TOPOLOGICAL SPACES ASSOCIATED TO HIGHER-RANK GRAPHS

ALEX KUMJIAN, DAVID PASK, AIDAN SIMS, AND MICHAEL F. WHITTAKER

Abstract. We investigate which topological spaces can be constructed as topological realisations of higher-rank graphs. We describe equivalence relations on higher-rank graphs for which the quotient is again a higher-rank graph, and show that identifying isomorphic cohereditary subgraphs in a disjoint union of two rank-k graphs gives rise to pullbacks of the associated C^*-algebras. We describe a combinatorial version of the connected-sum operation and apply it to the rank-2-graph realisations of the four basic surfaces to deduce that every compact 2-manifold is the topological realisation of a rank-2 graph. We also show how to construct k-spheres and wedges of k-spheres as topological realisations of rank-k graphs.

1. Introduction

Higher-rank graphs, also called k-graphs, were introduced by Kumjian and Pask [7] as combinatorial models for higher-rank Cuntz-Krieger algebras. Since then, the resulting class of C^*-algebras has been studied in detail. More recently, in [15, 16, 6, 8], an investigation of k-graphs from a topological point of view was begun. Definition 3.2 of [6] associates to each k-graph Λ a topological realisation X_Λ whose fundamental group and homology are the same as the fundamental group and cubical homology of Λ. The motivation for the current article was to investigate the range of topological spaces which can be constructed as topological realisations of k-graphs. We began with two goals: obtain all compact 2-manifolds as topological realisations of 2-graphs; and, more generally, obtain all triangularisable k-manifolds as topological realisations of k-graphs.

Our approach to the first goal was to exploit the classification of compact 2-manifolds as spheres, n-holed tori, or connected-sums of the latter with the Klein bottle or projective plane (see for example [12, Theorem I.7.2]). Examples of 2-graphs whose topological realisations were homeomorphic to each of the four basic surfaces were presented in [6]. So our aim was to develop a combinatorial connected-sum operation for 2-graphs, and to show that it can be applied to finite disjoint unions of the four 2-graphs just mentioned so as to construct any desired connected sum of their topological realisations. We achieve this in Section 4.

Our approach to the second, more general, goal consisted of four steps. Step 1 was to determine which equivalence relations on k-graphs have the property that the quotient category itself forms a k-graph. Step 2 was to invoke [6, Proposition 5.3] — which shows that topological realisation is a functor from k-graphs to topological spaces — to see that, for k-graphs, topological realisations of quotients coincide with quotients of topological realisations. Step 3 was to construct k-graphs Σ_k whose topological realisations are naturally homeomorphic to k-simplices. Step 4 was to show that equivalence relations corresponding to desired identifications amongst the $(k-1)$-faces of a disjoint union of copies of Σ_k are of the sort developed in Step 1; and then deduce that arbitrary triangularisable manifolds could be realised as the topological realisations of appropriate quotients of disjoint unions of copies of Σ_k. We have achieved steps 1–3, but the combinatorics of k-graphs place significant constraints on the ways in which faces in a disjoint union of copies of the k-graphs Σ_k from Step 3 can be identified so as to produce a new k-graph, so we are as yet unable to realise arbitrary triangularisable

Date: October 24, 2013.
2010 *Mathematics Subject Classification.* Primary: 05C20; Secondary: 46L05,57M50,18D99.
Key words and phrases. Higher-rank graph; C^*-algebra; connected-sum; simplex; topological realisation.
This research was supported by the ARC.

manifolds. However, our construction is flexible enough so that we can glue two k-simplices on their boundaries to obtain a k-sphere. The details of this appear in section 5.

The paper is organised as follows. In Section 2 we identify those equivalence relations $\sim$ on a k-graph Λ for which the quotient $\Lambda/{\sim}$ forms a k-graph. Proposition 2.3 shows that the quotient operation is well-behaved with respect to the topological realisation of a k-graph. In Section 3 we investigate the properties of the quotients of k-graphs at the level of their associated C^*-algebras. Specifically, given k-graphs Λ_1 and Λ_2 with a partially-defined isomorphism ϕ between the complements in the Λ_i of hereditary subsets H_i, the map ϕ induces an equivalence relation $\sim_\phi$ on the disjoint union of Λ_1 and Λ_2 and we show that the quotient $(\Lambda_1 \sqcup \Lambda_2)/{\sim_\phi}$ forms a k-graph. We then show that the Toeplitz algebra $\mathcal{T}C^*(\Lambda_1 \sqcup \Lambda_2)/{\sim_\phi})$ is a pullback of $\mathcal{T}C^*(\Lambda_1)$ and $\mathcal{T}C^*(\Lambda_2)$ over the C^*-algebra of the common subgraph (see Theorem 3.3). If the H_i are also saturated, then this result also descends to Cuntz-Kreiger algebras (Corollary 3.4).

In section 4 we define the connected-sum operation on 2-graphs and show that it corresponds to the connected-sum operation on their topological realisations. Then, after recalling from [6] how to realise the four basic surfaces, we show that every compact surface is the topological realisation of a 2-graph.

In section 5 we construct, for each $k \in \mathbb{N}$, a k-graph Σ_k whose topological realisation is a k-dimensional simplex (see Theorem 5.8). The combinatorics involved in this construction are interesting in their own right — the vertices of Σ_k are indexed by *placing functions* which can be thought of as the possible outcomes of a horse race involving $k+1$ horses (see the On-Line Encyclopedia of Integer Sequences [13, Sequence A000670]); alternatively, they can be regarded as ordered partitions of a $(k+1)$-element set. Using suitable equivalence relations, two (or more) copies of Σ_k can be glued along their common boundary to produce a new k-graph. In Theorem 5.1 we use this construction to realise all k-spheres as the topological realisations of k-graphs; we also show that for each $n \geq 1$ there is a finite k-graph Λ whose topological realisation is a wedge of n k-spheres.

Background and notation. We regard $\mathbb{N}^k$ as a semigroup under addition, with identity 0 and generators $e_1, \dots, e_k$. For $m, n \in \mathbb{N}^k$, we write m_i for the i^{th} coordinate of m, and we define $m \vee n \in \mathbb{N}^k$ by $(m \vee n)_i = \max\{m_i, n_i\}$. We write $m \leq n$ if and only if $m_i \leq n_i$ for all i.

Let Λ be a countable small category and $d : \Lambda \to \mathbb{N}^k$ a functor. Write $\Lambda^n := d^{-1}(n)$ for each $n \in \mathbb{N}^k$. Then Λ is a k-graph if d satisfies the *factorisation property*: $(\mu, \nu) \mapsto \mu\nu$ is a bijection of $\{(\mu, \nu) \in \Lambda^m \times \Lambda^n : s(\mu) = r(\nu)\}$ onto Λ^{m+n} for each $m, n \in \mathbb{N}^k$ (see [7]). We then have $\Lambda^0 = \{\mathrm{id}_o : o \in \mathrm{Obj}(\Lambda)\}$, and so we regard the domain and codomain maps as maps $s, r : \Lambda \to \Lambda^0$.

If Λ_i is a k_i-graph for $i = 1, 2$ then the *cartesian product* $\Lambda_1 \times \Lambda_2$ with the natural product structure forms a $(k_1 + k_2)$-graph (see [7, Proposition 1.8]).

A *k-graph morphism* between k-graphs (Λ, d_1) and (Σ, d_2) is a functor $\phi : \Lambda \to \Sigma$ such that $d_1(\lambda) = d_2(\phi(\lambda))$ for all $\lambda \in \Lambda$.

Recall from [15] that for $v, w \in \Lambda^0$ and $X \subseteq \Lambda$, we write

$$vX := \{\lambda \in X : r(\lambda) = v\}, \quad Xw := \{\lambda \in X : s(\lambda) = w\}, \quad \text{and} \quad vXw = vX \cap Xw.$$

If $V \subset \Lambda^0$, then $V\Lambda = r^{-1}(V)$ and $\Lambda V = s^{-1}(V)$. A k-graph Λ has *no sources* if $0 < |v\Lambda^n|$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

For $\lambda \in \Lambda$ and $0 \leq m \leq n \leq d(\lambda)$, the factorisation property yields unique elements $\alpha \in \Lambda^m$, $\beta \in \Lambda^{n-m}$ and $\gamma \in \Lambda^{d(\lambda)-n}$ such that $\lambda = \alpha\beta\gamma$. Define $\lambda(m, n) := \beta$. We then have $\lambda(0, m) = \alpha$ and $\lambda(n, d(\lambda)) = \gamma$. In particular, $\lambda = \lambda(0, m)\lambda(m, d(\lambda))$ for each $0 \leq m \leq d(\lambda)$.

Given $\mu, \nu \in \Lambda$, we define $\mathrm{MCE}(\mu, \nu) := \{\lambda \in \Lambda^{d(\mu) \vee d(\nu)} : \lambda = \mu\mu' = \nu\nu' \text{ for some } \mu', \nu'\}$. If $|\mathrm{MCE}(\mu, \nu)| < \infty$ for all $\mu, \nu \in \Lambda$ then we say that Λ is *finitely-aligned.*

We define $\Lambda^{*2} = \{(\lambda, \mu) \in \Lambda \times \Lambda : s(\lambda) = r(\mu)\}$, the collection of composable pairs in Λ. If Λ_1, Λ_2 are k-graphs, then the disjoint union $\Lambda_1 \sqcup \Lambda_2$ naturally forms a k-graph. We will allow for the possibility of 0-graphs with the convention that $\mathbb{N}^0$ is the trivial semigroup $\{0\}$. We insist that all k-graphs are nonempty. See [7] for further details regarding the basic structure of k-graphs.

A set of vertices $H \subset \Lambda^0$ is *hereditary* if $s(H\Lambda) \subseteq H$; similarly, H is *co-hereditary* $r(\Lambda H) \subseteq H$. A set H is hereditary if and only if $\Lambda^0 \setminus H$ is co-hereditary. A subgraph $\Gamma \subset \Lambda$ is said to be *hereditary* (resp. *co-hereditary*) if $\Gamma = \Gamma^0\Lambda$ (resp. $\Gamma = \Lambda\Gamma^0$). If Γ is a hereditary subgraph, then Γ^0 is a hereditary subset of Λ^0. Applying this result to the opposite category Λ^{op}, which is also a k-graph under the same degree map, yields the corresponding statement for a co-hereditary subgraph.

We now review the construction of the topological realisation of a k-graph for $k \geq 1$ given in [6]. Given $t \in \mathbb{R}^k$, we will write $\lceil t \rceil$ for the least element of $\mathbb{Z}^k$ which is coordinatewise greater than or equal to t and $\lfloor t \rfloor$ for the greatest element of $\mathbb{Z}^k$ which is coordinatewise less than or equal to t. Let $\mathbf{1}_k := (1, 1, \ldots, 1) \in \mathbb{N}^k$. Then $\lfloor t \rfloor \leq t \leq \lceil t \rceil \leq \lfloor t \rfloor + \mathbf{1}_k$ for all $t \in \mathbb{R}^k$.

Given $p \leq q \in \mathbb{N}^k$, we denote by $[p, q]$ the *closed interval* $\{t \in \mathbb{R}^k : p \leq t \leq q\}$, and we denote by (p, q) the *relatively open interval* $\{t \in [p, q] : p_i < t_i < q_i \text{ whenever } p_i < q_i\}$. Observe that (p, q) is not open in $\mathbb{R}^k$ unless $p_i < q_i$ for all i, but it is open as a subspace of $[p, q]$. The set (p, q) is never empty: for example if $p = q$ then $(p, q) = [p, q] = \{p\}$. In general, as a subset of $\mathbb{R}^k$, the dimension of (p, q) is $|\{i \leq k : p_i < q_i\}|$. If $p_i < q_i$ then the i^{th}-coordinate projection of (p, q) is (p_i, q_i), and if $p_i = q_i$ then the i^{th}-coordinate projection of (p, q) is $\{p_i\}$. If $m \in \mathbb{N}^k$ with $m \leq \mathbf{1}_k$, then $\lfloor t \rfloor = 0$ and $\lceil t \rceil = m$ for all $t \in (0, m)$.

We define a relation on the topological disjoint union $\bigsqcup_{\lambda \in \Lambda} \{\lambda\} \times [0, d(\lambda)]$ by

$$(1) \qquad (\mu, s) \sim (\nu, t) \iff \mu(\lfloor s \rfloor, \lceil s \rceil) = \nu(\lfloor t \rfloor, \lceil t \rceil) \text{ and } s - \lfloor s \rfloor = t - \lfloor t \rfloor.$$

It is straightforward to see that this is an equivalence relation.

Definition 1.1 (cf. [6, Definition 3.2])**.** Let Λ be a k-graph. With notation as above, we define the *topological realisation* X_Λ of Λ to be the quotient space

$$\Big(\bigsqcup_{\lambda \in \Lambda} \{\lambda\} \times [0, d(\lambda)] \Big) \Big/ \sim .$$

We will need to discuss topological realisations of 0-graphs in Section 5. To avoid discussing separate cases, we take the convention that $\mathbf{1}_0 = 0 \in \{0\} = \mathbb{N}^0$. If Λ is a 0-graph, then $\Lambda = \Lambda^0$, the relation $\sim$ of equation (1) is trivial, and we have $X_\Lambda = \Lambda^0 \times \{0\} \cong \Lambda^0$.

Remark 1.2. One can view the morphisms in a k-graph whose degrees are smaller than $\mathbf{1}_k$ as a cubical set in a natural way [8, Theorem A.4], and then the topological realisation X_Λ of Λ is the same as the topological realisation of the associated cubical set (see, for example, [4]).

2. Quotients of k-graphs

In this section we identify the equivalence relations $\sim$ on k-graphs Λ for which the quotient set $\Lambda/\sim$ itself becomes a k-graph. We describe the topological realisation of $\Lambda/\sim$ as a quotient of the topological realisation of Λ.

Proposition 2.1. *Let Λ be a k-graph. Suppose that $\sim$ is an equivalence relation on Λ with the following properties:*

(1) if $\mu \sim \nu$ then $d(\mu) = d(\nu)$;
(2) if $\alpha \sim \alpha'$ and $\beta \sim \beta'$ with $r(\beta) = s(\alpha)$ and $r(\beta') = s(\alpha')$, then $\alpha\beta \sim \alpha'\beta'$;
(3) if $\alpha\beta \sim \alpha'\beta'$ and $d(\alpha) = d(\alpha')$, then $\alpha \sim \alpha'$ and $\beta \sim \beta'$;
(4) if $s(\alpha) \sim r(\beta)$, then there exist α', β' such that $\alpha' \sim \alpha$, $\beta' \sim \beta$ and $s(\alpha') = r(\beta')$.

Then the structure maps on Λ descend to structure maps on $\Lambda/\sim$ under which the latter is a k-graph.

Proof. Let $[\lambda]$ denote the equivalence class containing λ and suppose that $\mu \in [\lambda]$. Note that $d(\lambda) = d(\mu)$ by (1). We have $\lambda = r(\lambda)\lambda$ and $\mu = r(\mu)\mu$, and since $d(r(\lambda)) = 0 = d(r(\mu))$, condition (3) implies that $r(\lambda) \sim r(\mu)$. Since $\lambda = \lambda s(\lambda)$, $\mu = \mu s(\mu)$ and $d(\lambda) = d(\mu)$, condition (3) implies that $s(\lambda) \sim s(\mu)$. Hence, the formulas

$$d([\mu]) := d(\mu), \quad r([\mu]) = [r(\mu)], \quad \text{ and } s([\mu]) = [s(\mu)]$$

are well defined.

If $[s(\lambda)] = [r(\mu)]$, then condition (4) implies that there exist $\alpha \in [\lambda]$ and $\beta \in [\mu]$ such that $s(\alpha) = r(\beta)$, and then condition (2) implies that $[\alpha\beta]$ does not depend on the choice of α and β; so we may define $[\lambda][\mu] := [\alpha\beta]$. To see that this is associative, suppose that $[\lambda], [\mu], [\nu]$ is a composable triple in $\Lambda/\sim$. By (4) there exist $\alpha \in [\lambda]$ and $\beta \in [\mu]$ with $s(\alpha) = r(\beta)$. Now $s([\alpha\beta]) = r([\nu])$, and so (4) again gives $\eta \in [\alpha\beta]$ and $\gamma \in [\nu]$ such that $s(\eta) = r(\nu)$. The factorisation property gives $\eta = \alpha'\beta'$ with $d(\alpha') = d(\alpha)$ and $d(\beta') = d(\beta)$, and then (3) implies that $\alpha' \sim \alpha$ and $\beta' \sim \beta$. We now have

$$([\lambda][\mu])[\nu] = ([\alpha'][\beta'])[\gamma] = [\alpha'\beta'][\gamma] = [\alpha'\beta'\gamma] = [\alpha'][\beta'\gamma] = [\lambda]([\mu][\nu]).$$

We have now established that $\Lambda/\sim$ is a category. For the unique factorisation property, suppose that $d([\lambda]) = m + n$. Then $d(\lambda) = m + n$, and the factorisation property in Λ allows us to write $\lambda = \mu\nu$ with $d(\mu) = m$ and $d(\nu) = n$. We then have $[\lambda] = [\mu][\nu]$ with $d([\mu]) = m$ and $d([\nu]) = n$, and condition (3) implies that this factorisation is unique. □

Example 2.2. Let Λ_1, Λ_2 be k-graphs and $\phi_i : \Gamma \to \Lambda_i$ be an injective k-graph homomorphism for $i = 1, 2$ such that $\phi_i(\Gamma)$ is hereditary (or cohereditary) in Λ_i for $i = 1, 2$. Routine checks show that the smallest equivalence relation $\sim_\phi$ on $\Lambda_1 \sqcup \Lambda_2$ such that $\phi_1(\gamma) \sim_\phi \phi_2(\gamma)$ for all $\gamma \in \Gamma$ satisfies the hypotheses of Proposition 2.1.

Proposition 2.3. *Under the hypotheses of Proposition 2.1, there is an equivalence relation $\approx$ on X_Λ such that $[\mu, s] \approx [\nu, t]$ if and only if $\mu(\lfloor s \rfloor, \lceil s \rceil) \sim \nu(\lfloor t \rfloor, \lceil t \rceil)$ and $s - \lfloor s \rfloor = t - \lfloor t \rfloor$. Let $[\![\lambda, t]\!]$ denote the equivalence class of $[\lambda, t] \in X_\Lambda$ under $\approx$, and let $[\lambda]_\sim$ denote the equivalence class of $\lambda \in \Lambda$ under $\sim$. Then there is a homeomorphism $X_\Lambda/\approx \;\cong X_{\Lambda/\sim}$ satisfying $[\![\lambda, t]\!] \mapsto [[\lambda]_\sim, t]$ for all λ, t.*

Proof. Recall that in X_Λ, we have $[\mu, s] = [\mu', s']$ if and only if $\mu(\lfloor s \rfloor, \lceil s \rceil) = \mu'(\lfloor s' \rfloor, \lceil s' \rceil)$ and $s - \lfloor s \rfloor = s' - \lfloor s' \rfloor$. So the formula for $\approx$ is well-defined and determines a relation on X_Λ. It is elementary to check that this is an equivalence relation.

The map $\lambda \mapsto [\lambda]_\sim$ is a surjective k-graph morphism from Λ to $\Lambda/\sim$, and so [6, Proposition 5.3] implies that there is a continuous surjection $\phi : X_\Lambda \to X_{\Lambda/\sim}$ satisfying $[\lambda, t] \mapsto \big[[\lambda]_\sim, t\big]$. We need to show that $\big[[\lambda]_\sim, t\big] = \big[[\lambda']_\sim, t'\big]$ if and only if $[\lambda, t] \approx [\lambda', t']$. Suppose that $\big[[\lambda]_\sim, t\big] = \big[[\lambda']_\sim, t'\big]$. Then $[\lambda]_\sim(\lfloor t \rfloor, \lceil t \rceil) = [\lambda']_\sim(\lfloor t' \rfloor, \lceil t' \rceil)$ and $t - \lfloor t \rfloor = t' - \lfloor t' \rfloor$. Hence,

$$[\lambda(\lfloor t \rfloor, \lceil t \rceil)] = [\lambda]_\sim(\lfloor t \rfloor, \lceil t \rceil) = [\lambda']_\sim(\lfloor t' \rfloor, \lceil t' \rceil) = [\lambda'(\lfloor t' \rfloor, \lceil t' \rceil)].$$

Thus $\lambda(\lfloor t \rfloor, \lceil t \rceil) \sim \lambda'(\lfloor t' \rfloor, \lceil t' \rceil)$, and so $[\lambda, t] \approx [\lambda', t']$. Reversing the steps above proves the converse. □

3. The Toeplitz algebras of quotients of k-graphs

In this section we study the C^*-algebras associated to quotients of k-graphs as discussed in the preceding section. There are two C^*-algebras associated to a finitely-aligned k-graph Λ with no sources: the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ and the "usual" k-graph algebra $C^*(\Lambda)$.

Recall from [18] that the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ of a finitely-aligned k-graph Λ is the universal C^*-algebra generated by elements $\{t_\lambda : \lambda \in \Lambda\}$ such that

(TCK1) $\{t_v : v \in \Lambda^0\}$ is a set of mutually orthogonal projections;
(TCK2) $t_\mu t_\nu = t_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;
(TCK3) $t_\mu^* t_\mu = t_{s(\mu)}$ for all μ;
(TCK4) for every $v \in \Lambda^0$, $n \in \mathbb{N}^k$ and finite $F \subseteq v\Lambda^n$, we have $t_v \geq \sum_{\mu \in F} t_\mu t_\mu^*$; and
(TCK5) $t_\mu t_\mu^* t_\nu t_\nu^* = \sum_{\lambda \in \mathrm{MCE}(\mu,\nu)} t_\lambda t_\lambda^*$ for all μ, ν (an empty sum is interpreted as zero).

A collection $t = \{t_\lambda : \lambda \in \Lambda\}$ satisfying (TCK1)–(TCK5) is called a *Toeplitz-Cuntz-Krieger Λ-family.*

To describe the "usual" k-graph algebra $C^*(\Lambda)$, first recall that for $v \in \Lambda^0$, a nonempty set $E \subseteq v\Lambda$ is *exhaustive* if for every $\mu \in v\Lambda$ there exists $\lambda \in E$ such that $\mathrm{MCE}(\mu, \lambda) \neq \emptyset$; equivalently, E is exhaustive if $\mu\Lambda \cap E\Lambda \neq \emptyset$ for all $\mu \in v\Lambda$. The C^*-algebra $C^*(\Lambda)$ is universal for Toeplitz-Cuntz-Krieger Λ-families satisfying the additional relation:

(CK) for every $v \in \Lambda^0$ and finite exhaustive $E \subseteq v\Lambda$, we have $\prod_{\lambda\in E}(s_v - s_\lambda s_\lambda^*) = 0$

(see [19]).

We now describe how inclusions of k-graphs induce homomorphisms of their Toeplitz algebras. The following is a generalisation of [14, Proposition 3.3], [1, Theorem 5.2][1] and [11, Lemma 2.3].

Lemma 3.1. *Let Λ be a finitely-aligned k-graph with no sources and suppose that $\Gamma \subseteq \Lambda$ is a subgraph such that $\mathrm{MCE}_\Lambda(\mu, \nu) \subseteq \Gamma$ for all $\mu, \nu \in \Gamma$. Let $\{s_\gamma : \gamma \in \Gamma\}$ denote the universal Toeplitz-Cuntz-Krieger Γ-family and let $\{t_\lambda : \lambda \in \Lambda\}$ denote the universal Toeplitz-Cuntz-Krieger Λ-family. There is an injective homomorphism $\iota : \mathcal{T}C^*(\Gamma) \to \mathcal{T}C^*(\Lambda)$ such that $\iota(s_\gamma) = t_\gamma$ for all $\gamma \in \Gamma$.*

Proof. The elements $\{t_\gamma : \gamma \in \Gamma\}$ form a Toeplitz-Cuntz-Krieger Γ-family in $\mathcal{T}C^*(\Lambda)$: the relations all follow from the same relations for Λ (the hypothesis that $\mathrm{MCE}_\Lambda(\mu, \nu) \subseteq \Gamma$ for all $\mu, \nu \in \Gamma$ ensures that condition (TCK5) holds). So the universal property of $\mathcal{T}C^*(\Gamma)$ induces a homomorphism $\iota : \mathcal{T}C^*(\Gamma) \to \mathcal{T}C^*(\Lambda)$ satisfying $\iota(s_\gamma) = t_\gamma$. Theorem 3.11 of [21] applied to Λ implies that $\prod_{\lambda\in E}(t_v - t_\lambda t_\lambda^*) \neq 0$ for every $v \in \Lambda^0$ and every finite $E \subseteq v\Lambda$. The same theorem applied to Γ then implies that ι is injective. □

Lemma 3.2. *Suppose that Λ is a finitely aligned k-graph with no sources and that $H \subseteq \Lambda^0$ is hereditary. Let $T = \Lambda^0 \setminus H$. Then $H\Lambda$ is a subgraph such that $\mathrm{MCE}_\Lambda(\mu, \nu) \subseteq H\Lambda$ for all $\mu, \nu \in H\Lambda$, and ΛT is subgraph of Λ. Let $\iota : \mathcal{T}C^*(H\Lambda) \to \mathcal{T}C^*(\Lambda)$ be the homomorphism of Lemma 3.1. There is a homomorphism $\pi : \mathcal{T}C^*(\Lambda) \to \mathcal{T}C^*(\Lambda T)$ satisfying*

$$\pi(t_\lambda) = \begin{cases} s_\lambda & \text{if } s(\lambda) \in T \\ 0 & \text{otherwise.} \end{cases}$$

The sum $\sum_{v\in H} t_v$ converges strictly to a full multiplier projection P_H of $\ker(\pi)$. We have

$$\ker(\pi) = \overline{\mathrm{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu) \in H\} \quad \text{and} \quad P_H \ker(\pi) P_H = \iota(\mathcal{T}C^*(H\Lambda)).$$

Proof. Establishing the properties of $H\Lambda$ and ΛT is straightforward. Let I_H be the ideal of $\mathcal{T}C^*(\Lambda)$ generated by $\{p_v : v \in H\}$. Theorem 4.4 of [21] applied with $\mathcal{E} = \emptyset$ and $c \equiv 1$ shows that there is an isomorphism $\mathcal{T}C^*(\Lambda)/I_H \cong \mathcal{T}C^*(\Lambda T)$ which carries $t_\lambda + I_H$ to s_λ for $\lambda \in \Lambda T$. Composing this with the quotient map from $\mathcal{T}C^*(\Lambda)$ to $\mathcal{T}C^*(\Lambda)/I_H$ gives the desired homomorphism π. It is routine to check that the sum $\sum_{v\in H} t_v$ converges to a multiplier of $\mathcal{T}C^*(\Lambda)$ (see [11, Lemma 2.1] or [2, Lemma 1.2]). So $\ker(\pi) = I_H$, and $P_H \ker(\pi) P_H$ is a full hereditary subalgebra of $\ker(\pi)$. Since H is hereditary, $\overline{\mathrm{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu) \in H\}$ is an ideal which is clearly contained in I_H and contains all its generators, so the two are equal. We have $P_H s_\lambda = s_\lambda$ if $r(\lambda) \in H$ and $P_H s_\lambda = 0$ otherwise, and so

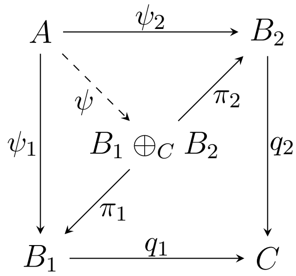
$$P_H \ker(\pi) P_H = P_H \mathcal{T}C^*(\Lambda) P_H = \overline{\mathrm{span}}\{s_\mu s_\nu^* : \mu, \nu \in H\Lambda\} = \iota(\mathcal{T}C^*(H\Lambda)). \qquad \square$$

Recall from [17, Section 2.2] (see also [3, §15.3]) that if B_1, B_2 are C^*-algebras and $q_i : B_i \to C$ is a homomorphism for each i, then the *pullback* $B_1 \oplus_C B_2$ is the subalgebra $\{(a, b) \in B_1 \oplus B_2 : q_1(a) = q_2(b)\}$ of $B_1 \oplus B_2$. It has the universal property that

(1) the canonical maps $\pi_i : B_1 \oplus_C B_2 \to B_i$ satisfy $q_1 \circ \pi_1 = q_2 \circ \pi_2$ and $\ker \pi_1 \cap \ker \pi_2 = \{0\}$; and

[1]there is a missing injectivity hypothesis in the statement of this result

(2) if $\psi_i : A \to B_i$ are homomorphisms such that $q_1 \circ \psi_1 = q_2 \circ \psi_2$, then there is a unique homomorphism $\psi : A \to B_1 \oplus_C B_2$ for which the following diagram commutes:



Theorem 3.3. *Let Λ_1, Λ_2 and Γ be finitely aligned k-graphs with no sources. Suppose that for $i = 1, 2$ we have an injective k-graph morphism $\phi_i : \Gamma \hookrightarrow \Lambda_i$ such that $\phi_i(\Gamma^0)$ is a co-hereditary subgraph of Λ_i. Let $\pi_i : \mathcal{T}C^*(\Lambda_i) \to \mathcal{T}C^*(\Gamma)$ be the homomorphism obtained from Lemma 3.2 and the isomorphism of Γ with $\phi_i(\Gamma)$, and form the pullback C^*-algebra $\mathcal{T}C^*(\Lambda_1) \oplus_{\mathcal{T}C^*(\Gamma)} \mathcal{T}C^*(\Lambda_2)$ with respect to π_1, π_2. Let $\phi : \phi_1(\Gamma) \to \phi_2(\Gamma)$ be the isomorphism $\phi_1(\lambda) \mapsto \phi_2(\lambda)$, and let $\sim_\phi$ be the equivalence relation of Example 2.2. For $i = 1, 2$, let $\{s^i_\lambda : \lambda \in \Lambda_i\}$ denote the universal Toeplitz-Cuntz-Krieger family in $\mathcal{T}C^*(\Lambda_i)$, and let $\{s_{[\lambda]} : \lambda \in (\Lambda_1 \sqcup \Lambda_2)/\sim_\phi\}$ be the universal generating family in $\mathcal{T}C^*((\Lambda_1 \sqcup \Lambda_2)/\sim_\phi)$. Then there is an isomorphism*

$$\theta : \mathcal{T}C^*((\Lambda_1 \sqcup \Lambda_2)/\sim_\phi) \to \mathcal{T}C^*(\Lambda_1) \oplus_{\mathcal{T}C^*(\Gamma)} \mathcal{T}C^*(\Lambda_2)$$

such that

$$\theta(s_{[\lambda]}) = \begin{cases} (s^1_\lambda, 0) & \text{if } \lambda \in \Lambda_1 \setminus \Gamma \\ (0, s^2_\lambda) & \text{if } \lambda \in \Lambda_2 \setminus \Gamma \\ (s^1_{\phi_1(\gamma)}, s^2_{\phi_2(\gamma)}) & \text{if } [\lambda] = \{\phi_1(\gamma), \phi_2(\gamma)\}. \end{cases} \tag{2}$$

Proof. The sets $H_1 := \{[v] : v \in \Lambda^0_1 \setminus \phi_1(\Gamma)\}$ and $H_2 := \{[v] : v \in \Lambda^0_2 \setminus \phi_2(\Gamma)\}$ are hereditary in Λ_1 and Λ_2 respectively. Let $T_i := \Sigma^0 \setminus H_i$ for $i = 1, 2$. Then $\lambda \mapsto [\lambda]$ is an isomorphism of Λ_i onto ΣT_{3-i} for $i = 1, 2$, and so Lemma 3.2 implies that there are homomorphisms $\psi_i : \mathcal{T}C^*(\Sigma) \to \mathcal{T}C^*(\Lambda_i)$ such that $\psi_i(t_{[\lambda]}) = s_\lambda$ for $\lambda \in \Lambda_i$. We then have

$$\pi_1 \circ \psi_1(t_{[\lambda]}) = \begin{cases} s_\lambda & \text{if } \lambda \in \Gamma \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

The universal property of the pullback now implies that there exists a homomorphism $\theta : \mathcal{T}C^*(\Sigma) \to \mathcal{T}C^*(\Lambda_1) \oplus_{\mathcal{T}C^*(\Gamma)} \mathcal{T}C^*(\Lambda_2)$ satisfying (2).

To see that θ is an isomorphism, we will invoke Proposition 3.1 of [17] to see that $\mathcal{T}C^*(\Sigma)$ is itself a pullback. We must show that

(1) $\ker(\psi_1) \cap \psi(\psi_2) = 0$,
(2) $\pi_2^{-1}(\pi_1(\mathcal{T}C^*(\Lambda_1)) = \psi_2(\mathcal{T}C^*(\Sigma))$, and
(3) $\psi_1(\ker(\psi_2)) = \ker(\pi_1)$.

For (1), observe first that since ψ_1 and ψ_2 are equivariant for the gauge actions on $\mathcal{T}C^*(\Sigma)$ and the $\mathcal{T}C^*(\Lambda_i)$, the ideal $\ker \psi_1 \cap \ker \psi_2$ is gauge invariant. Each $[v] \in \Sigma^0$ belongs to either Λ^0_1 or Λ^0_2, and so no $p_{[v]}$ belongs to $\ker \psi_1 \cap \ker \psi_2$. If $[v] \in \Sigma^0$ and $F \subseteq [v]\Sigma \setminus \{[v]\}$ is finite, then $v \in \Lambda^0_i$ for some i. Now

$$\psi_i\Big(\prod_{[\mu] \in F} (p_{[v]} - s_{[\mu]} s^*_{[\mu]})\Big) = \prod_{\mu \in \Lambda_1, [\mu] \in F} (p^i_v - s^i_\mu (s^i_\mu)^*) \neq 0.$$

So $\prod_{[\mu] \in F} (p_{[v]} - s_{[\mu]} s^*_{[\mu]}) \notin \ker \psi_1 \cap \ker \psi_2$. Theorem 4.6 of [21] implies that $\ker \psi_1 \cap \ker \psi_2 = \{0\}$.

For (2), the containment $\psi_2(\mathcal{T}C^*(\Sigma)) \subseteq \pi_2^{-1}(\pi_1(\mathcal{T}C^*(\Lambda_1)))$ is immediate because $\pi_1 \circ \psi_1 = \pi_2 \circ \psi_2$, and the reverse containment is clear because ψ_2 is surjective.

For (3), observe that Lemma 3.2 implies that

$$\ker(\psi_2) = \overline{\operatorname{span}}\big\{s_{[\mu]}s_{[\nu]}^* : s([\mu]) = s([\nu]) \in \{[v] : v \in \Lambda_1^0 \setminus \phi_1(\Gamma^0)\}\big\},$$

and that $\ker(\pi_1) = \overline{\operatorname{span}}\{s_\mu^1(s_\nu^1)^* : s(\mu) = s(\nu) \in \Lambda_1^0 \setminus \phi_1(\Gamma^0)\}$. If $s([\mu]) = s([\nu]) \in \{[v] : v \in \Lambda_1^0 \setminus \phi_1(\Gamma^0)\}$, then $\psi_1(s_{[\mu]}s_{[\nu]}^*) = s_\mu^1(s_\nu^1)^*$, so $\psi_1(\ker(\psi_2)) = \ker(\pi_1)$ as claimed.

We have now established the hypotheses of [17, Proposition 3.1], which then implies that $\mathcal{T}C^*(\Sigma)$ is a pullback of $\mathcal{T}C^*(\Lambda_1)$ and $\mathcal{T}C^*(\Lambda_2)$ over $\mathcal{T}C^*(\Gamma)$. The universal property of this pullback therefore yields a homomorphism $\eta : \mathcal{T}C^*(\Lambda_1) \oplus_{\mathcal{T}C^*(\Gamma)} \mathcal{T}C^*(\Lambda_2) \to \mathcal{T}C^*(\Sigma)$ which is inverse to θ. □

Recall from [20] that if Λ is a finitely aligned k-graph then a hereditary set $H \subseteq \Lambda^0$ is *saturated* if whenever $E \subseteq v\Lambda$ is finite exhaustive and $s(E) \subseteq H$ we have $v \in H$.

Corollary 3.4. *Let Λ_1 and Λ_2 and Γ be finitely aligned k-graphs with no sources. Suppose that for $i = 1, 2$, we have an injective k-graph morphism $\phi_i : \Gamma \hookrightarrow \Lambda_i$ such that $\phi_i(\Gamma^0)$ is a co-hereditary subgraph of Λ_i and $H_i := \Lambda_i^0 \setminus \phi_i(\Gamma^0)$ is saturated. Let $\pi_i : C^*(\Lambda_i) \to C^*(\Gamma)$ be the homomorphism obtained from Lemma 3.2, and form the pullback C^*-algebra $C^*(\Lambda_1) \oplus_{C^*(\Gamma)} C^*(\Lambda_2)$ with respect to π_1 and π_2. Let $\phi : \phi_1(\Gamma) \to \phi_2(\Gamma)$ be the isomorphism $\phi_1(\lambda) \mapsto \phi_2(\lambda)$, and let $\sim_\phi$ be the equivalence relation of Example 2.2. The isomorphism $\theta : \mathcal{T}C^*((\Lambda_1 \sqcup \Lambda_2)/{\sim_\phi}) \to \mathcal{T}C^*(\Lambda_1) \oplus_{\mathcal{T}C^*(\Gamma)} \mathcal{T}C^*(\Lambda_2)$ of Theorem 3.3 descends to an isomorphism*

$$\tilde{\theta} : C^*((\Lambda_1 \sqcup \Lambda_2)/{\sim_\phi}) \to C^*(\Lambda_1) \oplus_{C^*(\Gamma)} C^*(\Lambda_2).$$

Proof. Let θ be the isomorphism of Theorem 3.3, and let $q_0 : \mathcal{T}C^*(\Lambda_1) \oplus \mathcal{T}C^*(\Lambda_2) \to C^*(\Lambda_1) \oplus C^*(\Lambda_2)$ be the quotient map. Then q_0 restricts to a homomorphism

$$q : \mathcal{T}C^*(\Lambda_1) \oplus_{\mathcal{T}C^*(\Gamma)} \mathcal{T}C^*(\Lambda_2) \to C^*(\Lambda_1) \oplus_{C^*(\Gamma)} C^*(\Lambda_2).$$

Define $t_{[\lambda]} := q(\theta(s_{[\lambda]}))$ for $\lambda \in \Sigma := (\Lambda_1 \sqcup \Lambda_2)/{\sim_\phi}$. We claim that the $t_{[\lambda]}$ satisfy relation (CK). Suppose that $E \subseteq v\Sigma$ is finite exhaustive. If $v \notin \Gamma$, then E is finite exhaustive in Λ_i for some i and then (CK) follows from (CK) for Λ_i. Otherwise, $E \cap \Lambda_i$ is exhaustive for each of $i = 1, 2$, and then (CK) for Σ follows from (CK) for Λ_1 and Λ_2. So there is a homomorphism $\tilde{\theta}$ as claimed, and this $\tilde{\theta}$ is surjective because θ is.

To see that θ is injective, we apply the gauge-invariant uniqueness theorem. Let γ^i denote the gauge action on $C^*(\Lambda_i)$. Then the action $\gamma^1 \oplus \gamma^2$ of $\mathcal{T}^k$ on $C^*(\Lambda_1) \oplus C^*(\Lambda_2)$ restricts to an action β of $\mathcal{T}^k$ on the subalgebra $C^*(\Lambda_1) \oplus_{C^*(\Gamma)} C^*(\Lambda_2)$. The gauge action γ on $C^*\big((\Lambda_1 \sqcup \Lambda_2)/{\sim_\phi}\big)$ then satisfies $\beta_z \circ \tilde{\theta} = \tilde{\theta} \circ \gamma_z$ for all z. Since each $[v] \in (\Lambda_1 \sqcup \Lambda_2)/{\sim_\phi}$ has a representative v in either Λ_1^0 or Λ_2^0 we have $\tilde{\theta}(p_{[v]}) \neq 0$ for all $[v] \in \big((\Lambda_1 \sqcup \Lambda_2)/{\sim_\phi}\big)^0$. So the gauge-invariant uniqueness theorem [19, Theorem 3.1] implies that $\tilde{\theta}$ is injective. □

For the following result, observe that under the hypotheses of Corollary 3.4, the isomorphism $\theta : C^*((\Lambda_1 \sqcup \Lambda_2)/{\sim_\phi}) \to C^*(\Lambda_1) \oplus_{C^*(\Gamma)} C^*(\Lambda_2)$ determines an inclusion $\tilde{\theta} : C^*((\Lambda_1 \sqcup \Lambda_2)/{\sim_\phi}) \to C^*(\Lambda_1) \oplus C^*(\Lambda_2)$ satisfying the formula (3). Write $\pi_i : C^*(\Lambda_i) \to C^*(\Gamma)$ for the homomorphisms of Lemma 3.2. These induce homomorphisms $(\pi_i)_* : K_*(C^*(\Lambda_1)) \oplus K_*(C^*(\Lambda_2)) \to K_*(C^*(\Gamma))$.

Corollary 3.5. *With the hypotheses of Corollary 3.4, there is a 6-term exact sequence in K-theory as follows:*

$$\begin{array}{ccccc}
K_0\big(C^*((\Lambda_1 \sqcup \Lambda_2)/{\sim_\phi})\big) & \xrightarrow{\iota_*} & K_0(C^*(\Lambda_1)) \oplus K_0(C^*(\Lambda_2)) & \xrightarrow{(\pi_1)_* - (\pi_2)_*} & K_0(C^*(\Gamma)) \\
\uparrow & & & & \downarrow \\
K_1(C^*(\Gamma)) & \xleftarrow{(\pi_1)_* - (\pi_2)_*} & K_1(C^*(\Lambda_1)) \oplus K_0(C^*(\Lambda_2)) & \xleftarrow{\iota_*} & K_1\big(C^*((\Lambda_1 \sqcup \Lambda_2)/{\sim_\phi})\big)
\end{array}$$

Proof. This follows directly from Corollary 3.4 and the Meyer-Vietoris exact sequence for pullback C^*-algebras [3, Theorem 21.5.1]. □

4. Surfaces and the connected-sum operation

4.1. **Skeletons.** Recall from [5] that for $k \geq 1$, each k-graph Λ is completely determined by the k-coloured graph E_Λ with vertices Λ^0 and edges $\bigsqcup_{i=1}^k \Lambda^{e_i}$ coloured with k different colours $c_1, \dots, c_k$ (that is, λ has colour c_i if and only if $d(\lambda) = e_i$), together with the *factorisation rules* $ef = f'e'$ whenever $e, e' \in \Lambda^{e_i}$, $f, f' \in \Lambda^{e_j}$ and $ef = f'e'$ in Λ. This k-coloured graph is called the *skeleton* of Λ. Conversely, any k-coloured graph together with a set of bijections between ji-coloured paths and ij-coloured paths for distinct $i, j \leq k$, and satisfying the associativity condition of [5, §4] (the condition is vacuous when $k = 2$) determines a k-graph.

By convention, in a 2-coloured graph the edges of colour c_1 are drawn blue (or solid) and the edges of colour c_2 are drawn red (or dashed).

4.2. **The connected-sum operation.** We aim to prove the following Theorem:

Theorem 4.1. *For each compact 2-dimensional manifold M, there is a 2-graph Λ whose topological realisation X_Λ is homeomorphic to M.*

In order to prove the Theorem we develop a connected-sum operation on 2-graphs and apply the connected-sum to the four basic surfaces. The steps are given in 4.2.1-4.2.5

4.2.1. Let Λ_1 and Λ_2 be 2-graphs. Suppose that $u_i, v_i \in \Lambda_i^0$ have the property that $\Lambda_i u_i = \{u_i\}$ and $v_i \Lambda_i = \{v_i\}$ for $i = 1, 2$. Let $\sim$ be the smallest equivalence relation on $\Lambda_1 \sqcup \Lambda_2$ such that $u_1 \sim u_2$ and $v_1 \sim v_2$. Since there are no morphisms $\alpha_i \in \Lambda_i$ such that $s(\alpha_i) = u_i$ or $r(\alpha_i) = v_i$, the relation $\sim$ trivially satisfies properties (1)–(4) of Proposition 2.1, and so we may form the quotient 2-graph $(\Lambda_1 \sqcup \Lambda_2)/{\sim}$.

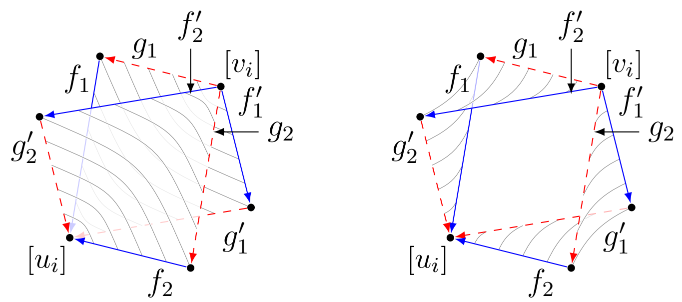
4.2.2. Now suppose that for $i = 1, 2$ there exist commuting squares $f_i g_i = g_i' f_i'$ in Λ_i with $f_i, f_i' \in \Lambda_i^{e_1}$ and $g_i, g_i' \in \Lambda_i^{e_2}$ such that $r(f_i) = u_i$, $s(g_i) = v_i$, and the vertices $u_i, s(f_i), s(g_i')$ and v_i are all distinct. Let $\mathcal{C}$ be the set of factorisation rules for $E_{(\Lambda_1 \cup \Lambda_2)/\sim}$ and specify a new set of factorisation rules $\mathcal{C}'$ by replacing $f_1 g_1 = g_1' f_1'$ and $f_2 g_2 = g_2' f_2'$ by $f_1 g_1 = g_2' f_2'$ and $f_2 g_2 = g_1' f_1'$. Since this new set of factorisation rules still specifies a range- and source-preserving bijection between red-blue paths and blue-red paths, and since the associativity condition of [5] is vacuous when $k = 2$, this is also a valid set of factorisation rules on $E_{(\Lambda_1 \cup \Lambda_2)/\sim}$. Hence Theorems 4.4 and 4.5 of [5] imply that there is a unique 2-graph $\Lambda_1 \# \Lambda_2$ with skeleton E and factorisation rules $\mathcal{C}'$, called the *connected-sum* of Λ_1 and Λ_2.

4.2.3. Proposition 2.3 implies that $X_{(\Lambda_1 \sqcup \Lambda_2)/\sim}$ is the surface formed by gluing the points $[u_1, 0]$ and $[v_1, 0]$ in X_{Λ_1} to the points $[u_2, 0]$ and $[v_2, 0]$ of X_{Λ_2}. Let $\alpha, \beta \in \big((\Lambda_1 \sqcup \Lambda_2)/{\sim}\big)^{(1,1)}$ be the elements $\alpha = f_1 g_1 = g_1' f_1'$ and $\beta = f_2 g_2 = g_2' f_2'$. Likewise, let $\eta = f_1 g_1 = g_2' f_2'$ and $\zeta = f_2 g_2 = g_1' f_1'$ in $(\Lambda_1 \# \Lambda_2)^{(1,1)}$. Then we have

$$X_{\Lambda_1 \# \Lambda_2} = X_{(\Lambda_1 \sqcup \Lambda_2)/\sim} \setminus \big(\{\alpha, \beta\} \times (0, (1,1))\big) \cup \big(\{\eta, \zeta\} \times (0, (1,1))\big).$$

The effect of this on the topological realisation is illustrated below: the squares corresponding to α and β in $(\Lambda_1 \sqcup \Lambda_2)/{\sim}$ are illustrated on the left, and those corresponding to η and ζ in $\Lambda_1 \# \Lambda_2$ are illustrated on the right.

$$f_1 g_1 = g_1' f_1' \qquad f_2 g_2 = g_2' f_2'$$

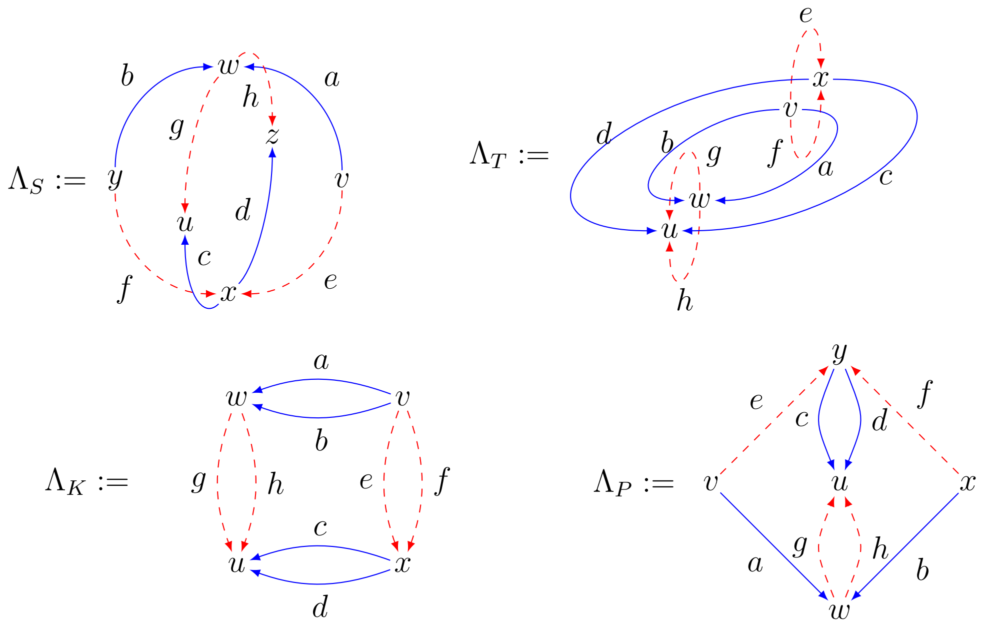


$$f_1 g_1 = g_2' f_2' \qquad f_2 g_2 = g_1' f_1'$$

The operation described in 4.2.3 deletes the interior of the square $f_1g_1 = g_1'f_1'$ in Λ_1 and the square $f_2g_2 = g_2'f_2'$ in Λ_2, and inserts a copy of a unit square bounded by f_1, g_1, f_2' and g_2' and another bounded by f_1', g_1', f_2 and g_2. Since the topological realisation of a 2-graph is obtained by pasting a unit square into each commuting square and identifying common edges (see the remark after the proof of Lemma 3.9 in [6]), we have shown that $X_{\Lambda_1\#\Lambda_2} \cong X_{\Lambda_1}\#X_{\Lambda_2}$.

4.2.4. In the connected-sum $\Lambda_1\#\Lambda_2$, the vertices $u = [u_1]$ and $v = [v_1]$ of $\Lambda_1\#\Lambda_2$ and the path $f_1g_1 = g_2'f_2'$ in $\Lambda^{(1,1)}$ have the properties required of u_1, v_1 in 4.2.1. So the process we have just described can be iterated.

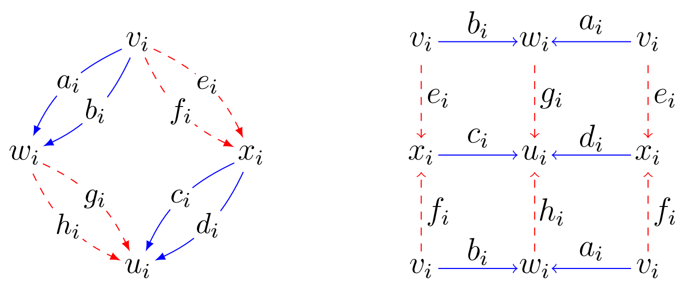
Examples 3.10–3.13 in Section 3.1 of [6] illustrate finite 2-graphs Λ_S, Λ_T, Λ_K, Λ_P whose topological realisations are the sphere, the torus, the Klein bottle and the projective plane respectively. Their skeletons are depicted below, each with the vertices labelled u and v satisfying the conditions in 4.2.1 and a commuting square $\alpha = ag = ec$ as in 4.2.2:



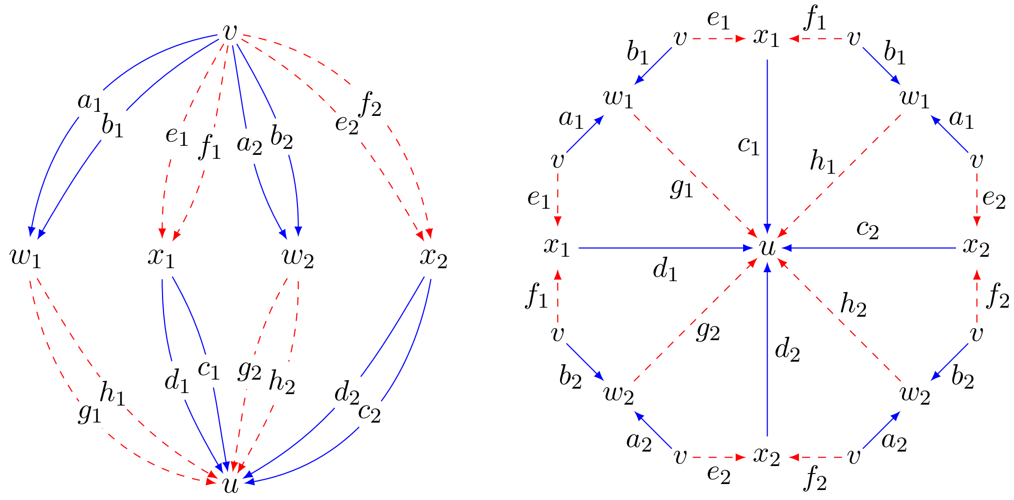
4.2.5. Fix a sequence Λ_i of 2-graphs, each of which is a copy of one of Λ_S, Λ_T, Λ_K or Λ_P. Let $\Gamma_1 = \Lambda_1$ and inductively construct $\Gamma_i = \Gamma_{i-1}\#\Lambda_i$ where the connected-sum construction is applied to $\alpha_b \in \Gamma_{i-1}$ and $\alpha_a \in \Lambda_i$. Then $\bigcup_{i=1}^{\infty}\bigcap_{j=i}^{\infty}\Gamma_i$ is a 2-graph and its topological realisation is the connected-sum of the Λ_i. So we can form any countable connected-sum of these four surfaces as the topological realisation of a 2-graph.

Proof of Theorem 4.1. The classification of compact 2-dimensional manifolds (see for example [12, Theorem I.7.2]) says that any such object is a sphere, a connected sum of n 2-tori, or the connected sum of the latter with either the Klein bottle or the projective plane. Fix such a decomposition, and then apply the proceedure 4.2.1–4.2.5 to the corresponding collection of k-graphs. The topological realisation of the resulting 2-graph is then the desired surface. □

Example 4.2. Let $\Lambda_i = \Lambda_T$, for $i = 1, 2$ where Λ_T is the 2-graph whose topological realisation is the 1-holed torus. The skeletons of the Λ_i both have the form of the skeleton on the left



with factorisation rules given by the square commuting diagrams on the right. To apply our construction we set $u = [u_i]$ and $v = [v_i]$ with the distinguished squares $d_1 f_1 = h_1 a_1$ of Λ_1 and $c_2 e_2 = g_2 b_2$ of Λ_2. Then the skeleton of the connected sum $\Lambda_1 \# \Lambda_2$ has the form of the diagram on the left with the same factorisation rules as above except that $d_1 f_1 = g_2 b_2$ and $h_1 a_1 = c_2 e_2$.



Organising this skeleton into the commuting diagram on the right (note that this is not the skeleton of the 2-graph, because some edges of the 2-graph appear more than once in the diagram), we recognise the standard octohedral planar diagram for a two-holed 2-torus.

5. Simplices and k-spheres from k-graphs

In this section we show how to realise a k-simplex as the topological realisation of a k-graph, Σ_k. Combining this with the results of Section 2 we prove the following result.

Theorem 5.1. *For each $k \geq 0$ there is a finite k-graph Γ whose topological realisation is homeomorphic to a k-sphere.*

Recall our convention that $\mathbf{1}_0 = 0$, the unique element of $\mathbb{N}^0$. We begin with the construction of Σ_k.

Definition 5.2. A function $f : \{0, \dots, k\} \to \{0, \dots, k\}$ is a *k-placing*, or just a *placing* if

$$f(j) = \big|\{i : f(i) < f(j)\}\big| \quad \text{for all } j \leq k. \tag{4}$$

We write P_k for set of k-placings, and for $f, g \in P_k$, we write $f \leq g$ if $f(i) \leq g(i)$ for all i.

Every permutation of $\{0, \dots, k\}$ is a placing, and these are precisely the maximal placings with respect to $\leq$. We write $P_k^{\max}$ for set of maximal placings. The zero function $0 : i \mapsto 0$ is the unique minimum k-placing.

We will construct a k-graph whose vertices are the k-placings. (An acknowledgement is in order: we found this parameterisation using The On-Line Encyclopedia of Integer Sequences [13] — we constructed the first three examples by hand, and then used the OEIS to seach for the sequence of the numbers of vertices appearing in these graphs.) To define the degree map on this k-graph, we first introduce the following height function on placings.

Definition 5.3. The *height function* $h : P_k \to \mathbb{N}^k$ is given by

$$h(f)_i = \begin{cases} 1 & \text{if } f^{-1}(i) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } 1 \leq i \leq k,$$

and satisfies $h(f) \leq \mathbf{1}_k$ for all $f \in P_k$.

Note that $h(0) = 0$, and if $\sigma \in P_k^{\max}$, then $h(\sigma) = \mathbf{1}_k$. We have $h(f) = e_i$ if and only if $\mathrm{range}(f) = \{0, i\}$, which in turn is equivalent to $|f^{-1}(0)| = i$ and $|f^{-1}(i)| = k + 1 - i$.

Proposition 5.4. *Let* $\Sigma_k = \{(f,g) \in P_k \times P_k : f \leq g\}$. *Define* $r(f,g) = (f,f)$, $s(f,g) = (g,g)$, $(f,g)(g,h) = (f,h)$, *and* $d(f,g) = h(g) - h(f)$. *With respect to these structure maps,* (Σ_k, d) *is a* k*-graph, and* d *maps* Σ_k *onto* $\{n \in \mathbb{N}^k : n \leq \mathbf{1}_k\}$.

To prove the proposition, we set aside a technical lemma.

Lemma 5.5. *Suppose* $f \in P_k$ *and* $h(f) \geq z \in \mathbb{N}^k$. *Define* $g : \{0,\dots,k\} \to \{0,\dots,k\}$ *by*

$$g(i) = \max\{j \leq f(i) : z_j = 1\}.$$

Then $g \in P_k$, $g \leq f$ *and* $h(g) = z$. *Moreover,* g *is the unique such element of* P_k, *and for* $n \in \mathrm{range}(g)$ *we have* $g^{-1}(\{1,\dots,n-1\}) = f^{-1}(\{1,\dots,n-1\})$.

Proof. It is clear that $g \leq f$ as functions.

For $i \leq k$, we claim that $\{j : g(j) < g(i)\} = \{j : f(j) < g(i)\}$. Since $g \leq f$, we certainly have $\{j : g(j) < g(i)\} \supseteq \{j : f(j) < g(i)\}$. For the reverse, we suppose that $f(j) \geq g(i)$ and show that $g(j) \geq g(i)$. We have $z_{g(i)} = 1$ by definition of g. So $f(j) \geq g(i)$ implies that

$$g(j) = \max\{l \leq f(j) : z_l = 1\} \geq \max\{l \leq g(i) : z_l = 1\} = g(i).$$

This proves the claim. Now to see that g is a placing, fix $i \leq k$. There exists $l \leq k$ such that $f(l) = g(i)$. Hence

$$g(i) = f(l) = \big|\{j : f(j) < f(l)\}\big| = \big|\{j : f(j) < g(i)\}\big| = \big|\{j : g(j) < g(i)\}\big|.$$

So g is a placing. To see that $h(g) = z$, observe that by definition of g we have $g^{-1}(i) \neq \emptyset$ if and only if $z_i = 1$. Hence, $h(g) = z$.

Suppose that $g' \leq f$ and $h(g') = z$. Then for $j \leq k$ we have $j \in \mathrm{range}(g') \iff z_j = 1$. Since $g \leq f$, we have $f^{-1}(\{0,\dots,n\}) \subseteq (g')^{-1}(\{0,\dots,n\})$ for each $n \leq k$. Suppose for contradiction that $f^{-1}(\{0,\dots,n-1\}) \subsetneq (g')^{-1}(\{0,\dots,n-1\})$ for some $n \in \mathrm{range}(g')$. Then there is a least such n; say $f(l) = g'(l') = n$. Then

$$\begin{aligned} n = f(l) &= \big|\{j : f(j) < f(l)\}\big| = \big|f^{-1}(\{0,\dots,n-1\})\big| \\ &< \big|(g')^{-1}(\{0,\dots,n-1\})\big| = \big|\{j : g'(j) < g'(l')\}\big| = n, \end{aligned}$$

a contradiction. So

$$f^{-1}(\{0,\dots,n-1\}) = (g')^{-1}(\{0,\dots,n-1\}) \quad \text{whenever } n \in \mathrm{range}(g'). \tag{5}$$

Let $n \in \mathrm{range}(g')$ and let $p = \max\{g'(j) : g'(j) < n\}$. Then

$$\begin{aligned} (g')^{-1}(\{0,\dots,p-1\}) \sqcup (g')^{-1}(\{p\}) &= (g')^{-1}(\{0,\dots,n-1\}) \\ &= f^{-1}(\{0,\dots,n-1\}) \\ &= f^{-1}(\{0,\dots,p-1\}) \sqcup f^{-1}(\{i : p \leq i < n,\}). \end{aligned}$$

So for $p \in \mathrm{range}(g')$ we have

$$(g')^{-1}(p) = f^{-1}\big(\{i \in \mathrm{range}(f) : p = \max\{j \in \mathrm{range}(g') : j \leq i\}\}\big).$$

Since $i \in \mathrm{range}(g')$ if and only if $z_i = 1$, we deduce that $g'(j) = \max\{f(l) : z_j = 1\}$ for all j; that is, $g' = g$. The final assertion now follows from (5). □

Proof of Proposition 5.4. Since $\leq$ is a partial order, Σ_k forms a category with identity morphisms (f,f). It is clear that d is a functor. So we just need to check the factorisation property.

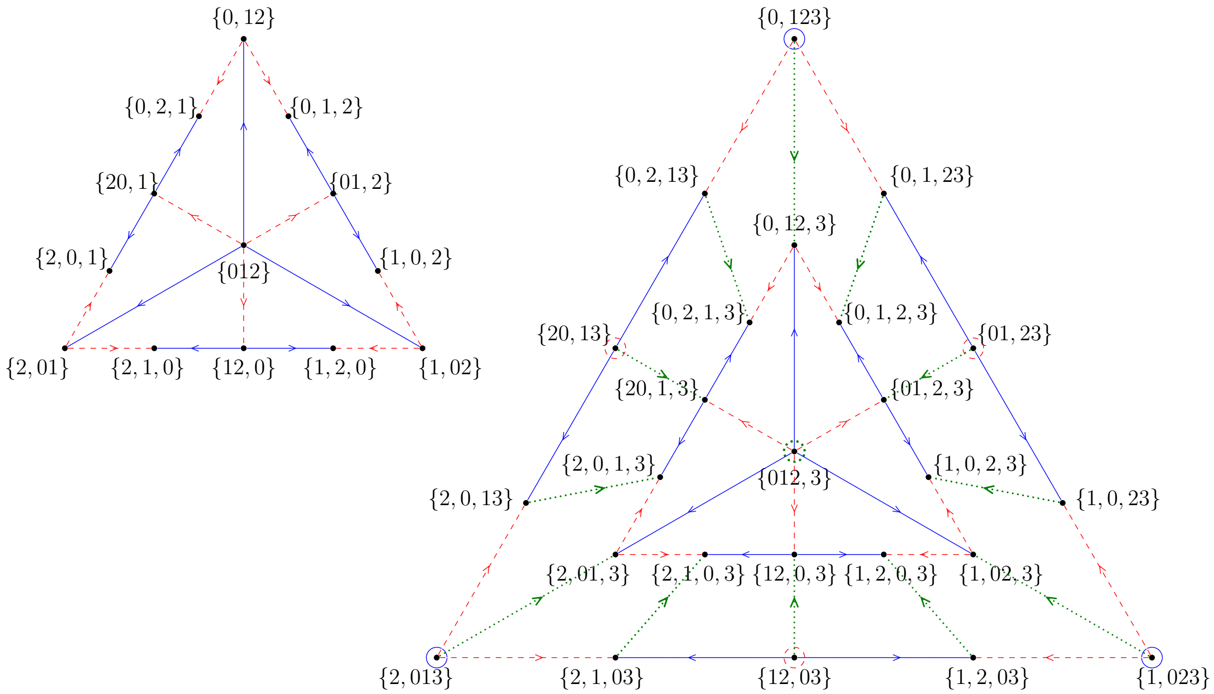
Suppose that $(f',f) \in \Sigma_k$ and $d(f',f) = n+m$. Let g be the placing constructed from f with $z = h(f') + n$ as in Lemma 5.5. The uniqueness assertion in Lemma 5.5 implies that

$$f'(i) = \max\{j < f(i) : h(f')_i = 1\} \leq \max\{j < f(i) : h(f')_i + n_i = 1\} = g(i)$$

So we have $(f',g), (g,f) \in \Sigma_k$ with $d(f',g) = n$ and $d(g,f) = m$, and $(f',g)(g,f) = n+m$. Uniqueness of this factorisation follows from the uniqueness assertion in Lemma 5.5.

The range of the degree map is contained in $\{n \in \mathbb{N}^k : n \leq \mathbf{1}_k\}$ because $h(f) \leq \mathbf{1}_k$ for all f. Each $f \in P_k^{\max}$ satisfies $h(f) = \mathbf{1}_k$, and so $d(0, f) = \mathbf{1}_k$. Then the factorisation property implies that the range of d is all of $\{n \in \mathbb{N}^k : n \leq \mathbf{1}_k\}$. □

Example 5.6. On the left side of the following picture, the 1-skeleton of the 2-simplex Σ_2 is shown, with vertices labelled as placing functions, where for example $\{2, 01\}$ denotes the placing function $f_{\{2,01\}}$ defined by $f_{\{2,01\}}(2) = 0$, $f_{\{2,01\}}(0) = 1$ and $f_{\{2,01\}}(1) = 1$. On the right hand side is one of the four faces of the one skeleton of the tetrahedron-shaped 3-simplex Σ_3. In a 3-graph it is customary to make the third colour in the 1-skeleton green (or dotted). The coloured circles around certain vertices indicates an edge of the same colour from that vertex to the central vertex labelled $\{0123\}$. The circles vertices are labelled by placing functions with exactly first and second place specified with circle colour blue (solid) if there is a unique first place, red (dashed) if there are ties for first and second place, and green (dotted) if there is a unique third place. For example, there is a blue edge from $\{0, 123\}$ to $\{0123\}$, a red edge from $\{20, 13\}$ to $\{0123\}$, and a green edge from $\{012, 3\}$ to $\{0123\}$.



We will show that the topological realisation of Σ_k can be identified with a k-simplex whose extreme points correspond to the $f \in \Sigma_k$ such that $\operatorname{range}(f) = \{0, 1\}$ for $k \geq 1$ (of course $\operatorname{range}(f) = \{0\}$ if $k = 0$, and then $\Sigma_0 = \Sigma_0^0$ is a point, as is its topological realisation). We use the following notation throughout the construction.

Notation 5.7. Let $\{\varepsilon_0, \dots, \varepsilon_k\}$ denote the usual basis vectors for $\mathbb{R}^{k+1}$. Let $v_0 = \frac{1}{k+1} \sum_{i=0}^{k} \varepsilon_i$, and for $f \in P_k$ and $n \in \operatorname{range}(f) \setminus \{0\}$, define $v_{f,n} \in \mathbb{R}^{k+1}$ by

$$v_{f,n} = \frac{1}{n} \sum_{f(j) < n} \varepsilon_j.$$

The defining property of placing functions ensures that $\|v_{f,n}\|_1 = 1$.

Recall that for $m \in \mathbb{N}^k$ we write $[0, m]$ for generalised interval $\{t \in \mathbb{R}^k : 0 \leq t_i \leq m_i$ for all $i \leq k\}$. Given $f \in P_k$, define a map $\phi_f : [0, h(f)] \to \mathbb{R}^{k+1}$ by $\phi_f(0) = v_0$ and

$$\phi_f(t) = (1 - \|t\|_\infty) v_0 + \frac{\|t\|_\infty}{\|t\|_1} \sum_{n \in \operatorname{range}(f) \setminus \{0\}} t_n v_{f,n} \tag{6}$$

for $t \neq 0$.

Theorem 5.8. *There is a homeomorphism* $i : X_{\Sigma_k} \to \operatorname{conv}\{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_k\}$ *such that*

$$i([(0,f),t]) = \phi_f(t) \quad \text{for all } f \in P_k \text{ and } t \in [0, h(f)]. \tag{7}$$

The reader is invited to use (6) to see that the formula (7) carries the topological realisation of Σ_2 to the simplex in $\mathbb{R}^3$ whose extreme points are the three unit basis vectors $\varepsilon_0, \varepsilon_1, \varepsilon_2$; for example, that the placing function $\{20, 1\}$ is sent to $(\varepsilon_0 + \varepsilon_2)/2$.

We need some technical lemmas to prove the theorem.

Lemma 5.9. *Let* $f \in P_k$ *and* $t \in [0, h(f)]$. *For* $i \le k$, *we have* $t_i \ne 0$ *if and only if there exists* $r \in \mathbb{R}$ *such that* $|\{j : \phi_f(t)_j > r\}| = i$. *Suppose that* $|\{j : \phi_f(t)_j > r\}| = i$, *and define* $g : \{0, \dots, k\} \to \{0, \dots, k\}$ *by*

$$g(j) = \begin{cases} i & \text{if } \phi_f(t)_j \le r \\ 0 & \text{if } \phi_f(t)_j > r. \end{cases}$$

Then $g \in P_k$, $h(g) = \varepsilon_i$ *and* $g \le f$.

Proof. Choose a permutation σ of $\{0, \dots, k\}$ such that $f \circ \sigma$ is nondecreasing. Then each $(v_{f,n})_{\sigma(i)} = 1/n$ if $i \le n$ and 0 if $i > n$, and $(v_0)_{\sigma(i)} = 1/(k+1)$ for all i. In particular, each $w \in \{v_0\} \cup \{v_{f,n} : n \le k\}$ satisfies $w_{\sigma(0)} \ge w_{\sigma(1)} \ge \dots \ge w_{\sigma(k)}$, and $w_{\sigma(i)} = w_{\sigma(i+1)}$ unless $w = v_{f,i+1}$. Hence each

$$\begin{aligned}\phi_f(t)_{\sigma(i)} &= (1 - \|t\|_\infty)(v_0)_{\sigma(i)} + \frac{\|t\|_\infty}{\|t\|_1} \sum_{n \in \operatorname{range}(f) \setminus \{0\}} t_n (v_{f,n})_{\sigma(i)} \\ &\ge (1 - \|t\|_\infty)(v_0)_{\sigma(i+1)} + \frac{\|t\|_\infty}{\|t\|_1} \sum_{n \in \operatorname{range}(f) \setminus \{0\}} t_n (v_{f,n})_{\sigma(i+1)},\end{aligned}$$

with strict inequality if and only if $t_{i+1} > 0$ (since $t \le h(f)$, this forces $i+1 \in \operatorname{range}(f) \setminus \{0\}$). This proves the first assertion.

To prove the second assertion, observe that $g \in P_k$ by choice of r, and $h(g) = e_i$ by definition of h. We claim that whenever $\phi_f(t)_{j'} > \phi_f(t)_j$, we have $f(j') < f(j)$. To prove this, first suppose that $j, j' \le k$ satisfy $\phi_f(t)_{\sigma(j')} > \phi_f(t)_{\sigma(j)}$. Then the preceding paragraph shows that $j' < j$ and $t_{l+1} > 0$ for some $j' \le l < j$. Since $f \circ \sigma$ is a nondecreasing placing function, it is dominated in the ordering on P_k by the identity permutation. So Lemma 5.5 implies that

$$f \circ \sigma(j') = \max\{n \le j' : h(f)_n = 1\} \le j' \le l.$$

Since $t < h(f)$, we also have

$$f \circ \sigma(j) = \max\{n \le j : h(f)_n = 1\} \ge \max\{n \le j : t_n > 0\} \ge l + 1.$$

In particular, $f \circ \sigma(j') < f \circ \sigma(j)$. Since σ is invertible, we deduce that $\phi_f(t)_{j'} > \phi_f(t)_j$ implies $f(j') < f(j)$ for all j, j'. This proves the claim.

To show that $g \le f$, observe that $g(j) = 0 \le f(j)$ whenever $\phi_f(t)_j > r$; and if $\phi_f(t)_j \le r$, then the claim gives

$$f(j) = |\{l : f(l) < f(j)\}| \ge |\{l : \phi_f(t)_l > \phi_f(t)_j\}| \ge |\{l : \phi_f(t)_l > r\}| = i = g(j). \qquad \square$$

For the next lemma, if Λ is a k-graph and F is a finite subset of Λ, then we define

$$\operatorname{MCE}(F) = \big\{\lambda \in \Lambda : \lambda \in \alpha\Lambda \text{ for all } \alpha \in F \text{ and } d(\lambda) = \textstyle\bigvee_{\alpha \in F} d(\alpha)\big\},$$

so we have $\operatorname{MCE}(\alpha, \beta) = \operatorname{MCE}(\{\alpha, \beta\})$ for all $\alpha, \beta \in \Lambda$. If F is a finite subset of Λ, $\lambda \in F$ and $G = F \setminus \{\lambda\}$, then

$$\operatorname{MCE}(F) = \bigcup_{\mu \in \operatorname{MCE}(G)} \operatorname{MCE}(\lambda, \mu). \tag{8}$$

Lemma 5.10. *For* $F \subseteq \{(0, f) : f \in P_k\} \subseteq \Sigma_k$, *we have* $|\operatorname{MCE}(F)| \le 1$.

Proof. We proceed by induction on $|F|$; if $|F| \leq 1$ then the statement is trivial.

Suppose that $|F| = 2$, say $F = \{(0, f), (0, g)\}$. We have $\mathrm{MCE}((0, f), (0, g)) = \{(0, a) : h(a) = h(f) \vee h(g) \text{ and } f, g \leq a\}$. So suppose that $a \in P_k$ satisfies $h(a) = h(f) \vee h(g)$ and $f, g \leq a$. Choose $\sigma \in P_k^{\max}$ such that $a \leq \sigma$. Since σ is surjective, Lemma 5.5 implies that for $j \leq k$,

$$f(j) = \max\{j \leq \sigma(i) : h(f)_j = 1\}, \quad g(j) = \max\{j \leq \sigma(i) : h(g)_j = 1\}, \quad \text{and}$$
$$a(j) = \max\{j \leq \sigma(i) : h(f)_j = 1 \text{ or } h(g)_j = 1\}.$$

So $a(j) = \max\{f(j), g(j)\}$ for all j, and in particular, a is uniquely determined by f, g.

Now suppose that $|\mathrm{MCE}(G)| \leq 1$ whenever $|G| < |F|$. Fix $\lambda \in F$ and let $G = F \setminus \{\lambda\}$. Property (8) above gives $\mathrm{MCE}(F) = \bigcup_{\mu \in \mathrm{MCE}(G)} \mathrm{MCE}(\lambda, \mu)$. The inductive hypothesis implies that there is at most one $\mu \in \mathrm{MCE}(G)$, and then the result follows from the base case. □

Lemma 5.11. *For $f \in P_k$ the map ϕ_f of Equation 6 is injective.*

Proof. We have $\phi_f(s) = v_0$ if and only if $s = 0$, so we suppose that $s, t \neq 0$ and $\phi_f(s) = \phi_f(t)$ and show that $s = t$. The vectors $v_0, v_{f,1}, \ldots, v_{f,n}$ are linearly independent, so $\phi_f(s) = \phi_f(t)$ implies $1 - \|s\|_\infty = 1 - \|t\|_\infty$ and $\frac{\|s\|_\infty}{\|s\|_1} s_n = \frac{\|t\|_\infty}{\|t\|_1} t_n$ for all n. In particular, $s = \lambda t$ for some $\lambda \in (0, \infty)$, and $\lambda\|t\|_\infty = \|\lambda t\|_\infty = \|s\|_\infty = \|t\|_\infty$. As $s \neq 0$ we have $\lambda = 1$ and hence $s = t$. □

Proof of Theorem 5.8. Each ϕ_f is a continuous map. By definition of X_{Σ_k}, each point in X_{Σ_k} has the form $[(g, f), t]$ where $g \leq f \in P_k$ and $h(g) \leq t \leq h(f)$. Since $((g, f), t) \sim ((0, f), t + h(g))$, it follows that every point in X_{Σ_k} can be expressed as $[(0, f), s]$ for some $f \in P_k$ and $s \in [0, h(f)]$.

So it suffices to show that $[(0, f), s] = [(0, g), t]$ if and only if $\phi_f(s) = \phi_g(t)$; for then (7) descends to a continuous bijection between X_{Σ_k} and $\mathrm{conv}\{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k\}$, which is then a homeomorphism because both sets are compact and Hausdorff.

First suppose that $[(0, f), s] = [(0, g), t]$. Then $s - \lfloor s \rfloor = t - \lfloor t \rfloor$, and $(0, f)(\lfloor s \rfloor, \lceil s \rceil) = (0, g)(\lfloor t \rfloor, \lceil t \rceil)$. Hence

$$\lfloor s \rfloor = h\big(r\big((0, f)(\lfloor s \rfloor, \lceil s \rceil)\big)\big) = h\big(r\big((0, g)(\lfloor t \rfloor, \lceil t \rceil)\big)\big) = \lfloor t \rfloor,$$

and combining this with $s - \lfloor s \rfloor = t - \lfloor t \rfloor$ gives $s = t$. By the factorisation property there is a unique factorisation $(0, f) = \alpha\beta$ in Σ_k with $d(\alpha) = \lceil s \rceil$. Define $f \wedge g$ to be the placing function for $s(\alpha)$. Then $f \wedge g \leq f, g$, and

$$[(0, f), s] = [(0, f \wedge g), s] = [(0, g), t].$$

So we may assume without loss of generality that $g \leq f$, and so $s \leq h(g) \leq h(f)$. Whenever $s_n \neq 0$ we have $h(g)_n = h(f)_n = 1$ and hence $n \in \mathrm{range}(g) \subseteq \mathrm{range}(f)$. So the final assertion of Lemma 5.5 implies that $g^{-1}(\{0, \ldots, n-1\}) = f^{-1}(\{0, \ldots, n-1\})$, and hence $v_{g,n} = v_{f,n}$. So $\phi_g(s) = \phi_f(s)$ as required.

Now suppose that $\phi_f(s) = \phi_g(t)$. The first assertion of Lemma 5.9 implies that $s_i > 0$ if and only if $t_i > 0$, and so $\lceil s \rceil = \lceil t \rceil$. The second assertion of Lemma 5.9 combined with the factorisation property in Σ_k implies that for each i such that $t_i > 0$ we have $d(0, f), d(0, g) \geq e_i$ and the unique edge $\alpha_i \in \Sigma_k^{e_i}$ such that $(0, f) = \alpha_i \tau_i$ for some τ_i also satisfies $(0, g) = \alpha_i \rho_i$ for some ρ_i. By Lemma 5.5 there are unique elements $f', g' \in P_k$ such that $h(f') = h(g') = \lceil s \rceil$ and $f' \leq f$ and $g' \leq g$. The factorisation property implies that $(0, f'), (0, g') \in \alpha_i \Sigma_k$ for all i. Since $d((0, f')) = d((0, g')) = \bigvee_{t_i > 0} d(\alpha_i)$, we then have $(0, f'), (0, g') \in \mathrm{MCE}(\{\alpha_i : t_i > 0\})$. So Lemma 5.10 implies that $|\mathrm{MCE}(\{\alpha_i : t_i > 0\})| \leq 1$, and we deduce that $f' = g'$. The argument of the preceding paragraph shows that

$$\phi_{f'}(s) = \phi_f(s) = \phi_g(t) = \phi_{g'}(t),$$

and since $f' = g'$, Lemma 5.11 implies that $s = t$. Hence

$$((0, f), s) \sim ((0, f'), s) = ((0, g'), t) \sim ((0, g), s).$$

□

Proof of Theorem 5.1. Consider the set $\{0,1\}$, regarded as a 0-graph, and the k-graph Σ_k described in Theorem 5.8. Form the cartesian product k-graph $\Lambda = \{0,1\} \times \Sigma_k$. Define a relation $\sim$ on Λ by $(i,(f,g)) \sim (j,(f',g'))$ if and only if $(f,g) = (f',g') \in \Sigma_k$ and $f \neq 0$. It is straightforward to check that this relation satisfies conditions (1)–(4) of Proposition 2.1, and so we may form the quotient k-graph $\Gamma = \Lambda/{\sim}$.

We show that X_Γ is homeomorphic to a k-sphere. Proposition 2.3 shows that X_Γ is the quotient of $\{0,1\} \times X_{\Sigma_k}$ by the equivalence relation $(i,[(f,g),t]) \sim (j,[(f,g),t])$ whenever $f \neq 0$.

We claim that the homeomorphism $i : X_{\Sigma_k} \to \overline{\text{conv}}\{\varepsilon_0,\dots,\varepsilon_k\} \subseteq \mathbb{R}^{k+1}$ carries $\{[(f,g),t] : f \neq 0\}$ to $\bigcup_{i=1}^k \overline{\text{conv}}\{\varepsilon_0,\dots,\varepsilon_{i-1},\varepsilon_{i+1},\dots,\varepsilon_k\}$, which is the surface of the k-simplex with extreme points $\varepsilon_0,\dots,\varepsilon_k$. To see this, observe that (6) implies that for any $f \in P_k$ and $n \in \text{range}(f) \setminus \{0\}$, we have $v_{f,n} \in \overline{\text{conv}}\{\varepsilon_j : f(j) < \|f\|_\infty\}$. There exists some i_f such that $f(i_f) = \|f\|_\infty$, and so

$$\overline{\text{conv}}\{v_{f,n} : n \in \text{range}\, f \setminus \{0\}\} \subseteq \overline{\text{conv}}\{\varepsilon_0,\dots,\varepsilon_{i-1},\varepsilon_{i+1},\dots,\varepsilon_k\}.$$

Consider $i([(f,g),t])$ where $f \neq 0$. Since some $f(i) > 0$ we have $h(f) > 0$ and therefore $t_j = 1$ for some j. In particular $\|t\|_\infty = 1$ and so (6) implies that $i([(f,g),t])$ belongs to $\overline{\text{conv}}\{v_{f,n} : n \in \text{range}\, f \setminus \{0\}\}$. This gives $\{[(f,g),t] : f \neq 0\} \subseteq \bigcup_{i=1}^k \overline{\text{conv}}\{\varepsilon_0,\dots,\varepsilon_{i-1},\varepsilon_{i+1},\dots,\varepsilon_k\}$. For the reverse inclusion, fix $\sigma \in P_k^{\max}$ and $f \in P_k$ with $f \leq \sigma$ and $h(f) = \varepsilon_i$. Let $j = \sigma^{-1}(k)$. Using what we have already proved, we see that

$$\begin{aligned} &i(\{[(f,\sigma),t] : t \in [e_i, \mathbf{1}_k]\}) \\ &\qquad = \{t \in \overline{\text{conv}}\{\varepsilon_1,\dots,\varepsilon_{j-1},\varepsilon_{j+1},\dots,\varepsilon_k\} : t_{\sigma^{-1}(l)} \geq t_{\sigma^{-1}(l+1)} \text{ for all } l\}. \end{aligned}$$

For $\sigma \in P_k^{\max}$ and $j \leq k$, let $f_{\sigma,j}$ be the unique element of P_k with $f_{\sigma,j} \leq \sigma$ and $h(f_{\sigma,j}) = e_j$. Then

$$\begin{aligned} i(\{[(f,g),t] : f \neq 0\}) &= \bigcup_{\sigma \in P_k^{\max}} \bigcup_{j \leq k} i(\{[(f_{\sigma,j},\sigma),t] : t \in [e_j, \mathbf{1}_k]\}) \\ &= \bigcup_{i=1}^k \overline{\text{conv}}\{\varepsilon_0,\dots,\varepsilon_{i-1},\varepsilon_{i+1},\dots,\varepsilon_k\}, \end{aligned}$$

which proves the claim.

So X_Γ is the topological disjoint union of two rank-k simplices (which are homeomorphic to k-spheres) glued along their common $(k-1)$-dimensional boundary; that is, a k-sphere. □

Corollary 5.12. *Fix $n \geq 1$ and $k \geq 0$. There is a finite k-graph whose topological realisation is homeomorphic to a wedge of n k-spheres.*

Proof. Let Γ be the k-graph of Theorem 5.1. Regard $\{1,\dots,n\}$ as a 0-graph so that $\{0,\dots,n\} \times \Gamma$ is a k-graph. Each $\{i\} \times \Gamma$ is a quotient of $\{0,1\} \times \Sigma_k$. The vertex $(0,0)$ in Σ_k satisfies $(0,0)\Sigma_k = \{(0,0)\}$, and so the vertex $v := [0,(0,0)]$ of Γ has the same property. For each $i \leq n$, let $v_i := (i,v)$ be the copy of v in $\{i\} \times \Gamma$. Define an equivalence relation on $\{0,\dots,n\} \times \Gamma$ by $\alpha \sim \beta$ if and only if $\alpha = v_i$ and $\beta = v_j$ for some i,j. Again, this relation satisfies (1)–(4) of Proposition 2.1, and so we may form the quotient k-graph $\big(\{0,\dots,n\} \times \Gamma\big)/{\sim}$. Theorem 2.3 implies that the topological realisation of this quotient is the disjoint union of the $X_{\{i\}\times\Gamma}$ all glued at a single point. We saw in the proof of Theorem 5.1 that each $X_{\{i\}\times\Gamma}$ is a k-sphere, so $X_{(\{0,\dots,n\}\times\Gamma)/\sim}$ is a homeomorphic to a wedge of n k-spheres. □

References

[1] S. Allen, *A gauge-invariant uniqueness theorem for corners of k-graph C^*-algebras*, Rocky Mountain J. Math. **38** (2008), 1887–1907.

[2] T. Bates, D. Pask, I. Raeburn, and W. Szymański, *The C^*-algebras of row-finite graphs*, New York J. Math. **6** (2000), 307–324.

[3] B. Blackadar, K-theory for operator algebras, Springer-Verlag, New York, 1986, viii+338.
[4] M. Grandis *Directed combinatorial homology and noncommutative tori (The breaking of symmetries in algebraic topology)*, Math. Proc. Cambridge Philosophical Soc. **138** (2005), 233–262.
[5] R. Hazlewood, I. Raeburn, A. Sims and S. Webster, *Remarks on some fundamental results about higher-rank graphs and their C^*-algebras*, Proc. Edinburgh Math. Soc. **56** (2013), 575–597.
[6] S. Kaliszewski, A. Kumjian, J. Quigg and A. Sims, *Topological realizations and fundamental groups of higher-rank graphs*, preprint 2012. [arXiv:1205.2858].
[7] A. Kumjian and D. Pask, *Higher rank graph C^*-algebras*, New York J. Math. **6** (2000), 1–20.
[8] A. Kumjian, D. Pask and A. Sims, *Homology for higher-rank graphs and twisted C^*-algebras.* J. Funct. Anal. **263** (2012), 1539–1574.
[9] A. Kumjian, D. Pask and A. Sims, *On twisted higher-rank graph C^*-algebras*, Trans. Amer. Math. Soc. to appear [arXiv:1112.6233]
[10] A. Kumjian, D. Pask and A. Sims, *On the K-theory of twisted higher-rank-graph C^*-algebras*, J. Math. Anal. Appl. **401** (2013), 104–113.
[11] B. Maloney, D. Pask and I. Raeburn, *Skew products of higher-rank graphs and crossed products by semigroups*, Semigroup Forum., to appear [arXiv:1203.4860]
[12] W. Massey, A basic course in algebraic topology, Graduate Texts in Mathematics **127** Springer-Verlag, Berlin-New York, 1991.
[13] The On-Line Encyclopedia of Integer Sequences (OEIS), Web address: http://oeis.org/A000670
[14] D. Pask, I. Raeburn and T. Yeend, *Actions of semigroups on directed graphs and their C^*-algebras*, J. Pure Appl. Algebra, **159** (2001), 297–313.
[15] D. Pask, J. Quigg and I. Raeburn, *Fundamental groupoids of k-graphs*, New York J. Math. **10** (2004), 195–207.
[16] D. Pask, J. Quigg and I. Raeburn, *Coverings of k-graphs*, J. Algebra **289** (2005), 161–191.
[17] G.K. Pederson, *Pullback and pushout constructions in C^*-algebra theory*, J. Funct. Anal. **167** (1999), 243–344.
[18] I. Raeburn and A. Sims, *Product systems of graphs and the Toeplitz algebras of higher-rank graphs*, J. Operator Theory **53** (2005), 399–429.
[19] I. Raeburn, A. Sims, and T. Yeend, *The C^*-algebras of finitely aligned higher-rank graphs*, J. Funct. Anal. **213** (2004), 206–240.
[20] A. Sims, *Gauge-invariant ideals in the C^*-algebras of finitely aligned higher-rank graphs*, Canad. J. Math. **58** (2006), 1268–1290.
[21] A. Sims, B. Whitehead and M.F. Whittaker, *Twisted C^*-algebras associated to finitely aligned higher-rank graphs*, in preparation.

Alex Kumjian, Department of Mathematics (084), University of Nevada, Reno NV 89557-0084, USA
E-mail address: alex@unr.edu

David Pask, Aidan Sims, Michael F. Whittaker, School of Mathematics and Applied Statistics, The University of Wollongong, NSW 2522, AUSTRALIA
E-mail address: dpask@uow.edu.au, asims@uow.edu.au, mfwhittaker@gmail.com