

**ON THE SPECTRAL PROPERTIES OF A CLASS OF
 H -SELFADJOINT RANDOM MATRICES AND THE
 UNDERLYING COMBINATORICS**

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ABSTRACT. An expansion of the Weyl function of a H -selfadjoint random matrix with one negative square is provided. It is shown that the coefficients converge to a certain generalization of Catalan numbers. Properties of this generalization are studied, in particular, a combinatorial interpretation is given.

INTRODUCTION

The main object of the present paper are the limit properties of the following product of a random and deterministic matrices

$$X_N^{(d)} = H_N^{(d)} W_N,$$

where W_N stands for the Wigner matrix,

$$H_N^{(d)} = \begin{bmatrix} d & 0 \\ 0 & I_N \end{bmatrix}$$

and d is a nonrandom real parameter. Note that for $d \geq 0$ the matrix $X_N^{(d)}$ is similar to the symmetric matrix

$$H_N^{(\sqrt{d})} W_N H_N^{(\sqrt{d})},$$

and hence, it has real spectrum only. The statistical properties of perturbed symmetric random matrices were studied in many papers e.g. [4, 5, 6, 10, 11, 14, 16] and the theory is well established. However, to our knowledge there is very less known about the spectral properties of $X_N^{(d)}$ in the case $d < 0$. Note that in this case $X_N^{(d)}$ is $H_N^{(1/d)}$ -selfadjoint, which allows to apply the indefinite linear algebra theory. We refer the reader to [15] as a basic reference for H -selfadjoint matrices and to [27], where the indefinite linear algebra was linked with randomness.

To track the (possibly complex) eigenvalues of the matrix $X_N^{(d)}$ the Weyl function

$$Q_{(d)}^N(z) = - \left(e_0^* H_N^{(d)} \left(X_N^{(d)} - z \right)^{-1} e_0 \right)^{-1} = - \frac{1}{d} \left(e_0^* \left(X_N^{(d)} - z \right)^{-1} e_0 \right)^{-1}.$$

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is considered. The zeros of $Q_{(d)}^N(z)$ are necessarily eigenvalues of $X_N^{(d)}$ by the Schur complement reasoning. This approach corresponds to the technique of Weyl function, or m -function in operator theory, see e.g. [12, 13, 17, 18, 20, 25] for the applications in the indefinite inner product setting.

Note that as $X_N^{(d)}$ is a random matrix, the function $Q_{(d)}^N(z)$ is random as well. The main object of the present paper are properties of the function $Q_{(d)}^N(z)$ for large matrices. In particular, Theorem 7 says that $Q_{(d)}^N(z)$ converges in probability with $N \rightarrow \infty$ to

$$Q_{(d)}(z) = \hat{\sigma}(z) + \frac{z}{d} = \frac{(2-d)z + d\sqrt{z^2 - 4}}{2d},$$

where $\hat{\mu}(z)$ denotes the Stieltjes transform of a measure μ and σ the Wigner semi-circle measure. This allows to determine the limit behavior of the eigenvalues of $X_N^{(d)}$ outside $[-2, 2]$ as a functions of the real parameter d . Namely, for $d \in (-\infty, 0)$ the equation $Q_{(d)}(z) = 0$ has two complex solutions

$$z_{(d)}^{\pm} = \pm \frac{d}{\sqrt{1-d}} i.$$

For $d \in [0, 2]$ the equation $Q_{(d)}(z) = 0$ has no solutions outside $[-2, 2]$. For $d \in (2, \infty)$ the equation $Q_{(d)}(z) = 0$ has two real solutions outside $[-2, 2]$

$$z_{(d)}^{\pm} = \pm \frac{d}{\sqrt{d-1}}.$$

Summarizing, it follows that for $d \in (-\infty, 0) \cup (2, \infty)$ there are precisely two eigenvalues of $X_N^{(d)}$ with limits in $\mathbb{C} \setminus [-2, 2]$, the limits being $z_{(d)}^{\pm}$, see Theorem 7 for details.

The idea of the present paper is to provide a combinatorial interpretation of $Q_{(d)}(z)$ in the spirit of the original Wigner's calculations. This includes, in particular, providing the following expansion

$$\frac{-1}{Q_{(d)}(z)} = -\frac{d}{z} \sum_{n=0}^{\infty} \frac{1}{z^{2n}} \pi_n^{(d)}.$$

The numbers $\pi_n^{(d)}$ are a generalization of the Catalan numbers and have a natural interpretation in terms of Dyck paths or noncrossing partitions. They appear in the study of the t -transformation of a measure or a free convolution [8, 9] and deformation of free Gaussian random variables [28, 29]. This issue is further discussed in Section 1 and in the closing remarks.

The paper is organized in a reverse order, compared to the presentation above. In Section 1 we define the numbers $\pi_n^{(d)}$ and show their basic properties: generating function, relation to Catalan numbers and closed formulas as polynomials in d . Section 2 is devoted to computing the limit in probability of the moments of the function $-1/Q_N^{(d)}$. Namely, it is shown in Theorem 5 that $e_0^* \left(X_N^{(d)}\right)^n e_0$ converges to zero if n is odd and to $\pi_{n/2}^{(d)}$ if n is even. The result is then used in Section 3 to prove the aforementioned Theorem 7 on the limit of $Q_{(d)}^N$ and the behavior of the eigenvalues of $X_N^{(d)}$ for large N . In the last section we discuss the limitations of this method of study of spectra of H -selfadjoint random matrices. The main results of the paper are Theorem 5 and Theorem 7.

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1. THE NUMBERS $\pi_n^{(d)}$ AND THEIR RELATION TO CATALANS

A *Dyck path* of order n is a walk from $(0, 0)$ to $(2n, 0)$ in the upper-half plane, consisting of vectors $[1, 1]$ and $[1, -1]$. The set of all Dyck paths of order n will be denoted by D_n . A classical result says that the cardinality c_n of D_n satisfies

$$(1) \quad c_n = \frac{1}{n+1} \binom{2n}{n},$$

which can be seen by the recurrence formula

$$(2) \quad c_0 = 1, \quad c_n = \sum_{j=1}^n c_{j-1} c_{n-j}, \quad n = 1, 2, \dots$$

The numbers c_n are called the *Catalan numbers*.

If w is a Dyck path then by $\xi(w)$ we denote the number of meetings of w with the x -axes, excluding the point $(2n, 0)$. We define

$$(3) \quad \pi_0^{(d)} = 1, \quad \pi_n^{(d)} = \sum_{w \in D_n} d^{\xi(w)}, \quad n = 1, 2, \dots$$

Clearly, $\pi_n^{(1)} = c_n$, for $\pi_n^{(-1)}$ see the end of this section.

The numbers $\pi_n^{(d)}$ can be interpreted in the language of non-crossing 2-partitions as follow. Let $\mathcal{NC}_2(2n)$ denote the set of all non-crossing partitions with two-element blocks only. For a non-crossing partition $\nu \in \mathcal{NC}_2(2n)$ with blocks $\nu = \{B_1, B_2, \dots, B_n\}$ a block $B_j = \{s_1, s_2\}$ is called *outer* if there is no block $B_i = \{s_3, s_4\}$ such that $s_3 < s_1, s_2 < s_4$. Blocks which do not enjoy this property are called *inner*. By $\sharp \text{ou}(\nu)$ ($\sharp \text{in}(\nu)$) we denote the number of outer (resp. inner) blocks of a partition ν . Using this notations we can write

$$(4) \quad \pi_n^{(d)} = \sum_{\nu \in \mathcal{NC}_2(2n)} d^{\sharp \text{ou}(\nu)}.$$

In [8] the numbers

$$(5) \quad C_n(d) = \sum_{\nu \in \mathcal{NC}_2(2n)} d^{\sharp \text{in}(\nu)}$$

were considered as moments of the central limit measure for the t -transformed free convolution, see [7] for a general form of a non-commutative central limit theorem and [9] for other examples. One sees that both $\pi_n^{(d)}$ and $C_n(d)$ are polynomials in d . Furthermore, by $\text{in}(w) + \text{ou}(w) = n$ one has

$$(6) \quad \pi_n^{(d)} = \text{rev}(C_n(d)),$$

where $\text{rev}(\sum_{i=0}^n a_i d^i) = \sum_{i=0}^n a_i d^{n-i}$.

The basic recurrence relation for the numbers $\pi_n^{(d)}$ is the following.

Lemma 1. For $d \in \mathbb{R} \setminus \{0\}$

$$(7) \quad \pi_n^{(d)} = d \sum_{k=1}^n c_{k-1} \pi_{n-k}^{(d)}, \quad n = 1, 2, \dots$$

Proof. We use a standard argument of considering the first intersection with the x -axes. Each Dyck path $w \in D_n$ is uniquely determined as a concatenation a Dyck path w_1 of order $k \leq n$ with $\xi(w_1) = 1$ and a Dyck path w_2 from $(2k, 0)$ to $(2n, 0)$. The path w_1 consists of the vector $[1, 1]$, some Dyck path w'_1 from $(1, 1)$ to $(2k - 1, 1)$ of order $k - 1$ and the vector $[1, -1]$. The Dyck paths $w'_1 \in D_{k-1}$ and $w_2 \in D_{n-k}$ uniquely determine w . Hence

$$\pi_n^{(d)} = \sum_{k=1}^n \sum_{w'_1 \in D_{k-1}} \sum_{w_2 \in D_{n-k}} d \cdot d^{\xi(w_2)} = d \sum_{k=1}^n c_{k-1} \pi_{n-k}^{(d)}.$$

□

The following result can be easily obtained from the results in [28], where similar calculations were derived for d in the operator theory setting, see also [29] for generalizations. For the completeness of the presentation we include an elementary proof.

Proposition 2. *Let $d \in \mathbb{R} \setminus \{0\}$. The generating function $G_{(d)}(z)$ of the sequence $(\pi_n^{(d)})_{n=0}^\infty$ satisfies*

$$G_{(d)}(z) = \frac{1}{1 - z d F(z)} = \frac{2}{2 - d + d\sqrt{1 - 4z}},$$

where $F(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$ is the generating function for the Catalan numbers.

Proof. Using the formula (7) we obtain

$$G_{(d)}(z) - 1 = \sum_{n=1}^{\infty} \pi_n^{(d)} z^n = d \sum_{n=1}^{\infty} \sum_{k=1}^n \pi_{n-k}^{(d)} c_{k-1} z^{n-1} z = d G_{(d)}(z) F(z) z.$$

□

Using the Lemma 1 we can show a simpler recurrence formula for $\pi_n^{(d)}$.

Lemma 3. *For $d \in \mathbb{R} \setminus \{0, 1\}$ one has*

$$(8) \quad \pi_n^{(d)} = \frac{-d^2}{1-d} \pi_{n-1}^{(d)} + \frac{d}{1-d} c_{n-1}, \quad n = 1, 2, \dots$$

Proof. First observe that

$$\frac{-d^2}{1-d} \pi_0^{(d)} + \frac{d}{1-d} c_0 = \frac{-d^2}{1-d} + \frac{d}{1-d} = d = \pi_1^{(d)}.$$

Now let us assume that (8) holds for all $j \leq n$, where $n \geq 1$ is fixed. Then

$$\begin{aligned} \pi_{n+1}^{(d)} &= d \sum_{j=0}^n c_{n-j} \pi_j^{(d)} = d \sum_{j=1}^n c_{n-j} \pi_j^{(d)} + d c_n = \\ &= d \sum_{j=1}^n c_{n-j} \left(\frac{-d^2}{1-d} \pi_{j-1}^{(d)} + \frac{d}{1-d} c_{j-1} \right) + d c_n = \\ &= \frac{-d^2}{1-d} d \sum_{j=1}^n c_{n-j} \pi_{j-1}^{(d)} + d \frac{d}{1-d} \sum_{j=1}^n c_{n-j} c_{j-1} + d c_n = \\ &= \frac{-d^2}{1-d} \pi_n^{(d)} + d \frac{d}{1-d} c_n + d c_n = \frac{-d^2}{1-d} \pi_n^{(d)} + \frac{d}{1-d} c_n. \end{aligned}$$

□

The following proposition shows a closed formula for $\pi_n^{(d)}$. The Catalan triangle $[t_{n,k}]_{n,k=0}^\infty$ (see [24, A009766]) is defined as

$$t_{0,k} = \delta_{0,k}, \quad t_{n,k} = \sum_{j=0}^k t_{n-1,j} = \left(\binom{n+k}{k+1} - \binom{n+k}{k} \right).$$

Proposition 4. *The numbers $\pi_n^{(d)}$ satisfy*

$$\pi_0^{(d)} = 1, \quad \pi_n^{(d)} = \sum_{k=1}^n t_{n-1,n-k} d^k \quad (n > 0).$$

The proof can be obtained from the relation (6) and the formula

$$C_n(d) = 1 + \sum_{k=0}^{n-2} d^{k+1} t_{n,k},$$

see Proposition 6.1 in [9]. However, we present a simple argument using Lemma 3.

Proof. By formula (7) it is clear that

$$\pi_n^{(d)} = \sum_{k=1}^n a_{n-1,n-k} d^k \quad (n > 0),$$

with some coefficients $a_{n,k}$. Since $\pi_{n-1}^{(1)} = c_{n-1}$, we have

$$\sum_{k=1}^{n-1} a_{n-2,n-1-k} = c_{n-1}, \quad n > 1.$$

Using this and Lemma 3 we obtain for $n > 1$

$$\begin{aligned} \pi_n^{(d)} &= \frac{-d^2}{1-d} \pi_{n-1}^{(d)} + \frac{d}{1-d} c_{n-1} = \\ &= \frac{-d^2}{1-d} \sum_{k=1}^{n-1} a_{n-2,n-1-k} d^k + \frac{d}{1-d} c_{n-1} = \\ &= \sum_{k=1}^{n-1} \left(\frac{-d^{k+2}}{1-d} + \frac{d}{1-d} \right) a_{n-2,n-1-k} = \\ &= \sum_{k=1}^{n-1} (d + d^2 + \dots + d^{k+1}) a_{n-2,n-1-k}. \end{aligned}$$

Comparing the coefficients of polynomials on the left and right hand side of the above we obtain for $l = 1, \dots, n$

$$a_{n-1,n-l} = \sum_{k=l-1}^{n-1} a_{n-2,n-1-k} = \sum_{j=0}^{n-l} a_{n-2,j}.$$

This, together with the information that $\pi_1^{(d)} = d$, proves the result. □

To finish the section, let us show that $\pi_n^{(-1)} = (-1)^n a_n$ ($n = 0, 1 \dots$), where

$$a_n = \left(\frac{1}{2}\right)^n \left(1 + \sum_{k=0}^{n-1} c_k (-2)^k\right), \quad n = 0, 1 \dots,$$

are the generalized Catalan numbers $C(-1, n)$ (see [3] and the OEIS database [24] number A064310). Indeed, for $d = -1$ the formula (8) is of the form

$$\pi_n^{(-1)} = -\frac{\pi_{n-1}^{(-1)} + c_{n-1}}{2}.$$

On the other hand, we have

$$\begin{aligned} (-1)^n a_n &= -\frac{1}{2} \left(\left(-\frac{1}{2}\right)^{n-1} \left(1 + \sum_{k=0}^{n-2} c_k (-2)^k\right) + \left(-\frac{1}{2}\right)^{n-1} c_{n-1} (-2)^{n-1} \right) = \\ &= -\frac{(-1)^{n-1} a_{n-1} + c_{n-1}}{2}. \end{aligned}$$

Finally, to conclude $\pi_n^{(-1)} = (-1)^n a_n$ it is enough to see that $\pi_0^{(-1)} = 1 = a(0)$.

2. WIGNER MATRICES WITH ONE NEGATIVE SQUARE

In what follows

$$W_N = \frac{1}{\sqrt{N}} [x_{ij}]_{ij=0}^N$$

stands for the Wigner matrix, that is a random matrix with entries x_{ij} ($0 \leq i \leq j \leq N$) being real, independent, zero mean, the off-diagonal entries x_{ij} ($0 \leq i < j \leq N$) being identically distributed, and the diagonal entries x_{ii} ($0 \leq i \leq N$) being identically distributed. For simplicity we assume that the variance of the off-diagonal entries is one. Moreover, we assume that

$$(9) \quad r_k := \max \{|x_{00}|^k, |x_{11}|^k\} < +\infty$$

We set

$$H_N^{(d)} = \begin{bmatrix} d & 0 \\ 0 & I_N \end{bmatrix}, \quad X_N^{(d)} = H_N^{(d)} W_N,$$

where d is a nonrandom, nonzero, real parameter.

Theorem 5. *Let the random matrix $X_N^{(d)}$ satisfy the above probabilistic assumptions. Then, for $d \in \mathbb{R} \setminus \{0\}$*

$$e_0^* \left(X_N^{(d)} \right)^n e_0 \rightarrow \begin{cases} \pi_{n/2}^{(d)} & : n \text{ even} \\ 0 & : n \text{ odd} \end{cases} \quad (N \rightarrow \infty)$$

in $L^2(\mathbb{P})$ and, in particular, in probability.

A basis for our considerations is the combinatorial proof of the Wigner's result, mainly as presented in [2]. Before the proof of Theorem 5 we review the classical proof, also introducing notations needed later on. Lemma 2.1.6 of [2] shows that

$$\mathbb{E} \frac{\text{tr}(W_N)^n}{N} \rightarrow \begin{cases} c_{n/2} & : n \text{ even} \\ 0 & : n \text{ odd} \end{cases} \quad (N \rightarrow \infty).$$

The proof is based on passage to the limit in the formula

$$(10) \quad \mathbb{E} \frac{\text{tr}(W_N)^n}{N} = \frac{1}{N} \sum_{i_1, i_2, \dots, i_n=0}^N \mathbb{E} W_N(i_1, i_2) W_N(i_2, i_3) \dots W_N(i_n, i_1).$$

To analyze the above expression the set \mathcal{W} of all closed words (i_1, \dots, i_n, i_1) over the alphabet $\{1, \dots, N\}$ is introduced. On this set an equivalence relation \sim is defined by saying that two words are equivalent if there exists a bijection on the alphabet that maps one word to the other. We denote the set of all equivalence classes by $[\mathcal{W}]_{\sim}$. The *weight* of a word is the number of its distinct letters. Note that all words in one equivalence class $A \in [\mathcal{W}]_{\sim}$ have the same weight, and the number of elements of the class equals

$$C_{N,t} = N(N-1) \cdots (N-t+1) = \mathcal{O}(N^t),$$

where t is the common weight of the words.

Observe that we can rewrite the right hand side of (10) as

$$(11) \quad \frac{1}{N} \sum_{A \in [\mathcal{W}]_{\sim}} \sum_{(i_1, i_2, \dots, i_n, i_1) \in A} \mathbb{E} W_N(i_1, i_2) W_N(i_2, i_3) \dots W_N(i_n, i_1).$$

It is shown in [2, Subsection 2.1.3] that the number of equivalence classes is bounded from above for all N . In consequence, the limit of (11) is zero for n odd. For n even the passage to infinity with N in formula (11) is survived only by the equivalence classes that are in one-to-one correspondence with Dyck paths, the canonical correspondence is described in [2, p.15]. Each of such classes contains words of weight $n/2 + 1$ and consequently its power equals $C_{N, n/2+1} = \mathcal{O}(N^{n/2+1})$. In consequence,

$$(12) \quad \lim_{N \rightarrow \infty} \mathbb{E} \frac{\text{tr}(W_N)^n}{N} = \lim_{N \rightarrow \infty} \frac{1}{N} \frac{1}{N^{n/2}} \sum_{w \in D_{n/2}} C_{N, n/2+1} E(x_{1,2}^2)^{n/2} = \sum_{w \in D_{n/2}} 1 = c_{n/2}.$$

Having this classical argument recalled, we are ready to prove Theorem 5.

Proof of Theorem 5. The proof consists of three usual steps.

Step 1.

$$\mathbb{E} e_0^* \left(X_N^{(d)} \right)^n e_0 \rightarrow \begin{cases} \pi_{\frac{n}{2}}^{(d)}, & \text{for } n \text{ even,} \\ 0, & \text{for } n \text{ odd.} \end{cases}$$

We begin the proof with an analogue of (10)

$$\begin{aligned} \mathbb{E} (e_0^* (X_N)^n e_0) &= \sum_{i_1, i_2, \dots, i_{n-1}=0}^N \mathbb{E} X_N(0, i_1) X_N(i_1, i_2) \dots X_N(i_{n-1}, 0) \\ &= \sum_{i_1, i_2, \dots, i_{n-1}=0}^N d^{\eta(0, i_1, \dots, i_{n-1})} \mathbb{E} W_N(0, i_1) W_N(i_1, i_2) \dots W_N(i_{n-1}, 0), \end{aligned}$$

where $\eta(j_1, \dots, j_k)$ is the number of zeros in the sequence (j_1, \dots, j_k) . We introduce \mathcal{W}_0 as the set of words over $\{0, \dots, N\}$ of the form $(0, i_1, \dots, i_{n-1}, 0)$. Note that all words in one equivalence class $A \in [\mathcal{W}_0]_{\sim}$ have the same weight, and the number

of elements of the class equals $C_{N,t-1}$, where t is the common weight of the words. Then, analogously to (11), one has

$$(13) \quad \mathbb{E}e_0^* \left(X_N^{(d)} \right)^n e_0 = \sum_{A \in [\mathcal{W}_0]_{\sim}} \sum_{(0, i_1, i_2, \dots, i_{n-1}, 0) \in A} d^{\eta(0, i_1, \dots, i_{n-1})} \mathbb{E} (W_N(0, i_1) W_N(i_1, i_2) \dots W_N(i_{n-1}, 0)).$$

As in the proof of the Wigners result, the limit in the case n is odd is zero, and in the case n is even the passage to infinity with N in formula (13) is survived only by the equivalence classes that are in one-to-one correspondence with Dyck paths of order $n/2$. Each of such classes contains words of weight $n/2 + 1$ and consequently has precisely $C_{N, n/2}$ elements. Hence,

$$(14) \quad \lim_{N \rightarrow \infty} \mathbb{E}e_0^* \left(X_N^{(d)} \right)^n e_0 = \lim_{N \rightarrow \infty} \frac{1}{N^{n/2}} \sum_{w \in D_{n/2}} d^{\xi(w)} C_{N, n/2} \mathbb{E}(x_{1,2}^2)^{n/2}.$$

We used in the above equality the fact, that for all words $(0, i_1, \dots, i_{n-1}, 0)$ from class $A \in [\mathcal{W}_0]_{\sim}$ we have

$$(15) \quad \eta(0, i_1, \dots, i_{n-1}) = \xi(w),$$

where w is the Dyck word corresponding to the class A . Indeed, in the canonical bijection described in [2] meeting of the Dyck path with the x -axes corresponds to a zero on the corresponding position in the word. Finally, by (14),

$$\lim_{N \rightarrow \infty} \mathbb{E}e_0^* \left(X_N^{(d)} \right)^n e_0 = \sum_{w \in D_{n/2}} d^{\xi(w)} = \pi_{n/2}.$$

Step 2.

$$\mathbb{E} \left(e_0^* \left(X_N^{(d)} \right)^n e_0 \right)^2 \rightarrow \begin{cases} (\pi_{\frac{n}{2}}^{(d)})^2, & \text{for } n \text{ even,} \\ 0, & \text{for } n \text{ odd.} \end{cases}$$

We start the proof similarly as in Step 1

$$\begin{aligned} & \mathbb{E} (e_0^* (W_N)^n e_0)^2 = \\ & \sum_{i_1, \dots, i_{n-1}=0}^N \sum_{j_1, \dots, j_{n-1}=0}^N \mathbb{E} (X_N(0, i_1) \cdots X_N(i_{n-1}, 0) X_N(0, j_1) \cdots X_N(j_{n-1}, 0)) \\ & = \sum_{i_1, \dots, i_{n-1}=0}^N \sum_{j_1, \dots, j_{n-1}=0}^N d^{\eta(0, i_1, \dots, i_{n-1}, 0, j_1, \dots, j_{n-1})} \\ & \cdot \mathbb{E} (W_N(0, i_1) \cdots W_N(i_{n-1}, 0) W_N(0, j_1) \cdots W_N(j_{n-1}, 0)). \end{aligned}$$

We introduce \mathcal{W}_{00} as the set of words over $\{0, \dots, N\}$ of the form

$$(0, i_1, \dots, i_{n-1}, 0, j_1, \dots, j_{n-1}, 0).$$

Note that the power of each equivalence class $A \in [\mathcal{W}_{00}]_{\sim}$ equals $C_{N,t-1}$, where t is the common weight of the words in A . Furthermore,

$$(16) \quad \mathbb{E} \left(e_0^* \left(X_N^{(d)} \right)^n e_0 \right)^2 = \sum_{A \in [\mathcal{W}_{00}]_{\sim}} \sum_{I \in A} d^{\eta(0, i_1, \dots, i_{n-1}, 0, j_1, \dots, j_{n-1})} \cdot \mathbb{E} (W_N(0, i_1) \cdots W_N(i_{n-1}, 0) W_N(0, j_1) \cdots W_N(j_{n-1}, 0)),$$

with

$$I = (0, i_1, \dots, i_{n-1}, 0, j_1, \dots, j_{n-1}, 0).$$

The passage with N to infinity is again survived by the equivalence classes corresponding to Dyck paths D_n^0 of order n , which meet the x axes after n steps. Since there is no word in \mathcal{W}_{00} of length $2n + 1$ corresponding to a Dyck path, the limit with $N \rightarrow \infty$ of the expression above is zero for n odd. For n even the words in equivalence classes that survive passage $N \rightarrow \infty$ have weight $n + 1$. Consequently

$$(17) \quad \lim_{N \rightarrow \infty} \mathbb{E} \left(e_0^* \left(X_N^{(d)} \right)^n e_0 \right)^2 = \lim_{N \rightarrow \infty} \frac{1}{N^n} \sum_{w \in D_n^0} d^{\xi(w)} C_{N,n} E(x_{1,2}^2)^n = \sum_{w \in D_n^0} d^{\xi(w)}.$$

Note that each Dyck path $w \in D_n^0$ is a concatenation of two Dyck paths $w_1, w_2 \in D_{n/2}$ and $\xi(w) = \xi(w_1) + \xi(w_2)$. Hence,

$$(18) \quad \lim_{N \rightarrow \infty} \mathbb{E} \left(e_0^* \left(X_N^{(d)} \right)^n e_0 \right)^2 = \sum_{w_1, w_2 \in D_{n/2}} d^{\xi(w_1) + \xi(w_2)} = \left(\pi_{n/2}^{(d)} \right)^2.$$

Step 3. The application of Chebyshev's inequality finishes the proof. \square

3. REPRESENTATION AND CONVERGENCE OF THE WEYL FUNCTIONS

As it was mentioned in the Introduction, we define the Weyl function as

$$(19) \quad Q_{(d)}^N(z) = - \left(e_0^* H_N^{(d)} \left(X_N^{(d)} - z \right)^{-1} e_0 \right)^{-1},$$

where e_0 is the first vector of the canonical basis of \mathbb{C}^{N+1} and $H_N^{(d)}, X_N^{(d)}$ are defined as in Section 2. It is clear that the zeros of $Q_{(d)}^N(z)$ are eigenvalues of $X_N^{(d)}$, the converse is not necessarily true. Nevertheless, the Weyl function will allow us to determine the part of the spectrum of $X_N^{(d)}$ lying in $\mathbb{C} \setminus [-2, 2]$ for large values of N . For this aim we prove that $Q_{(d)}^N(z)$ converges in probability to the function

$$(20) \quad Q_{(d)}(z) := \hat{\sigma}(z) + \frac{z}{d} = \frac{(2-d)z + d\sqrt{z^2 - 4}}{2d},$$

where σ is the Wigner semicircle measure.

The function $\frac{-1}{Q_{(d)}(z)}$ is for $d > 0$ an ordinary Nevanlinna function and for $d < 0$ a generalized Nevanlinna function with one negative square. In both cases the general theory (see e.g. [1, 19]) admits an expansion at infinity. More precisely, we have the following.

Lemma 6. *For $Q_{(d)}(z)$ defined by (20) one has*

$$(21) \quad \frac{-1}{Q_{(d)}(z)} = -\frac{d}{z} \sum_{n=0}^{\infty} \frac{1}{z^{2n}} \pi_n^{(d)},$$

where the series absolutely converges for $z \in \mathbb{C}$ such that

$$|z| > \begin{cases} 2 & : |d| < 2 \\ \frac{|d|}{\sqrt{|d-1|}} & : |d| \geq 2 \end{cases}.$$

Proof. Note that by Proposition 2 the right hand side of (21) equals

$$-\frac{d}{z}G^{(d)}\left(\frac{1}{z^2}\right) = \frac{-2d}{(2-d)z + d\sqrt{z^2-4}} = \frac{-1}{Q_{(d)}(z)}.$$

The claim on convergence follows from the fact that the function $\frac{-1}{Q_{(d)}(z)}$ is holomorphic in $\mathbb{C} \setminus \left([-2, 2] \cup \{z_{(d)}^\pm\}\right)$ for

$$z_{(d)}^\pm = \begin{cases} \pm \frac{d}{\sqrt{1-d}} i & : d < 0 \\ \pm \frac{d}{\sqrt{d-1}} & : d > 2 \\ 0 & : d \in [0, 2] \end{cases},$$

see Introduction. □

We formulate now the second main theorem of the paper. Statement (i) is a new result, (ii), (iii) and (iv) were already proved in [27] with a different method that allowed to omit the assumption (22). A perturbation problem, similar to the one described in (v), is widely discussed in the literature, see e.g. [4, 5, 6, 10, 11, 14, 16]. Nevertheless, (v) is stated for completeness of the analysis of the change of the spectrum of $X_N^{(d)}$ with the parameter d .

Theorem 7. *Assume that $X_N^{(d)}$ satisfies the probabilistic assumption of Section 2, then*

(i) *the function $Q_{(d)}^N(z)$ defined by (19) has the following representation*

$$Q_{(d)}^N(z) = -a_N + \frac{z}{d} + \hat{\mu}_N(z),$$

where a_N is a real, random variable, $a_N \rightarrow 0$ with $N \rightarrow \infty$ in probability, μ_N is a random, discrete, probability measure on \mathbb{R} and

$$\int_{\mathbb{R}} t^n d\mu_N(t) \rightarrow \int_{\mathbb{R}} t^n d\sigma(t), \quad n = 0, 1, \dots,$$

with $N \rightarrow \infty$ in probability.

If, additionally, the moments r_k defined by (9) satisfy

$$(22) \quad r_k \leq k^{Ck}, \quad k = 1, 2, \dots,$$

for some constant $C \geq 0$ then

- (ii) $\mu_N \rightarrow \sigma$ with $N \rightarrow \infty$ weakly in probability;
- (iii) for $z \in \mathbb{C}^+$ the number $Q_{(d)}^N(z)$ converges in probability with $N \rightarrow \infty$ to $Q_{(d)}(z)$;
- (iv) if $d < 0$ the (unique) eigenvalue of nonpositive type of $X_N^{(d)}$ converges in probability to $z_{(d)}^+$;
- (v) if $d > 2$ then the minimal (maximal) eigenvalue of $X_N^{(d)}$ converges with $N \rightarrow \infty$ to $z_{(d)}^-(z_{(d)}^+)$, respectively).

Proof. (i) Writing $X_N^{(d)}$ as

$$\begin{bmatrix} da_N & db_N^* \\ b_N & C_N \end{bmatrix}$$

and using the Schur complement argument we see, that

$$Q_{(d)}^N(z) = - \left(e_0^* H_N^{(d)} (X_N^{(d)} - z)^{-1} e_0 \right)^{-1} = -a_N + \frac{z}{d} + b_N^* (C_N - z)^{-1} b_N.$$

Observe that with a discrete, random measure

$$\mu_N := \sum_{j=1}^N |f_j^N|^2 \delta_{\lambda_j^N},$$

where $U_N C_N U_N^* = \text{diag}(\lambda_1^N, \dots, \lambda_j^N)$ is the unitary diagonalization of C_N and $f_N = [f_1^N, \dots, f_N^N]^\top = U_N b_N$, one has

$$(23) \quad Q_{(d)}^N(z) = \frac{z}{d} - a_N + \hat{\mu}_N(z).$$

Expanding the Stieltjes transform of $\hat{\mu}(z)$ at infinity

$$\hat{\mu}_N(z) = \sum_{n=1}^{\infty} \alpha_n^N z^{-n}$$

with (random) real coefficients

$$\alpha_n^N = \int t^{n-1} d\mu_N(t), \quad n = 1, 2, \dots$$

Consequently, we obtain the following Laurent series expansion at infinity of $Q_{(d)}^N(z)$

$$(24) \quad Q_{(d)}^N(z) = \sum_{n=-1}^{\infty} \alpha_n^N z^{-n},$$

with $\alpha_{-1}^N := \frac{1}{d}$, $\alpha_0^N := -a_N$ ($N = 1, 2, \dots$). On the other hand consider the expansion of $-1/Q_{(d)}^N(z)$

$$(25) \quad -1/Q_{(d)}^N(z) = d e_0^* \left(X_N^{(d)} - z \right)^{-1} e_0 = \sum_{n=1}^{\infty} \gamma_n^N z^{-n},$$

with

$$\gamma_n^N := -d e_0^* (X_N^{(d)})^{n-1} e_0.$$

Observe that, by the Cauchy product rule, the random sequences $(\alpha_n^N)_{n=-1}^{\infty}$, $(\gamma_n^N)_{n=1}^{\infty}$ satisfy surely for each $N = 0, 1, \dots$ the following equalities

$$(26) \quad \alpha_{-1}^N \gamma_1^N = 1, \quad \sum_{i=0}^k \alpha_{i-1}^N \gamma_{k-i+1}^N = 0 \quad k = 1, 2, \dots$$

Furthermore, the nonrandom sequences $(\alpha_n)_{n=-1}^{\infty}$, $(\gamma_n)_{n=1}^{\infty}$ defined by $\alpha_{-1} := \frac{1}{d}$, $\alpha_0 := 0$

$$\alpha_n := \begin{cases} c_{(n-1)/2} & : n \text{ odd} \\ 0 & : n \text{ even} \end{cases} \quad \gamma_n := \begin{cases} \pi_{(n-1)/2}^{(d)} & : n \text{ odd} \\ 0 & : n \text{ even} \end{cases}, \quad n = 1, 2, \dots$$

satisfy by Lemma 6

$$(27) \quad \alpha_{-1}\gamma_1 = 1, \quad \sum_{i=0}^k \alpha_{i-1}\gamma_{k-i+1} = 0, \quad k = 1, 2, \dots$$

By Theorem 5, for each $n = 0, 1, \dots$

$$(28) \quad \gamma_n^N \rightarrow \gamma_n \quad (N \rightarrow \infty), \text{ in probability.}$$

Employing (26), (27) and (28) in a simple induction argument with respect to n we obtain that for each $n = 0, 1, \dots$

$$\alpha_n^N \rightarrow \alpha_n \quad (N \rightarrow \infty), \text{ in probability.}$$

Since $\alpha_n = \int_{\mathbb{R}} t^{n-1} d\sigma(t)$ the proof of (i) is complete.

(ii) Note that by [2, Theorem 2.1.22]

$$\mathbb{P}(\text{supp } \mu_N \subset [-3, 3]) \rightarrow 1, \quad N \rightarrow \infty.$$

Consequently, for each $\varepsilon > 0$

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\int_{|x| > 3} |x|^k d\mu_N > \varepsilon \right) \rightarrow 0.$$

A standard approximation argument (cf. e.g. [2] formula (2.1.9) and below) shows that μ_N converges to σ weakly in probability. This, together with (20) and (23), finishes the proof of (ii).

Statement (iii) follows directly from (ii), statement (iv) follows from (ii) and the continuity of the eigenvalue of nonpositive type as a function of μ_N and a_N , see [27]. To see (v) assume $d > 2$. Let M denote the set of pairs (μ, a) of a positive, finite measures μ on \mathbb{R} with

$$(29) \quad \text{supp } \mu \subset \left[\frac{-2 + z_{(d)}^-}{2}, \frac{2 + z_{(d)}^+}{2} \right],$$

and $a \in \mathbb{R}$ such that the equation $a + \frac{z}{d} + \hat{\mu}(z) = 0$ has a solution in each of the half axes $(-\infty, \min \text{supp } \mu)$, $(\max \text{supp } \mu, +\infty)$. Note that these both solutions are necessarily unique, we denote them by ζ_N^- and ζ_N^+ , respectively. Note that $(\sigma, 0) \in M$ and that the pair (μ_N, a) belongs to M for some (equivalently: for every) $a \in \mathbb{R}$ if and only if (29) is satisfied. We endow M with the topology inherited from the product of the weak and natural topology. Thanks to (29) the mapping $M \ni (\mu, a) \mapsto \zeta_N^-$ is continuous at $(\sigma, 0)$.

Let Ξ_N stand for the event that $\mu = \mu_N$ satisfies (29). Observe that

$$P \left(|\zeta_N^- - z_{(d)}^-| > \varepsilon \right) \leq P(\Xi_N^c) + P \left(\Xi_N, |\zeta_N^- - z_{(d)}^-| > \varepsilon \right),$$

and the first summand converges with $N \rightarrow \infty$ to zero by [2, Theorem 2.1.22]. The second summand converges to zero since, by (i), (μ_N, a_N) converges to $(\sigma, 0)$ in M . Thanks to the afore-mentioned continuity at $(\sigma, 0)$, ζ_N^- converges in probability to $z_{(d)}^-$. Analogously one obtains the convergence of ζ_N^+ . □

4. FINAL REMARKS

One should mention that infinite tridiagonal matrices of the form

$$(30) \quad J_{(d)} = \begin{bmatrix} & \sqrt{d} & & & \\ \sqrt{d} & & 1 & & \\ & 1 & & 1 & \\ & & 1 & & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

are in some sense operator analogues of the matrices $H_N^{(\sqrt{d})}W_NH_N^{(\sqrt{d})}$, namely the functions $e_0^*(J_{(d)} - z)^{-1}e_0$ and $-1/Q_{(d)}(z)$ coincide. The family (30) was studied in [28], see therein and [8, 9] for a relation with noncommutative probability and t -transformations of convolutions. Also the representation of the function $-1/Q_{(d)}(z)$ for $d > 0$ as a Stieltjes transform of a measure was derived in [28].

However, for random matrices, contrary to the operator case (30), the function $-1/Q_{(d)}(z)$ does *not* provide information about the limit of the empirical measure of eigenvalues. Namely, using the intertwining principle one may easily show that the empirical measure of spectrum of the matrix $X_N^{(d)}$ converges for all $d > 0$ to the Wigner measure σ , while $-1/Q_{(d)}(z) \neq \hat{\sigma}(z)$.

Now let us discuss the limitations of the combinatorial methods in computing the spectra of H -selfadjoint Wigner matrices. For a fixed $k \in \mathbb{N}$ consider the following matrices

$$H_N^{(d),k} = \begin{bmatrix} I_k & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & I_{N-k} \end{bmatrix} \in \mathbb{C}^{(N+1) \times (N+1)}, \quad N = k, k+1, \dots$$

First observe that the spectrum of $X_N^{(d),k} = H_N^{(d),k}W_N$ coincides for large N with the spectrum of $X_N^{(d)}$. Indeed, the unitary matrix

$$U_{k,N} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & I_{k-1} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{N-k} \end{bmatrix}$$

satisfies $U_{k,N}H_N^{(d)}U_{k,N} = H_N^{(d),k}$. Hence, the spectrum of $X_N^{(d),k}$ coincides with the spectrum of $H_N^{(d)}U_{k,N}W_NU_{k,N}$. Since $U_{k,N}W_NU_{k,N}$ is again Wigner matrix, we get that the limit distributions of real eigenvalues as well as the limit of the eigenvalue of nonpositive type of $X_N^{(d),k}$ coincide with, respectively, the limit distributions of real eigenvalues and the limit of the eigenvalue of nonpositive type of $X_N^{(d)}$.

However, for the above matrices the combinatorial interpretation given in the proof of Theorem 5 is no longer true. Namely, repeating the proof it is not possible to show that after dividing into equivalence classes and passing to the limit each Dyck path $w \in D_n$ contributes precisely $d^{\xi(w)}$ to the total sum, since formula (15) is no longer true. Although for the above ensemble of matrices $X_N^{(d),k}$ this does not seem to be a large drawback, the real problem appears while considering matrices

as

$$H_N^{(d_1, \dots, d_\kappa)} = \text{diag}(d_1, \dots, d_\kappa) \oplus I_{N-\kappa+1}, \quad X_N^{(d_1, \dots, d_\kappa)} = H_N^{(d_1, \dots, d_\kappa)} W_N.$$

For the calculation of spectrum of those matrices one needs to develop a different method, the topic will be treated in a subsequent paper.

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