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EUCLID-EULER HEURISTICS FOR (ODD) PERFECT NUMBERS

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Dedicated to Dr. Severino V. Gervacio, for his suggestion of tackling Sorli's conjecture first

Abstract

An odd perfect number N is said to be given in *Eulerian form* if $N = q^k n^2$ where q is prime with $q \equiv k \equiv 1 \pmod 4$ and $\gcd(q, n) = 1$. Similarly, an even perfect number M is said to be given in *Euclidean form* if $M = (2^p - 1) \cdot 2^{p-1}$ where p and $2^p - 1$ are primes. In this article, we show how simple considerations surrounding the differences between the underlying properties of the Eulerian and Euclidean forms of perfect numbers give rise to what we will call the *Euclid-Euler heuristics* for (odd) perfect numbers.

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1. Introduction

If J is a positive integer, then we write $\sigma(J)$ for the sum of the divisors of J. A number L is *perfect* if $\sigma(L) = 2L$.

An even perfect number M is said to be given in *Euclidean form* if

$$M = (2^p - 1) \cdot 2^{p-1}$$

where p and $2^p - 1$ are primes. We call $M_p = 2^p - 1$ the *Mersenne prime* factor of M. Currently, there are only 48 known Mersenne primes, which correspond to 48 even perfect numbers.

An odd perfect number N is said to be given in Eulerian form if

$$N = q^k n^2$$

where q is prime with $q \equiv k \equiv 1 \pmod 4$ and $\gcd(q, n) = 1$. We call q^k the *Euler part* of N while n^2 is called the *non-Euler part* of N.

It is currently unknown whether there are infinitely many even perfect numbers, or whether any odd perfect numbers exist. It is widely believed that there is an infinite number of even perfect numbers. On the other hand, no examples for an odd perfect number have been found (despite extensive computer searches), nor has a proof for their nonexistence been established.

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Ochem and Rao [15] recently proved that $N > 10^{1500}$. In a recent preprint, Nielsen [13] has obtained the lower bound $\omega(N) \geq 10$ for the number of *distinct* prime factors of N, improving on his last result $\omega(N) \geq 9$ (see [14]).

Sorli conjectured in [17] that $k = \nu_q(N) = 1$ always holds. Dris conjectured in [8] and [11] that the divisors q^k and n are related by the inequality $q^k < n$. This conjecture was made on the basis of the result $I(q^k) < \sqrt[3]{2} < I(n)$.

Broughan, et. al. [3] recently showed that for any odd perfect number $N = q^k n^2$, the ratio of the non-Euler part n^2 to the Euler part q^k is greater than $315/2$. This improves on a result of Dris [8].

In a recent paper, Chen and Chen [4] improves on Broughan, et. al.'s results, and poses a related (open) problem.

We denote the abundancy index I of the positive integer x as $I(x) = \sigma(x)/x$.

2. The General Multiplicative Form of All Perfect Numbers

Suppose that $N = q^k n^2$ is an odd perfect number given in Eulerian form. Since prime powers are deficient and $\gcd(q, n) = 1$, we know that $q^k \neq n$. (In particular, it is also true that $q \neq n$.) Consequently, we know that either $q^k < n$ or $n < q^k$ is true.

Observe that the Euclidean form $M = (2^p - 1) \cdot 2^{p-1}$ for an even perfect number M possesses a multiplicative structure that is *almost* similar to that of the Eulerian form $N = q^k n^2$ for an odd perfect number N. Here is a table comparing and contrasting the underlying properties of these two forms, which we shall refer to as the *Euclid-Euler heuristics* for (odd) perfect numbers:

(E-1) (Euclid-Euler Theorem) The Mersenne primes M_p are in one-to-one correspondence with the even perfect numbers.	(O-1) (Conjecture, 2010 [10]) The Euler primes q are in one-to-one correspondence with the odd perfect numbers.
(E-2) The Mersenne primes M_p satisfy $M_p \equiv 3 \pmod 4$. (Trivial)	(O-2) The Euler primes q satisfy $q \equiv 1 \pmod 4$. (Trivial)
(E-3) The exponent $s = \nu_{M_p}(M)$ is one. (Trivial)	(O-3) The exponent $k = \nu_q(N)$ is one. (Sorli's Conjecture, 2003 [17])
(E-4) If $M \equiv 0 \pmod 2$ is perfect, then given the Euclidean form $M = 2^{p-1}(2^p - 1) = \prod_{i=1}^{2} {p_i}^{\alpha_i}$, then ${p_i}^{\alpha_i}\sigma({p_i}^{\alpha_i})/M = i$, for $i = 1, 2$. (Observation, Dris 2011)	(O-4) If $N \equiv 1 \pmod 2$ is perfect, then given the Eulerian form $N = q^k n^2 = \prod_{j=1}^{\omega(N)} {q_j}^{\beta_j}$, then ${q_j}^{\beta_j}\sigma({q_j}^{\beta_j})/N \le 2/3 < j$, for all j, $1 \le j \le \omega(N)$. (Theorem, Dris 2008 [11])
(E-5) There are infinitely many even perfect numbers. (EPN Conjecture)	(O-5) There do not exist any odd perfect numbers. (OPN Conjecture)
(E-6) The density of even perfect numbers is zero. (Kanold 1954 [12])	(O-6) The density of odd perfect numbers is zero. (Kanold 1954 [12])
(E-7) $1 < I(M_p) \le \frac{8}{7}$ for $p \ge 3$. $\frac{7}{4} \le \frac{2}{I(M_p)} = I(\frac{M_p+1}{2}) < 2$ In particular, $\frac{8}{7} < \sqrt{\frac{7}{4}} < I(\sqrt{\frac{M_p+1}{2}}) < 2$.	(O-7) $1 < I(q^k) < \frac{5}{4}$ for $q \ge 5$. $\frac{8}{5} < \frac{2}{I(q^k)} = I(n^2) < 2$ In particular, $\frac{5}{4} < \sqrt{\frac{8}{5}} < I(n) < 2$.
(E-8) An even perfect number M has exactly two distinct prime factors. (i.e., $\omega(M) = 2$)	(O-8) An odd perfect number N has more than two distinct prime factors. (In fact, we know that $\omega(N) \ge 9$ [14].)
(E-9) $\gcd(2^p - 1, 2^{p-1}) = 1$ (Trivial)	(O-9) $\gcd(q^k, n^2) = \gcd(q, n) = 1$ (Euler)

Remark 2.1. *We excluded $p_1 = 2$ from (E-7) because* $\frac{M_{p_1} + 1}{2} = 2^{p_1 - 1} = 2^{2-1} = 2$ *is squarefree.*

In the next section, we give some *known* relationships between the divisors of even and odd perfect numbers. We will also discuss a conjectured relationship between certain divisors of odd perfect numbers, which first appeared in the M. Sc. thesis [11] that was completed in August of 2008.

3. Inequalities Relating the Divisors of Perfect Numbers

From Section 2, note that the heuristic (E-4), upon setting $Q = 2^p - 1$, $K = 1$, and $\overline{n}^2 = 2^{p-1}$, actually gives

$$\frac{\sigma(Q^K)}{\overline{n}^2} = \frac{\sigma(2^p - 1)}{2^{p-1}} = \frac{(2^p - 1) + 1}{2^{p-1}} = \frac{2^p}{2^{p-1}} = 2,$$

and

$$\frac{\sigma(\overline{n}^2)}{Q^K} = \frac{\sigma(2^{p-1})}{2^p - 1} = \frac{2^{(p-1)+1} - 1}{(2-1)(2^p - 1)} = \frac{2^p - 1}{2^p - 1} = 1.$$

We state this result as our first lemma for this section.

Lemma 3.1. *If $M = Q^K\overline{n}^2 = (2^p - 1) \cdot 2^{p-1}$ is an even perfect number given in Euclidean form, then we have the inequality*

$$\frac{\sigma(\overline{n}^2)}{Q^K} = 1 < 2 = \frac{\sigma(Q^K)}{\overline{n}^2}.$$

Remark 3.2. *Except for the case of the first even perfect number $M = 6$ (as was pointed out in Remark 2.1), the abundancy indices of the divisors of an even perfect number given in the Euclidean form $M = Q^K\overline{n}^2 = (2^p - 1) \cdot 2^{p-1}$ (where the relabelling is done to mimic the appearance of the variables in the Eulerian form of an odd perfect number $N = q^k n^2$) satisfy the inequality*

$$1 < I(Q^K) \le \frac{8}{7} < \frac{7}{4} \le I(\overline{n}^2) < 2,$$

as detailed out in heuristic (E-7). In particular, the inequality $I(Q^K) < I(\overline{n}^2)$, together with Lemma 3.1, imply the inequality

$$\overline{n}^2 < Q^K.$$

(In other words, we have the inequality

$$2^{p-1} < 2^p - 1$$

where p and $M_p = 2^p - 1$ are both primes. Compare this with the inequality

$$q^k < n^2$$

for the divisors of an odd perfect number given in the Eulerian form $N = q^k n^2$ [see [8], [11]].)

The following result is taken from [8] and [11].

Lemma 3.3. *If $N = q^k n^2$ is an odd perfect number given in Eulerian form, then we have the inequality*

$$\frac{\sigma(q^k)}{n^2} \le \frac{2}{3} < 3 \le \frac{\sigma(n^2)}{q^k}.$$

Going back to the relabelling $Q = 2^p - 1$, $K = 1$, and $\overline{n}^2 = 2^{p-1}$ for an even perfect number $M = Q^K\overline{n}^2 = (2^p - 1)\cdot 2^{p-1}$ given in Euclidean form, we now compute:

$$\frac{\sigma(Q^K)}{\overline{n}} = \frac{\sigma(2^p - 1)}{2^{\frac{p-1}{2}}} = \frac{(2^p - 1) + 1}{2^{\frac{p-1}{2}}} = \frac{2^p}{2^{\frac{p-1}{2}}} = 2^{p-(\frac{p-1}{2})} = 2^{\frac{p+1}{2}},$$

and

$$\frac{\sigma(\overline{n})}{Q^K} = \frac{\sigma(2^{\frac{p-1}{2}})}{2^p - 1} = \frac{2^{\frac{p-1}{2}+1} - 1}{(2-1)(2^p - 1)} = \frac{2^{\frac{p+1}{2}} - 1}{2^p - 1}.$$

Now observe that

$$2^{\frac{p+1}{2}} - 1 < 2^p - 1$$

since $1 < \frac{p+1}{2} < p$, while we also have

$$2^{\frac{p+1}{2}} \geq 4$$

because $p \geq 3$. (Again, we excluded the first (even) perfect number $M = 6$ from this analysis because it is *squarefree*.)

These preceding numerical inequalities imply that

$$\frac{\sigma(Q^K)}{\overline{n}} = 2^{\frac{p+1}{2}} \geq 4 > 1 > \frac{2^{\frac{p+1}{2}} - 1}{2^p - 1} = \frac{\sigma(\overline{n})}{Q^K}.$$

We state the immediately preceding result as our third lemma for this section.

Lemma 3.4. *If $M = Q^K\overline{n}^2 = (2^p-1)\cdot 2^{p-1}$ is an even perfect number given in Euclidean form (and $M \neq 6$), then we have the inequality*

$$\frac{\sigma(Q^K)}{\overline{n}} \geq 4 > 1 > \frac{\sigma(\overline{n})}{Q^K}.$$

Remark 3.5. *In particular, observe that the inequality*

$$\frac{\sigma(\overline{n})}{Q^K} < 1$$

from Lemma 3.4 implies that

$$\overline{n} < Q^K,$$

which, of course, trivially *follows from the inequality*

$$\overline{n}^2 < Q^K$$

in Remark 3.2.

Likewise, compare the inequality (from Lemma 3.4)

$$\frac{\sigma(\overline{n})}{Q^K} < \frac{\sigma(Q^K)}{\overline{n}},$$

for an even perfect number $M = Q^K \overline{n}^2 = (2^p - 1) \cdot 2^{p-1}$ *given in Euclidean form, with the inequality*

$$\frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k},$$

for an odd perfect number $N = q^k n^2$ *given in Eulerian form. This second inequality was originally conjectured in [9], and in fact, it has been recently shown (see [5]) to be equivalent to the conjecture* $q^k < n$*, which originally appeared in the M. Sc. thesis [11].*

Details for the proof of the biconditional

$$q^k < n \iff \sigma(q^k) < \sigma(n) \iff \frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}$$

are clarified online in `http://math.stackexchange.com/questions/548528`*.*

The next section will explain our motivation for pursuing a proof for the following conjecture:

Conjecture 3.6. *If* $N = q^k n^2$ *is an odd perfect number given in Eulerian form, then the conjunction*

$$\{k = \nu_q(N) = 1\} \wedge \{q^k < n\}$$

always holds.

4. On the Conjectures of Sorli and Dris Regarding Odd Perfect Numbers

We begin this section with a recapitulation of the two main conjectures on odd perfect numbers that have been mentioned earlier in this article.

Conjecture 4.1. *Sorli's conjecture states that if* $N = q^k n^2$ *is an odd perfect number given in Eulerian form, then*

$$k = \nu_q(N) = 1$$

always holds.

Remark 4.2. *Dris gave a sufficient condition for Sorli's conjecture in [8]. Some errors, however, were found in the initial published version of that article, and Dris had to retract his claim that the biconditional*

$$k = \nu_q(N) = 1 \iff n < q$$

always holds. (The current published version of [8] contains a proof only for the one-sided implication

$$n < q \implies k = \nu_q(N) = 1.$$

In two [continually] evolving papers [see [5] and [6]], work is in progress to try to disprove the converse

$$k = \nu_q(N) = 1 \implies n < q$$

and thereby get a proof for Conjecture 3.6.)

Moreover, Acquaah and Konyagin [1] almost *disproves $n < q$ by obtaining the estimate $q < n\sqrt{3}$ under the assumption $k = \nu_q(N) = 1$. (Since the contrapositive of the implication $n < q \implies k = 1$ is $k > 1 \implies q < n$, we know that Acquaah and Konyagin's estimate for the Euler prime q implies that the inequality*

$$q < n\sqrt{3}$$

holds unconditionally.)

Curiously enough, the two papers [8] and [7] by Dris are cited in OEIS sequence A228059 [16], whose description is reproduced below:

Odd numbers of the form $r^{1+4L}s^2$, where r is prime of the form $1 + 4m$, $s > 1$, and $\gcd(r, s) = 1$ *that are* closer to being perfect than previous terms.

Coincidentally, *the "Euler prime" of the first 9 terms in this OEIS sequence all have exponent 1:*

$$45 = 5 \cdot 3^2$$
$$405 = 5 \cdot 3^4$$
$$2205 = 5 \cdot (3 \cdot 7)^2$$
$$26325 = 13 \cdot (3^2 \cdot 5)^2$$
$$236925 = 13 \cdot (3^3 \cdot 5)^2$$
$$1380825 = 17 \cdot (3 \cdot 5 \cdot 19)^2$$
$$1660725 = 61 \cdot (3 \cdot 5 \cdot 11)^2$$
$$35698725 = 61 \cdot (3^2 \cdot 5 \cdot 17)^2$$
$$3138290325 = 53 \cdot (3^4 \cdot 5 \cdot 19)^2.$$

CONJECTURE 4.3. *Dris's conjecture states that if $N = q^k n^2$ is an odd perfect number given in Eulerian form, then*

$$q^k < n$$

always holds.

REMARK 4.4. *Prior to the paper [1] by Acquaah and Konyagin, and the data from OEIS sequence A228059 [16] as detailed out in Remark 4.2, the only heuristic available to justify Dris's conjecture that $q^k < n$ was the inequality*

$$I(q^k) < \frac{5}{4} < \sqrt[3]{2} < \sqrt{\frac{8}{5}} < I(n).$$

(See the paper [8] for a proof.) In particular, the heuristic justification *is that the divisibility constraint* $\gcd(q^k, n) = \gcd(q, n) = 1$ *appears to induce an "ordering property"*

between certain divisors of an odd perfect number related via an appropriate inequality between their abundancy indices. That is, Dris expects his conjecture $q^k < n$ to follow from the last inequality above, in the sense that the inequality $q^k < n^2$ appears to have followed from the related inequality

$$I(q^k) < \frac{5}{4} < \sqrt{2} < \frac{8}{5} < I(n^2).$$

Additionally, note that all of the 8 *terms (apart from the first one) in the OEIS sequence mentioned in Remark 4.2 satisfy Dris's conjecture.*

5. Conclusion

To conclude, a recent e-mail correspondence of the author with Brian D. Beasley of Presbyterian College revealed the following information, quoted verbatim from page 25 of [2]:

"Before proceeding with Euler's proof, we pause to note that his result was not quite what Descartes and Frenicle had conjectured, as they believed that $k = 1$, but it came very close. In fact, current research continues in an effort to prove $k = 1$. For example, Dris has made progress in this direction, although his paper refers to Descartes' and Frenicle's claim (that $k = 1$) as Sorli's conjecture; Dickson has documented Descartes's conjecture as occurring in a letter to Marin Mersenne in 1638, with Frenicle's subsequent observation occurring in 1657."

It might be wise (at this point) to delve deeper into this little bit of history in mathematics, to attempt to answer the particular question of whether Descartes and Frenicle used *similar* or totally different methods to arrive at what we have come to call as Sorli's conjecture on odd perfect numbers. Perhaps they *both* used methods similar to the ones used in this article - who knows? Besides, Mersenne's predictions for succeeding primes p for which $2^p - 1$ turned out to be a Mersenne prime were already stunning as they were. Did Mersenne use an *algorithm*, for testing primality of Mersenne prime-number candidates, that remains unknown to the rest of us to this day? Only time can tell.

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References

[1] P. Acquaah, S. Konyagin, On prime factors of odd perfect numbers, *Int. J. Number Theory*, **08** (2012), 1537, doi:http://dx.doi.org/10.1142/S1793042112500935.

[2] B. D. Beasley, Euler and the ongoing search for odd perfect numbers, ACMS 19th Biennial Conference Proceedings, Bethel University, May 29 to Jun. 1, 2013, http://godandmath.files.wordpress.com/2013/07/acms-2013-proceedings.pdf, pages 21-31.

[3] K. A. Broughan, D. Delbourgo and Q. Zhou, Improving the Chen and Chen result for odd perfect numbers, *Integers*, **13** (2013), Article #A39, http://www.emis.de/journals/INTEGERS/papers/n39/n39.pdf, ISSN 1867-0652.

[4] F. J. Chen, Y. G. Chen, On the index of an odd perfect number, *Colloq. Math.*, **136** (2014), 41-49, doi:http://dx.doi.org/10.4064/cm136-1-4.

[5] J. A. B. Dris, New results for Sorli's conjecture on odd perfect numbers, (Mar. 2015), preprint, http://arxiv.org/abs/1302.5991.

[6] J. A. B. Dris, New results for Sorli's conjecture on odd perfect numbers - Part II, (Aug. 2013), preprint, http://arxiv.org/abs/1303.2329.

[7] J. A. B. Dris, A short "proof" for Sorli's conjecture on odd perfect numbers, (Oct. 2014), preprint, http://arxiv.org/abs/1308.2156.

[8] J. A. B. Dris, The abundancy index of divisors of odd perfect numbers, *J. Integer Seq.*, **15** (Sep. 2012), Article 12.4.4, https://cs.uwaterloo.ca/journals/JIS/VOL15/Dris/dris8.html, ISSN 1530-7638.

[9] J. A. B. Dris, Some new results on bounds for the abundancy indices of the components of odd perfect numbers, (Mar. 2011), preprint, http://arxiv.org/pdf/1103.1090v1.pdf.

[10] J. A. B. Dris, Perfect numbers - On Mersenne and Euler primes, http://mathoverflow.net/questions/48953, asked on 12/10/2010.

[11] J. A. B. Dris, Solving the Odd Perfect Number Problem: Some Old and New Approaches, M. Sc. thesis, De La Salle University, Manila, Philippines, 2008, http://arxiv.org/abs/1204.1450.

[12] H.-J. Kanold, Uber die Dichten der Mengen der vollkommenen und der befreundeten Zahlen, *Math. Z.*, **61** (1954), 180-185, doi:http://dx.doi.org/10.1007/BF01181341.

[13] P. P. Nielsen, Odd perfect numbers, Diophantine equations, and upper bounds, preprint, http://math.byu.edu/~pace/BestBound_web.pdf.

[14] P. P. Nielsen, Odd perfect numbers have at least nine distinct prime factors, *Math. Comp.*, **76** (2007), 2109-2126, doi:http://dx.doi.org/10.1090/S0025-5718-07-01990-4.

[15] P. Ochem, M. Rao, Odd perfect numbers are greater than 10^{1500}, *Math. Comp.*, **81** (2012), 1869-1877, doi:http://dx.doi.org/10.1090/S0025-5718-2012-02563-4.

[16] T. D. Noe, Online Encyclopedia of Integer Sequences, sequence A228059, http://oeis.org/A228059, last accessed: 10/12/2013.

[17] R. M. Sorli, Algorithms in the Study of Multiperfect and Odd Perfect Numbers, Ph. D. Thesis, University of Technology, Sydney, 2003, http://epress.lib.uts.edu.au/research/handle/10453/20034.

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