ASYMPTOTIC FORMULA FOR SYMMETRIC INVOLUTIONS

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The sequence A000898 in OEIS [5]:

1, 2, 6, 20, 76, 312, 1384, 6512, 32400, 168992, ...

enumerates the numbers of symmetric involutions I_{2n}^{rc}, see Egge [2]. It satisfies an obvious recurrence relation:

$$I_0 = 1, \quad I_1 = 2, \quad I_n = 2\big(I_{n-1} + (n-1)I_{n-2}\big), \ \forall\, n \geq 2. \tag{1}$$

Using standard techniques it is easy to convert the recurrence relation to the exponential generating function:

$$F(z) = \exp\big(z^2 + 2z\big). \tag{2}$$

Down the webpage, OEIS refers its asymptotic formula to Robinson's paper [4], but that formula did not yield satisfactory results. The computation for $n = 1000$ shows that Robinson's formula gives the number 8.480×10^{1442}, which is about 3.160×10^{-10} times the actual number 2.684×10^{1452}. In this short note we will give the correct asymptotic formula for I_n.

The main tool we use here is the theorem of Hayman [3], in which one finds the definition for *admissible* functions:

Theorem 1 (Hayman). *If $f(z)$ is an admissible entire function, with power series $\sum a_n z^n$, then*

$$a_n \sim \frac{f(r_n)}{r_n^n \sqrt{2\pi b(r_n)}}, \tag{3}$$

where $r_n (> 0)$ and b are defined by

$$a(r_n) = n, \quad a(r) = r \frac{\mathrm{d}}{\mathrm{d}r} \log f(r), \quad b(r) = r\, a'(r).$$

Here our function $F(z)$ is admissible, hence Theorem 1 is applicable. We have $a(r) = 2r^2 + 2r$, therefore r_n is found to be:

$$r_n = \frac{\sqrt{2n+1} - 1}{2} = \sqrt{\frac{n}{2}} - \frac{1}{2} + \frac{1}{4\sqrt{2n}} + O(n^{-3/2}) \tag{4}$$

as $n \to \infty$.

From this formula (4) for r_n, we compute $f(r_n)$, r_n^n, and $b(r_n)$ as follows:

$$\begin{aligned} f(r_n) &= \exp(r_n^2 + 2r_n) = \exp\Big(\frac{1}{2}(2r_n^2 + 2r_n) + r_n\Big) \\ &= \exp\Big(\frac{n}{2} + \sqrt{\frac{n}{2}} - \frac{1}{2}\Big)\big(1 + \frac{1}{4\sqrt{2n}} + O(n^{-1})\big). \end{aligned}$$

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$$\begin{aligned}\log r_n^n &= n\log\Big(\sqrt{\frac{n}{2}}-\frac12+\frac{1}{4\sqrt{2n}}+O(n^{-3/2})\Big)\\ &= n\Big(\log\sqrt{\frac{n}{2}}+\log\big(1-\frac{1}{\sqrt{2n}}+\frac{1}{4n}+O(n^{-2})\big)\Big)\\ &= n\log\sqrt{\frac{n}{2}}-\sqrt{\frac{n}{2}}+\frac{1}{12\sqrt{2n}}+O(n^{-1}).\end{aligned}$$

$$\begin{aligned}\sqrt{b(r_n)} &= \sqrt{4r_n^2+2r_n}=\sqrt{2(2r_n^2+2r_n)-2r_n}\\ &= \sqrt{2n-\sqrt{2n}+1+O(n^{-1/2})}\\ &= \sqrt{2n}\big(1-\frac{1}{2\sqrt{2n}}+O(n^{-1})\big).\end{aligned}$$

Finally we put everything back and make use of the Stirling's formula to derive from (3) that

$$I_n = \frac{\mathrm{e}^{\sqrt{2n}}}{\sqrt{2\mathrm{e}}}\Big(\frac{2n}{\mathrm{e}}\Big)^{n/2}\big(1+\frac{\sqrt2}{3\sqrt n}+O(n^{-1})\big). \tag{5}$$

Denote the asymptotic formula on the right-hand side of (5) by I_n^*. The following table compares I_n and I_n^* for $n = 10^2, 10^3, 10^4$, and 10^5:

n	10^2	10^3	10^4	10^5
I_n	$1.3506\cdot 10^{99}$	$2.6836\cdot 10^{1452}$	$5.3760\cdot 10^{19394}$	$4.276309\cdot 10^{243530}$
I_n^*	$1.3520\cdot 10^{99}$	$2.6839\cdot 10^{1452}$	$5.3761\cdot 10^{19394}$	$4.276313\cdot 10^{243530}$

The order of the errors should be $O(n^{-1})$.

References

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2. Eric S Egge, *Restricted symmetric permutations*, Annals of Combinatorics **11** (2007), no. 3-4, 405–434.
3. W. K. Hayman, *A generalisation of Stirling's formula*, J. Reine Angew. Math. **196** (1956), 67–95. MR 0080749 (18,293f)
4. Robert W. Robinson, *Counting arrangements of bishops*, Combinatorial mathematics, IV (Proc. Fourth Australian Conf., Univ. Adelaide, Adelaide, 1975), Springer, Berlin, 1976, pp. 198–214. Lecture Notes in Math., Vol. 560. MR 0434835 (55 #7799)
5. N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, A000898, $a(n) = 2(a(n-1)+(n-1)a(n-2))$.

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