ARITHMETIC PROPERTIES OF APÉRY-LIKE NUMBERS

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Abstract. We provide lower bounds for p-adic valuations of multisums of factorial ratios which satisfy an Apéry-like recurrence relation: these include Apéry, Domb, Franel numbers, the numbers of abelian squares over a finite alphabet, and constant terms of powers of certain Laurent polynomials. In particular, we prove Beukers' conjectures on the p-adic valuation of Apéry numbers. Furthermore, we give an effective criterion for a sequence of factorial ratios to satisfy the p-Lucas property for almost all primes p.

1. Introduction

1.1. **Beukers' conjectures on Apéry numbers.** For all $n \in \mathbb{N}$, we set

$$A_1(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \quad \text{and} \quad A_2(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}.$$

Those sequences were used in 1979 by Apéry in his proofs of the irrationality of $\zeta(3)$ and $\zeta(2)$ (see [2]). In the 1980's, several congruences satisfied by those sequences were demonstrated (see for example [4], [5], [10], [15], [18]). In particular, Gessel proved in [15] that A_1 satisfies the p-Lucas property for all prime numbers p, that is, for any prime p, all $v \in \{0, \dots, p-1\}$ and all $n \in \mathbb{N}$, we have

$$A_1(v + np) \equiv A_1(v) A_1(n) \mod p.$$

Thereby, if $n = \sum_{k=0}^{N} n_k p^k$ with $n_k \in \{0, \dots, p-1\}$, then we obtain

$$A_1(n) \equiv A_1(n_0) \cdots A_1(n_N) \mod p. \tag{1.1}$$

In particular, p divides $A_1(n)$ if and only if there exists $k \in \{0, \dots, N\}$ such that p divides $A_1(n_k)$. Beukers stated in [3] two conjectures, when $p = 5$ or 11, which generalize this property ([1]). Before stating these conjectures, we observe that the set of all $v \in \{0, \dots, 4\}$ (respectively $v \in \{0, \dots, 10\}$) satisfying $A_1(v) \equiv 0 \mod 5$ (respectively $A_1(v) \equiv 0 \mod 11$) is $\{1, 3\}$ (respectively $\{5\}$).

Conjecture A (Beukers, [3])**.** *Let $n \in \mathbb{N}$, $n = \sum_{k=0}^{N} n_k 5^k$ with $n_k \in \{0, \dots, 4\}$. Let α be the number of $k \in \{0, \dots, N\}$ such that $n_k \in \{1, 3\}$. Then 5^α divides $A_1(n)$.*

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[1]When $p \in \{2, 3, 7\}$, for all $v \in \{0, \dots, p-1\}$, $A_1(v)$ is coprime to p so that, according to (1.1), for all $n \in \mathbb{N}$, $A_1(n)$ is coprime to p.

Conjecture B (Beukers, [3])**.** *Let $n \in \mathbb{N}$, $n = \sum_{k=0}^{N} n_k 11^k$ with $n_k \in \{0, \ldots, 10\}$. Let α be the number of $k \in \{0, \ldots, N\}$ such that $n_k = 5$. Then 11^α divides $A_1(n)$.*

Similarly, Sequence A_2 satisfies the p-Lucas property for all primes p. Furthermore, Beukers and Stienstra proved in [6] that, if $p \equiv 3 \mod 4$, then $A_2\left(\frac{p-1}{2}\right) \equiv 0 \mod p$, and Beukers stated in [3] the following conjecture.

Conjecture C. *Let p be a prime number satisfying $p \equiv 3 \mod 4$. Let $n \in \mathbb{N}$, $n = \sum_{k=0}^{N} n_k p^k$ with $n_k \in \{0, \ldots, p-1\}$. Let α be the number of $k \in \{0, \ldots, N\}$ such that $n_k = \frac{p-1}{2}$. Then p^α divides $A_2(n)$.*

The main aim of this article is to prove Theorem 1, stated in Section 1.3, which demonstrates and generalizes Conjectures A-C. First, we introduce some notations which we use throughout this article.

1.2. **Notations.** For all primes p, we write $\mathbb{Z}_p$ for the ring of p-adic integers. If d is a positive integer and if $A = \big(A(\mathbf{n})\big)_{\mathbf{n} \in \mathbb{N}^d}$ is a $\mathbb{Z}_p$-valued family, then we say that A satisfies the p-Lucas property if and only if, for all $\mathbf{v} \in \{0, \ldots, p-1\}^d$ and all $\mathbf{n} \in \mathbb{N}^d$, we have

$$A(\mathbf{v} + \mathbf{n}p) \equiv A(\mathbf{v})A(\mathbf{n}) \mod p\mathbb{Z}_p. \tag{1.2}$$

In (1.2) and in the sequel of this article, if $\mathbf{m} = (m_1, \ldots, m_d)$ and $\mathbf{n} = (n_1, \ldots, n_d)$ belong to $\mathbb{R}^d$ and if $\lambda \in \mathbb{R}$, then we set $\mathbf{m}+\mathbf{n} := (m_1+n_1, \ldots, m_d+n_d)$, $\mathbf{m}\cdot\mathbf{n} := m_1n_1+\cdots+m_dn_d$ and $\mathbf{m}\lambda := (m_1\lambda, \ldots, m_d\lambda)$. For all $k \in \{1, \ldots, d\}$, we write $\mathbf{m}^{(k)}$ for m_k. We write $\mathbf{m} \geq \mathbf{n}$ if and only if, for all $k \in \{1, \ldots, d\}$, we have $m_k \geq n_k$. We set $\mathbf{0} := (0, \ldots, 0) \in \mathbb{R}^d$, $\mathbf{1} := (1, \ldots, 1)$ and, for all $k \in \{1, \ldots, d\}$, we write $\mathbf{1}_k$ for the vector in $\mathbb{N}^d$, all of whose coordinates equal zero except the k-th which is 1. Furthermore, we write f_A for the generating function of A defined by $f_A(\mathbf{z}) := \sum_{\mathbf{n} \in \mathbb{N}^d} A(\mathbf{n})\mathbf{z}^{\mathbf{n}}$, where, if $\mathbf{z} = (z_1, \ldots, z_d)$ is a vector of variables and $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$, $\mathbf{z}^{\mathbf{n}}$ denotes $z_1^{n_1} \cdots z_d^{n_d}$.

In addition, we write $\mathcal{Z}_p(A)$ for the set of all $\mathbf{v} \in \{0, \ldots, p-1\}^d$ such that $A(\mathbf{v}) \in p\mathbb{Z}_p$. For all $\mathbf{n} \in \mathbb{N}^d$, $\mathbf{n} = \sum_{k=0}^{N} \mathbf{n}_k p^k$ with $\mathbf{n}_k \in \{0, \ldots, p-1\}^d$, $N \in \mathbb{N}$ and $\mathbf{n}_N \neq \mathbf{0}$, we write $\alpha_p(A, \mathbf{n})$ for the number of $k \in \{0, \ldots, N\}$ such that $\mathbf{n}_k \in \mathcal{Z}_p(A)$. Thereby, to prove Conjectures A-C, we have to show that $A_i(n) \in p^{\alpha_p(A_i, n)}\mathbb{Z}$ with $i = 1, p \in \{5, 11\}$ and $i = 2$, $p \equiv 3 \mod 4$.

Given tuples of vectors in $\mathbb{N}^d$, $e = (\mathbf{e}_1, \ldots, \mathbf{e}_u)$ and $f = (\mathbf{f}_1, \ldots, \mathbf{f}_v)$, we write $|e| = \sum_{i=1}^{u} \mathbf{e}_i$ and, for all $\mathbf{n} \in \mathbb{N}^d$ and all $m \in \mathbb{N}$, we set

$$\mathcal{Q}_{e,f}(\mathbf{n}) := \frac{\prod_{i=1}^{u} (\mathbf{e}_i \cdot \mathbf{n})!}{\prod_{i=1}^{v} (\mathbf{f}_i \cdot \mathbf{n})!} \quad \text{and} \quad \mathfrak{S}_{e,f}(m) := \sum_{\mathbf{n} \in \mathbb{N}^d, |\mathbf{n}| = m} \mathcal{Q}_{e,f}(\mathbf{n}).$$

For every positive integer r, we say that e is *r-admissible* if there exists $s \in \{0, \ldots, r\}$ such that e satisfies the following conditions:

- $s = 0$ or there exist $1 \leq i_1 < \cdots < i_s \leq u$, such that $\mathbf{e}_{i_j} \geq \mathbf{1}$ for all $j \in \{1, \ldots, s\}$.
- $s = r$ or, for all $k \in \{1, \ldots, d\}$, there exist $1 \leq i_1 < \cdots < i_{r-s} \leq u$, such that $\mathbf{e}_{i_j} \geq d\mathbf{1}_k$ for all $j \in \{1, \ldots, r-s\}$.

For all primes p, we write $\mathfrak{F}_p^d$ for the set of all functions $g : \mathbb{N}^d \to \mathbb{Z}_p$ such that, for all $K \in \mathbb{N}$, there exists a sequence $(P_{K,k})_{k\geq 0}$ of polynomial functions with coefficients in $\mathbb{Z}_p$ which converges pointwise to g on $\{0, \dots, K\}^d$. For all tuples e and f of vectors in $\mathbb{N}^d$, all $g \in \mathfrak{F}_p^d$ and all $m \in \mathbb{N}$, we set

$$\mathfrak{S}_{e,f}^g(m) := \sum_{\mathbf{n}\in\mathbb{N}^d, |\mathbf{n}|=m} \mathcal{Q}_{e,f}(\mathbf{n})g(\mathbf{n}).$$

Finally, we set $\theta := z\frac{d}{dz}$ and we say that a differential operator $\mathcal{L} \in \mathbb{Z}_p[z, \theta]$ is of

- *type I* if $\mathcal{L} = \sum_{k=0}^{q} z^k P_k(\theta)$ with $q \in \mathbb{N}$, $P_k(X) \in \mathbb{Z}_p[X]$, $P_0(\mathbb{Z}_p^\times) \subset \mathbb{Z}_p^\times$ and if, for all $k \in \{2, \dots, q\}$, we have $P_k(X) \in \prod_{i=1}^{k-1}(X + i)^2\mathbb{Z}_p[X]$;
- *type II* if $\mathcal{L} = \sum_{k=0}^{2} z^k P_k(\theta)$ with $P_k(X) \in \mathbb{Z}_p[X]$, $P_0(\mathbb{Z}_p^\times) \subset \mathbb{Z}_p^\times$ and $P_2(X) \in (X + 1)\mathbb{Z}_p[X]$.

1.3. **Main results.** The main result of this article is the following.

Theorem 1. *Let e and $f = (\mathbf{1}_{k_1}, \dots, \mathbf{1}_{k_v})$ be two disjoint tuples of vectors in $\mathbb{N}^d$ such that $|e| = |f|$, for all $i \in \{1, \dots, v\}$, $k_i \in \{1, \dots, d\}$, and e is 2-admissible. Let p be a fixed prime. Assume that $f_{\mathfrak{S}_{e,f}}$ is canceled by a differential operator $\mathcal{L} \in \mathbb{Z}_p[z, \theta]$ such that at least one of the following conditions holds:*

- *$\mathcal{L}$ is of type I.*
- *$\mathcal{L}$ is of type II and $p - 1 \in \mathcal{Z}_p(\mathfrak{S}_{e,f})$.*

Then, for all $n \in \mathbb{N}$ and all $g \in \mathfrak{F}_p^d$, we have

$$\mathfrak{S}_{e,f}(n) \in p^{\alpha_p(\mathfrak{S}_{e,f},n)}\mathbb{Z} \quad \text{and} \quad \mathfrak{S}_{e,f}^g \in p^{\alpha_p(\mathfrak{S}_{e,f},n)-1}\mathbb{Z}_p.$$

In Section 1.5, we show that Theorem 1 applies to many classical sequences. In particular, Theorem 1 implies Conjectures A-C. Indeed, we have $A_1 = \mathfrak{S}_{e_1,f_1}$ and $A_2 = \mathfrak{S}_{e_2,f_2}$ with $d = 2$, $e_1 = \big((2, 1), (2, 1)\big)$, $f_1 = \big((1, 0), (1, 0), (1, 0), (1, 0), (0, 1), (0, 1)\big)$, $e_2 = \big((2, 1), (1, 1)\big)$ and $f_2 = \big((1, 0), (1, 0), (1, 0), (0, 1), (0, 1)\big)$. Furthermore, it is well known that f_{A_1}, respectively f_{A_2}, is canceled by the differential operator $\mathcal{L}_1$, respectively $\mathcal{L}_2$, defined by

$$\mathcal{L}_1 = \theta^3 - z(34\theta^3 + 51\theta^2 + 27\theta + 5) + z^2(\theta + 1)^3$$

and

$$\mathcal{L}_2 = \theta^2 - z(11\theta^2 + 11\theta + 3) - z^2(\theta + 1)^2.$$

Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are of type I for all primes p, the conditions of Theorem 1 are satisfied by A_1 and A_2, and Conjectures A-C hold. In addition, for all primes p and all $n, \alpha \in \mathbb{N}$, we obtain

$$\sum_{k=0}^{n} k^\alpha \binom{n}{k}^2 \binom{n+k}{k}^2 \in p^{\alpha_p(A_1,n)-1}\mathbb{Z} \quad \text{and} \quad \sum_{k=0}^{n} k^\alpha \binom{n}{k}^2 \binom{n+k}{k} \in p^{\alpha_p(A_2,n)-1}\mathbb{Z}.$$

We provide a similar result which applies to the constant terms of powers of certain Laurent polynomials.

Theorem 2. *Let p be a fixed prime. Let $\Lambda(\mathbf{x})\in\mathbb{Z}_p[x_1^{\pm},\dots,x_d^{\pm}]$ be a Laurent polynomial, and consider the sequence of the constant terms of powers of Λ defined, for all $n\in\mathbb{N}$, by*

$$A(n):=\big[\Lambda(\mathbf{x})^n\big]_{\mathbf{0}}.$$

Assume that the Newton polyhedron of Λ contains the origin as its only interior integral point, and that f_A is canceled by a differential operator $\mathcal{L}\in\mathbb{Z}_p[z,\theta]$ such that at least one of the following conditions holds:

- *$\mathcal{L}$ is of type I.*
- *$\mathcal{L}$ is of type II and $p-1\in\mathcal{Z}_p(A)$.*

Then, for all $n\in\mathbb{N}$, we have

$$A(n)\in p^{\alpha_p(A,n)}\mathbb{Z}_p.$$

By a result of Mellit and Vlasenko [17, Theorem 1], if $\Lambda(\mathbf{x})\in\mathbb{Z}_p[x_1^{\pm},\dots,x_d^{\pm}]$ contains the origin as its only interior integral point, then $\big([\Lambda(\mathbf{x})^n]_{\mathbf{0}}\big)_{n\geq 0}$ satisfies the p-Lucas property, which is essential for the proof of Theorem 2. Likewise, the proof of Theorem 1 rests on the fact that $\mathfrak{S}_{e,f}$ satisfies the p-Lucas property when $|e|=|f|$, e is 2-admissible and $f=(\mathbf{1}_{k_1},\dots,\mathbf{1}_{k_v})$. Since those results deal with multisums of factorial ratios, it seems natural to study similar arithmetic properties for simpler numbers such as families of factorial ratios. To that purpose, we prove Theorem 3 below which gives an effective criterion for $\mathcal{Q}_{e,f}$ to satisfy the p-Lucas property for almost all primes p ([2]). Furthermore, Theorem 3 shows that if $A:=\mathcal{Q}_{e,f}$ satisfies the p-Lucas property for almost all primes p, then, for all $n\in\mathbb{N}$ and every prime p, we have $A(n)\in p^{\alpha_p(A,n)}\mathbb{Z}$.

To state this result, we introduce some additional notations. For all tuples e and f of vectors in $\mathbb{N}^d$, we write $\Delta_{e,f}$ for Landau's function defined, for all $\mathbf{x}\in\mathbb{R}^d$, by

$$\Delta_{e,f}(\mathbf{x}):=\sum_{i=1}^{u}\lfloor\mathbf{e}_i\cdot\mathbf{x}\rfloor-\sum_{i=1}^{v}\lfloor\mathbf{f}_i\cdot\mathbf{x}\rfloor.$$

Therefore, according to Landau's criterion [16] and a precision of the author [11], we have the following dichotomy.

- If, for all $\mathbf{x}\in[0,1]^d$, we have $\Delta_{e,f}(\mathbf{x})\geq 0$, then $\mathcal{Q}_{e,f}$ is a family of integers;
- if there exists $\mathbf{x}\in[0,1]^d$ such that $\Delta_{e,f}(\mathbf{x})\leq -1$, then there are only finitely many primes p such that $\mathcal{Q}_{e,f}$ is a family of p-adic integers.

In the rest of the article, we write $\mathcal{D}_{e,f}$ for the semi-algebraic set of all $\mathbf{x}\in[0,1)^d$ such that there exists a component $\mathbf{d}$ of e or f satisfying $\mathbf{d}\cdot\mathbf{x}\geq 1$. Observe that $\Delta_{e,f}$ vanishes on the nonempty set $[0,1)^d\setminus\mathcal{D}_{e,f}$.

Theorem 3. *Let e and f be disjoint tuples of vectors in $\mathbb{N}^d$ such that $\mathcal{Q}_{e,f}$ is a family of integers. Then we have the following dichotomy.*

(1) *If $|e|=|f|$ and if, for all $\mathbf{x}\in\mathcal{D}_{e,f}$, we have $\Delta_{e,f}(\mathbf{x})\geq 1$, then for all primes p, $\mathcal{Q}_{e,f}$ satisfies the p-Lucas property;*

[2]Throughout this article, we say that an assertion $\mathcal{A}_p$ is true for almost all primes p if there exists a constant $C\in\mathbb{N}$ such that $\mathcal{A}_p$ holds for all primes $p\geq C$.

(2) *if* $|e| \neq |f|$ *or if there exists* $\mathbf{x} \in \mathcal{D}_{e,f}$ *such that* $\Delta_{e,f}(x) = 0$*, then there are only finitely many primes* p *such that* $\mathcal{Q}_{e,f}$ *satisfies the* p*-Lucas property.*

Furthermore, if $\mathcal{Q}_{e,f}$ *satisfies the* p*-Lucas property for all primes* p*, then, for all* $\mathbf{n} \in \mathbb{N}^d$ *and every prime* p*, we have*

$$\mathcal{Q}_{e,f}(\mathbf{n}) \in p^{\alpha_p(\mathcal{Q}_{e,f},\mathbf{n})}\mathbb{Z}.$$

Remark. Theorem 3 implies that $\mathcal{Q}_{e,f}$ satisfies the p-Lucas property for all primes p if and only if all Taylor coefficients at the origin of the associated mirror maps $z_{e,f,k}$, $1 \leq k \leq d$, are integers (see Theorems 1 and 3 in [11]). Indeed, if $\Delta_{e,f} \geq 0$ on $[0,1]^d$ and if $|e| \neq |f|$, then there exists $k \in \{1, \dots, d\}$ such that $|e|^{(k)} > |f|^{(k)}$.

Coster proved in [9] similar results to Theorems 1-3 for the coefficients of certain algebraic power series. Namely, given a prime $p \geq 3$, $a_1, \dots, a_{p-1} \in \mathbb{Z}$, and a sequence A such that

$$f_A(z) = (1 + a_1 z + \cdots + a_{p-1} z^{p-1})^{\frac{1}{1-p}},$$

Coster proved that, for all $n \in \mathbb{N}$, we have

$$v_p\big(A(n)\big) \geq \left\lfloor \frac{\alpha_p(A,n)+1}{2} \right\rfloor.$$

1.4. **Auxiliary results.** The proof of Theorem 1 rests on three results which may be useful to study other sequences.

Proposition 1. *Let* p *be a fixed prime and* A *a* $\mathbb{Z}_p$*-valued sequence satisfying the* p*-Lucas property with* $A(0) \in \mathbb{Z}_p^\times$*. Let* $\mathfrak{A}$ *be the* $\mathbb{Z}_p$*-module spanned by* A*. Assume that*

(a) *there exists a set* $\mathfrak{B}$ *of* $\mathbb{Z}_p$*-valued sequences with* $\mathfrak{A} \subset \mathfrak{B}$ *such that, for all* $B \in \mathfrak{B}$*, all* $v \in \{0, \dots, p-1\}$ *and all positive integers* n*, there exist* $A' \in \mathfrak{A}$ *and a sequence* $(B_k)_{k \geq 0}$*,* $B_k \in \mathfrak{B}$*, such that*

$$B(v + np) = A'(n) + \sum_{k=0}^{\infty} p^{k+1} B_k(n-k);$$

(b) $f_A(z)$ *is canceled by a differential operator* $\mathcal{L} \in \mathbb{Z}_p[z,\theta]$ *such that at least one of the following conditions holds:*
- $\mathcal{L}$ *is of type I.*
- $\mathcal{L}$ *is of type II and* $p-1 \in \mathcal{Z}_p(A)$*.*

Then, for all $B \in \mathfrak{B}$ *and all* $n \in \mathbb{N}$*, we have*

$$A(n) \in p^{\alpha_p(A,n)}\mathbb{Z}_p \quad \text{and} \quad B(n) \in p^{\alpha_p(A,n)-1}\mathbb{Z}_p.$$

In Proposition 1 and throughout this article, if $\big(A(n)\big)_{n\geq 0}$ is a sequence taking its values in $\mathbb{Z}$ or $\mathbb{Z}_p$, then, for all negative integers n, we set $A(n) := 0$. Therefore, to prove Theorem 1, it suffices to demonstrate that $\mathfrak{S}_{e,f}$ satisfies the p-Lucas property and Condition (a) of Proposition 1 with $\mathfrak{B} = \{\mathfrak{S}_{e,f}^g \,:\, g \in \mathfrak{F}_p^d\}$. To that purpose, we shall prove the following results.

Proposition 2. *Let e and f be disjoint tuples of vectors in $\mathbb{N}^d$ such that $|e| = |f|$ and, for all $\mathbf{x} \in \mathcal{D}_{e,f}$, $\Delta_{e,f}(\mathbf{x}) \geq 1$. Assume that e is 1-admissible. Then, $\mathfrak{S}_{e,f}$ is integer-valued and satisfies the p-Lucas property for all primes p.*

Proposition 3. *Let p be a fixed prime. We write Γ_p for the p-adic Gamma function. Then, there exists $g \in \mathfrak{F}_p^2$ such that, for all $n, m \in \mathbb{N}$, we have*

$$\frac{\Gamma_p\big((m+n)p\big)}{\Gamma_p(mp)\Gamma_p(np)} = 1 + g(m,n)p.$$

1.5. **Application of Theorem 1.** By applying Theorem 1, we obtain similar results to Conjectures A-C for numbers satisfying Apéry-like recurrence relations which we list below. Characters in brackets in the last column of the following table form the sequence number in the Online Encyclopedia of Integer Sequences [20].

Sequence	$\mathcal{Q}_{e,f}(n_1,n_2)$	$\mathcal{L}$	Reference
$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$	$\frac{(2n_1+n_2)!^2}{n_1!^4 n_2!^2}$	[1, (γ)]	Apéry numbers (A005259)
$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}$	$\frac{(2n_1+n_2)!(n_1+n_2)!}{n_1!^3 n_2!^2}$	[21, **D**]	Apéry numbers (A005258)
$\binom{2n}{n}=\sum_{k=0}^{n}\binom{n}{k}^2$	$\frac{(n_1+n_2)!^2}{n_1!^2 n_2!^2}$	type I	Central binomial coefficients (A000984)
$\sum_{k=0}^{n}\binom{n}{k}^3$	$\frac{(n_1+n_2)!^3}{n_1!^3 n_2!^3}$	[21, **A**]	Franel numbers (A000172)
$\sum_{k=0}^{n}\binom{n}{k}^4$	$\frac{(n_1+n_2)!^4}{n_1!^4 n_2!^4}$	[13],[14]	(A005260)
$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}$	$\frac{(n_1+n_2)!(2n_1)!(2n_2)!}{n_1!^3 n_2!^3}$	[1, (d)]	(A081085)
$\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}$	$\frac{(n_1+n_2)!^2(2n_1)!}{n_1!^4 n_2!^2}$	[21, **C**]	Number of abelian squares of length $2n$ over an alphabet with 3 letters (A002893)
$\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$	$\frac{(n_1+n_2)!^2(2n_1)!(2n_2)!}{n_1!^4 n_2!^4}$	[1, (α)]	Domb numbers (A002895)
$\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2$	$\frac{(2n_1)!^2(2n_2)!^2}{n_1!^4 n_2!^4}$	[1, (β)]	(A036917)

All differential operators listed in the above table are of type I for all primes p, except the one associated with $A_5(n) := \sum_{k=0}^{n} \binom{n}{k}^4$ which reads

$$\mathcal{L}_5 = \theta^3 - z2(2\theta+1)(3\theta^2+3\theta+1) - z^2 4(\theta+1)(4\theta+5)(4\theta+3).$$

Hence $\mathcal{L}_5$ is of type II for all primes p. By a result of Calkin [8, Proposition 3], for all primes p, we have $A_5(p-1) \equiv 0 \mod p$, *i.e.* $p-1 \in \mathcal{Z}_p(A_5)$. Thus we can apply Theorem 1 to A_5.

Observe that the generating function of the central binomial coefficients is canceled by the differential operator $\mathcal{L} = \theta - z(4\theta+2)$ which is of type I for all primes p.

According to the recurrence relation found by Almkvist and Zudilin (see Case (d) in [1]), $A_6(n) := \sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}$ is also Sequence **E** in Zagier's list [21], that is

$$A_6(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} 4^{n-2k} \binom{n}{2k}\binom{2k}{k}^2.$$

Furthermore, according to [19], Domb numbers $A_8(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}\binom{2(n-k)}{n-k}$ are also the numbers of abelian squares of length $2n$ over an alphabet with 4 letters.

Now we consider the numbers $C_i(n)$ of abelian squares of length $2n$ over an alphabet with i letters which, for all positive integers $i \geq 2$, satisfy (see [19])

$$C_i(n) = \sum_{\substack{k_1+\cdots+k_i=n \\ k_1,\dots,k_i \in \mathbb{N}}} \left(\frac{n!}{k_1! \cdots k_i!}\right)^2.$$

According to [7], $C_i(n)$ is also the $2n$-th moment of the distance to the origin after i steps traveled by a walk in the plane with unit steps in random directions.

To apply Theorem 1 to C_i, it suffices to show that f_{C_i} is canceled by a differential operator of type I for all primes p. Indeed, by Proposition 1 and Theorem 2 in [7], for all $j \geq 2$, $C_j(n)$ satisfies the recurrence relation of order $\lceil j/2 \rceil$ with polynomial coefficients of degree $j-1$:

$$n^{j-1}C_j(n) + \sum_{i\geq 1} \left(n^{j-1} \sum_{\alpha_1,\dots,\alpha_i} \prod_{k=1}^{i} (-\alpha_k)(j+1-\alpha_k) \left(\frac{n-k}{n-k+1}\right)^{\alpha_k - 1} \right) C_j(n-i) = 0, \tag{1.3}$$

where the sum is over all sequences $\alpha_1, \dots, \alpha_i$ satisfying $1 \leq \alpha_k \leq j$ and $\alpha_{k+1} \leq \alpha_k - 2$. We consider $i \geq 2$ and $1 \leq \alpha_1, \dots, \alpha_i \leq j$, $\alpha_{k+1} \leq \alpha_k - 2$. We have

$$n^{j-1} \prod_{k=1}^{i} \left(\frac{n-k}{n-k+1}\right)^{\alpha_k - 1} = \frac{n^{j-1}}{n^{\alpha_1 - 1}} \left(\prod_{k=1}^{i-1} (n-k)^{\alpha_k - \alpha_{k+1}} \right) (n-i)^{\alpha_i - 1},$$

with $j-\alpha_1 \geq 0$, $\alpha_k - \alpha_{k+1} \geq 2$ and $\alpha_i - 1 \geq 0$. Then, $f_{C_j}(z)$ is canceled by a differential operator $\mathcal{L} = \sum_{k=0}^{q} z^k P_k(\theta)$ with $P_0(\theta) = \theta^{j-1}$ and, for all $i \geq 2$,

$$P_i(\theta) \in \prod_{k=1}^{i-1} (\theta + i - k)^2 \mathbb{Z}[\theta] \subset \prod_{k=1}^{i-1} (\theta + k)^2 \mathbb{Z}[\theta],$$

so that $\mathcal{L}$ is of type I for all primes p, as expected.

1.6. **Structure of the article.** In Section 2, we use several results of [11] to prove Theorem 3. Section 3 is devoted to the proofs of Theorem 2 and Proposition 1. In particular, we prove Lemma 1 which points out the role played by the differential operators in our proofs. In Section 4, we prove Theorem 1 by applying Proposition 1 to $\mathfrak{S}_{e,f}$. This is the most technical part of this article.

2. Proof of Theorem 3

First, we prove that if $|e| = |f|$, then, for all primes p, all $\mathbf{a} \in \{0, \dots, p-1\}^d$ and all $\mathbf{n} \in \mathbb{N}^d$, we have

$$\frac{\mathcal{Q}_{e,f}(\mathbf{a} + \mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n})} \in \frac{\prod_{i=1}^{u} \prod_{j=1}^{\lfloor \mathbf{e}_i \cdot \mathbf{a}/p \rfloor} \left(1 + \frac{\mathbf{e}_i \cdot \mathbf{n}}{j}\right)}{\prod_{i=1}^{v} \prod_{j=1}^{\lfloor \mathbf{f}_i \cdot \mathbf{a}/p \rfloor} \left(1 + \frac{\mathbf{f}_i \cdot \mathbf{n}}{j}\right)} (1 + p\mathbb{Z}_p). \tag{2.1}$$

Indeed, we have

$$\frac{\mathcal{Q}_{e,f}(\mathbf{a} + \mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n})} = \frac{\mathcal{Q}_{e,f}(\mathbf{a} + \mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n}p)} \cdot \frac{\mathcal{Q}_{e,f}(\mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{n})}.$$

Since $|e| = |f|$, we can apply [11, Lemma 7] ([3]) with $\mathbf{c} = \mathbf{0}$, $\mathbf{m} = \mathbf{n}$ and $s = 0$ which yields

$$\frac{\mathcal{Q}_{e,f}(\mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{n})} \in 1 + p\mathbb{Z}_p.$$

Furthermore, we have

$$\begin{aligned}
\frac{\mathcal{Q}_{e,f}(\mathbf{a} + \mathbf{n}p)}{\mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n}p)} &= \frac{1}{\mathcal{Q}_{e,f}(\mathbf{a})} \frac{\prod_{i=1}^{u} \prod_{j=1}^{\mathbf{e}_i \cdot \mathbf{a}} (j + \mathbf{e}_i \cdot \mathbf{n}p)}{\prod_{i=1}^{v} \prod_{j=1}^{\mathbf{f}_i \cdot \mathbf{a}} (j + \mathbf{f}_i \cdot \mathbf{n}p)} \\
&= \frac{\prod_{i=1}^{u} \prod_{j=1}^{\mathbf{e}_i \cdot \mathbf{a}} \left(1 + \frac{\mathbf{e}_i \cdot \mathbf{n}p}{j}\right)}{\prod_{i=1}^{v} \prod_{j=1}^{\mathbf{f}_i \cdot \mathbf{a}} \left(1 + \frac{\mathbf{f}_i \cdot \mathbf{n}p}{j}\right)} \\
&\in \frac{\prod_{i=1}^{u} \prod_{j=1}^{\lfloor \mathbf{e}_i \cdot \mathbf{a}/p \rfloor} \left(1 + \frac{\mathbf{e}_i \cdot \mathbf{n}}{j}\right)}{\prod_{i=1}^{v} \prod_{j=1}^{\lfloor \mathbf{f}_i \cdot \mathbf{a}/p \rfloor} \left(1 + \frac{\mathbf{f}_i \cdot \mathbf{n}}{j}\right)} (1 + p\mathbb{Z}_p),
\end{aligned}$$

[3] The proof of this lemma uses a lemma of Lang which contains an error. Fortunately, Lemma 7 remains true. Details of this correction are presented in [12, Section 2.4].

because, if p does not divide j, then $1 + (\mathbf{e}_i \cdot \mathbf{n}p)/j \in 1 + p\mathbb{Z}_p$. This finishes the proof of (2.1).

Now we prove Assertion (1) in Theorem 3. Let p be a fixed prime number. It is well known that, for all $n \in \mathbb{N}$, we have

$$v_p(n!) = \sum_{\ell=1}^{\infty} \left\lfloor \frac{n}{p^\ell} \right\rfloor.$$

Thus, for all $\mathbf{n} \in \mathbb{N}^d$, we have

$$v_p\big(\mathcal{Q}_{e,f}(\mathbf{n})\big) = \sum_{\ell=1}^{\infty} \Delta_{e,f}\left(\frac{\mathbf{n}}{p^\ell}\right).$$

Let $\mathbf{n} \in \mathbb{N}^d$ and $\mathbf{a} \in \{0, \dots, p-1\}^d$ be fixed. Let $\{\cdot\}$ denote the fractional part function. For all $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, we set $\{\mathbf{x}\} := (\{x_1\}, \dots, \{x_d\})$. Since $|e| = |f|$, we have

$$v_p\big(\mathcal{Q}_{e,f}(\mathbf{a} + \mathbf{n}p)\big) = \sum_{\ell=1}^{\infty} \Delta_{e,f}\left(\left\{\frac{\mathbf{a} + \mathbf{n}p}{p^\ell}\right\}\right) \geq \Delta_{e,f}\left(\frac{\mathbf{a}}{p}\right),$$

because $\Delta_{e,f}$ is nonnegative on $[0,1]^d$. On the one hand, if $\mathbf{a}/p \in \mathcal{D}_{e,f}$, then we have both $\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{n}p) \equiv 0 \mod p$ and $\mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n}) \equiv 0 \mod p$. On the other hand, if $\mathbf{a}/p \notin \mathcal{D}_{e,f}$, then, for all $\mathbf{d}$ in e or f, we have $\lfloor \mathbf{d} \cdot \mathbf{a}/p \rfloor = 0$ so that (2.1) yields

$$\mathcal{Q}_{e,f}(\mathbf{a} + \mathbf{n}p) \equiv \mathcal{Q}_{e,f}(\mathbf{a})\mathcal{Q}_{e,f}(\mathbf{n}) \mod p\mathbb{Z}_p,$$

as expected. This proves Assertion (1) in Theorem 3.

Now we prove Assertion (2) in Theorem 3. If $|e| \neq |f|$ then, since $\Delta_{e,f}$ is nonnegative on $[0,1]^d$, there exists $k \in \{1, \dots, d\}$ such that $|e|^{(k)} - |f|^{(k)} = \Delta_{e,f}(\mathbf{1}_k) \geq 1$. Thereby, for almost all primes p, we have

$$v_p\big(\mathcal{Q}_{e,f}(\mathbf{1}_k + \mathbf{1}_k p)\big) = \sum_{\ell=1}^{\infty} \Delta_{e,f}\left(\frac{\mathbf{1}_k + \mathbf{1}_k p}{p^\ell}\right) \geq \Delta_{e,f}\left(\mathbf{1}_k + \frac{\mathbf{1}_k}{p}\right) \geq 1,$$

but $v_p\big(\mathcal{Q}_{e,f}(\mathbf{1}_k)\big) = 0$ so that $\mathcal{Q}_{e,f}$ does not satisfy the p-Lucas property.

Throughout the rest of this proof, we assume that $|e| = |f|$. According to Section 7.3.2 in [11], there exist $k \in \{1, \dots, d\}$ and a rational fraction $R(X) \in \mathbb{Q}(X)$, $R(X) \neq 1$, such that, for all large enough prime numbers p, we can choose $\mathbf{a}_p \in \{0, \dots, p-1\}^d$ satisfying $\mathcal{Q}_{e,f}(\mathbf{a}_p) \in \mathbb{Z}_p^\times$, and such that, for all $n \in \mathbb{N}$, we have (see [11, (7.10)])

$$\mathcal{Q}_{e,f}(\mathbf{a}_p + \mathbf{1}_k np) \in R(n)\mathcal{Q}_{e,f}(\mathbf{a}_p)\mathcal{Q}_{e,f}(\mathbf{1}_k n)(1 + p\mathbb{Z}_p).$$

We fix $n \in \mathbb{N}$ satisfying $R(n) \neq 1$. For almost all primes p, $R(n)$, $\mathcal{Q}_{e,f}(\mathbf{1}_k n)$ and $\mathcal{Q}_{e,f}(\mathbf{a}_p)$ belong to $\mathbb{Z}_p^\times$, and $R(n) \not\equiv 1 \mod p\mathbb{Z}_p$. Thus, we obtain

$$\mathcal{Q}_{e,f}(\mathbf{a}_p + \mathbf{1}_k np) \not\equiv \mathcal{Q}_{e,f}(\mathbf{a}_p)\mathcal{Q}_{e,f}(\mathbf{1}_k n) \mod p\mathbb{Z}_p,$$

which finishes the proof of Assertion (2) in Theorem 3.

Now we assume that $|e| = |f|$ and that, for all $\mathbf{x} \in \mathcal{D}_{e,f}$, we have $\Delta_{e,f}(\mathbf{x}) \geq 1$. Hence, for every prime p, we have

$$\mathcal{Z}_p(\mathcal{Q}_{e,f}) = \big\{\mathbf{v} \in \{0,\dots,p-1\}^d \,:\, \mathbf{v}/p \in \mathcal{D}_{e,f}\big\}.$$

Furthermore, if $\mathbf{v}/p \in \mathcal{D}_{e,f}$, then, for all positive integers N and all $\mathbf{a}_0,\dots,\mathbf{a}_{N-1} \in \{0,\dots,p-1\}^d$, we have

$$\frac{\mathbf{v}}{p} \leq \left\{\frac{\mathbf{a}_0 + \mathbf{a}_1 p + \cdots + \mathbf{a}_{N-1}p^{N-1} + \mathbf{v}p^N}{p^{N+1}}\right\} \in \mathcal{D}_{e,f},$$

so that, for all $\mathbf{n} \in \mathbb{N}^d$, $\mathbf{n} = \sum_{k=0}^{\infty} \mathbf{n}_k p^k$ with $\mathbf{n}_k \in \{0,\dots,p-1\}^d$, we have

$$v_p\big(\mathcal{Q}_{e,f}(\mathbf{n})\big) = \sum_{\ell=1}^{\infty} \Delta_{e,f}\left(\left\{\frac{\sum_{k=0}^{\ell-1}\mathbf{n}_k p^k}{p^\ell}\right\}\right) \geq \alpha_p(\mathcal{Q}_{e,f},\mathbf{n}),$$

and Theorem 3 is proved.

3. Proofs of Theorem 2 and Proposition 1

3.1. **Induction *via* Apéry-like recurrence relations.** In this section, we fix a prime p. If A is a $\mathbb{Z}_p$-valued sequence, then, for all $r \in \mathbb{N}$, we write $\mathcal{U}_A(r)$ for the assertion "For all $n, i \in \mathbb{N}$, $i \leq r$, if $\alpha_p(A,n) \geq i$, then $A(n) \in p^i\mathbb{Z}_p$". As a first step, we shall prove the following result.

Lemma 1. *Let A be a $\mathbb{Z}_p$-valued sequence satisfying the p-Lucas property with $A(0) \in \mathbb{Z}_p^\times$. Assume that f_A is canceled by a differential operator $\mathcal{L} \in \mathbb{Z}_p[z,\theta]$ such that at least one of the following conditions holds:*

- *$\mathcal{L}$ is of type I.*
- *$\mathcal{L}$ is of type II and $p-1 \in \mathcal{Z}_p(A)$.*

Let $r \in \mathbb{N}$ be such that $\mathcal{U}_A(r)$ holds. Then, for all $n_0 \in \mathcal{Z}_p(A)$ and all $m \in \mathbb{N}$ satisfying $\alpha_p(A,m) \geq r$, we have

$$A(n_0 + mp) \in p^{r+1}\mathbb{Z}_p.$$

Proof. Since A satisfies the p-Lucas property, we can assume that $r \geq 1$. The series $f_A(z)$ is canceled by a differential operator $\mathcal{L} = \sum_{k=0}^{q} z^k P_k(\theta)$ with $q \in \mathbb{N}$, $P_k(X) \in \mathbb{Z}_p[X]$ and $P_0(\mathbb{Z}_p^\times) \subset \mathbb{Z}_p^\times$. Thus, for all $n \in \mathbb{N}$, we have

$$\sum_{k=0}^{q} P_k(n-k)A(n-k) = 0. \tag{3.1}$$

We fix $m \in \mathbb{N}$ satisfying $\alpha_p(A,m) \geq r$. In particular, since $r \geq 1$ and $A(0) \in \mathbb{Z}_p^\times$, we have $m \geq 1$. Furthermore, for all $v \in \{0,\dots,p-1\}$, we also have $\alpha_p(A, v+mp) \geq r$. According to $\mathcal{U}_A(r)$, we obtain that, for all $v \in \{0,\dots,p-1\}$, we have $A(v+mp) \in p^r\mathbb{Z}_p$ so that $A(v+mp) =: \beta(v,m)p^r$, with $\beta(v,m) \in \mathbb{Z}_p$.

By (3.1), for all $v \in \{q, \dots, p-1\}$, we have

$$0 = \sum_{k=0}^{q} P_k(v-k+mp)A(v-k+mp) = p^r \sum_{k=0}^{q} P_k(v-k+mp)\beta(v-k,m)$$
$$\equiv p^r \sum_{k=0}^{q} P_k(v-k)\beta(v-k,m) \mod p^{r+1}\mathbb{Z}_p,$$

because, for all $P \in \mathbb{Z}_p[X]$ and all $a, c \in \mathbb{Z}$, we have $P(a+cp) \equiv P(a) \mod p\mathbb{Z}_p$. Thus, for all $v \in \{q, \dots, p-1\}$, we obtain

$$\sum_{k=0}^{q} P_k(v-k)\beta(v-k,m) \equiv 0 \mod p\mathbb{Z}_p. \tag{3.2}$$

We claim that if $v \in \{1, \dots, q-1\}$, then, for all $k \in \{v+1, \dots, q\}$, we have

$$P_k(v+mp-k)A(v+mp-k) \in p^{r+1}\mathbb{Z}_p. \tag{3.3}$$

Indeed, on the one hand, if $\mathcal{L}$ is of type II, then we have $q = 2$ and $P_2(X) \in (X+1)\mathbb{Z}_p[X]$ which yields

$$P_2(-1+mp)A(-1+mp) \in pA\big(p-1+(m-1)p\big)\mathbb{Z}_p.$$

Since $0 \notin \mathcal{Z}_p(A)$, we have $\alpha_p(A, m-1) \geq r-1$ which, together with $p-1 \in \mathcal{Z}_p(A)$, leads to

$$\alpha_p\big(A, p-1+(m-1)p\big) \geq r.$$

According to $\mathcal{U}_A(r)$, we obtain $pA\big(p-1+(m-1)p\big) \in p^{r+1}\mathbb{Z}_p$, as expected. On the other hand, if $\mathcal{L}$ is of type I, then for all $v \in \{1, \dots, q-1\}$ and all $k \in \{v+1, \dots, q\}$, we have

$$v_p\big(P_k(v+mp-k)\big) \geq v_p\left(\prod_{i=1}^{k-1}(v+mp-k+i)^2\right).$$

Writing $k - v = a + bp$ with $a \in \{0, \dots, p-1\}$ and $b \in \mathbb{N}$, we obtain $k-1 \geq a+bp$ so that

$$v_p\left(\prod_{i=1}^{k-1}(mp+i-a-bp)\right) \geq \begin{cases} b & \text{if } a = 0; \\ b+1 & \text{if } a \geq 1. \end{cases}.$$

Thus, it is enough to prove that

$$A(v+mp-k) \in \begin{cases} p^{r+1-2b}\mathbb{Z}_p & \text{if } a = 0; \\ p^{r-1-2b}\mathbb{Z}_p & \text{if } a \geq 1. \end{cases}. \tag{3.4}$$

We have $v+mp-k = -a+(m-b)p$. If $-a+(m-b)p < 0$, then $A(v+mp-k) = 0$ and (3.4) holds. If $m-b \geq 0$, then we have $\alpha_p(A, m-b) \geq r-b$. Thus, we have either $a = 0$ and $\alpha_p(A, v+mp-k) \geq r-b$, or $a, m-b \geq 1$ and

$$\alpha_p(A, v+mp-k) = \alpha_p\big(A, p-a+(m-b-1)p\big) \geq r-b-1.$$

Hence Assertion $\mathcal{U}_A(r)$ yields

$$A(v+mp-k) \in \begin{cases} p^{r-b}\mathbb{Z}_p & \text{if } a=0; \\ p^{r-1-b}\mathbb{Z}_p & \text{if } a \geq 1. \end{cases}.$$

If $a=0$, then $b \geq 1$ so that (3.4) holds and (3.3) is proved.

By (3.3), for all $v \in \mathbb{N}$, $1 \leq v \leq \min(q-1, p-1)$, we have

$$\begin{aligned} 0 &= \sum_{k=0}^{q} P_k(v-k+mp)A(v-k+mp) \\ &\equiv \sum_{k=0}^{v} P(v-k+mp)A(v-k+mp) \mod p^{r+1}\mathbb{Z}_p \\ &\equiv p^r \sum_{k=0}^{v} P_k(v-k+mp)\beta(v-k,m) \mod p^{r+1}\mathbb{Z}_p \\ &\equiv p^r \sum_{k=0}^{v} P_k(v-k)\beta(v-k,m) \mod p^{r+1}\mathbb{Z}_p. \end{aligned}$$

Thus, for all $v \in \mathbb{N}$, $1 \leq v \leq \min(q-1, p-1)$, we have

$$\sum_{k=0}^{v} P_k(v-k)\beta(v-k,m) \equiv 0 \mod p\mathbb{Z}_p. \tag{3.5}$$

Both sequences $\big(\beta(v,m)\big)_{0\leq v\leq p-1}$ and $\big(A(v)\big)_{0\leq v\leq p-1}$ satisfy Equations (3.2) and (3.5). Furthermore, for all $v \in \{1,\dots,p-1\}$, we have $P_0(v), A(0) \in \mathbb{Z}_p^{\times}$. Hence there exists $\gamma(m) \in \{0,\dots,p-1\}$ such that, for all $v \in \{0,\dots,p-1\}$, we have $\beta(v,m) \equiv A(v)\gamma(m) \mod p\mathbb{Z}_p$ so that

$$A(v+mp) \equiv A(v)\gamma(m)p^r \mod p^{r+1}\mathbb{Z}_p.$$

Since $n_0 \in \mathcal{Z}_p(A)$, we obtain $A(n_0+mp) \in p^{r+1}\mathbb{Z}_p$ and Lemma 1 is proved. □

3.2. **Proof of Theorem 2.** Let p be a fixed prime number. For every positive integer n, we set $\ell(n) := \lfloor \log_p(n) \rfloor + 1$ the length of the expansion of n to the base p, and $\ell(0) := 1$. For all positive integers $r, n_1, \dots, n_r$, we set

$$n_1 * \cdots * n_r := n_1 + n_2 p^{\ell(n_1)} + \cdots + n_r p^{\ell(n_1)+\cdots+\ell(n_{r-1})},$$

so that the expansion of $n_1 * \cdots * n_r$ to the base p is the concatenation of the respective expansions of $n_1, \dots, n_r$. Then, by a result of Mellit and Vlasenko [17, Lemma 1], there exists a $\mathbb{Z}_p$-valued sequence $(c_n)_{n\geq 0}$ such that, for all $n \geq 1$, we have

$$A(n) = \sum_{\substack{n_1*\cdots*n_r=n \\ 1\leq r\leq \ell(n),\, n_i\geq 0}} c_{n_1}\cdots c_{n_r} \quad \text{and} \quad c_n \equiv 0 \mod p^{\ell(n)-1}\mathbb{Z}_p. \tag{3.6}$$

For all $r \in \mathbb{N}$, we write $\mathcal{U}(r)$ for the assertion: "For all $n, i \in \mathbb{N}$, $i \le r$, if $\alpha_p(A,n) \ge i$, then $A(n), c_n \in p^i\mathbb{Z}_p$". To prove Theorem 2, it suffices to show that, for all $r \in \mathbb{N}$, Assertion $\mathcal{U}(r)$ holds.

First we prove $\mathcal{U}(1)$. By Theorem 1 of [17], A satisfies the p-Lucas property. In addition, if $v \in \mathcal{Z}_p(A)$, then $v \neq 0$ because $A(0) = 1$, and by (3.6) we have $c_v = A(v) \in p\mathbb{Z}_p$. Now, if $n \in \mathbb{N}$ satisfies $\ell(n) = 2$ and $\alpha_p(A,n) \ge 1$, then Equation (3.6) yields $A(n) \equiv c_n \mod p\mathbb{Z}_p$, so that $c_n \in p\mathbb{Z}_p$. Hence, by induction on $\ell(n)$, we obtain that, for all $n \in \mathbb{N}$ satisfying $\alpha_p(A,n) \ge 1$, we have $c_n \in p\mathbb{Z}_p$, so that $\mathcal{U}(1)$ holds.

Let r be a positive integer such that $\mathcal{U}(r)$ holds. We shall prove that $\mathcal{U}(r+1)$ is true. For all positive integers M, we write $\mathcal{U}_M(r+1)$ for the assertion: "For all $n, i \in \mathbb{N}$, $n \le M$, $i \le r+1$, if $\alpha_p(A,n) \ge i$, then $A(n), c_n \in p^i\mathbb{Z}_p$". Hence $\mathcal{U}_M(r+1)$ is true if $\ell(M) \le r$. Let M be a positive integer such that $\mathcal{U}_M(r+1)$ holds. We shall prove $\mathcal{U}_{M+1}(r+1)$. By Assertions $\mathcal{U}(r)$ and $\mathcal{U}_M(r+1)$, it suffices to prove that if $\alpha_p(A, M+1) \ge r+1$, then $A(M+1), c_{M+1} \in p^{r+1}\mathbb{Z}_p$. In the rest of the proof, we assume that $\alpha_p(A, M+1) \ge r+1$.

If $2 \le u \le \ell(M+1)$ and $n_1, \dots, n_u \in \mathbb{N}$ satisfy $n_1 * \cdots * n_u = M+1$, then, for all $i \in \{1, \dots, u\}$, we have $n_i \le M$ and $\alpha_p(A,n_1) + \cdots + \alpha_p(A,n_u) = \alpha_p(A, M+1) \ge r+1$. Then there exist $1 \le a_1 < \cdots < a_k \le u$, $1 \le i_1, \dots, i_k \le r+1$, such that $\alpha_p(A, n_{a_j}) \ge i_j$ and $i_1 + \cdots + i_k \ge r+1$. Thereby, Assertion $\mathcal{U}_M(r+1)$ yields $c_{n_1} \cdots c_{n_u} \in p^{r+1}\mathbb{Z}_p$, so that

$$\sum_{\substack{n_1 * \cdots * n_u = M+1 \\ 2 \le u \le \ell(M+1),\, n_i \ge 0}} c_{n_1} \cdots c_{n_u} \in p^{r+1}\mathbb{Z}_p.$$

By (3.6), we obtain

$$A(M+1) \equiv c_{M+1} \mod p^{r+1}\mathbb{Z}_p \quad \text{and} \quad c_{M+1} \equiv 0 \mod p^{\ell(M+1)-1}\mathbb{Z}_p.$$

Hence it suffices to consider the case $\ell(M+1) = r+1$. In particular, we have $M+1 = v + mp$ with $v \in \mathcal{Z}_p(A)$ and $m \in \mathbb{N}$, $\alpha_p(A,m) = r$. Since $\mathcal{U}(r)$ holds, Lemma 1 yields $A(M+1) \in p^{r+1}\mathbb{Z}_p$. Thus we also have $c_{M+1} \in p^{r+1}\mathbb{Z}_p$ and Assertion $\mathcal{U}_{M+1}(r+1)$ holds. This finishes the proof of $\mathcal{U}(r+1)$ so that of Theorem 2. □

3.3. **Proof of Proposition 1.** Let p be a prime and A a $\mathbb{Z}_p$-valued sequence satisfying hypothesis of Proposition 1. For all $n \in \mathbb{N}$, we write $\alpha(n)$, respectively $\mathcal{Z}$, as a shorthand for $\alpha_p(A,n)$, respectively for $\mathcal{Z}_p(A)$. For all $r \in \mathbb{N}$, we define assertions

$$\mathcal{U}(r): \text{ "For all } n, i \in \mathbb{N},\ i \le r, \text{ if } \alpha(n) \ge i, \text{ then } A(n) \in p^i\mathbb{Z}_p.\text{"},$$

and

$$\mathcal{V}(r): \text{ "For all } n, i \in \mathbb{N},\ i \le r, \text{ and all } B \in \mathfrak{B}, \text{ if } \alpha(n) \ge i, \text{ then } B(n) \in p^{i-1}\mathbb{Z}_p\text{"}.$$

To prove Proposition 1, we have to show that, for all $r \in \mathbb{N}$, Assertions $\mathcal{U}(r)$ and $\mathcal{V}(r)$ are true. We shall prove those assertions by induction on r.

Observe that Assertions $\mathcal{U}(0)$, $\mathcal{V}(0)$ and $\mathcal{V}(1)$ are trivial. Furthermore, since A satisfies the p-Lucas property, Assertion $\mathcal{U}(1)$ holds. Let r_0 be a fixed positive integer, $r_0 \ge 2$, such that Assertions $\mathcal{U}(r_0 - 1)$ and $\mathcal{V}(r_0 - 1)$ are true. First, we prove Assertion $\mathcal{V}(r_0)$.

Let $B \in \mathfrak{B}$ and $m \in \mathbb{N}$ be such that $\alpha(m) \geq r_0$. We write $m = v + np$ with $v \in \{0, \dots, p-1\}$. Since $r_0 \geq 2$ and $0 \notin \mathcal{Z}$, we have $n \geq 1$ and, by Assertion (a) in Proposition 1, there exist $A' \in \mathfrak{A}$ and a sequence $(B_k)_{k \geq 0}$, $B_k \in \mathfrak{B}$, such that

$$B(v + np) = A'(n) + \sum_{k=0}^{\infty} p^{k+1} B_k(n-k). \tag{3.7}$$

In addition, we have $\alpha(n) \geq r_0 - 1$ and, since $0 \notin \mathcal{Z}$, we have $\alpha(n-1) \geq r_0 - 2$. By induction, for all $k \in \mathbb{N}$, $k \leq n$, we have $\alpha(n-k) \geq r_0 - 1 - k$. Thus, by (3.7) in combination with $\mathcal{U}(r_0 - 1)$ and $\mathcal{V}(r_0 - 1)$, we obtain

$$A'(n) \in p^{r_0 - 1} \quad \text{and} \quad p^{k+1} B_k(n-k) \in p^{k+1+r_0-2-k}\mathbb{Z}_p \subset p^{r_0-1}\mathbb{Z}_p,$$

so that $B(v + np) \in p^{r_0 - 1}\mathbb{Z}_p$ and $\mathcal{V}(r_0)$ is true.

Now we prove Assertion $\mathcal{U}(r_0)$. We write $\mathcal{U}_N(r_0)$ for the assertion: "For all $n, i \in \mathbb{N}$, $n \leq N$, $i \leq r_0$, if $\alpha(n) \geq i$, then $A(n) \in p^i\mathbb{Z}_p$". We shall prove $\mathcal{U}_N(r_0)$ by induction on N. Assertion $\mathcal{U}_1(r_0)$ holds. Let N be a positive integer such that $\mathcal{U}_N(r_0)$ is true. Let $n := n_0 + mp \leq N + 1$ with $n_0 \in \{0, \dots, p-1\}$ and $m \in \mathbb{N}$. We can assume that $\alpha(n) \geq r_0$.

If $n_0 \in \mathcal{Z}$, then we have $\alpha(m) \geq r_0 - 1$ and, by Lemma 1, we obtain $A(n) \in p^{r_0}\mathbb{Z}_p$ as expected. It remains to consider the case $n_0 \notin \mathcal{Z}$. In this case, we have $\alpha(m) \geq r_0$. By Assertion (a) in Proposition 1, there exist $A' \in \mathfrak{A}$ and a sequence $(B_k)_{k \geq 0}$ with $B_k \in \mathfrak{B}$ such that

$$A(n) = A'(m) + \sum_{k=0}^{\infty} p^{k+1} B_k(m-k).$$

We have $m \leq N$, $\alpha(m) \geq r_0$ and $\alpha(m-k) \geq r_0 - k$, hence, by Assertions $\mathcal{U}_N(r_0)$ and $\mathcal{V}(r_0)$, we obtain $A(n) \in p^{r_0}\mathbb{Z}_p$. This finishes the induction on N and proves $\mathcal{U}(r_0)$. Therefore, by induction on r_0, Proposition 1 is proved. □

4. Proof of Theorem 1

To prove Theorem 1, we shall apply Proposition 1 to $\mathfrak{S}_{e,f}$. As a first step, we prove that this sequence satisfies the p-Lucas property.

Proof of Proposition 2. For all $\mathbf{x} \in [0,1]^d$, we have $\Delta_{e,f}(\mathbf{x}) = \Delta_{e,f}(\{\mathbf{x}\}) \geq 0$ so that, by Landau's criterion, $\mathcal{Q}_{e,f}$ is integer-valued. Let p be a fixed prime, $v \in \{0, \dots, p-1\}$ and $n \in \mathbb{N}$. We have

$$\mathfrak{S}_{e,f}(v + np) = \sum_{\substack{k_1 + \cdots + k_d = v + np \\ k_i \in \mathbb{N}}} \mathcal{Q}_{e,f}(k_1, \dots, k_d).$$

Write $k_i = a_i + m_i p$ with $a_i \in \{0, \dots, p-1\}$ and $m_i \in \mathbb{N}$. If $a_1 + \cdots + a_d \neq v$, then we have $a_1 + \cdots + a_d \geq p$ and there exists $i \in \{1, \dots, d\}$ such that $a_i \geq p/d$. Since e is 1-admissible, we have $(a_1, \dots, a_d)/p \in \mathcal{D}_{e,f}$ so that $\Delta_{e,f}\big((a_1, \dots, a_p)/p\big) \geq 1$ and $\mathcal{Q}_{e,f}(k_1, \dots, k_d) \in p\mathbb{Z}_p$.

In addition, by Theorem 3, $\mathcal{Q}_{e,f}$ satisfies the p-Lucas property for all primes p. Hence we obtain

$$\begin{aligned}\mathfrak{S}_{e,f}(v+np) &\equiv \sum_{\substack{a_1+\cdots+a_d=v\\0\leq a_i\leq p-1}}\sum_{\substack{m_1+\cdots+m_d=n\\m_i\in\mathbb{N}}}\mathcal{Q}_{e,f}(a_1+m_1p,\dots,a_d+m_dp)\mod p\mathbb{Z}_p\\&\equiv \sum_{\substack{a_1+\cdots+a_d=v\\0\leq a_i\leq p-1}}\sum_{\substack{m_1+\cdots+m_d=n\\m_i\in\mathbb{N}}}\mathcal{Q}_{e,f}(a_1,\dots,a_d)\mathcal{Q}_{e,f}(m_1,\dots,m_d)\mod p\mathbb{Z}_p\\&\equiv \mathfrak{S}_{e,f}(v)\mathfrak{S}_{e,f}(n)\mod p\mathbb{Z}_p.\end{aligned}$$

This finishes the proof of Proposition 2. □

If e is 2-admissible then e is also 1-admissible. Furthermore, if $f=(\mathbf{1}_{k_1},\dots,\mathbf{1}_{k_v})$, then, for all $\mathbf{x}\in\mathcal{D}_{e,f}$, we have

$$\Delta_{e,f}(\mathbf{x})=\sum_{i=1}^{u}\lfloor\mathbf{e}_i\cdot\mathbf{x}\rfloor\geq 1.$$

Hence, if e and f satisfy the conditions of Theorem 1, then Proposition 2 implies that, for all primes p, $\mathfrak{S}_{e,f}$ has the p-Lucas property and $\mathfrak{S}_{e,f}(0)=1\in\mathbb{Z}_p^\times$. Thereby, to prove Theorem 1, it remains to prove that $\mathfrak{S}_{e,f}$ satisfies Condition (a) in Proposition 1 with

$$\mathfrak{B}=\{\mathfrak{S}_{e,f}^g\,:\,g\in\mathfrak{F}_p^d\}.$$

First we prove that some special functions belong to $\mathfrak{F}_p$.

4.1. **Special functions in $\mathfrak{F}_p$.** For all primes p, we write $|\cdot|_p$ for the ultrametric norm on $\mathbb{Q}_p$ (the field of p-adic numbers) defined by $|a|_p:=p^{-v_p(a)}$. Note that $(\mathbb{Z}_p,|\cdot|_p)$ is a compact space. Furthermore, if $(c_n)_{n\geq0}$ is a $\mathbb{Z}_p$-valued sequence, then $\sum_{n=0}^\infty c_n$ is convergent in $(\mathbb{Z}_p,|\cdot|_p)$ if and only if $|c_n|_p\underset{n\to\infty}{\longrightarrow}0$. In addition, if $\sum_{n=0}^\infty c_n$ converges, then $(c_n)_{n\in\mathbb{N}}$ is a summable family in $(\mathbb{Z}_p,|\cdot|_p)$.

In the rest of the article, for all primes p and all positive integers k, we set $\Psi_{p,k,0}(0)=1$, $\Psi_{p,k,i}(0)=0$ for $i\geq1$ and, for all $i,m\in\mathbb{N}$, $m\geq1$, we set

$$\Psi_{p,k,i}(m):=(-1)^i\sigma_{m,i}\left(\frac{1}{k},\frac{1}{k+p},\dots,\frac{1}{k+(m-1)p}\right),$$

where $\sigma_{m,i}$ is the i-th elementary symmetric polynomial of m variables. Let us remind to the reader that, for all positive integers m and all $i\in\mathbb{N}$, $i>m$, we have $\sigma_{m,i}=0$.

The aim of this section is to prove that, for all primes p, all $k\in\{1,\dots,p-1\}$ and all $i\in\mathbb{N}$, we have

$$i!\Psi_{p,k,i}\in\mathfrak{F}_p. \tag{4.1}$$

Proof of (4.1). Throughout this proof, we fix a prime number p and an integer $k\in\{1,\dots,p-1\}$. Furthermore, for all nonnegative integers i, we use Ψ_i as a shorthand for $\Psi_{p,k,i}$ and $\mathbb{N}_{\geq i}$ as a shorthand for the set of integers larger than or equal to i. We shall prove (4.1) by induction on i. To that end, for all $i\in\mathbb{N}$, we write $\mathcal{A}_i$ for the following assertion:

"There exists a sequence $(T_{i,r})_{r\geq 0}$ of polynomial functions with coefficients in $\mathbb{Z}_p$ which converges uniformly to $i!\Psi_i$ on $\mathbb{N}$".

First, observe that, for all $m\in\mathbb{N}$, we have $\Psi_0(m)=1$, so that assertion $\mathcal{A}_0$ is true. Let i be a fixed positive integer such that assertions $\mathcal{A}_0,\dots,\mathcal{A}_{i-1}$ are true. According to the Newton-Girard formulas, for all $m\in\mathbb{N}$, $m\geq i$, we have

$$i(-1)^i\sigma_{m,i}(X_1,\dots,X_m) = -\sum_{t=1}^{i}(-1)^{i-t}\sigma_{m,i-t}(X_1,\dots,X_m)\Lambda_t(X_1,\dots,X_m),$$

where $\Lambda_t(X_1,\dots,X_m) := \sum_{s=1}^{m} X_s^t$. Thereby, for all $m\geq i$, we have

$$i\Psi_i(m) = -\sum_{t=1}^{i}\Psi_{i-t}(m)\Lambda_t\left(\frac{1}{k},\dots,\frac{1}{k+(m-1)p}\right). \tag{4.2}$$

For all $j,t\in\mathbb{N}$, we have

$$\frac{1}{(k+jp)^t} = \frac{1}{k^t}\frac{1}{(1+\frac{j}{k}p)^t} = \frac{1}{k^t} + \sum_{s=1}^{\infty}\frac{(-1)^s}{k^t}\binom{t-1+s}{s}\left(\frac{j}{k}\right)^s p^s, \tag{4.3}$$

where the right hand side of (4.3) is a convergent series in $(\mathbb{Z}_p,|\cdot|_p)$ because $k\in\mathbb{Z}_p^\times$. Therefore, we obtain that

$$\begin{aligned}\Lambda_t\left(\frac{1}{k},\dots,\frac{1}{k+(m-1)p}\right) &= \frac{m}{k^t} + \sum_{j=0}^{m-1}\sum_{s=1}^{\infty}\frac{(-1)^s}{k^t}\binom{t-1+s}{s}\left(\frac{j}{k}\right)^s p^s\\ &= \frac{m}{k^t} + \sum_{s=1}^{\infty}\frac{(-1)^s}{k^{t+s}}\binom{t-1+s}{s}p^s\left(\sum_{j=0}^{m-1}j^s\right). \end{aligned}\tag{4.4}$$

According to Faulhaber's formula, for all positive integers s, we have

$$p^s\sum_{j=0}^{m-1}j^s = \sum_{c=1}^{s+1}(-1)^{s+1-c}\binom{s+1}{c}p^s\frac{B_{s+1-c}}{s+1}(m-1)^c,$$

where B_k is the k-th first Bernoulli number. For all positive integers s and t, we set $R_{0,t}(X) := X/k^t$ and

$$R_{s,t}(X) := \frac{1}{k^{t+s}}\binom{t-1+s}{s}\sum_{c=1}^{s+1}(-1)^{1-c}\binom{s+1}{c}p^s\frac{B_{s+1-c}}{s+1}(X-1)^c,$$

so that

$$\Lambda_t\left(\frac{1}{k},\dots,\frac{1}{k+(m-1)p}\right) = \sum_{s=0}^{\infty}R_{s,t}(m).$$

In the rest of this article, for all polynomials $P(X)=\sum_{n=0}^{N}a_nX^n\in\mathbb{Z}_p[X]$, we set

$$\|P\|_p := \max\left\{|a_n|_p \,:\, 0\leq n\leq N\right\}.$$

We claim that, for all $s, t \in \mathbb{N}$, $t \geq 1$, we have

$$R_{s,t}(X) \in \mathbb{Z}_p[X],\ \|R_{s,t}\|_p \underset{s\to\infty}{\longrightarrow} 0 \quad \text{and} \quad R_{s,t}(0) = 0. \tag{4.5}$$

Indeed, on the one hand, if $p = 2$ and $s = 1$, then we have

$$R_{1,t}(X) = \frac{-t}{k^{t+1}}\left(X - 1 + (X-1)^2\right) \in X\mathbb{Z}_2[X].$$

On the other hand, if $p \geq 3$ or $s \geq 2$, then we have $p^s > s+1$ so that $v_p(s+1) \leq s-1$. Furthermore, according to the von Staudt-Clausen theorem, we have $v_p(B_{s+1-c}) \geq -1$. Thus, we obtain that $R_{s,t}(X) \in \mathbb{Z}_p[X]$. To be more precise, we have $v_p(s+1) \leq \log_p(s+1)$, so that $\|R_{s,t}\|_p \underset{s\to\infty}{\longrightarrow} 0$ as expected. In addition, we have

$$\begin{aligned}
R_{s,t}(0) &= -\frac{p^s}{(s+1)k^{t+s}}\binom{t-1+s}{s}\sum_{c=1}^{s+1}\binom{s+1}{c}B_{s+1-c} \\
&= -\frac{p^s}{(s+1)k^{t+s}}\binom{t-1+s}{s}\sum_{d=0}^{s}\binom{s+1}{d}B_d = 0,
\end{aligned}$$

where we used the well known reccurence relation satisfied by the Bernoulli numbers

$$\sum_{d=0}^{s}\binom{s+1}{d}B_d = 0,\ (s \geq 1).$$

According to $\mathcal{A}_0, \dots, \mathcal{A}_{i-1}$, for all $j \in \{0, \dots, i-1\}$, there exists a sequence $(T_{j,r})_{r\geq 0}$ of polynomial functions with coefficients in $\mathbb{Z}_p$ which converges uniformly to $j!\Psi_j$ on $\mathbb{N}$. According to (4.2) and (4.5), for all $N \in \mathbb{N}$, there exists $S_N \in \mathbb{N}$ such that, for all $r \geq S_N$ and all $m \geq i$, we have

$$i!\Psi_i(m) \equiv -\sum_{t=1}^{i}\frac{(i-1)!}{(i-t)!}T_{i-t,r}(m)\sum_{s=0}^{r}R_{s,t}(m) \mod p^N\mathbb{Z}_p.$$

Thus, the sequence $(T_{i,r})_{r\geq 0}$ of polynomial functions with coefficients in $\mathbb{Z}_p$, defined by

$$T_{i,r}(x) := -\sum_{t=1}^{i}\frac{(i-1)!}{(i-t)!}T_{i-t,r}(x)\sum_{s=0}^{r}R_{s,t}(x),\ (x, r \in \mathbb{N}), \tag{4.6}$$

converges uniformly to $i!\Psi_i$ on $\mathbb{N}_{\geq i}$. To prove $\mathcal{A}_i$, it suffices to show that, for all $m \in \{0, \dots, i-1\}$, we have

$$T_{i,r}(m) \underset{r\to\infty}{\longrightarrow} 0. \tag{4.7}$$

Observe that Equations (4.6) and (4.5) lead to $T_{i,r}(0)=0$. In particular, if $i=1$, then (4.7) holds. Now we assume that $i\geq 2$. For all $m\geq 2$, we have

$$\begin{aligned}\sum_{j=0}^{m}\Psi_j(m)X^j &= \prod_{w=0}^{m-1}\left(1-\frac{X}{k+wp}\right)\\ &= \left(1-\frac{X}{k+(m-1)p}\right)\prod_{w=0}^{m-2}\left(1-\frac{X}{k+wp}\right)\\ &= \left(1-\frac{X}{k+(m-1)p}\right)\sum_{j=0}^{m-1}\Psi_j(m-1)X^j.\end{aligned}$$

Thereby, for all $j\in\{1,\dots,m-1\}$, we obtain that

$$\Psi_j(m)=\Psi_j(m-1)-\frac{\Psi_{j-1}(m-1)}{k+(m-1)p},$$

with

$$\frac{1}{k+(m-1)p}=\sum_{s=0}^{\infty}\frac{(-1)^s}{k^{s+1}}p^s(m-1)^s.$$

Thus, there exists a sequence $(U_r)_{r\geq 0}$ of polynomials with coefficients in $\mathbb{Z}_p$ such that, for all positive integers N, there exits $S_N\in\mathbb{N}$ such that, for all $r\geq S_N$ and all $m\geq i+1$, we have

$$T_{i,r}(m)\equiv T_{i,r}(m-1)-T_{i-1,r}(m-1)U_r(m-1)\mod p^N\mathbb{Z}_p. \tag{4.8}$$

But, if $V_1(X),V_2(X)\in\mathbb{Z}_p[X]$ and if there exists $a\in\mathbb{N}$ such that, for all $m\geq a$, we have $V_1(m)\equiv V_2(m)\mod p^N\mathbb{Z}_p$, then, for all $n\in\mathbb{Z}$, we have $V_1(n)\equiv V_2(n)\mod p^N\mathbb{Z}_p$. Indeed, let n be an integer, there exists $v\in\mathbb{N}$ satisfying $n+vp^N\geq a$. Thus, we obtain that $V_1(n)\equiv V_1(n+vp^N)\equiv V_2(n+vp^N)\equiv V_2(n)\mod p^N\mathbb{Z}_p$. In particular, Equation (4.8) also holds for all positive integers m.

Furthermore, according to $\mathcal{A}_{i-1}$, for all $m\in\{0,\dots,i-2\}$, we have $T_{i-1,r}(m)\underset{r\to\infty}{\longrightarrow}0$. Thus, for all positive integers N, there exists $S_N\in\mathbb{N}$, such that, for all $r\geq S_N$ and all $m\in\{1,\dots,i-1\}$, we have

$$T_{i,r}(m)\equiv T_{i,r}(m-1)\mod p^N\mathbb{Z}_p.$$

Since $T_{i,r}(0)=0$, we obtain that $T_{i,r}(m)\equiv 0\mod p^N\mathbb{Z}_p$ for all $m\in\{0,\dots,i-1\}$, so that (4.7) holds. This finishes the induction on i and proves (4.1). □

4.2. **On the p-adic Gamma function.** For every prime p, we write Γ_p for the p-adic Gamma function, so that, for all $n\in\mathbb{N}$, we have

$$\Gamma_p(n)=(-1)^n\prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{n-1}\lambda.$$

The aim of this section is to prove Proposition 3.

Proof of Proposition 3. Let p be a fixed prime number. For all $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
\frac{\Gamma_p\big((m+n)p\big)}{\Gamma_p(mp)\Gamma_p(np)} &= \left(\prod_{\substack{\lambda=np \\ p\nmid\lambda}}^{(m+n)p} \lambda\right) \Bigg/ \left(\prod_{\substack{\lambda=1 \\ p\nmid\lambda}}^{mp} \lambda\right) \\
&= \left(\prod_{\substack{\lambda=1 \\ p\nmid\lambda}}^{mp} (np+\lambda)\right) \Bigg/ \left(\prod_{\substack{\lambda=1 \\ p\nmid\lambda}}^{mp} \lambda\right) \\
&= \prod_{\substack{\lambda=1 \\ p\nmid\lambda}}^{mp} \left(1 + \frac{np}{\lambda}\right). \qquad (4.9)
\end{aligned}$$

Let $X, T_1, \dots, T_m$ be $m+1$ variables. Then, we have

$$\prod_{j=1}^{m}(X - T_j) = X^m + \sum_{i=1}^{\infty}(-1)^i \sigma_{m,i}(T_1, \dots, T_m) X^{m-i}.$$

Therefore, we obtain

$$\begin{aligned}
\prod_{\substack{\lambda=1 \\ p\nmid\lambda}}^{mp} \left(1 + \frac{np}{\lambda}\right) &= \prod_{k=1}^{p-1} \prod_{\omega=0}^{m-1} \left(1 + \frac{np}{k + \omega p}\right) \\
&= \prod_{k=1}^{p-1} \left(1 + \sum_{i=1}^{\infty} (-1)^i \sigma_{m,i}\left(\frac{-np}{k}, \cdots, \frac{-np}{k + (m-1)p}\right)\right) \\
&= \prod_{k=1}^{p-1} \left(1 + \sum_{i=1}^{\infty} (-1)^i n^i p^i \Psi_{p,k,i}(m)\right). \qquad (4.10)
\end{aligned}$$

Let $k \in \{1, \dots, p-1\}$ be fixed. By (4.1), for all positive integers i, there exists a sequence $(P_{i,\ell})_{\ell \geq 0}$ of polynomial functions with coefficients in $\mathbb{Z}_p$ which converges pointwise to $i!\Psi_{p,k,i}$. We fix $K \in \mathbb{N}$. For all positive integers N, we set

$$f_N(x, y) := 1 + \sum_{i=1}^{K+1} (-1)^i x^i \frac{p^i}{i!} P_{i,N}(y).$$

If $n, m \in \{0, \dots, K\}$, then we have

$$\begin{aligned}
R_N :&= 1 + \sum_{i=1}^{\infty} (-1)^i n^i p^i \Psi_{p,k,i}(m) - f_N(n, m) \\
&= \sum_{i=1}^{K+1} (-1)^i n^i \frac{p^i}{i!} \big(i!\Psi_{p,k,i}(m) - P_{i,N}(m)\big) \underset{N\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Furthermore, we have $f_N(x,y) \in 1 + p\mathbb{Z}_p[x,y]$. Indeed, if $i = i_0 + i_1 p + \cdots + i_a p^a$ with $i_j \in \{0,\dots,p-1\}$, then we set $\mathfrak{s}_p(i) := i_0 + \cdots + i_a$ so that, for all positive integers i, we have

$$i - v_p(i!) = i - \frac{i - \mathfrak{s}_p(i)}{p-1} = \frac{i(p-2) + \mathfrak{s}_p(i)}{p-1} > 0.$$

Hence, by (4.10), we obtain that there exists $g \in \mathfrak{F}_p^2$ such that, for all $n, m \in \mathbb{N}$, we have

$$\prod_{\substack{\lambda=1\\ p\nmid\lambda}}^{mp} \Big(1 + \frac{np}{\lambda}\Big) = 1 + g(n,m)p,$$

which, together with (4.9), finishes the proof of Proposition 3. □

4.3. **Last step in the proof of Theorem 1.** Let $\mathfrak{A}$ be the $\mathbb{Z}_p$-module spanned by $\mathfrak{S}_{e,f}$. We set $\mathfrak{B} = \{\mathfrak{S}_{e,f}^g,\ g \in \mathfrak{F}_p^d\}$. We shall prove that $\mathfrak{S}_{e,f}$ and $\mathfrak{B}$ satisfy Condition (a) of Proposition 1. Obviously, $\mathfrak{B}$ is constituted of $\mathbb{Z}_p$-valued sequences and we have $\mathfrak{A} \subset \mathfrak{B}$. Let $g \in \mathfrak{F}_p^d$, $v \in \{0,\dots,p-1\}$ and $n \in \mathbb{N}$ be fixed. For all $\mathbf{a} \in \{0,\dots,p-1\}^d$ and $\mathbf{m} \in \mathbb{N}^d$, we have

$$\mathcal{Q}_{e,f}(\mathbf{a} + \mathbf{m}p) = \frac{\prod_{i=1}^u (\mathbf{e}_i \cdot \mathbf{m}p)! \prod_{k=1}^{\mathbf{e}_i\cdot\mathbf{a}} (\mathbf{e}_i \cdot \mathbf{m}p + k)}{\prod_{i=1}^v (\mathbf{f}_i \cdot \mathbf{m}p)! \prod_{k=1}^{\mathbf{f}_i\cdot\mathbf{a}} (\mathbf{f}_i \cdot \mathbf{m}p + k)},$$

with

$$\frac{\prod_{i=1}^u (\mathbf{e}_i \cdot \mathbf{m}p)!}{\prod_{i=1}^v (\mathbf{f}_i \cdot \mathbf{m}p)!} = p^{(|e|-|f|)\cdot\mathbf{m}} \mathcal{Q}_{e,f}(\mathbf{m}) \frac{\prod_{i=1}^u (-1)^{\mathbf{e}_i\cdot\mathbf{m}p}\Gamma_p(\mathbf{e}_i \cdot \mathbf{m}p)}{\prod_{i=1}^v (-1)^{\mathbf{f}_i\cdot\mathbf{m}p}\Gamma_p(\mathbf{f}_i \cdot \mathbf{m}p)}$$

and

$$\frac{\prod_{i=1}^u \prod_{k=1}^{\mathbf{e}_i\cdot\mathbf{a}} (\mathbf{e}_i \cdot \mathbf{m}p + k)}{\prod_{i=1}^v \prod_{k=1}^{\mathbf{f}_i\cdot\mathbf{a}} (\mathbf{f}_i \cdot \mathbf{m}p + k)} = \frac{\prod_{i=1}^u \prod_{k=1, p\nmid k}^{\mathbf{e}_i\cdot\mathbf{a}} (\mathbf{e}_i \cdot \mathbf{m}p + k)}{\prod_{i=1}^v \prod_{k=1, p\nmid k}^{\mathbf{f}_i\cdot\mathbf{a}} (\mathbf{f}_i \cdot \mathbf{m}p + k)} \cdot p^{\Delta_{e,f}(\mathbf{a}/p)} \frac{\prod_{i=1}^u \prod_{k=1}^{\lfloor \mathbf{e}_i\cdot\mathbf{a}/p\rfloor} (\mathbf{e}_i \cdot \mathbf{m} + k)}{\prod_{i=1}^v \prod_{k=1}^{\lfloor \mathbf{f}_i\cdot\mathbf{a}/p\rfloor} (\mathbf{f}_i \cdot \mathbf{m} + k)}.$$

Since $|e| = |f|$, we have

$$\frac{\prod_{i=1}^u (-1)^{\mathbf{e}_i\cdot\mathbf{m}p}\Gamma_p(\mathbf{e}_i \cdot \mathbf{m}p)}{\prod_{i=1}^v (-1)^{\mathbf{f}_i\cdot\mathbf{m}p}\Gamma_p(\mathbf{f}_i \cdot \mathbf{m}p)} = \frac{\prod_{i=1}^u \Gamma_p(\mathbf{e}_i \cdot \mathbf{m}p)}{\prod_{i=1}^v \Gamma_p(\mathbf{f}_i \cdot \mathbf{m}p)}.$$

By Proposition 3, there exists $h \in \mathfrak{F}_p^d$ such that, for all $m_1, \dots, m_d \in \mathbb{N}$, we have

$$\frac{\Gamma_p\big((m_1 + \cdots + m_d)p\big)}{\Gamma_p(m_1 p)\cdots\Gamma_p(m_d p)} = \prod_{i=2}^d \frac{\Gamma_p\big((m_1 + \cdots + m_{i-1})p + m_i p\big)}{\Gamma_p\big((m_1 + \cdots + m_{i-1})p\big)\Gamma_p(m_i p)} = 1 + h(m_1,\dots,m_d)p.$$

Since f is only constituted by vectors $\mathbf{1}_k$, there exists $g' \in \mathfrak{F}_p^d$ such that, for all $\mathbf{m} \in \mathbb{N}^d$, we have

$$\frac{\prod_{i=1}^u \Gamma_p(\mathbf{e}_i \cdot \mathbf{m}p)}{\prod_{i=1}^v \Gamma_p(\mathbf{f}_i \cdot \mathbf{m}p)} = 1 + g'(\mathbf{m})p.$$

Furthermore, for all $\mathbf{a} \in \{0,\dots,p-1\}^d$, there exist $\lambda_{\mathbf{a}} \in \mathbb{Z}_p$ and $g_{\mathbf{a}} \in \mathfrak{F}_p^d$ such that, for all $\mathbf{m} \in \mathbb{N}^d$, we have

$$\frac{\prod_{i=1}^u \prod_{k=1, p\nmid k}^{\mathbf{e}_i\cdot\mathbf{a}} (\mathbf{e}_i \cdot \mathbf{m}p + k)}{\prod_{i=1}^v \prod_{k=1, p\nmid k}^{\mathbf{f}_i\cdot\mathbf{a}} (\mathbf{f}_i \cdot \mathbf{m}p + k)} = \lambda_{\mathbf{a}} + g_{\mathbf{a}}(\mathbf{m})p.$$

Since f is only constituted by vectors $\mathbf{1}_k$, for all $i \in \{1,\dots,v\}$, we have $\lfloor \mathbf{f}_i \cdot \mathbf{a}/p \rfloor = 0$. Thereby, for all $\mathbf{a} \in \{0,\dots,p-1\}^d$, there exists $h_{\mathbf{a}} \in \mathbb{Z}_p + \mathfrak{F}_p^d$, such that, for all $\mathbf{m} \in \mathbb{N}^d$, we have

$$\mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{m}p) = \mathcal{Q}_{e,f}(\mathbf{m}) h_{\mathbf{a}}(\mathbf{m}) p^{\Delta_{e,f}(\mathbf{a}/p)} \prod_{i=1}^{u} \prod_{k=1}^{\lfloor \mathbf{e}_i \cdot \mathbf{a}/p \rfloor} (\mathbf{e}_i \cdot \mathbf{m} + k).$$

For all $\mathbf{a} \in \{0,\dots,p-1\}^d$ and $\mathbf{m} \in \mathbb{N}^d$, we set $\tau_{\mathbf{a}}(\mathbf{m}) := g(\mathbf{a}+\mathbf{m}p) h_{\mathbf{a}}(\mathbf{m})$, so that $\tau_{\mathbf{a}} \in \mathbb{Z}_p + p\mathfrak{F}_p^d$. Therefore, we have

$$\begin{aligned}\mathfrak{S}_{e,f}^{g}(v+np) &= \sum_{\mathbf{0} \le \mathbf{a} \le \mathbf{1}(p-1)} \sum_{|\mathbf{a}+\mathbf{m}p| = v+np} g(\mathbf{a}+\mathbf{m}p) \mathcal{Q}_{e,f}(\mathbf{a}+\mathbf{m}p) \\ &= \sum_{\mathbf{0} \le \mathbf{a} \le \mathbf{1}(p-1)} \sum_{|\mathbf{a}+\mathbf{m}p| = v+np} \mathcal{Q}_{e,f}(\mathbf{m}) \tau_{\mathbf{a}}(\mathbf{m}) p^{\Delta_{e,f}(\mathbf{a}/p)} \prod_{i=1}^{u} \prod_{k=1}^{\lfloor \mathbf{e}_i \cdot \mathbf{a}/p \rfloor} (\mathbf{e}_i \cdot \mathbf{m} + k).\end{aligned}$$

If $|\mathbf{a}+\mathbf{m}p| = v+np$, then we have $|\mathbf{a}| = v+jp$ with

$$0 \le j \le \min\left(n, \left\lfloor \frac{d(p-1)-v}{p} \right\rfloor\right) =: M.$$

Furthermore, we have $\lfloor |\mathbf{a}|/p \rfloor = j$ and there is $k \in \{1,\dots,d\}$ such that $\mathbf{a}^{(k)} \ge (v+jp)/d$. Since e is 2-admissible and $f = (\mathbf{1}_{k_1},\dots,\mathbf{1}_{k_v})$, we obtain that

$$\Delta_{e,f}(\mathbf{a}/p) = \sum_{i=1}^{u} \left\lfloor \frac{\mathbf{e}_i \cdot \mathbf{a}}{p} \right\rfloor \ge 2j.$$

In addition, we have either

$$p^{\Delta_{e,f}(\mathbf{a}/p)} \prod_{i=1}^{u} \prod_{k=1}^{\lfloor \mathbf{e}_i \cdot \mathbf{a}/p \rfloor} (\mathbf{e}_i \cdot \mathbf{m} + k) = 1, \ (\forall \mathbf{m} \in \mathbb{N}^d),$$

or

$$\mathbf{m} \mapsto p^{\Delta_{e,f}(\mathbf{a}/p)} \prod_{i=1}^{u} \prod_{k=1}^{\lfloor \mathbf{e}_i \cdot \mathbf{a}/p \rfloor} (\mathbf{e}_i \cdot \mathbf{m} + k) \in p\mathfrak{F}_p^d.$$

Hence, for all $\mathbf{a} \in \{0,\dots,p-1\}^d$, there exist a function $f_{\mathbf{a}}$ wich is either constant or in $p\mathfrak{F}_p^d$, and $g_{\mathbf{a}} \in \mathfrak{F}_p^d$ such that

$$\begin{aligned}\mathfrak{S}_{e,f}^{g}(v+np) = &\sum_{\substack{\mathbf{0} \le \mathbf{a} \le \mathbf{1}(p-1) \\ |\mathbf{a}| = v}} \sum_{|\mathbf{m}| = n} \mathcal{Q}_{e,f}(\mathbf{m}) \tau_{\mathbf{a}}(\mathbf{m}) f_{\mathbf{a}}(\mathbf{m}) \\ &+ \sum_{j=1}^{M} p^{2j} \sum_{\substack{\mathbf{0} \le \mathbf{a} \le \mathbf{1}(p-1) \\ |\mathbf{a}| = v+jp}} \sum_{|\mathbf{m}| = n-j} \mathcal{Q}_{e,f}(\mathbf{m}) \tau_{\mathbf{a}}(\mathbf{m}) g_{\mathbf{a}}(\mathbf{m}).\end{aligned}$$

Therefore, there exist $A' \in \mathfrak{A}$ and a sequence $(B_k)_{k\geq 0}$, $B_k \in \mathfrak{B}$, such that

$$\mathfrak{S}_{e,f}^{g}(v+np) = A'(n) + pB_0(n) + \sum_{k=1}^{\infty} p^{k+1} B_k(n-k).$$

This shows that $\mathfrak{S}_{e,f}$ and $\mathfrak{B}$ satisfy Condition (a) in Proposition 1, so that Theorem 1 is proved. □

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