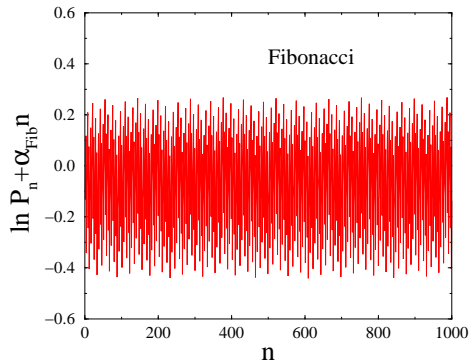


Figure 6: Plot of $\ln P_n + \alpha_{\text{Fib}}n$ against length n for Fibonacci patterns.

Thue-Morse sequence.

It is generated by the substitution

$$S_{\text{TM}} : \begin{cases} A \rightarrow AB \\ B \rightarrow BA \end{cases} \quad (74)$$

which again acts on two letters. Starting with A and iterating the above rules, we obtain the Thue-Morse sequence $ABBABAABBAABABBA \dots$. This sequence has many specific properties, including a purely singular continuous Fourier transform [38]. Figure 7 shows a plot of the quantity $\ln P_n + \alpha_{\text{TM}}n$ for the patterns thus defined, with

$$\alpha_{\text{TM}} = 0.583018\dots \quad (75)$$

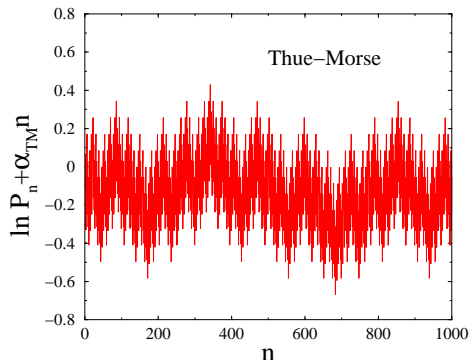


Figure 7: Plot of $\ln P_n + \alpha_{\text{TM}}n$ against length n for Thue-Morse patterns.

Rudin-Shapiro sequence.

It is generated by the substitution

$$S_{\text{RS}} : \begin{cases} A \rightarrow AC \\ B \rightarrow DC \\ C \rightarrow AB \\ D \rightarrow DB \end{cases} \quad (76)$$

acting on four letters. Starting with A and iterating the above rules, we obtain the Rudin-Shapiro sequence $ACABACDCACABDBAB\dots$. The rule is now to read every A or C as a rise ($\varepsilon = +$) and every B or D as a fall ($\varepsilon = -$). The binary sequence thus obtained again has many peculiar properties [38]. Figure 8 shows a plot of the quantity $\ln P_n + \alpha_{\text{RS}}n$ for the patterns thus defined, with

$$\alpha_{\text{RS}} = 0.780693\dots \quad (77)$$

10. Chirping patterns

We have seen that the two straight patterns exhibit a factorial decay of the probabilities P_n (see (21)), formally corresponding to an infinite decay rate. More generally, patterns consisting mostly of rises, whereas falls become more and more scarce (or vice versa), can be expected to yield a super-exponential decay of the probabilities. We refer to these patterns as chirping, because the density of falls slowly goes to zero. Besides being a birdsong, a *chirp* is indeed also a signal whose frequency varies slowly in time.

Consider a chirping pattern consisting mostly of rises. The position $n(k)$ of the k th fall grows faster than linearly in k , and so the distance $p(k)$ between the k th fall and the $(k-1)$ st one grows indefinitely with k . If p were a large fixed number, the pattern would be periodic, and so the probabilities P_n would decay exponentially, with a rate $\alpha \approx \ln p - 1$ growing logarithmically with the

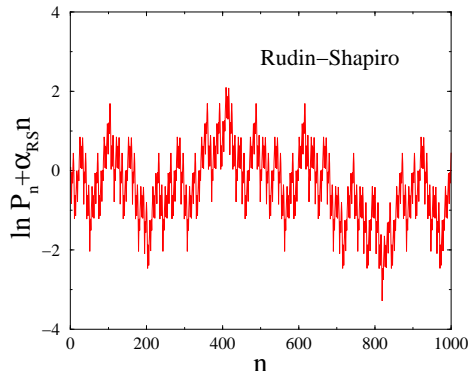


Figure 8: Plot of $\ln P_n + \alpha_{RS} n$ against length n for Rudin-Shapiro patterns.

period p (see (56)). Now, in the presence of a slowly varying ‘period’ $p(k)$, it seems natural to estimate the probabilities as

$$\ln P_n \approx - \sum_{n(k) < n} p(k) (\ln p(k) - 1). \quad (78)$$

This estimate generalizes the result (68). It also agrees with an exact upper bound for P_n , which has been conjectured to be asymptotically exact in the regime where all the distances $p(k)$ are large [25].

The above prediction can be made more precise in the case of a power-law scaling

$$n(k) \approx C k^b, \quad (79)$$

with a scaling exponent $b > 1$, so that the distance $p(k)$ diverges itself as a power law:

$$p(k) \approx bC k^{b-1}. \quad (80)$$

Evaluating the sum in (78) as an integral, we thus obtain a super-exponential decay of the form

$$\ln P_n \approx -n \left(\frac{b-1}{b} \ln n + \ln b - 2 + \frac{1 + \ln C}{b} \right). \quad (81)$$

The associated rate formally diverges logarithmically, as

$$\alpha(n) \approx \frac{b-1}{b} \ln n. \quad (82)$$

As the scaling exponent b can take any value in the range $b > 1$, there is a continuum of logarithmically divergent effective rates $\alpha(n)$, bounded by the worst case of the straight patterns (see (22)).

We have checked the above prediction against exact numerical data in the following two cases.

Square chirping patterns.

Falls occur at places given by the squares of the integers: $n = k^2$ ($k = 1, 2, \dots$). We have $b = 2$, $C = 1$, and so (81) reads

$$\ln P_n \approx -\frac{n}{2}(\ln(4n) - 3). \quad (83)$$

Triangular chirping patterns.

Falls occur at places given by the triangular numbers: $n = k(k + 1)/2$ ($k = 1, 2, \dots$). We have $b = 2$, $C = 1/2$, and so (81) reads

$$\ln P_n \approx -\frac{n}{2}(\ln(2n) - 3). \quad (84)$$

Figure 9 shows a logarithmic plot of the probabilities P_n in both cases. The data exhibit a super-exponential decay which is correctly described by the asymptotic results (83), (84) (dashed lines), together with an undulation induced by the distribution of falls.

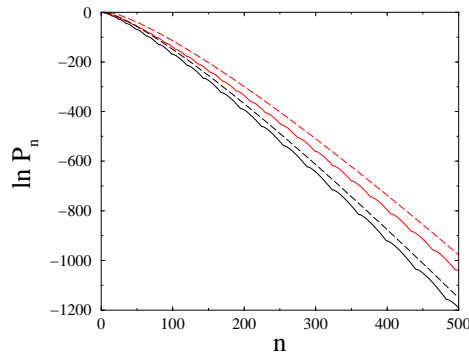


Figure 9: Logarithmic plot of the probabilities P_n of chirping patterns against length n . Full undulating lines: exact data. Lower (black): square patterns. Upper (red): triangular patterns. Dashed lines: asymptotic results (83), (84).

11. Random patterns

11.1. Uniform patterns: multifractal properties

We now turn to the statistical analysis of the probabilities P_n of observing random patterns chosen in various ensembles. In this Section we consider a pattern chosen uniformly among the 2^n patterns $\varepsilon_1 \dots \varepsilon_n$ of fixed length n .

Figure 10 shows a plot of $-\ln P_{12}$ for the 4096 patterns of length $n = 12$, listed in lexicographical order. This plot gives a picture of the behavior of the rate α as a function of the pattern. We have indeed $-\ln P_{12} \approx 12\alpha$. The rate α is observed to exhibit a very erratic behavior, with structures at all scales.

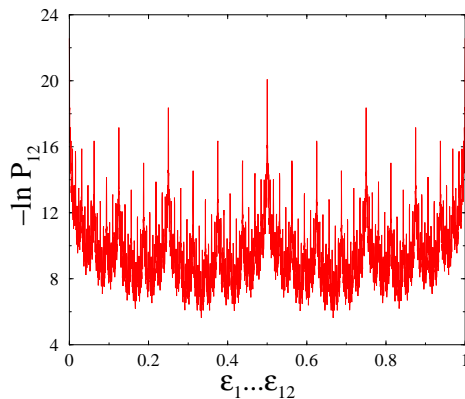


Figure 10: Plot of $-\ln P_{12} \approx 12\alpha$ of all patterns of length $n = 12$ in lexicographical order.

This suggests that *multifractal analysis* [41, 42, 43] may provide the appropriate framework for a quantitative analysis of these data.

Let us recall the basics of the multifractal formalism in a framework adapted to the present case. The key object is the partition function

$$Z_n(q) = 2^n \langle P_n^q \rangle_{\epsilon_1 \dots \epsilon_n} = \sum_{\epsilon_1 \dots \epsilon_n} P_n(\epsilon_1 \dots \epsilon_n)^q, \quad (85)$$

where we have introduced the notation

$$\langle X \rangle_{\epsilon_1 \dots \epsilon_n} = \frac{1}{2^n} \sum_{\epsilon_1 \dots \epsilon_n} X(\epsilon_1 \dots \epsilon_n). \quad (86)$$

The real parameter q plays the role of the inverse temperature β , while the role of the energy of the configuration $\epsilon_1 \dots \epsilon_n$ is played by $-\ln P_n(\epsilon_1 \dots \epsilon_n)$.

If $q = 2, 3, \dots$ is a positive integer, $Z_n(q)$ can be interpreted as the probability that q random permutations, chosen independently and uniformly, have the same up-down signature, i.e., yield the same pattern [21].

The set $\{P_n\}$ is said to be multifractal if the partition function obeys an exponential law of the form

$$Z_n(q) \sim 2^{-n\tau(q)}, \quad (87)$$

at least in some range of values of q . The function $\tau(q)$ is the analogue of a free energy. The normalization of the probabilities implies $Z_n(1) = 1$, hence the obvious result

$$\langle P_n \rangle_{\epsilon_1 \dots \epsilon_n} = 2^{-n}, \quad (88)$$

and $\tau(1) = 0$. We set

$$\tau(q) = (q - 1)D_q, \quad (89)$$

where D_q is the generalized (Rényi) dimension of order q . If all the 2^n patterns of length n had equal probabilities $P_n = 2^{-n}$, we would have $Z_n(q) = 2^{-n(q-1)}$, and therefore $D_q = 1$ for all q .

The scaling law (87) is commonly interpreted in terms of a multifractal spectrum of rates α . For a fixed α , consider the set $\mathcal{N}(\alpha, d\alpha)$ of patterns $\varepsilon_1 \dots \varepsilon_n$ such that $n\alpha < -\ln P_n(\varepsilon_1 \dots \varepsilon_n) < n(\alpha + d\alpha)$. In a generic multifractal situation, the set $\mathcal{N}(\alpha, d\alpha)$ has a well-defined dimension $f(\alpha)$, meaning that its size (the number of its elements) grows exponentially as

$$|\mathcal{N}(\alpha, d\alpha)| \sim 2^{nf(\alpha)}. \quad (90)$$

The partition function may thus be estimated as

$$Z_n(q) \sim \int_0^\infty e^{-nq\alpha} 2^{nf(\alpha)} d\alpha. \quad (91)$$

Evaluating the integral by the saddle-point method, we obtain the property that the functions $\tau(q)$ and $f(\alpha)$ are related to each other by a Legendre transform:⁴

$$\tau(q) + f(\alpha) = \frac{q\alpha}{\ln 2}, \quad q = \ln 2 f'(\alpha), \quad \alpha = \ln 2 \tau'(q). \quad (92)$$

In the present case, exact numerical results demonstrate in an unambiguous way that the scaling law (87) holds for $q > 0$, i.e., positive temperatures. This observation corroborates and extends the findings of Mallows and Shepp [21], who have established rigorously that the scaling law (87) holds whenever $q = 2, 3, \dots$ is a positive integer.

For negative temperatures, i.e., $q < 0$, the growth of the partition function is asymptotically governed by the patterns whose probabilities are the smallest, i.e., the two straight ones (see (21)). The partition function therefore grows super-exponentially as $Z_n(q) \approx 2((n+1)!)^{|q|}$, for any negative value of q . This behavior leads to the breakdown of the multifractal formalism. This is not an artifact which could be circumvented easily. Indeed, there is actually a continuum of patterns, including all the chirping ones, which yield a logarithmically diverging effective rate $\alpha(n)$ (see (82)). More generally, any quantity which has a high sensitivity to the smallest of the probabilities P_n will be affected by logarithmic violations to scaling.

Let us proceed and describe quantitative results.

For $q = 0$, we have $Z_n(0) = 2^n$, hence $\tau(0) = -1$, and so the support dimension takes the obvious value $D_0 = 1$. More interestingly, taking the derivative of (87) at $q = 0$, we obtain

$$\langle \ln P_n \rangle_{\varepsilon_1 \dots \varepsilon_n} = 2^{-n} Z_n'(0) \approx -n\alpha_0. \quad (93)$$

The very accurate numerical value

$$\alpha_0 = \tau'(0) \ln 2 = 0.806361 \dots \quad (94)$$

⁴Here and in the following, primes denote derivatives.

has been obtained by an exact numerical evaluation of $\langle \ln P_n \rangle_{\varepsilon_1 \dots \varepsilon_n}$ up to $n = 30$. The very fast convergence of the data toward the asymptotic linear law (93) is illustrated in Figure 11. A similar kind of convergence is observed for all subsequent quantities.

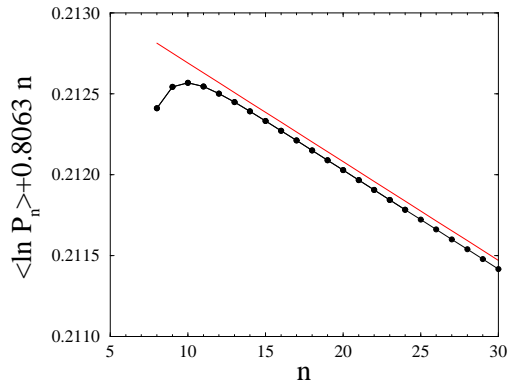


Figure 11: Plot of $\langle \ln P_n \rangle_{\varepsilon_1 \dots \varepsilon_n} + 0.8063n$ against n up to $n = 30$. The red line has slope -61×10^{-6} .

Considering higher-order derivatives of (87) at $q = 0$, we conclude that all the cumulants of $\ln P_n$ are *extensive*, in the sense that they grow asymptotically linearly with n . We have in particular

$$\text{var}_{\varepsilon_1 \dots \varepsilon_n} \ln P_n = \langle (\ln P_n)^2 \rangle_{\varepsilon_1 \dots \varepsilon_n} - \langle \ln P_n \rangle_{\varepsilon_1 \dots \varepsilon_n}^2 \approx nw_0, \quad (95)$$

with

$$w_0 = -\tau''(0) \ln 2 = 0.435600 \dots \quad (96)$$

As a consequence, and in physical terms, the typical value α_0 of the rate is self-averaging. In other words, we have

$$\ln P_n \approx -n\alpha_0 \quad (97)$$

for almost all (long enough) patterns.

Interestingly enough, the typical rate α_0 can also be interpreted as the Lyapunov exponent of the random dynamical system defined by the recursion (4). The *mean value* of the function $f_n(x)$, obtained by averaging at each step the recursion (4) over both values of ε_n , has the simple expression $\langle f_n(x) \rangle_{\varepsilon_1 \dots \varepsilon_n} = 2^{-n}$, in agreement with the simple result $\langle P_n \rangle_{\varepsilon_1 \dots \varepsilon_n} = 2^{-n}$ (see (88)). The *typical value* of the function $f_n(x)$ however keeps fluctuating in a non-trivial way, and it falls off as $f_n(x) \sim e^{-n\alpha_0}$, i.e., exponentially faster than the mean value, as $\alpha_0 - \ln 2 = 0.113214 \dots > 0$.

For $q = 1$, as already mentioned, the normalization of the probabilities P_n ensures $Z_n(1) = 1$. More interestingly, taking the derivative of (87) at $q = 1$,

we predict that the entropy Σ_n of the set $\{P_n\}$ grows as

$$\Sigma_n = - \sum_{\varepsilon_1 \dots \varepsilon_n} P_n(\varepsilon_1 \dots \varepsilon_n) \ln P_n(\varepsilon_1 \dots \varepsilon_n) \approx n D_1 \ln 2, \quad (98)$$

with

$$D_1 = \tau'(1) = 0.904475 \dots \quad (99)$$

This number is referred to as the entropy (information) dimension.

Higher integer values of the index q are also of interest. Indeed, as recalled above, the partition function $Z_n(q)$ is equal to the probability that q independent random permutations yield the same pattern. We have $\tau(2) = 0.856199 \dots$ (hence $D_2 = 0.856199 \dots$), $\tau(3) = 1.647144 \dots$ (hence $D_3 = 0.823572 \dots$), and so on. The case of pairs of permutations ($q = 2$) has been investigated by analytical means by Mallows and Shepp [21]. These authors have determined the value of D_2 in terms of the smallest zero of an explicit entire series. Their approach however does not extend to higher values of q .

In the $q \rightarrow \infty$ limit, the growth of the partition function is governed by the most probable patterns, i.e., the alternating patterns, with rate $\alpha_{\min} = \ln(\pi/2)$ (see (41)). We thus get

$$D_\infty = \frac{\alpha_{\min}}{\ln 2} = \frac{\ln(\pi/2)}{\ln 2} = 0.651496 \dots \quad (100)$$

The main outcome of multifractal analysis is given in Figure 12, showing (left) the generalized dimensions D_q (for $q > 0$) against $q/(q+1)$ and (right) the multifractal spectrum $f(\alpha)$ against α . The latter only makes sense in the range $\alpha_{\min} \leq \alpha \leq \alpha_0$, where $f(\alpha)$ grows from $f(\alpha_{\min}) = 0$ to $f(\alpha_0) = 1$.

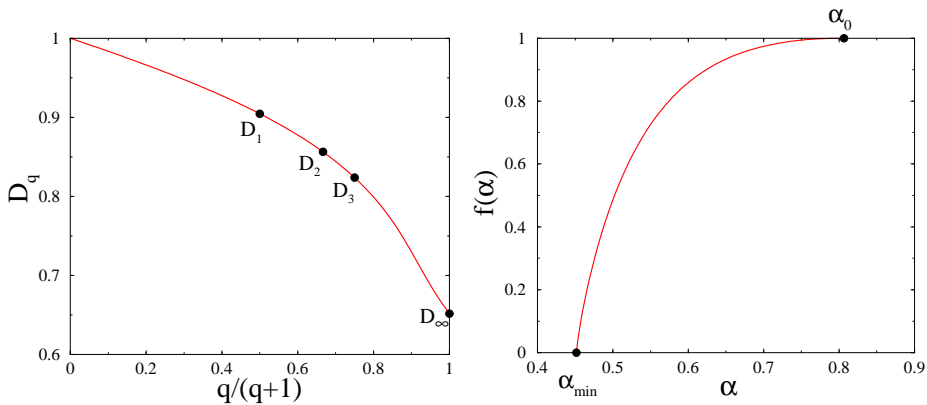


Figure 12: Left: generalized dimensions D_q against $q/(q+1)$. Right: multifractal spectrum $f(\alpha)$ against α in the range $\alpha_{\min} < \alpha < \alpha_0$.

11.2. Patterns at fixed concentration of rises

For uniformly chosen random patterns of length n , we have seen that the logarithm of the probability P_n is self-averaging and characterized by the typical rate α_0 . The same self-averaging property holds for more general ensembles of random patterns.

A first interesting example consists in imposing the concentration c of rises, i.e., in choosing at every place either a rise with probability c , or a fall with the complementary probability:

$$\varepsilon_n = \begin{cases} + & \text{with probability } c, \\ - & \text{with probability } 1 - c. \end{cases} \quad (101)$$

Within this ensemble, the logarithm of the probability P_n is again self-averaging, i.e., we have

$$\ln P_n \approx -n\beta(c), \quad (102)$$

where the effective typical rate $\beta(c)$ now depends on the concentration c of rises. Figure 13 shows a plot of this quantity. Each data point is obtained by averaging $\ln P_n$ over 10^5 independent patterns of length $n = 200$.

For $c = 1/2$, the uniform ensemble is recovered, and so the rate takes its minimal value $\beta(1/2) = \alpha_0 = 0.806361\dots$. As c goes to 0 (resp. 1), distances between consecutive rises (resp. falls) become large. More precisely, these distances are exponentially distributed, with a mean value approximately equal to $1/c$ (resp. $1/(1-c)$). Following the line of thought which led us to (78), we thus obtain

$$\beta(c) \approx -\ln(c(1-c)) - \mathbf{C} \quad (c \rightarrow 0 \text{ or } 1), \quad (103)$$

where \mathbf{C} denotes Euler's constant. This estimate provides a surprisingly good description of the effective rate over the whole range of concentrations.

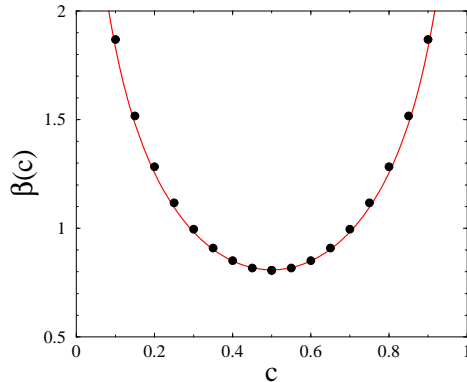


Figure 13: Effective typical rate $\beta(c)$ of the ensemble of random patterns at fixed concentration c of rises, against c . Black symbols: numerical data. Full red line: estimate (103).

11.3. Symmetric Markovian patterns

Another interesting example consists of the patterns where rises and falls are equally probable but correlated. The null model for this case is the symmetric Markovian ensemble, where ε_n is equal to ε_{n-1} with some persistence probability r , and to its opposite with the complementary probability:

$$\varepsilon_n = \begin{cases} +\varepsilon_{n-1} & \text{with probability } r, \\ -\varepsilon_{n-1} & \text{with probability } 1 - r. \end{cases} \quad (104)$$

The logarithm of the probability P_n is again self-averaging, i.e., we have

$$\ln P_n \approx -n\gamma(r), \quad (105)$$

where the effective typical rate $\gamma(r)$ depends on the persistence probability r . Figure 14 shows a plot of this quantity.

For $r = 1/2$, the uniform ensemble is again recovered, and so we have $\gamma(1/2) = \alpha_0 = 0.806361\dots$. As r goes to 0, a rise is followed by a fall with very high probability, and vice versa. As a consequence, a typical pattern of the ensemble consists of long alternating stretches, and so $\gamma(r)$ goes to $\alpha_{\min} = \ln(\pi/2)$ (see (41)). In the opposite limit ($r \rightarrow 1$), a typical pattern consists of long ordered stretches of rises and falls, whose lengths are again exponentially distributed, with a mean value scaling as $1/(1-r)$, and so

$$\gamma(r) \approx -\ln(1-r) - \mathbf{C} \quad (r \rightarrow 1), \quad (106)$$

where \mathbf{C} again denotes Euler's constant.

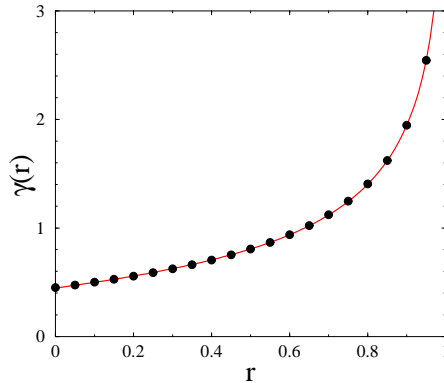


Figure 14: Effective typical rate $\gamma(r)$ of the symmetric Markovian ensemble of random patterns, against the persistence probability r . Black symbols: numerical data. Full red line: two-parameter fit incorporating the estimate (106).

Appendix A. Explicit correspondence between the probabilistic and combinatorial approaches

This appendix presents the explicit correspondence between the probabilistic approach and the combinatorial one. For a given pattern $\varepsilon_1 \dots \varepsilon_n$ of n rises and falls, the probabilistic approach (see Section 3) involves the n th degree polynomial $f_n(x)$, which has $n+1$ coefficients, while the combinatorial one (see Section 4) involves the $n+1$ integers $a_{n,j}$. These two sets of numbers carry the same information.

The precise correspondence (15) between both descriptions can be established as follows. Consider a permutation σ yielding the pattern $\varepsilon_1 \dots \varepsilon_n$ and such that $\sigma_n = j$. The number x_n is therefore the $(j+1)$ st one of the random numbers x_0, \dots, x_n written in increasing order. In other words, $f_{n,j}(x) dx$ is the probability of the following event: j numbers are in the interval $[0, x]$, one number is in the interval $[x, x+dx]$, and the remaining $n-j$ numbers are in the interval $[x+dx, 1]$. The latter probability is given by the multinomial formula

$$f_{n,j}(x) dx = \frac{(n+1)!}{j!1!(n-j)!} x^j (dx)^1 (1-x-dx)^{n-j}, \quad (\text{A.1})$$

i.e. (to first order in dx),

$$f_{n,j}(x) = \frac{(n+1)!}{j!(n-j)!} x^j (1-x)^{n-j}. \quad (\text{A.2})$$

By definition, there are $a_{n,j}$ permutations σ such that $\sigma_n = j$, among a total of $(n+1)!$. Summing the expression (A.2) over j with the weights $a_{n,j}/(n+1)!$, we get the result announced in (15), i.e.,

$$f_n(x) = \sum_{j=0}^n a_{n,j} \frac{x^j (1-x)^{n-j}}{j!(n-j)!}. \quad (\text{A.3})$$

The inverse formula reads

$$a_{n,j} = j!(n-j)! \oint \frac{dx}{2\pi i} \frac{f_n(x)}{x^{j+1}(1-x)^{n-j+1}}, \quad (\text{A.4})$$

where the integration contour circles once around the point $x=0$.

In order to be complete, let us check explicitly that the recursions (4) and (10) are equivalent to each other. Assume the probabilistic recursion (4) holds, and consider the case where $\varepsilon_n = +$ (the other case can be dealt with in a similar way). We have $f_n(0) = 0$, implying $a_{n,0} = 0$, and $f_{n-1}(x) = f'_n(x)$. Now consider the difference $d_{n,j} = a_{n,j} - a_{n,j-1}$ for $j = 1, \dots, n$. The formula (A.4) yields

$$d_{n,j} = \oint \frac{dx}{2\pi i} f_n(x) \left(\frac{j!(n-j)!}{x^{j+1}(1-x)^{n-j+1}} - \frac{(j-1)!(n-j+1)!}{x^j(1-x)^{n-j+2}} \right)$$

$$\begin{aligned}
&= \oint \frac{dx}{2\pi i} f_n(x) \left(-\frac{d}{dx} \frac{(j-1)!(n-j)!}{x^j(1-x)^{n-j+1}} \right) \\
&= \oint \frac{dx}{2\pi i} \left(\frac{d}{dx} f_n(x) \right) \frac{(j-1)!(n-j)!}{x^j(1-x)^{n-j+1}} \quad (\text{by parts}) \\
&= \oint \frac{dx}{2\pi i} f_{n-1}(x) \frac{(j-1)!(n-j)!}{x^j(1-x)^{n-j+1}} \\
&= a_{n-1, j-1}. \tag{A.5}
\end{aligned}$$

This completes the explicit check of the equivalence between the probabilistic approach and the combinatorial one.

Appendix B. Generalized hyperbolic and trigonometric functions

In this appendix we gather formulas and results on generalized hyperbolic and trigonometric functions, which are used in the study of periodic patterns (sections 7 and 8). Both families of functions are sometimes [16, 44] referred to as Olivier functions [45]. Generalized hyperbolic functions are also described in [46].

Generalized hyperbolic functions.

These functions provide a useful basis of solutions to the differential equation

$$f^{(p)} = f, \tag{B.1}$$

where $p \geq 2$ is a given integer. Looking for a solution of the form e^{ax} , we are left with the condition $a^p = 1$. We thus obtain a basis of p exponential solutions:

$$A_j(x) = e^{x\omega^j} \quad (j = 0, \dots, p-1), \tag{B.2}$$

corresponding to $a = \omega^j$, where

$$\omega = e^{2\pi i/p} \tag{B.3}$$

is the first p th root of unity.

It is advantageous to introduce the linear combinations

$$H_{p,q}(x) = \frac{1}{p} \sum_{j=0}^{p-1} \omega^{-qj} e^{x\omega^j} \quad (q = 0, \dots, p-1). \tag{B.4}$$

The generalized hyperbolic functions thus defined provide another basis of solutions to (B.1). They obey the first-order differential equations

$$H'_{p,q} = H_{p,q-1} \quad (q = 1, \dots, p-1), \quad H'_{p,0} = H_{p,p-1}. \tag{B.5}$$

The power-series expressions

$$H_{p,q}(x) = \sum_{k \geq 0} \frac{x^{kp+q}}{(kp+q)!} \tag{B.6}$$

are obtained by expanding the exponentials in (B.4). Another advantage of these functions is the simple expressions of their Laplace transforms:

$$\widehat{H}_{p,q}(s) = \frac{s^{p-q-1}}{s^p - 1}. \quad (\text{B.7})$$

The first few generalized hyperbolic functions read

$$\begin{aligned} H_{2,0}(x) &= \cosh x, \\ H_{2,1}(x) &= \sinh x, \end{aligned} \quad (\text{B.8})$$

$$\begin{aligned} H_{3,0}(x) &= \frac{1}{3} \left(e^x + 2 e^{-x/2} \cos \frac{x\sqrt{3}}{2} \right), \\ H_{3,1}(x) &= \frac{1}{3} \left(e^x - e^{-x/2} \cos \frac{x\sqrt{3}}{2} + \sqrt{3} e^{-x/2} \sin \frac{x\sqrt{3}}{2} \right), \\ H_{3,2}(x) &= \frac{1}{3} \left(e^x - e^{-x/2} \cos \frac{x\sqrt{3}}{2} - \sqrt{3} e^{-x/2} \sin \frac{x\sqrt{3}}{2} \right), \end{aligned} \quad (\text{B.9})$$

$$\begin{aligned} H_{4,0}(x) &= \frac{1}{2} (\cosh x + \cos x), \\ H_{4,1}(x) &= \frac{1}{2} (\sinh x + \sin x), \\ H_{4,2}(x) &= \frac{1}{2} (\cosh x - \cos x), \\ H_{4,3}(x) &= \frac{1}{2} (\sinh x - \sin x). \end{aligned} \quad (\text{B.10})$$

Generalized trigonometric functions.

The above construction can be transposed to the differential equation

$$f^{(p)} = -f. \quad (\text{B.11})$$

Looking again for a solution of the form e^{ax} , we are left with the condition $a^p = -1$. We thus obtain a basis of p exponential solutions:

$$B_j(x) = e^{x\omega^{j+1/2}} \quad (j = 0, \dots, p-1), \quad (\text{B.12})$$

corresponding to $a = \omega^{j+1/2}$.

We introduce the linear combinations

$$T_{p,q}(x) = \frac{1}{p} \sum_{j=0}^{p-1} \omega^{-q(j+1/2)} e^{x\omega^{j+1/2}} \quad (q = 0, \dots, p-1). \quad (\text{B.13})$$

The generalized trigonometric functions thus defined provide another basis of solutions to (B.11). They obey the first-order differential equations

$$T'_{p,q} = T_{p,q-1} \quad (q = 1, \dots, p-1), \quad T'_{p,0} = -T_{p,p-1}. \quad (\text{B.14})$$

The power-series expressions

$$T_{p,q}(x) = \sum_{k \geq 0} (-1)^k \frac{x^{kp+q}}{(kp+q)!} \quad (\text{B.15})$$

are obtained by expanding the exponentials in (B.13). Finally, their Laplace transforms are

$$\widehat{T}_{p,q}(s) = \frac{s^{p-q-1}}{s^p + 1}. \quad (\text{B.16})$$

The first few generalized trigonometric functions read

$$\begin{aligned} T_{2,0}(x) &= \cos x, \\ T_{2,1}(x) &= \sin x, \end{aligned} \quad (\text{B.17})$$

$$\begin{aligned} T_{3,0}(x) &= \frac{1}{3} \left(e^{-x} + 2e^{x/2} \cos \frac{x\sqrt{3}}{2} \right), \\ T_{3,1}(x) &= \frac{1}{3} \left(-e^{-x} + e^{x/2} \cos \frac{x\sqrt{3}}{2} + \sqrt{3} e^{-x/2} \sin \frac{x\sqrt{3}}{2} \right), \\ T_{3,2}(x) &= \frac{1}{3} \left(e^{-x} - e^{x/2} \cos \frac{x\sqrt{3}}{2} + \sqrt{3} e^{-x/2} \sin \frac{x\sqrt{3}}{2} \right), \end{aligned} \quad (\text{B.18})$$

$$\begin{aligned} T_{4,0}(x) &= \cosh \frac{x}{\sqrt{2}} \cos \frac{x}{\sqrt{2}}, \\ T_{4,1}(x) &= \frac{1}{\sqrt{2}} \left(\cosh \frac{x}{\sqrt{2}} \sin \frac{x}{\sqrt{2}} + \sinh \frac{x}{\sqrt{2}} \cos \frac{x}{\sqrt{2}} \right), \\ T_{4,2}(x) &= \sinh \frac{x}{\sqrt{2}} \sin \frac{x}{\sqrt{2}}, \\ T_{4,3}(x) &= \frac{1}{\sqrt{2}} \left(\cosh \frac{x}{\sqrt{2}} \sin \frac{x}{\sqrt{2}} - \sinh \frac{x}{\sqrt{2}} \cos \frac{x}{\sqrt{2}} \right). \end{aligned} \quad (\text{B.19})$$

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