

Multiplicative Arithmetic Functions of Several Variables: A Survey

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Abstract

We survey general properties of multiplicative arithmetic functions of several variables and related convolutions, including the Dirichlet convolution and the unitary convolution. We introduce and investigate a new convolution, called gcd convolution. We define and study the convolutes of arithmetic functions of several variables, according to the different types of convolutions. We discuss the multiple Dirichlet series and Bell series and present certain arithmetic and asymptotic results of some special multiplicative functions arising from problems in number theory, group theory and combinatorics. We give a new proof to obtain the asymptotic density of the set of ordered r -tuples of positive integers with pairwise relatively prime components and consider a similar question related to unitary divisors.

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Contents

1	Introduction	2
2	Notations	2
3	Multiplicative functions of several variables	3
3.1	Multiplicative functions	3
3.2	Firmly multiplicative functions	4
3.3	Completely multiplicative functions	4
3.4	Examples	5
4	Convolutions of arithmetic functions of several variables	7
4.1	Dirichlet convolution	7
4.2	Unitary convolution	8
4.3	Gcd convolution	9
4.4	Lcm convolution	9
4.5	Binomial convolution	10
5	Generating series	11
5.1	Dirichlet series	11
5.2	Bell series	13

6	Convolutives of arithmetic functions of several variables	13
6.1	General results	15
6.2	Special cases	18
7	Asymptotic properties	19
7.1	Mean values	19
7.2	Asymptotic densities	20
7.3	Asymptotic formulas	22
8	Acknowledgement	24

1 Introduction

Multiplicative arithmetic functions of a single variable are very well known in the literature. Their various properties were investigated by several authors and they represent an important research topic up to now. Less known are multiplicative arithmetic functions of several variables of which detailed study was carried out by R. VAIDYANATHASWAMY [74] more than eighty years ago. Since then many, sometimes scattered results for the several variables case were published in papers and monographs, and some authors of them were not aware of the paper [74]. In fact, there are two different notions of multiplicative functions of several variables, used in the last decades, both reducing to the usual multiplicativity in the one variable case. For the other concept we use the term firmly multiplicative function.

In this paper we survey general properties of multiplicative arithmetic functions of several variables and related convolutions, including the Dirichlet convolution and the unitary convolution. We introduce and investigate a new convolution, called gcd convolution. We define and study the convolutives of arithmetic functions of several variables, according to the different types of convolutions. The concept of the convolute of a function with respect to the Dirichlet convolution was introduced by R. VAIDYANATHASWAMY [74]. We also discuss the multiple Dirichlet series and Bell series. We present certain arithmetic and asymptotic results of some special multiplicative functions arising from problems in number theory, group theory and combinatorics. We give a new proof to obtain the asymptotic density of the set of ordered r -tuples of positive integers with pairwise relatively prime components. Furthermore, we consider a similar question, namely the asymptotic density of the set of ordered r -tuples with pairwise *unitary relatively prime* components, that is, the greatest common unitary divisor of each two distinct components is 1.

For general properties of (multiplicative) arithmetic functions of a single variable see, e.g., the books of T. M. APOSTOL [4], G. H. HARDY, E. M. WRIGHT [27], P. J. MC.CARTHY [42], W. SCHWARZ, J. SPILKER [53] and R. SIVARAMAKRISHNAN [55]. For algebraic properties of the ring of arithmetic functions of a single variable with the Dirichlet convolution we refer to H. N. SHAPIRO [54]. Incidence algebras and semilattice algebras concerning arithmetic functions of a single variable were investigated by D. A. SMITH [57]. For properties of certain subgroups of the group of multiplicative arithmetic functions of a single variable under the Dirichlet convolution we refer to the papers by T. B. CARROLL, A. A. GIOIA [7], P.-O. DEHAYE [14], J. E. DELANY [17], T. MACHENRY [41] and R. W. RYDEN [48]. Algebraical and topological properties of the ring of arithmetic functions of a single variable with the unitary convolution were given by J. SNELLMAN [58, 59]. See also J. SÁNDOR, B. CRSTICI [50, Sect. 2.2] and H. SCHEID [51].

2 Notations

Throughout the paper we use the following notations.
General notations:

- $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$,
- the prime power factorization of $n \in \mathbb{N}$ is $n = \prod_p p^{\nu_p(n)}$, the product being over the primes p , where all but a finite number of the exponents $\nu_p(n)$ are zero,
- $d \parallel n$ means that d is a unitary divisor of n , i.e., $d \mid n$ and $\gcd(d, n/d) = 1$,
- $\text{gcd}(n_1, \dots, n_k)$ denotes the greatest common unitary divisor of $n_1, \dots, n_k \in \mathbb{N}$,
- $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the additive group of residue classes modulo n ,
- ζ is the Riemann zeta function,
- γ is Euler's constant.

Arithmetic functions of a single variable:

- δ is the arithmetic function given by $\delta(1) = 1$ and $\delta(n) = 0$ for $n > 1$,
- id is the function $\text{id}(n) = n$ ($n \in \mathbb{N}$),
- ϕ_k is the Jordan function of order k given by $\phi_k(n) = n^k \prod_{p \mid n} (1 - 1/p^k)$ ($k \in \mathbb{C}$),
- $\phi = \phi_1$ is Euler's totient function,
- ψ is the Dedekind function given by $\psi(n) = n \prod_{p \mid n} (1 + 1/p)$,
- μ is the Möbius function,
- τ_k is the Piltz divisor function of order k , $\tau_k(n)$ representing the number of ways of expressing n as a product of k factors,
- $\tau(n) = \tau_2(n)$ is the number of divisors of n ,
- $\sigma_k(n) = \sum_{d \mid n} d^k$ ($k \in \mathbb{C}$),
- $\sigma(n) = \sigma_1(n)$ is the sum of divisors of n ,
- $\omega(n) = \#\{p : \nu_p(n) \neq 0\}$ stands for the number of distinct prime divisors of n ,
- $\mu^\times(n) = (-1)^{\omega(n)}$
- $\Omega(n) = \sum_p \nu_p(n)$ is the number of prime power divisors of n ,
- $\lambda(n) = (-1)^{\Omega(n)}$ is the Liouville function,
- $\xi(n) = \prod_p \nu_p(n)!$,
- $c_n(k) = \sum_{1 \leq q \leq n, \gcd(q, n)=1} \exp(2\pi i qk/n)$ ($n, k \in \mathbb{N}$) is the Ramanujan sum, which can be viewed as a function of two variables.

Arithmetic functions of several variables:

- \mathcal{A}_r is the set of arithmetic functions of r variables ($r \in \mathbb{N}$), i.e., of functions $f: \mathbb{N}^r \rightarrow \mathbb{C}$,
 - $\mathcal{A}_r^{(1)} = \{f \in \mathcal{A}_r : f(1, \dots, 1) \neq 0\}$,
 - $\mathbf{1}_r$ is the constant 1 function in \mathcal{A}_r , i.e., $\mathbf{1}_r(n_1, \dots, n_r) = 1$ for every $n_1, \dots, n_r \in \mathbb{N}$,
 - $\delta_r(n_1, \dots, n_r) = \delta(n_1) \cdots \delta(n_r)$, that is $\delta_r(1, \dots, 1) = 1$ and $\delta_r(n_1, \dots, n_r) = 0$ for $n_1 \cdots n_r > 1$,
 - for $f \in \mathcal{A}_r$ the function $\bar{f} \in \mathcal{A}_1$ is given by $\bar{f}(n) = f(n, \dots, n)$ for every $n \in \mathbb{N}$.
- Other notations will be fixed inside the paper.

3 Multiplicative functions of several variables

In what follows we discuss the notions of multiplicative, firmly multiplicative and completely multiplicative functions. We point out that properties of firmly and completely multiplicative functions of several variables reduce to those of multiplicative, respectively completely multiplicative functions of a single variable. Multiplicative functions can not be reduced to functions of a single variable. We also present examples of such functions.

3.1 Multiplicative functions

A function $f \in \mathcal{A}_r$ is said to be multiplicative if it is not identically zero and

$$f(m_1 n_1, \dots, m_r n_r) = f(m_1, \dots, m_r) f(n_1, \dots, n_r)$$

holds for any $m_1, \dots, m_r, n_1, \dots, n_r \in \mathbb{N}$ such that $\gcd(m_1 \cdots m_r, n_1 \cdots n_r) = 1$.

If f is multiplicative, then it is determined by the values $f(p^{\nu_1}, \dots, p^{\nu_r})$, where p is prime and $\nu_1, \dots, \nu_r \in \mathbb{N}_0$. More exactly, $f(1, \dots, 1) = 1$ and for any $n_1, \dots, n_r \in \mathbb{N}$,

$$f(n_1, \dots, n_r) = \prod_p f(p^{\nu_p(n_1)}, \dots, p^{\nu_p(n_r)}).$$

If $r = 1$, i.e., in the case of functions of a single variable we reobtain the familiar notion of multiplicativity: $f \in \mathcal{A}_1$ is multiplicative if it is not identically zero and $f(mn) = f(m)f(n)$ for every $m, n \in \mathbb{N}$ such that $\gcd(m, n) = 1$.

Let \mathcal{M}_r denote the set of multiplicative functions in r variables.

3.2 Firmly multiplicative functions

We call a function $f \in \mathcal{A}_r$ *firmly multiplicative* (following P. HAUKKANEN [28]) if it is not identically zero and

$$f(m_1 n_1, \dots, m_r n_r) = f(m_1, \dots, m_r) f(n_1, \dots, n_r)$$

holds for any $m_1, \dots, m_r, n_1, \dots, n_r \in \mathbb{N}$ such that $\gcd(m_1, n_1) = \dots = \gcd(m_r, n_r) = 1$. Let \mathcal{F}_r denote the set of firmly multiplicative functions in r variables.

A firmly multiplicative function is completely determined by its values at $(1, \dots, 1, p^\nu, 1, \dots, 1)$, where p runs through the primes and $\nu \in \mathbb{N}_0$. More exactly, $f(1, \dots, 1) = 1$ and for any $n_1, \dots, n_r \in \mathbb{N}$,

$$f(n_1, \dots, n_r) = \prod_p \left(f(p^{\nu_p(n_1)}, 1, \dots, 1) \cdots f(1, \dots, 1, p^{\nu_p(n_r)}) \right).$$

If a function $f \in \mathcal{A}_r$ is firmly multiplicative, then it is multiplicative. Also, if $f \in \mathcal{F}_r$, then $f(n_1, \dots, n_r) = f_1(n_1, 1, \dots, 1) \cdots f_r(1, \dots, 1, n_r)$ for every $n_1, \dots, n_r \in \mathbb{N}$. This immediately gives the following property:

Proposition 1. *A function $f \in \mathcal{A}_r$ is firmly multiplicative if and only if there exist multiplicative functions $f_1, \dots, f_r \in \mathcal{M}_1$ (each of a single variable) such that $f(n_1, \dots, n_r) = f_1(n_1) \cdots f_r(n_r)$ for every $n_1, \dots, n_r \in \mathbb{N}$. In this case $f_1(n) = f(n, 1, \dots, 1)$, ..., $f_r(n) = f(1, \dots, 1, n)$ for every $n \in \mathbb{N}$.*

In the case of functions of a single variable the notion of firmly multiplicative function reduces to that of multiplicative function. For $r > 1$ the concepts of multiplicative and firmly multiplicative functions are different.

3.3 Completely multiplicative functions

A function $f \in \mathcal{A}_r$ is called completely multiplicative if it is not identically zero and

$$f(m_1 n_1, \dots, m_r n_r) = f(m_1, \dots, m_r) f(n_1, \dots, n_r)$$

holds for any $m_1, \dots, m_r, n_1, \dots, n_r \in \mathbb{N}$. Note that R. VAIDYANATHASWAMY [74] used for such a function the term 'linear function'.

Let \mathcal{C}_r denote the set of completely multiplicative functions in r variables. If $f \in \mathcal{C}_r$, then it is determined by its values at $(1, \dots, 1, p, 1, \dots, 1)$, where p runs through the primes. More exactly, $f(1, \dots, 1) = 1$ and for any $n_1, \dots, n_r \in \mathbb{N}$,

$$f(n_1, \dots, n_r) = \prod_p \left(f(p, 1, \dots, 1)^{\nu_p(n_1)} \cdots f(1, \dots, 1, p)^{\nu_p(n_r)} \right).$$

In the case of functions of a single variable we reobtain the familiar notion of completely multiplicative function: $f \in \mathcal{A}_1$ is completely multiplicative if it is not identically zero and $f(mn) = f(m)f(n)$ for every $m, n \in \mathbb{N}$.

It is clear that if a function $f \in \mathcal{A}_r$ is completely multiplicative, then it is firmly multiplicative. Also, similar to Proposition 1:

Proposition 2. A function $f \in \mathcal{A}_r$ is completely multiplicative if and only if there exist completely multiplicative functions $f_1, \dots, f_r \in \mathcal{C}_1$ (each of a single variable) such that $f(n_1, \dots, n_r) = f_1(n_1) \cdots f_r(n_r)$ for every $n_1, \dots, n_r \in \mathbb{N}$. In this case $f_1(n) = f(n, 1, \dots, 1)$, \dots , $f_r(n) = f(1, \dots, 1, n)$ for every $n \in \mathbb{N}$.

3.4 Examples

The functions $(n_1, \dots, n_r) \mapsto \gcd(n_1, \dots, n_r)$ and $(n_1, \dots, n_r) \mapsto \text{lcm}(n_1, \dots, n_r)$ are multiplicative for every $r \in \mathbb{N}$, but not firmly multiplicative for $r \geq 2$.

The functions $(n_1, \dots, n_r) \mapsto \tau(n_1) \cdots \tau(n_r)$, $(n_1, n_2) \mapsto \tau(n_1)\sigma(n_2)$ are firmly multiplicative, but not completely multiplicative.

The functions $(n_1, \dots, n_r) \mapsto n_1 \cdots n_r$, $(n_1, n_2) \mapsto n_1 \lambda(n_2)$ are completely multiplicative.

According to Propositions 1 and 2 firmly multiplicative and completely multiplicative functions reduce to multiplicative, respectively completely multiplicative functions of a single variable. There is no similar characterization for multiplicative functions of several variables.

Let $h \in \mathcal{M}_1$. Then the functions $(n_1, \dots, n_r) \mapsto h(\gcd(n_1, \dots, n_r))$, $(n_1, \dots, n_r) \mapsto h(\text{lcm}(n_1, \dots, n_r))$ are multiplicative. The product and the quotient of (nonvanishing) multiplicative functions are multiplicative.

If $f \in \mathcal{M}_r$ is multiplicative and we fix one (or more, say s) variables, then the resulting function of $r - 1$ (or $r - s$) variables is not necessarily multiplicative. For example, $(k, n) \mapsto c_n(k)$ is multiplicative as a function of two variables, see Section 4.1, but for a fixed n the function $k \mapsto c_n(k)$ is in general not multiplicative (it is multiplicative if and only if $\mu(n) = 1$).

If $f \in \mathcal{M}_r$ is multiplicative, then the function \bar{f} of a single variable is multiplicative.

Other examples from number theory, group theory and combinatorics:

Example 1. Let $N_{g_1, \dots, g_r}(n_1, \dots, n_r)$ denote the number of solutions $x \pmod{n}$, with $n = \text{lcm}(n_1, \dots, n_r)$, of the simultaneous congruences $g_1(x) \equiv 0 \pmod{n_1}$, \dots , $g_r(x) \equiv 0 \pmod{n_r}$, where g_1, \dots, g_r are polynomials with integer coefficients. Then the function $(n_1, \dots, n_r) \mapsto N_{g_1, \dots, g_r}(n_1, \dots, n_r)$ is multiplicative. See L. TÓTH [65, Sect. 2] for a proof.

Example 2. For a fixed integer $r \geq 2$ let

$$\varrho(n_1, \dots, n_r) = \begin{cases} 1, & \text{if } n_1, \dots, n_r \text{ are pairwise relatively prime,} \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

This function is multiplicative, which follows from the definition, and for every $n_1, \dots, n_r \in \mathbb{N}$,

$$\varrho(n_1, \dots, n_r) = \sum_{d_1 | n_1, \dots, d_r | n_r} \tau(d_1 \cdots d_r) \mu(n_1/d_1) \cdots \mu(n_r/d_r), \quad (2)$$

cf. Section 4.1.

Example 3. For $r \geq 2$ let

$$\varrho^\times(n_1, \dots, n_r) = \begin{cases} 1, & \text{if } \text{gcd}(n_i, n_j) = 1 \text{ for every } i \neq j, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

This is the characteristic function of the set of ordered r -tuples $(n_1, \dots, n_r) \in \mathbb{N}^r$ such that n_1, \dots, n_r are pairwise unitary relatively prime, i.e., such that for every prime p there are no $i \neq j$ with $\nu_p(n_i) = \nu_p(n_j) \geq 1$. This function is also multiplicative (by the definition).

Example 4. Consider the group $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$. Let $s(n_1, \dots, n_r)$ and $c(n_1, \dots, n_r)$ denote the total number of subgroups of the group G and the number of its cyclic subgroups, respectively. Then the functions $(n_1, \dots, n_r) \mapsto s(n_1, \dots, n_r)$ and $(n_1, \dots, n_r) \mapsto c(n_1, \dots, n_r)$

are multiplicative. For every $n_1, \dots, n_r \in \mathbb{N}$,

$$c(n_1, \dots, n_r) = \sum_{d_1 | n_1, \dots, d_r | n_r} \frac{\phi(d_1) \cdots \phi(d_r)}{\phi(\text{lcm}(d_1, \dots, d_r))}, \quad (4)$$

see L. TÓTH [65, Th. 3], [66, Th. 1]. In the case $r = 2$ this gives

$$c(n_1, n_2) = \sum_{d_1 | n_1, d_2 | n_2} \phi(\text{gcd}(d_1, d_2)).$$

Also,

$$s(n_1, n_2) = \sum_{d_1 | n_1, d_2 | n_2} \text{gcd}(d_1, d_2), \quad (5)$$

for every $n_1, n_2 \in \mathbb{N}$. See M. HAMPEJS, N. HOLIGHAUS, L. TÓTH, C. WIESMEYR [25] and M. HAMPEJS, L. TÓTH [26].

Example 5. We define the sigma function of r variables by

$$\sigma(n_1, \dots, n_r) = \sum_{d_1 | n_1, \dots, d_r | n_r} \text{gcd}(d_1, \dots, d_r) \quad (6)$$

having the representation

$$\sigma(n_1, \dots, n_r) = \sum_{d | \text{gcd}(n_1, \dots, n_r)} \phi(d) \tau(n_1/d) \cdots \tau(n_r/d), \quad (7)$$

valid for every $n_1, \dots, n_r \in \mathbb{N}$. Note that for $r = 1$ this function reduces to the sum-of-divisors function and in the case $r = 2$ we have $\sigma(m, n) = s(m, n)$ given in Example 4.

We call $(n_1, \dots, n_r) \in \mathbb{N}^r$ a *perfect r -tuple* if $\sigma(n_1, \dots, n_r) = 2 \text{gcd}(n_1, \dots, n_r)$. If $2^r - 1 = p$ is a Mersenne prime, then (p, p, \dots, p) is a perfect r -tuple. For example, $(3, 3)$ is a perfect pair and $(7, 7, 7)$ is perfect triple. We formulate as an open problem: Which are all the perfect r -tuples?

Example 6. The Ramanujan sum $(k, n) \mapsto c_n(k)$ having the representation

$$c_n(k) = \sum_{d | \text{gcd}(k, n)} d \mu(n/d) \quad (8)$$

is multiplicative as a function of two variables. This property was pointed out by K. R. JOHNSON [34], see also Section 4.1.

Example 7. For $n_1, \dots, n_r \in \mathbb{N}$ let $n := \text{lcm}(n_1, \dots, n_r)$. The function of r variables

$$E(n_1, \dots, n_r) = \frac{1}{n} \sum_{j=1}^n c_{n_1}(j) \cdots c_{n_r}(j)$$

has combinatorial and topological applications, and was investigated in the papers of V. A. LISKOVEYS [40] and L. TÓTH [67]. All values of $E(n_1, \dots, n_r)$ are nonnegative integers and the function E is multiplicative. Furthermore, it has the following representation ([67, Prop. 3]):

$$E(n_1, \dots, n_r) = \sum_{d_1 | n_1, \dots, d_r | n_r} \frac{d_1 \mu(n_1/d_1) \cdots d_r \mu(n_r/d_r)}{\text{lcm}(d_1, \dots, d_r)}, \quad (9)$$

valid for every $n_1, \dots, n_r \in \mathbb{N}$. See also L. TÓTH [68, 71] for generalizations of the function E .

Example 8. Another multiplicative function, similar to E is

$$A(n_1, \dots, n_r) = \frac{1}{n} \sum_{k=1}^n \gcd(k, n_1) \cdots \gcd(k, n_r),$$

where $n_1, \dots, n_r \in \mathbb{N}$ and $n := \text{lcm}(n_1, \dots, n_r)$, as above. One has for every $n_1, \dots, n_r \in \mathbb{N}$,

$$A(n_1, \dots, n_r) = \sum_{d_1 | n_1, \dots, d_r | n_r} \frac{\phi(d_1) \cdots \phi(d_r)}{\text{lcm}(d_1, \dots, d_r)}, \quad (10)$$

see L. TÓTH [64, Eq. (45)], [67, Prop. 12].

See the paper of M. PETER [47] for properties of recurrent multiplicative arithmetical functions of several variables.

4 Convolutions of arithmetic functions of several variables

In this Section we survey the basic properties of the Dirichlet and unitary convolutions of arithmetic functions of several variables. We also define and discuss the gcd, lcm and binomial convolutions, not given in the literature in the several variables case. We point out that the gcd convolution reduces to the unitary convolution in the one variable case, but they are different for r variables with $r > 1$. For other convolutions we refer to the papers of J. SÁNDOR, A. BEGE [49], E. D. SCHWAB [52] and M. V. SUBBARAO [61].

4.1 Dirichlet convolution

For every $r \in \mathbb{N}$ the set \mathcal{A}_r of arithmetic functions of r variables is a \mathbb{C} -linear space with the usual linear operations. With the Dirichlet convolution defined by

$$(f * g)(n_1, \dots, n_r) = \sum_{d_1 | n_1, \dots, d_r | n_r} f(d_1, \dots, d_r) g(n_1/d_1, \dots, n_r/d_r)$$

the space \mathcal{A}_r forms a unital commutative \mathbb{C} -algebra, the unity being the function δ_r , and $(\mathcal{A}_r, +, *)$ is an integral domain. Moreover, $(\mathcal{A}_r, +, *)$ is a unique factorization domain, as pointed out by T. ONOZUKA [45]. In the case $r = 1$ this was proved by E. D. CASHWELL, C. J. EVERETT [9]. The group of invertible functions is $\mathcal{A}_r^{(1)}$. The inverse of f will be denoted by f^{-1*} . The inverse of the constant 1 function $\mathbf{1}_r$ is μ_r , given by $\mu_r(n_1, \dots, n_r) = \mu(n_1) \cdots \mu(n_r)$ (which is firmly multiplicative, where μ is the classical Möbius function).

The Dirichlet convolution preserves the multiplicativity of functions. This property, well known in the one variable case, follows easily from the definitions. Using this fact the multiplicativity of the functions $c(n_1, \dots, n_r)$, $s(n_1, n_2)$, $\sigma(n_1, \dots, n_r)$, $E(n_1, \dots, n_r)$ and $A(n_1, \dots, n_r)$ is a direct consequence of the convolutional representations (4), (5), (6), (9) and (10), respectively. The multiplicativity of the Ramanujan sum $(k, n) \mapsto c_n(k)$ follows in a similar manner from (8), showing that $c_n(k)$ is the convolution of the multiplicative functions f and g defined by $f(m, n) = m$ for $m = n$, $f(m, n) = 0$ for $m \neq n$ and $g(m, n) = \mu(m)$ for every $m, n \in \mathbb{N}$.

If $f, g \in \mathcal{F}_r$, then

$$(f * g)(n_1, \dots, n_r) = (f_1 * g_1)(n_1) \cdots (f_r * g_r)(n_r)$$

and

$$f^{-1*}(n_1, \dots, n_r) = f_1^{-1*}(n_1) \cdots f_r^{-1*}(n_r),$$

with the notations of Proposition 1, hence $f * g \in \mathcal{F}_r$ and $f^{-1*} \in \mathcal{F}_r$. We deduce

Proposition 3. *One has the following subgroup relations:*

$$(\mathcal{F}_r, *) \leq (\mathcal{M}_r, *) \leq (\mathcal{A}_r^{(1)}, *).$$

The set \mathcal{C}_r does not form a group under the Dirichlet convolution. If $f \in \mathcal{C}_r$, then $f^{-1*} = \mu_r f$ (well known in the case $r = 1$). Note that for every $f \in \mathcal{C}_r$ one has $(f * f)(n_1, \dots, n_r) = f(n_1, \dots, n_r) \tau(n_1) \cdots \tau(n_r)$, in particular $(\mathbf{1}_r * \mathbf{1}_r)(n_1, \dots, n_r) = \tau(n_1) \cdots \tau(n_r)$.

R. VAIDYANATHASWAMY [74] called the Dirichlet convolution 'composition of functions'.

Other convolutional properties known in the one variable case, for example Möbius inversion can easily be generalized. As mentioned in Section 3.4, Example 2 the characteristic function ϱ of the set of ordered r -tuples $(n_1, \dots, n_r) \in \mathbb{N}^r$ such that $n_1, \dots, n_r \in \mathbb{N}$ are pairwise relatively prime is multiplicative. One has for every $n_1, \dots, n_r \in \mathbb{N}$,

$$\sum_{d_1 | n_1, \dots, d_r | n_r} \varrho(d_1, \dots, d_r) = \tau(n_1 \cdots n_r), \quad (11)$$

since both sides are multiplicative and in the case of prime powers $n_1 = p^{\nu_1}, \dots, n_r = p^{\nu_r}$ both sides of (11) are equal to $1 + \nu_1 + \dots + \nu_r$. Now, Möbius inversion gives the formula (2).

For further algebraic properties of the \mathbb{C} -algebra \mathcal{A}_r and more generally, of the R -algebra $A_r(R) = \{f : \mathbb{N}^r \mapsto R\}$, where R is an integral domain and using the concept of firmly multiplicative functions see E. ALKAN, A. ZAHARESCU, M. ZAKI [1]. That paper includes, among others, constructions of a class of derivations and of a family of valuations on $A_r(R)$. See also P. HAUKKANEN [28] and A. ZAHARESCU, M. ZAKI [75].

4.2 Unitary convolution

The linear space \mathcal{A}_r forms another unital commutative \mathbb{C} -algebra with the unitary convolution defined by

$$(f \times g)(n_1, \dots, n_r) = \sum_{d_1 | n_1, \dots, d_r | n_r} f(d_1, \dots, d_r) g(n_1/d_1, \dots, n_r/d_r).$$

Here the unity is the function δ_r again. Note that $(\mathcal{A}_r, +, \times)$ is not an integral domain, there exist divisors of zero. The group of invertible functions is again $\mathcal{A}_r^{(1)}$. The unitary r variables Möbius function μ_r^\times is defined as the inverse of the function $\mathbf{1}_r$. One has $\mu_r^\times(n_1, \dots, n_r) = \mu^\times(n_1) \cdots \mu^\times(n_r) = (-1)^{\omega(n_1) + \dots + \omega(n_r)}$ (and it is firmly multiplicative). Similar to Proposition 3,

Proposition 4. *One has the following subgroup relations:*

$$(\mathcal{F}_r, \times) \leq (\mathcal{M}_r, \times) \leq (\mathcal{A}_r^{(1)}, \times).$$

If $f \in \mathcal{F}_r$, then its inverse is $f^{-1 \times} = \mu_r^\times f$ and

$$(f \times f)(n_1, \dots, n_r) = f(n_1, \dots, n_r) 2^{\omega(n_1) + \dots + \omega(n_r)},$$

in particular $(\mathbf{1}_r \times \mathbf{1}_r)(n_1, \dots, n_r) = 2^{\omega(n_1) + \dots + \omega(n_r)}$.

R. VAIDYANATHASWAMY [74] used the term 'compounding of functions' for the unitary convolution.

For further algebraic properties of the R -algebra $A_r(R) = \{f : \mathbb{N}^r \mapsto R\}$, where R is an integral domain with respect to the unitary convolution and using the concept of firmly multiplicative functions see E. ALKAN, A. ZAHARESCU, M. ZAKI [2].

4.3 Gcd convolution

We define a new convolution for functions $f, g \in \mathcal{A}_r$, we call it *gcd convolution*, given by

$$(f \odot g)(n_1, \dots, n_r) = \sum_{\substack{d_1 e_1 = n_1, \dots, d_r e_r = n_r \\ \gcd(d_1 \cdots d_r, e_1 \cdots e_r) = 1}} f(d_1, \dots, d_r) g(e_1, \dots, e_r), \quad (12)$$

which is in concordance with the definition of multiplicative functions.

In the case $r = 1$ the unitary and gcd convolutions are identic, i.e., $f \times g = f \odot g$ for every $f, g \in \mathcal{A}_1$, but they differ for $r > 1$.

Main properties: \mathcal{A}_r forms a unital commutative \mathbb{C} -algebra with the gcd convolution defined by (12). The unity is the function δ_r and there exist divisors of zero. The group of invertible functions is again $\mathcal{A}_r^{(1)}$. Here the inverse of the constant 1 function is $\mu_r^\odot(n_1, \dots, n_r) = (-1)^{\omega(n_1 \cdots n_r)}$. More generally, the inverse $f^{-1 \odot}$ of an arbitrary multiplicative function f is given by $f^{-1 \odot}(n_1, \dots, n_r) = (-1)^{\omega(n_1 \cdots n_r)} f(n_1, \dots, n_r)$. Also, if $f \in \mathcal{M}_r$, then

$$(f \odot f)(n_1, \dots, n_r) = f(n_1, \dots, n_r) 2^{\omega(n_1 \cdots n_r)}.$$

Proposition 5. *One has*

$$(\mathcal{M}_r, \odot) \leq (\mathcal{A}_r^{(1)}, \odot).$$

The set \mathcal{F}_r does not form a group under the gcd convolution. To see this note that $(\mathbf{1}_r \odot \mathbf{1}_r)(n_1, \dots, n_r) = 2^{\omega(n_1 \cdots n_r)}$, but this function is not firmly multiplicative, since $\omega(n_1 \cdots n_r) = \omega(n_1) + \dots + \omega(n_r)$ does not hold for every $n_1, \dots, n_r \in \mathbb{N}$ (cf. Proposition 1).

4.4 Lcm convolution

We define the lcm convolution of functions of r variables by

$$(f \oplus g)(n_1, \dots, n_r) = \sum_{\substack{d_1 e_1 = n_1, \dots, d_r e_r = n_r \\ \text{lcm}(d_1, e_1) = n_1, \dots, \text{lcm}(d_r, e_r) = n_r}} f(d_1, \dots, d_r) g(e_1, \dots, e_r).$$

In the case $r = 1$ this convolution originates by R. D. VON STERNECK [60], was investigated and generalized by D. H. LEHMER [37, 38, 39], and is also called von Sterneck-Lehmer convolution. See also R. G. BUSCHMAN [5].

Note that the lcm convolution can be expressed by the Dirichlet convolution. More exactly,

Proposition 6. *For every $f, g \in \mathcal{A}_r$,*

$$f \oplus g = (f * \mathbf{1}_r)(g * \mathbf{1}_r) * \mu_r. \quad (13)$$

Proof. Write

$$\begin{aligned} & \sum_{a_1 | n_1, \dots, a_r | n_r} (f \oplus g)(a_1, \dots, a_r) \\ &= \sum_{\substack{d_1 e_1 = n_1, \dots, d_r e_r = n_r \\ \text{lcm}(d_1, e_1) | n_1, \dots, \text{lcm}(d_r, e_r) | n_r}} f(d_1, \dots, d_r) g(e_1, \dots, e_r) \\ &= \sum_{d_1 | n_1, \dots, d_r | n_r} f(d_1, \dots, d_r) \sum_{e_1 | n_1, \dots, e_r | n_r} g(e_1, \dots, e_r) \\ &= (f * \mathbf{1}_r)(n_1, \dots, n_r) (g * \mathbf{1}_r)(n_1, \dots, n_r), \end{aligned}$$

and by Möbius inversion we obtain (13). \square

In the case $r = 1$ Proposition 6 is due to R. D. VON STERNECK [60] and D. H. LEHMER [38].

Note that $(\mathbf{1}_r \oplus \mathbf{1}_r)(n_1, \dots, n_r) = \tau(n_1^2) \cdots \tau(n_r^2)$. It turns out that the lcm convolution preserves the multiplicativity of functions, but \mathcal{M}_r does not form a group under the lcm convolution. The unity for the lcm convolution is the function δ_r again. Here $(\mathcal{A}_r, +, \oplus)$ is a unital commutative ring having divisors of zero. The group of invertible functions is $\tilde{\mathcal{A}}_r = \{f \in \mathcal{A}_r : (f * \mathbf{1}_r)(n_1, \dots, n_r) \neq 0 \text{ for every } (n_1, \dots, n_r) \in \mathbb{N}^r\}$ and we deduce

Proposition 7. *i) Let $\tilde{\mathcal{M}}_r = \{f \in \mathcal{M}_r : (f * \mathbf{1}_r)(n_1, \dots, n_r) \neq 0 \text{ for every } (n_1, \dots, n_r) \in \mathbb{N}^r\}$. Then $\tilde{\mathcal{M}}_r$ is a subgroup of $\tilde{\mathcal{A}}_r$ with respect to the lcm convolution.*

*ii) The inverse of the function $\mathbf{1}_r \in \tilde{\mathcal{A}}_r$ is $\mu_r^\oplus = \mu_r * 1/(\mathbf{1}_r * \mathbf{1}_r)$. The function μ_r^\oplus is multiplicative and for every prime powers $p^{\nu_1}, \dots, p^{\nu_r}$,*

$$\mu_r^\oplus(p^{\nu_1}, \dots, p^{\nu_r}) = \frac{(-1)^r}{\nu_1(\nu_1 + 1) \cdots \nu_r(\nu_r + 1)}.$$

Proof. Here (ii) follows from $(\mathbf{1}_r * \mathbf{1}_r)(n_1, \dots, n_r) = \tau(n_1) \cdots \tau(n_r)$, already mentioned in Section 4.1. \square

4.5 Binomial convolution

We define the binomial convolution of the functions $f, g \in \mathcal{A}_r$ by

$$\begin{aligned} & (f \circ g)(n_1, \dots, n_r) \\ &= \sum_{d_1 | n_1, \dots, d_r | n_r} \left(\prod_p \binom{\nu_p(n_1)}{\nu_p(d_1)} \cdots \binom{\nu_p(n_r)}{\nu_p(d_r)} \right) f(d_1, \dots, d_r) g(n_1/d_1, \dots, n_r/d_r), \end{aligned}$$

where $\binom{a}{b}$ is the binomial coefficient. It is remarkable that the binomial convolution preserves the complete multiplicativity of arithmetical functions, which is not the case for the Dirichlet convolution and other convolutions. Let ξ_r be the firmly multiplicative function given by $\xi_r(n_1, \dots, n_r) = \xi(n_1) \cdots \xi(n_r)$, that is $\xi_r(n_1, \dots, n_r) = \prod_p (\nu_p(n_1)! \cdots \nu_p(n_r)!)$. Then for every $f, g \in \mathcal{A}_r$,

$$f \circ g = \xi_r \left(\frac{f}{\xi_r} * \frac{g}{\xi_r} \right), \quad (14)$$

leading to the next result.

Proposition 8. *The algebras $(\mathcal{A}_r, +, \circ, \mathbb{C})$ and $(\mathcal{A}_r, +, *, \mathbb{C})$ are isomorphic under the mapping $f \mapsto \frac{f}{\xi_r}$.*

Formula (14) also shows that the binomial convolution preserves the multiplicativity of functions. Furthermore, for any fixed $r \in \mathbb{N}$ the structure $(\mathcal{A}_r, +, \circ)$ is an integral domain with unity δ_r . The group of invertible functions is again $\mathcal{A}_r^{(1)}$. If $f, g \in \mathcal{F}_r$, then with the notations of Proposition 1,

$$(f \circ g)(n_1, \dots, n_r) = (f_1 \circ g_1)(n_1) \cdots (f_r \circ g_r)(n_r)$$

and the inverse of f is

$$f^{-1 \circ}(n_1, \dots, n_r) = f_1^{-1 \circ}(n_1) \cdots f_r^{-1 \circ}(n_r),$$

hence $f \circ g \in \mathcal{F}_r$ and $f^{-1 \circ} \in \mathcal{F}_r$. The inverse of the function $\mathbf{1}_r$ under the binomial convolution is the function λ_r given by $\lambda_r(n) = \lambda(n_1) \cdots \lambda(n_r)$, i.e., $\lambda_r(n) = (-1)^{\Omega(n_1) + \dots + \Omega(n_r)}$. We deduce

Proposition 9. *One has the following subgroup relations:*

$$(\mathcal{C}_r, \circ) \leq (\mathcal{F}_r, \circ) \leq (\mathcal{M}_r, \circ) \leq (\mathcal{A}_r^{(1)}, \circ).$$

In the case $r = 1$ properties of this convolution were discussed in the paper by L. TÓTH, P. HAUKKANEN [72]. The proofs are similar in the multivariable case.

5 Generating series

As generating series for multiplicative arithmetic functions of r variables we present certain properties of the multiple Dirichlet series, used earlier by several authors and the Bell series, which constituted an important tool of R. VAIDYANATHASWAMY [74].

5.1 Dirichlet series

The multiple Dirichlet series of a function $f \in \mathcal{A}_r$ is given by

$$D(f; z_1, \dots, z_r) = \sum_{n_1, \dots, n_r=1}^{\infty} \frac{f(n_1, \dots, n_r)}{n_1^{z_1} \cdots n_r^{z_r}}.$$

Similar to the one variable case, if $D(f; z_1, \dots, z_r)$ is absolutely convergent in $(s_1, \dots, s_r) \in \mathbb{C}^r$, then it is absolutely convergent in every $(z_1, \dots, z_r) \in \mathbb{C}^r$ with $\Re z_j \geq \Re s_j$ ($1 \leq j \leq r$).

Proposition 10. *Let $f, g \in \mathcal{A}_r$. If $D(f; z_1, \dots, z_r)$ and $D(g; z_1, \dots, z_r)$ are absolutely convergent, then $D(f * g; z_1, \dots, z_r)$ is also absolutely convergent and*

$$D(f * g; z_1, \dots, z_r) = D(f; z_1, \dots, z_r)D(g; z_1, \dots, z_r).$$

Also, if $f \in \mathcal{A}_r^{(1)}$, then

$$D(f^{-1*}; z_1, \dots, z_r) = D(f; z_1, \dots, z_r)^{-1},$$

formally or in the case of absolute convergence.

If $f \in \mathcal{M}_r$ is multiplicative, then its Dirichlet series can be expanded into a (formal) Euler product, that is,

$$D(f; z_1, \dots, z_r) = \prod_p \sum_{\nu_1, \dots, \nu_r=0}^{\infty} \frac{f(p^{\nu_1}, \dots, p^{\nu_r})}{p^{\nu_1 z_1 + \dots + \nu_r z_r}}, \quad (15)$$

the product being over the primes p . More exactly,

Proposition 11. *Let $f \in \mathcal{M}_r$. For every $(z_1, \dots, z_r) \in \mathbb{C}^r$ the series $D(f; z_1, \dots, z_r)$ is absolutely convergent if and only if*

$$\prod_p \sum_{\substack{\nu_1, \dots, \nu_r=0 \\ \nu_1 + \dots + \nu_r \geq 1}}^{\infty} \frac{|f(p^{\nu_1}, \dots, p^{\nu_r})|}{p^{\nu_1 \Re z_1 + \dots + \nu_r \Re z_r}} < \infty$$

and in this case the equality (15) holds.

Next we give the Dirichlet series representations of certain special functions discussed above. These follow from the convolutional identities given in Section 3.4. For every $g \in \mathcal{A}_1$ we have formally,

$$\sum_{n_1, \dots, n_r=1}^{\infty} \frac{g(\gcd(n_1, \dots, n_r))}{n_1^{z_1} \cdots n_r^{z_r}} = \frac{\zeta(z_1) \cdots \zeta(z_r)}{\zeta(z_1 + \dots + z_r)} \sum_{n=1}^{\infty} \frac{g(n)}{n^{z_1 + \dots + z_r}}. \quad (16)$$

In particular, taking $g(n) = n$ (16) gives for $r \geq 2$ and $\Re z_1 > 1, \dots, \Re z_r > 1$,

$$\sum_{n_1, \dots, n_r=1}^{\infty} \frac{\gcd(n_1, \dots, n_r)}{n_1^{z_1} \cdots n_r^{z_r}} = \frac{\zeta(z_1) \cdots \zeta(z_r) \zeta(z_1 + \dots + z_r - 1)}{\zeta(z_1 + \dots + z_r)}$$

and taking $g = \delta$ one obtains for $r \geq 2$ and $\Re z_1 > 1, \dots, \Re z_r > 1$,

$$\sum_{\substack{n_1, \dots, n_r=1 \\ \gcd(n_1, \dots, n_r)=1}}^{\infty} \frac{1}{n_1^{z_1} \cdots n_r^{z_r}} = \frac{\zeta(z_1) \cdots \zeta(z_r)}{\zeta(z_1 + \dots + z_r)}, \quad (17)$$

where the identity (17) is the Dirichlet series of the characteristic function of the set of points in \mathbb{N}^r , which are visible from the origin (cf. T. M. APOSTOL [4, Page 248, Ex. 15]).

The Dirichlet series of the characteristic function ϱ concerning r pairwise relatively prime integers is

$$\begin{aligned} & \sum_{n_1, \dots, n_r=1}^{\infty} \frac{\varrho(n_1, \dots, n_r)}{n_1^{z_1} \cdots n_r^{z_r}} \\ &= \zeta(z_1) \cdots \zeta(z_r) \prod_p \left(\prod_{j=1}^r \left(1 - \frac{1}{p^{z_j}} \right) + \sum_{j=1}^r \frac{1}{p^{z_j}} \prod_{\substack{k=1 \\ k \neq j}}^r \left(1 - \frac{1}{p^{z_k}} \right) \right) \end{aligned} \quad (18)$$

$$= \zeta(z_1) \cdots \zeta(z_r) \prod_p \left(1 + \sum_{j=2}^r (-1)^{j-1} (j-1) \sum_{1 \leq i_1 < \dots < i_j \leq r} \frac{1}{p^{z_{i_1} + \dots + z_{i_j}}} \right), \quad (19)$$

valid for $\Re z_1 > 1, \dots, \Re z_r > 1$. Here (18) follows at once from the definition (1) of the function ϱ . For (19) see L. TÓTH [69, Eq. (4.2)]. Note that in the paper [69] certain other Dirichlet series representations are also given, leading to generalizations of the Busche-Ramanujan identities.

Concerning the Ramanujan sum $c_n(k)$ and the functions $s(m, n)$ and $c(m, n)$, cf. Section 3.4, Example 4 we have for $\Re z > 1, \Re w > 1$,

$$\sum_{k, n=1}^{\infty} \frac{c_n(k)}{k^z n^w} = \frac{\zeta(w) \zeta(z+w-1)}{\zeta(z)},$$

$$\sum_{m, n=1}^{\infty} \frac{s(m, n)}{m^z n^w} = \frac{\zeta^2(z) \zeta^2(w) \zeta(z+w-1)}{\zeta(z+w)}, \quad (20)$$

$$\sum_{m, n=1}^{\infty} \frac{c(m, n)}{m^z n^w} = \frac{\zeta^2(z) \zeta^2(w) \zeta(z+w-1)}{\zeta^2(z+w)}. \quad (21)$$

The formulas (20) and (21) were derived by W. G. NOWAK, L. TÓTH [44]. Also, for $\Re z > 2, \Re w > 2$,

$$\sum_{m, n=1}^{\infty} \frac{\text{lcm}(m, n)}{m^z n^w} = \frac{\zeta(z-1) \zeta(w-1) \zeta(z+w-1)}{\zeta(z+w-2)}.$$

As a generalization of (20), for $\Re z_1 > 1, \dots, \Re z_r > 1$,

$$\sum_{n_1, \dots, n_r=1}^{\infty} \frac{\sigma(n_1, \dots, n_r)}{n_1^{z_1} \cdots n_r^{z_r}} = \frac{\zeta^2(z_1) \cdots \zeta^2(z_r) \zeta(z_1 + \dots + z_r - 1)}{\zeta(z_1 + \dots + z_r)}.$$

5.2 Bell series

If f is a multiplicative function of r variables, then its (formal) Bell series to the base p (p prime) is defined by

$$f_{(p)}(x_1, \dots, x_r) = \sum_{e_1, \dots, e_r=0}^{\infty} f(p^{e_1}, \dots, p^{e_r}) x_1^{e_1} \cdots x_r^{e_r},$$

where the constant term is 1. The main property is the following: for every $f, g \in \mathcal{M}_r$,

$$(f * g)_{(p)}(x_1, \dots, x_r) = f_{(p)}(x_1, \dots, x_r) g_{(p)}(x_1, \dots, x_r).$$

The connection of Bell series to Dirichlet series and Euler products is given by

$$D(f; z_1, \dots, z_r) = \prod_p f_{(p)}(p^{-z_1}, \dots, p^{-z_r}), \quad (22)$$

valid for every $f \in \mathcal{M}_r$. For example, the Bell series of the gcd function $f(n_1, \dots, n_r) = \gcd(n_1, \dots, n_r)$ is

$$f_{(p)}(x_1, \dots, x_r) = \frac{1 - x_1 \cdots x_r}{(1 - x_1) \cdots (1 - x_r)(1 - px_1 \cdots x_r)}.$$

The Bell series of other multiplicative functions, in particular of $c(m, n)$, $s(m, n)$, $\sigma(n_1, \dots, n_r)$ and $c_n(k)$ can be given from their Dirichlet series representations and using the relation (22).

Note that in the one variable case the Bell series to a fixed prime of the unitary convolution of two multiplicative functions is the sum of the Bell series of the functions, that is

$$(f \times g)_{(p)}(x_1) = f_{(p)}(x_1) + g_{(p)}(x_1),$$

see R. VAIDYANATHASWAMY [74, Th. XII]. This is not valid in the case of r variables with $r > 1$.

6 Convolutives of arithmetic functions of several variables

Let $f \in \mathcal{A}_r$. By choosing $n_1 = \dots = n_r = n$ we obtain the function of a single variable $n \mapsto \bar{f}(n) = f(n, \dots, n)$. If $f \in \mathcal{M}_r$, then $\bar{f} \in \mathcal{M}_1$, as already mentioned. Less trivial ways to retrieve from f functions of a single variable is to consider for $r > 1$,

$$\Psi_{\text{dir}}(f)(n) = \sum_{d_1 \cdots d_r = n} f(d_1, \dots, d_r), \quad (23)$$

$$\Psi_{\text{unit}}(f)(n) = \sum_{\substack{d_1 \cdots d_r = n \\ \gcd(d_i, d_j) = 1, i \neq j}} f(d_1, \dots, d_r), \quad (24)$$

$$\Psi_{\text{gcd}}(f)(n) = \sum_{\substack{d_1 \cdots d_r = n \\ \gcd(d_1, \dots, d_r) = 1}} f(d_1, \dots, d_r), \quad (25)$$

$$\Psi_{\text{lcm}}(f)(n) = \sum_{\text{lcm}(d_1, \dots, d_r) = n} f(d_1, \dots, d_r), \quad (26)$$

$$\Psi_{\text{binom}}(f)(n) = \sum_{d_1 \cdots d_r = n} \left(\prod_p \binom{\nu_p(n)}{\nu_p(d_1), \dots, \nu_p(d_r)} \right) f(d_1, \dots, d_r), \quad (27)$$

where the sums are over all ordered r -tuples $(d_1, \dots, d_r) \in \mathbb{N}^r$ with the given additional conditions, the last one involving multinomial coefficients. For (24) the condition is that $d_1 \cdots d_r = n$ and d_1, \dots, d_r are pairwise relatively prime. Note that (24) and (25) are the same for $r = 2$, but they differ in the case $r > 2$.

Assume that there exist functions $g_1, \dots, g_r \in \mathcal{A}_1$ (each of a single variable) such that $f(n_1, \dots, n_r) = g_1(n_1) \cdots g_r(n_r)$ (in particular this holds if $f \in \mathcal{F}_r$ by Proposition 1). Then (23), (24), (26) and (27) reduce to the Dirichlet convolution, unitary convolution, lcm convolution and binomial convolution, respectively, of the functions g_1, \dots, g_r . For $r = 2$ we have the corresponding convolutions of two given functions of a single variable.

Note the following special case of (26):

$$\sum_{\text{lcm}(d_1, \dots, d_r) = n} \phi(d_1) \cdots \phi(d_r) = \phi_r(n) \quad (n \in \mathbb{N}),$$

due to R. D. VON STERNECK [60].

We remark that (23), with other notation, appears in [74, p. 591-592], where $\Psi_{\text{dir}}(f)$ is called the 'convolute' of f (obtained by the convolution of the arguments). We will call $\Psi_{\text{dir}}(f)$, $\Psi_{\text{unit}}(f)$, $\Psi_{\text{gcd}}(f)$, $\Psi_{\text{lcm}}(f)$ and $\Psi_{\text{binom}}(f)$ the *Dirichlet convolute*, *unitary convolute*, *gcd convolute*, *lcm convolute* and *binomial convolute*, respectively of the function f .

Some special cases of convolutes of functions which are not the product of functions of a single variable are the following. Special Dirichlet convolutes are

$$g_r(n) = \sum_{d_1 \cdots d_r = n} \text{gcd}(d_1, \dots, d_r), \quad (28)$$

$$\ell_r(n) = \sum_{d_1 \cdots d_r = n} \text{lcm}(d_1, \dots, d_r). \quad (29)$$

$$N(n) = \sum_{d|n} \phi(\text{gcd}(d, n/d)). \quad (30)$$

For $r = 2$ (28) and (29) are sequences A055155 and A057670, respectively in [56]. The function $N(n)$ given by (30) represents the number of parabolic vertices of $\gamma_0(n)$ (sequence A001616 in [56]), cf. S. FINCH [23].

In the case $r = 2$ the lcm convolute of the gcd function

$$c(n) = \sum_{\text{lcm}(d,e)=n} \text{gcd}(d, e), \quad (31)$$

represents the number of cyclic subgroups of the group $\mathbb{Z}_n \times \mathbb{Z}_n$, as shown by A. PAKAPONGPUN, T. WARD [46, Ex. 2]. That is, $c(n) = c(n, n)$ for every $n \in \mathbb{N}$, with the notation of Section 3.4, Example 4 (it is sequence A060648 in [56]).

The Dirichlet, unitary and lcm convolutes of the Ramanujan sums are

$$a(n) = \sum_{d|n} c_d(n/d), \quad (32)$$

$$b(n) = \sum_{d||n} c_d(n/d), \quad (33)$$

$$h(n) = \sum_{\text{lcm}(d,e)=n} c_d(e). \quad (34)$$

All the functions $g_r, \ell_r, N, c, a, b, h$ defined above are multiplicative, as functions of a single variable. See Corollary 1.

6.1 General results

The Dirichlet, unitary, gcd, lcm and binomial convolutions preserve the multiplicativity of functions of a single variable, cf. Section 4. As a generalization of this property, we prove the next result.

Proposition 12. *Let $f \in \mathcal{M}_r$ be an arbitrary multiplicative function. Then all the functions $\Psi_{\text{dir}}(f)$, $\Psi_{\text{unit}}(f)$, $\Psi_{\text{gcd}}(f)$, $\Psi_{\text{lcm}}(f)$ and $\Psi_{\text{binom}}(f)$ are multiplicative.*

Note that for the Dirichlet convolute this property was pointed out by R. VAIDYANATHASWAMY in [74, p. 591-592].

Proof. By the definitions. Let $n, m \in \mathbb{N}$ such that $\gcd(n, m) = 1$. If $d_1 \cdots d_r = nm$, then there exist unique integers $a_1, b_1, \dots, a_r, b_r \in \mathbb{N}$ such that $a_1, \dots, a_r \mid n$, $b_1, \dots, b_r \mid m$ and $d_1 = a_1 b_1, \dots, d_r = a_r b_r$. Here $\gcd(a_1 \cdots a_r, b_1 \cdots b_r) = 1$. Using the multiplicativity of f we obtain

$$\begin{aligned} \Psi_{\text{dir}}(f)(nm) &= \sum_{\substack{a_1 \cdots a_r = n \\ b_1 \cdots b_r = m}} f(a_1 b_1, \dots, a_r b_r) \\ &= \sum_{a_1 \cdots a_r = n} f(a_1, \dots, a_r) \sum_{b_1 \cdots b_r = m} f(b_1, \dots, b_r) \\ &= \Psi_{\text{dir}}(f)(n) \Psi_{\text{dir}}(f)(m), \end{aligned}$$

showing the multiplicativity of $\Psi_{\text{dir}}(f)$. The proof in the case of the other functions is similar. For the function $\Psi_{\text{gcd}}(f)$,

$$\Psi_{\text{gcd}}(f)(nm) = \sum_{\substack{a_1 \cdots a_r = n \\ b_1 \cdots b_r = m \\ \gcd(a_1 b_1, \dots, a_r b_r) = 1}} f(a_1 b_1, \dots, a_r b_r),$$

where $1 = \gcd(a_1 b_1, \dots, a_r b_r) = \gcd(a_1, \dots, a_r) \gcd(b_1, \dots, b_r)$, since the gcd function in r variables is multiplicative. Hence

$$\begin{aligned} \Psi_{\text{gcd}}(f)(nm) &= \sum_{\substack{a_1 \cdots a_r = n \\ \gcd(a_1, \dots, a_r) = 1}} f(a_1, \dots, a_r) \sum_{\substack{b_1 \cdots b_r = m \\ \gcd(b_1, \dots, b_r) = 1}} f(b_1, \dots, b_r) \\ &= \Psi_{\text{gcd}}(f)(n) \Psi_{\text{gcd}}(f)(m). \end{aligned}$$

In the case of the function $\Psi_{\text{lcm}}(f)$ use that the lcm function in r variables is multiplicative, whence $nm = \text{lcm}(a_1 b_1, \dots, a_r b_r) = \text{lcm}(a_1, \dots, a_r) \text{lcm}(b_1, \dots, b_r)$ and it follows that $\text{lcm}(a_1, \dots, a_r) = n$, $\text{lcm}(b_1, \dots, b_r) = m$. \square

Remark 1. Alternative proofs for the multiplicativity of $\Psi_{\text{unit}}(f)$, $\Psi_{\text{gcd}}(f)$ and $\Psi_{\text{lcm}}(f)$ can be given as follows. In the first two cases the property can be reduced to that of $\Psi_{\text{dir}}(f)$. Let

$$f^{\flat}(n_1, \dots, n_r) = \begin{cases} f(n_1, \dots, n_r), & \text{if } \gcd(n_1, \dots, n_r) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\Psi_{\text{gcd}}(f) = \Psi_{\text{dir}}(f^{\flat})$. If f is multiplicative, then f^{\flat} is also multiplicative and the multiplicativity of $\Psi_{\text{gcd}}(f)$ follows by the same property of the Dirichlet convolute. Similar for $\Psi_{\text{unit}}(f)$. Furthermore,

$$\Psi_{\text{lcm}}(f) = \overline{(f * \mathbf{1}_r)} * \mu. \quad (35)$$

Indeed, we have similar to the proof of (13) given above,

$$\sum_{d \mid n} \Psi_{\text{lcm}}(f)(d) = \sum_{\text{lcm}(d_1, \dots, d_r) \mid d} f(d_1, \dots, d_r) = \sum_{d_1 \mid n, \dots, d_r \mid n} f(d_1, \dots, d_r)$$

$$= (f * \mathbf{1}_r)(n, \dots, n) = \overline{(f * \mathbf{1}_r)}(n).$$

If f is multiplicative, so is $f * \mathbf{1}_r$ (as a function of r variables). Therefore, $\overline{f * \mathbf{1}_r}$ is multiplicative (as a function of a single variable) and deduce by (35) that $\Psi_{\text{lcm}}(f)$ is multiplicative.

Remark 2. Note that if f is completely multiplicative, then $\Psi_{\text{binom}}(f)$ is also completely multiplicative.

Corollary 1. The functions $g_r, \ell_r, N, c, a, b, h$ defined by (28)-(34) are multiplicative.

Proposition 13. The convolutes of the function $f = \mathbf{1}_r$, that is the number of terms of the sums defining the convolutes are the following multiplicative functions:

- i) $\Psi_{\text{dir}}(\mathbf{1}_r) = \tau_r$, the Piltz divisor function of order r given by $\tau_r(n) = \prod_p (\nu_p(n) + r - 1)$,
- ii) $\Psi_{\text{unit}}(\mathbf{1}_r) = H_r$ given by $H_r(n) = r^{\omega(n)}$,
- iii) $\Psi_{\text{gcd}}(\mathbf{1}_r) = N_r$, where $N_r(n) = \sum_{a^r b = n} \mu(a) \tau_r(b) = \prod_p \left((\nu_p(n) + r - 1) - (\nu_p(n) - 1) \right)$ with $\binom{\nu - 1}{r - 1} = 0$ for $\nu < r$,
- iv) $\Psi_{\text{lcm}}(\mathbf{1}_r) = M_r$, where $M_r(n) = \sum_{ab = n} \mu(a) \tau(b)^r = \prod_p ((\nu_p(n) + 1)^r - \nu_p(n)^r)$,
- v) $\Psi_{\text{binom}}(\mathbf{1}_r) = Q_r$, where $Q_r(n) = r^{\Omega(n)}$.

Proof. i) and ii) are immediate from the definitions.

iii) By the property of the Möbius function,

$$\Psi_{\text{gcd}}(\mathbf{1}_r)(n) = \sum_{d_1 \cdots d_r = n} \sum_{a | \gcd(d_1, \dots, d_r)} \mu(a) = \sum_{a^r b_1 \cdots b_r = n} \mu(a) = \sum_{a^r b = n} \mu(a) \tau_r(b).$$

iv) Follows from (35) in the case $f = \mathbf{1}_r$.

v) By Remark 2. See also L. TÓTH, P. HAUKKANEN [72, Cor. 3.2]. \square

Here i) and iv) of Proposition 13 can be generalized as follows.

Proposition 14. Assume that there is a function $g \in \mathcal{A}_1$ (of a single variable) such that $f(n_1, \dots, n_r) = g(\gcd(n_1, \dots, n_r))$ for every $n_1, \dots, n_r \in \mathbb{N}$. Then for every $n \in \mathbb{N}$,

$$\Psi_{\text{dir}}(f)(n) = \sum_{d_1 \cdots d_r = n} g(\gcd(d_1, \dots, d_r)) = \sum_{a^r b = n} (\mu * g)(a) \tau_r(b), \quad (36)$$

$$\Psi_{\text{lcm}}(f)(n) = \sum_{\text{lcm}(d_1, \dots, d_r) = n} g(\gcd(d_1, \dots, d_r)) = (g * \mu * \mu * \tau^r)(n). \quad (37)$$

Proof. The identity (36) is given by E. KRÄTZEL, W. G. NOWAK, L. TÓTH [35, Prop. 5.1]. We recall its proof, which is simple and similar to that of iii) of Proposition 13: for $f(n_1, \dots, n_r) = g(\gcd(n_1, \dots, n_r))$,

$$\begin{aligned} \Psi_{\text{dir}}(f)(n) &= \sum_{d_1 \cdots d_r = n} \sum_{a | \gcd(d_1, \dots, d_r)} (\mu * g)(a) \\ &= \sum_{a^r b_1 \cdots b_r = n} (\mu * g)(a) = \sum_{a^r b = n} (\mu * g)(a) \tau_r(b). \end{aligned}$$

Now for (37),

$$\begin{aligned} \Psi_{\text{lcm}}(f)(n) &= \sum_{\text{lcm}(d_1, \dots, d_r) = n} \sum_{a | \gcd(d_1, \dots, d_r)} (\mu * g)(a) \\ &= \sum_{a | n} (\mu * g)(a) \sum_{\text{lcm}(b_1, \dots, b_r) = n/a} 1 = \sum_{a | n} (\mu * g)(a) (\mu * \tau^r)(n/a) \\ &= (g * \mu * \mu * \tau^r)(n), \end{aligned}$$

using iv) of Proposition 13. \square

Proposition 15. For every $f, g \in \mathcal{A}_r$,

$$\begin{aligned}\Psi_{\text{dir}}(f * g) &= \Psi_{\text{dir}}(f) * \Psi_{\text{dir}}(g), \\ \Psi_{\text{unit}}(f \times g) &= \Psi_{\text{unit}}(f) \times \Psi_{\text{unit}}(g), \\ \Psi_{\text{gcd}}(f \odot g) &= \Psi_{\text{gcd}}(f) \times \Psi_{\text{gcd}}(g), \\ \Psi_{\text{lcm}}(f \oplus g) &= \Psi_{\text{lcm}}(f) \oplus \Psi_{\text{lcm}}(g), \\ \Psi_{\text{binom}}(f \circ g) &= \Psi_{\text{binom}}(f) \circ \Psi_{\text{binom}}(g).\end{aligned}$$

Proof. For the Dirichlet convolute Ψ_{dir} ,

$$\begin{aligned}\Psi_{\text{dir}}(f * g)(n) &= \sum_{d_1 \cdots d_r = n} (f * g)(d_1, \dots, d_r) \\ &= \sum_{d_1 \cdots d_r = n} \sum_{a_1 b_1 = d_1, \dots, a_r b_r = d_r} f(a_1, \dots, a_r) g(b_1, \dots, b_r) \\ &= \sum_{a_1 b_1 \cdots a_r b_r = n} f(a_1, \dots, a_r) g(b_1, \dots, b_r) \\ &= \sum_{xy=n} \sum_{a_1 \cdots a_r = x} f(a_1, \dots, a_r) \sum_{b_1 \cdots b_r = y} g(b_1, \dots, b_r) \\ &= \sum_{xy=n} \Psi_{\text{dir}}(f)(x) \Psi_{\text{dir}}(g)(y) = (\Psi_{\text{dir}}(f) * \Psi_{\text{dir}}(g))(n),\end{aligned}$$

and similar for the other ones. \square

Proposition 16. Let $r \geq 2$. The following maps are surjective algebra homomorphisms:

$$\begin{aligned}\Psi_{\text{dir}}: (\mathcal{A}_r, +, \cdot, *) &\rightarrow (\mathcal{A}_1, +, \cdot, *), \quad \Psi_{\text{unit}}: (\mathcal{A}_r, +, \cdot, \times) \rightarrow (\mathcal{A}_1, +, \cdot, \times), \\ \Psi_{\text{gcd}}: (\mathcal{A}_r, +, \cdot, \odot) &\rightarrow (\mathcal{A}_1, +, \cdot, \times), \quad \Psi_{\text{lcm}}: (\mathcal{A}_r, +, \cdot, \oplus) \rightarrow (\mathcal{A}_1, +, \cdot, \oplus), \\ \Psi_{\text{binom}}: (\mathcal{A}_r, +, \cdot, \circ) &\rightarrow (\mathcal{A}_1, +, \cdot, \circ).\end{aligned}$$

Proof. Use Proposition 15. For the surjectivity: for a given $f \in \mathcal{A}_1$ consider $F \in \mathcal{A}_r$ defined by $F(n, 1, \dots, 1) = f(n)$ for every $n \in \mathbb{N}$ and $F(n_1, \dots, n_r) = 0$ otherwise, i.e., for every $n_1, \dots, n_r \in \mathbb{N}$ with $n_2 \cdots n_r > 1$. Then $\Psi_{\text{dir}}(F) = \Psi_{\text{unit}}(F) = \Psi_{\text{gcd}}(F) = \Psi_{\text{lcm}}(F) = \Psi_{\text{binom}}(F) = f$. \square

Corollary 2. Let $r \geq 2$.

- i) The maps $\Psi_{\text{dir}}: (\mathcal{M}_r, *) \rightarrow (\mathcal{M}_1, *)$, $\Psi_{\text{unit}}: (\mathcal{M}_r, \times) \rightarrow (\mathcal{M}_1, \times)$, $\Psi_{\text{gcd}}: (\mathcal{M}_r, \odot) \rightarrow (\mathcal{M}_1, \times)$ and $\Psi_{\text{binom}}: (\mathcal{M}_r, \circ) \rightarrow (\mathcal{M}_1, \circ)$ are surjective group homomorphisms.
- ii) The maps $\Psi_{\text{dir}}: (\mathcal{F}_r, *) \rightarrow (\mathcal{M}_1, *)$, $\Psi_{\text{unit}}: (\mathcal{F}_r, \times) \rightarrow (\mathcal{M}_1, \times)$ and $\Psi_{\text{binom}}: (\mathcal{F}_r, \circ) \rightarrow (\mathcal{M}_1, \circ)$ are surjective group homomorphisms.
- iii) The map $\Psi_{\text{binom}}: (\mathcal{C}_r, \circ) \rightarrow (\mathcal{C}_1, \circ)$ is a surjective group homomorphism.

Proof. Follows from Propositions 12, 15 and from the fact that for every (completely) multiplicative f the function F constructed in the proof of Proposition 16 is (completely) multiplicative. For iii) use also Remark 2. \square

6.2 Special cases

We present identities for the convolutes of some special functions. For Dirichlet convolutes we have the next result.

Corollary 3. *For every $k \in \mathbb{C}$,*

$$\begin{aligned} \sum_{d_1 \cdots d_r = n} (\gcd(d_1, \dots, d_r))^k &= \sum_{a^r b = n} \phi_k(a) \tau_r(b). \\ \sum_{d_1 \cdots d_r = n} \sigma_k(\gcd(d_1, \dots, d_r)) &= \sum_{a^r b = n} a^k \tau_r(b). \end{aligned} \quad (38)$$

Proof. Follows from the first identity of Proposition 14. \square

The function (38) is for $r = 2$ and $k = 0$ the sequence A124315 in [56], and for $r = 2$, $k = 1$ it is sequence A124316 in [56]. See also [35, Sect. 5].

Special cases of lcm convolutes which do not seem to be known are the following. Let $\beta = \text{id} * \lambda$ be the alternating sum-of-divisors function, see L. TÓTH [70].

Corollary 4. *For every $n \in \mathbb{N}$,*

$$\sum_{\text{lcm}(d_1, \dots, d_r) = n} \gcd(d_1, \dots, d_r) = (\phi * M_r)(n),$$

where the function M_r is given in Proposition 13,

$$\begin{aligned} \sum_{\text{lcm}(d_1, \dots, d_r) = n} \tau(\gcd(d_1, \dots, d_r)) &= \tau(n)^r, \\ \sum_{\text{lcm}(d, e) = n} \phi(\gcd(d, e)) &= \psi(n), \\ \sum_{\text{lcm}(d, e) = n} \sigma(\gcd(d, e)) &= (\tau * \psi)(n) = (\phi * \tau^2)(n), \\ \sum_{\text{lcm}(d, e) = n} \beta(\gcd(d, e)) &= \sigma(n), \\ \sum_{\text{lcm}(d, e) = n} \mu(\gcd(d, e)) &= \mu^2(n), \\ \sum_{\text{lcm}(d, e) = n} \lambda(\gcd(d, e)) &= 1. \end{aligned} \quad (39)$$

Proof. Follow from the second identity of Proposition 14. \square

Here (39) is $s(n, n)$, representing the number of all subgroups of the group $\mathbb{Z}_n \times \mathbb{Z}_n$ (sequence A060724 in [56]).

For the convolutes of the Ramanujan sum we have

Proposition 17.

$$\sum_{de=n} c_d(e) = \begin{cases} \sqrt{n}, & n \text{ perfect square,} \\ 0, & \text{otherwise,} \end{cases} \quad (40)$$

$$\sum_{\substack{de=n \\ \gcd(d, e)=1}} c_d(e) = \begin{cases} 1, & n \text{ squarefull,} \\ 0, & \text{otherwise,} \end{cases} \quad (41)$$

$$\sum_{\text{lcm}(d, e) = n} c_d(e) = \phi(n) \quad (n \in \mathbb{N}). \quad (42)$$

Proof. According to Corollary 1 all these functions are multiplicative and it is enough to compute their values for prime powers. Alternatively, the formula (8) can be used. \square

Formulas (40) and (41) are well known, see, e.g., P. J. MCCARTHY [42, p. 191–192], while (42) seems to be new.

7 Asymptotic properties

We discuss certain results concerning the mean values of arithmetic functions of r variables and the asymptotic densities of some sets in \mathbb{N}^r . We also present asymptotic formulas for special multiplicative functions given in the previous sections.

7.1 Mean values

Let $f \in \mathcal{A}_r$. The mean value of f is

$$M(f) = \lim_{x_1, \dots, x_r \rightarrow \infty} \frac{1}{x_1 \cdots x_r} \sum_{n_1 \leq x_1, \dots, n_r \leq x_r} f(n_1, \dots, n_r),$$

where x_1, \dots, x_r tend to infinity independently, provided that this limit exists. As a generalization of Wintner's theorem (valid for the case $r = 1$), N. USHIROYA [73, Th. 1] proved the next result.

Proposition 18. *If $f \in \mathcal{A}_r$ ($r \geq 1$) and*

$$\sum_{n_1, \dots, n_r=1}^{\infty} \frac{|(\mu_r * f)(n_1, \dots, n_r)|}{n_1 \cdots n_r} < \infty,$$

then the mean value $M(f)$ exists and

$$M(f) = \sum_{n_1, \dots, n_r=1}^{\infty} \frac{(\mu_r * f)(n_1, \dots, n_r)}{n_1 \cdots n_r}.$$

For multiplicative functions we have the following result due to N. USHIROYA [73, Th. 4] in a slightly different form.

Proposition 19. *Let $f \in \mathcal{M}_r$ ($r \geq 1$). Assume that*

$$\sum_p \sum_{\substack{\nu_1, \dots, \nu_r=0 \\ \nu_1 + \dots + \nu_r \geq 1}}^{\infty} \frac{|(\mu_r * f)(p^{\nu_1}, \dots, p^{\nu_r})|}{p^{\nu_1 + \dots + \nu_r}} < \infty.$$

Then the mean value $M(f)$ exists and

$$M(f) = \prod_p \left(1 - \frac{1}{p}\right)^r \sum_{\nu_1, \dots, \nu_r=0}^{\infty} \frac{f(p^{\nu_1}, \dots, p^{\nu_r})}{p^{\nu_1 + \dots + \nu_r}}.$$

Corollary 5. (N. USHIROYA [73, Th. 7]) *Let $g \in \mathcal{M}_1$ be a multiplicative function and denote by a_g the absolute convergence abscissa of the Dirichlet series $D(g; z)$. Then for every $r > 1$, $r > a_g$ the main value of the function $(n_1, \dots, n_r) \mapsto g(\gcd(n_1, \dots, n_r))$ exists and*

$$M(f) = \frac{1}{\zeta(r)} \sum_{n=1}^{\infty} \frac{g(n)}{n^r}.$$

Proof. Follows from Proposition 18 and the identity (16). \square

For example, the mean value of the function $(n_1, \dots, n_r) \mapsto \gcd(n_1, \dots, n_r)$ is $\zeta(r-1)/\zeta(r)$ ($r \geq 3$), the mean value of the function $(n_1, \dots, n_r) \mapsto \phi(\gcd(n_1, \dots, n_r))$ is $\zeta(r-1)/\zeta^2(r)$ ($r \geq 3$).

The analog of Proposition 18 for the unitary convolution is the next result (see W. NARKIEWICZ [43] in the case $r = 1$).

Proposition 20. *If $f \in \mathcal{A}_r$ ($r \geq 1$) and*

$$\sum_{n_1, \dots, n_r=1}^{\infty} \frac{|(\mu_r^\times \times f)(n_1, \dots, n_r)|}{n_1 \cdots n_r} < \infty,$$

then the mean value $M(f)$ exists and

$$M(f) = \sum_{n_1, \dots, n_r=1}^{\infty} \frac{(\mu_r^\times \times f)(n_1, \dots, n_r) \phi(n_1) \cdots \phi(n_r)}{n_1^2 \cdots n_r^2}.$$

For further results on the mean values of multiplicative arithmetic functions of several variables and generalizations to the several variables case of results of G. HALÁSZ [24] we refer to the papers of O. CASAS [8], H. DELANGE [15, 16], E. HEPPNER [29, 30] and K.-H. INDLEKOFER [33]. See also E. ALKAN, A. ZAHARESCU, M. ZAKI [3].

7.2 Asymptotic densities

Let $S \subset \mathbb{N}^r$. The set S possesses the asymptotic density d_S if the characteristic function χ_S of S has the mean value $M(\chi_S) = d_S$. In what follows we consider the densities of certain special sets.

Let $g \in \mathcal{M}_1$ be a multiplicative function such that $g(n) \in \{0, 1\}$ for every $n \in \mathbb{N}$. Let $S_g = \{(n_1, \dots, n_r) \in \mathbb{N}^r : g(\gcd(n_1, \dots, n_r)) = 1\}$. It follows from Corollary 5 that for $r \geq 2$ the set S_g has the asymptotic density given by

$$d_{S_g} = \frac{1}{\zeta(r)} \sum_{n=1}^{\infty} \frac{g(n)}{n^r}.$$

In particular, for $r \geq 2$ the set of points $(n_1, \dots, n_r) \in \mathbb{N}^r$ which are visible from the origin, i.e., such that $\gcd(n_1, \dots, n_r) = 1$ holds has the density $1/\zeta(r)$ (the case $g = \delta$). Another example: the set of points $(n_1, \dots, n_r) \in \mathbb{N}^r$ such that $\gcd(n_1, \dots, n_r)$ is squarefree has the density $1/\zeta(2r)$ (the case $g = \mu^2$). These results are well known.

Now consider the set of points $(n_1, \dots, n_r) \in \mathbb{N}^r$ such that n_1, \dots, n_r are pairwise relatively prime. The next results was first proved by L. TÓTH [63], giving also an asymptotic formula for $\sum_{n_1, \dots, n_r \leq x} \varrho(n_1, \dots, n_r)$ and by J.-Y. CAI, E. BACH [6, Th. 3.3]. Here we give a simple different proof.

Proposition 21. *Let $r \geq 2$. The asymptotic density of the set of points in \mathbb{N}^r with pairwise relatively prime coordinates is*

$$A_r = \prod_p \left(1 - \frac{1}{p}\right)^{r-1} \left(1 + \frac{r-1}{p}\right). \quad (43)$$

Proof. Apply Proposition 18 for the function $f = \varrho$ defined by (1). Then according to the Dirichlet series representation (18), the density is

$$\sum_{n_1, \dots, n_r=1}^{\infty} \frac{(\mu_r * \varrho)(n_1, \dots, n_r)}{n_1 \cdots n_r} = \prod_p \left(\left(1 - \frac{1}{p}\right)^r + \frac{r}{p} \left(1 - \frac{1}{p}\right)^{r-1} \right),$$

which equals A_r , given by (43). \square

See the quite recent paper by J. HU [31] for a generalization of Proposition 21. See J. L. FERNÁNDEZ, P. FERNÁNDEZ [19, 20, 21, 22] for various statistical regularity properties concerning mutually relatively prime and pairwise relatively prime integers.

Unitary analogs of the problems of above are the following.

Proposition 22. *Let $r \geq 2$. The asymptotic density of the set of points $(n_1, \dots, n_r) \in \mathbb{N}^r$ such that $\text{gcd}(n_1, \dots, n_r) = 1$ is*

$$\prod_p \left(1 - \frac{(p-1)^r}{p^r(p^r-1)} \right).$$

Proof. The characteristic function of this set is given by

$$\delta(\text{gcd}(n_1, \dots, n_r)) = \sum_{d|\text{gcd}(n_1, \dots, n_r)} \mu^\times(d) = \sum_{d_1||n_r, \dots, d_r||n_r} G(d_1, \dots, d_r),$$

where

$$G(n_1, \dots, n_r) = \begin{cases} \mu^\times(n), & \text{if } n_1 = \dots = n_r = n, \\ 0, & \text{otherwise.} \end{cases}$$

We deduce from Proposition 20 that the density in question is

$$\begin{aligned} \sum_{n_1, \dots, n_r=1}^{\infty} \frac{G(n_1, \dots, n_r) \phi(n_1) \cdots \phi(n_r)}{n_1^2 \cdots n_r^2} &= \sum_{n=1}^{\infty} \frac{\mu^\times(n) \phi(n)^r}{n^{2r}} \\ &= \prod_p \left(1 - \frac{(p-1)^r}{p^r(p^r-1)} \right). \end{aligned}$$

□

Proposition 22 was proved by L. TÓTH [62] using different arguments. See [62] for other related densities and asymptotic formulas.

Corollary 6. ($r=2$) *The set of points $(m, n) \in \mathbb{N}^2$ such that $\text{gcd}(m, n) = 1$ has the density*

$$\prod_p \left(1 - \frac{p-1}{p^2(p+1)} \right).$$

Proposition 23. *Let $r \geq 2$. The asymptotic density of the set of points in \mathbb{N}^r with pairwise unitary relatively prime coordinates is*

$$A_r^\times = \prod_p \sum_{\nu_1, \dots, \nu_r=0}^{\infty} \frac{Q(p^{\nu_1}, \dots, p^{\nu_r}) \phi(p^{\nu_1}) \cdots \phi(p^{\nu_r})}{p^{2\nu_1 + \dots + 2\nu_r}},$$

where Q is the multiplicative function of r variables given as follows: Let $p^{\nu_1}, \dots, p^{\nu_r}$ be arbitrary powers of the prime p with $\nu_1, \dots, \nu_r \in \mathbb{N}_0$, $\nu_1 + \dots + \nu_r \geq 1$. Assume that the exponents ν_1, \dots, ν_r have q ($1 \leq q \leq r$) distinct positive values, taken t_1, \dots, t_q times ($1 \leq t_1 + \dots + t_q \leq r$). Then

$$Q(p^{\nu_1}, \dots, p^{\nu_r}) = (-1)^r (1 - t_1) \cdots (1 - t_q).$$

Proof. We use Proposition 20 for the function ϱ^\times defined by (3). The density is

$$A_r^\times = \sum_{n_1, \dots, n_r=1}^{\infty} \frac{Q(n_1, \dots, n_r) \phi(n_1) \cdots \phi(n_r)}{n_1^2 \cdots n_r^2},$$

where $Q_r = \mu_r^\times \times \varrho^\times$ and the Euler product formula can be used by the multiplicativity of the involved functions. □

Note that for $r = 2$,

$$Q(p^{\nu_1}, p^{\nu_2}) = \begin{cases} 1, & \nu_1 = \nu_2 = 0, \\ -1, & \nu_1 = \nu_2 \geq 1, \\ 0, & \text{otherwise} \end{cases}$$

and reobtain Corollary 6.

Corollary 7. ($r = 3, 4$)

$$A_3^\times = \zeta(2)\zeta(3) \prod_p \left(1 - \frac{4}{p^2} + \frac{7}{p^3} - \frac{9}{p^4} + \frac{8}{p^5} - \frac{2}{p^6} - \frac{3}{p^7} + \frac{2}{p^8} \right).$$

$$A_4^\times = \zeta^2(2)\zeta(3)\zeta(4) \prod_p \left(1 - \frac{8}{p^2} + \frac{3}{p^3} + \frac{27}{p^4} - \frac{24}{p^5} - \frac{14}{p^6} - \frac{3}{p^7} + \frac{37}{p^8} - \frac{30}{p^9} + \frac{42}{p^{10}} - \frac{33}{p^{11}} - \frac{41}{p^{12}} + \frac{78}{p^{13}} - \frac{44}{p^{14}} + \frac{9}{p^{15}} \right).$$

Proof. According to Proposition 23,

$$Q(p^{\nu_1}, p^{\nu_2}, p^{\nu_3}) = \begin{cases} 1, & \nu_1 = \nu_2 = \nu_3 = 0, \\ -1, & \nu_1 = \nu_2 \geq 1, \nu_3 = 0 \text{ and symmetric cases,} \\ 2, & \nu_1 = \nu_2 = \nu_3 \geq 1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$Q(p^{\nu_1}, p^{\nu_2}, p^{\nu_3}, p^{\nu_4}) = \begin{cases} 1, & \nu_1 = \nu_2 = \nu_3 = 0, \\ -1, & \nu_1 = \nu_2 \geq 1, \nu_3 = \nu_4 = 0 \text{ and symmetric cases,} \\ -2, & \nu_1 = \nu_2 = \nu_3 \geq 1, \nu_4 = 0 \text{ and symmetric cases,} \\ -3, & \nu_1 = \nu_2 = \nu_3 = \nu_4 \geq 1, \\ 1, & \nu_1 = \nu_2 > \nu_3 = \nu_4 \geq 1 \text{ and symmetric cases,} \\ 0, & \text{otherwise,} \end{cases}$$

and direct computations lead to the given infinite products. \square

We refer to the papers by J. CHRISTOPHER [12], H. DELANGE [15] and N. USHIROYA [73] for related density results.

7.3 Asymptotic formulas

One has

$$\sum_{m, n \leq x} \gcd(m, n) = \frac{x^2}{\zeta(2)} \left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta(2)}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) + O(x^{1+\theta+\varepsilon}), \quad (44)$$

for every $\varepsilon > 0$, where θ is the exponent appearing in Dirichlet's divisor problem, that is

$$\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + O(x^{\theta+\varepsilon}). \quad (45)$$

It is known that $1/4 \leq \theta \leq 131/416 \approx 0.3149$, where the upper bound, the best up to date, is the result of M. N. HUXLEY [32].

If $r \geq 3$, then

$$\sum_{n_1, \dots, n_r \leq x} \gcd(n_1, \dots, n_r) = \frac{\zeta(r-1)}{\zeta(r)} x^r + O(R_r(x)), \quad (46)$$

where $R_3(x) = x^2 \log x$ and $R_r(x) = x^{r-1}$ for $r \geq 4$, which follows from the representation

$$\sum_{n_1, \dots, n_r \leq x} \gcd(n_1, \dots, n_r) = \sum_{d \leq x} \phi(d) [x/d]^r.$$

Furthermore,

$$\sum_{m, n \leq x} \operatorname{lcm}(m, n) = \frac{\zeta(3)}{4\zeta(2)} x^4 + O(x^3 \log x). \quad (47)$$

The formulas (44), (46), (47) can be deduced by elementary arguments and go back to the work of E. CESÀRO [10], E. COHEN [13] and P. DIACONIS, P. ERDŐS [18]. See also L. TÓTH [64, Eq. (25)]. A formula similar to (44), with the same error term holds for $\sum_{m, n \leq x} g(\gcd(m, n))$, where $g = h * \operatorname{id}$, $h \in \mathcal{A}_1$ is bounded, including the cases $g = \phi, \sigma, \psi$. See L. TÓTH [64, p. 7]. See J. L. FERNÁNDEZ, P. FERNÁNDEZ [19, 20, 21] for statistical regularity properties of the gcd's and lcm's of positive integers.

Consider next the function $g_2(n) = \sum_{d|n} \gcd(d, n/d)$, which is the Dirichlet convolute of the gcd function for $r = 2$.

Proposition 24.

$$\sum_{n \leq x} g_2(n) = \frac{3}{2\pi^2} x (\log^2 x + c_1 \log x + c_2) + R(x),$$

where c_1, c_2 are constants and $R(x) = O(x^\theta (\log x)^{\theta'})$ with $\theta = \frac{547}{832} = 0.65745\dots$, $\theta' = \frac{26947}{8320}$.

This was proved using analytic tools (Huxley's method) by E. KRÄTZEL, W. G. NOWAK, L. TÓTH [35, Th. 3.5]. See M. KÜHLEITNER, W. G. NOWAK [36] for omega estimates on the function $g_2(n)$. The papers [35] and [36] contain also results for the function $g_r(n)$ ($r \geq 3$), defined by (28) and related functions.

For the function $\ell_2(n) = \sum_{d|n} \operatorname{lcm}(d, n/d)$, representing the Dirichlet convolute of the lcm function for $r = 2$ one can deduce the next asymptotics.

Proposition 25.

$$\sum_{n \leq x} \ell_2(n)/n = \frac{\zeta(3)}{\zeta(2)} x \left(\log x + 2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)} + \frac{2\zeta'(3)}{\zeta(3)} \right) + O(x^{\theta+\varepsilon}),$$

where θ is given by (45).

A similar formula can be given for the function $\ell_r(n)$ ($r \geq 3$) defined by (29).

For the functions $s(m, n)$ and $c(m, n)$ defined in Section 3.4, Example 4, W. G. NOWAK, L. TÓTH [44] proved the following asymptotic formulas.

Proposition 26. For every fixed $\varepsilon > 0$,

$$\sum_{m, n \leq x} s(m, n) = \frac{2}{\pi^2} x^2 (\log^3 x + a_1 \log^2 x + a_2 \log x + a_3) + O\left(x^{\frac{1117}{701} + \varepsilon}\right),$$

$$\sum_{m, n \leq x} c(m, n) = \frac{12}{\pi^4} x^2 (\log^3 x + b_1 \log^2 x + b_2 \log x + b_3) + O\left(x^{\frac{1117}{701} + \varepsilon}\right),$$

where $1117/701 \approx 1.5934$ and $a_1, a_2, a_3, b_1, b_2, b_3$ are explicit constants.

See the recent paper by T. H. CHAN, A. V. KUMCHEV [11] concerning asymptotic formulas for $\sum_{n \leq x, k \leq y} c_n(k)$.

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References

- [1] E. ALKAN, A. ZAHARESCU, M. ZAKI, Arithmetical functions in several variables, *Int. J. Number Theory*, **1** (2005), 383–399.
- [2] E. ALKAN, A. ZAHARESCU, M. ZAKI, Unitary convolution for arithmetical functions in several variables, *Hiroshima Math. J.*, **36** (2006), 113–124.
- [3] E. ALKAN, A. ZAHARESCU, M. ZAKI, Multidimensional averages and Dirichlet convolution, *Manuscripta Math.*, **123** (2007), 251–267.
- [4] T. M. APOSTOL, *Introduction to Analytic Number Theory*, Springer, 1976.
- [5] R. G. BUSCHMAN, lcm-products of number-theoretic functions revisited, *Kyungpook Math. J.*, **39** (1999), 159–164.
- [6] J.-Y. CAI, E. BACH, On testing for zero polynomials by a set of points with bounded precision, *Theoret. Comp. Sci.*, **296** (2003), 15–25.
- [7] T. B. CARROLL, A. A. GIOIA, On a subgroup of the group of multiplicative arithmetic functions, *J. Austral. Math. Soc.*, **20** (Series A) (1975), 348–358.
- [8] O. CASAS, Arithmetical functions in two variables. An analogue of a result of Delange, *Lect. Mat.*, **27** (2006), Número especial, 5–12.
- [9] E. D. CASHWELL, C. J. EVERETT, The ring of number-theoretic functions, *Pacific J. Math.*, **9** (1959), 975–985.
- [10] E. CESÀRO, Étude moyenne du plus grand commun diviseur de deux nombres, *Annali di Matematica Pura ed Applicata*, **13** (1885), 235–250.
- [11] T. H. CHAN, A. V. KUMCHEV, On sums of Ramanujan sums, *Acta Arith.*, **152** (2012), 1–10.
- [12] J. CHRISTOPHER, The asymptotic density of some k dimensional sets, *Amer. Math. Monthly*, **63** (1956), 399–401.
- [13] E. COHEN, Arithmetical functions of a greatest common divisor, III. Cesàro’s divisor problem, *Proc. Glasgow Math. Assoc.*, **5** (1961-1962), 67–75.
- [14] P.-O. DEHAYE, On the structure of the group of multiplicative arithmetical functions, *Bull. Belg. Math. Soc.*, **9** (2002), 15–21.
- [15] H. DELANGE, On some sets of pairs of positive integers, *J. Number Theory*, **1** (1969), 261–279.
- [16] H. DELANGE, Sur les fonctions multiplicatives de plusieurs entiers, *Enseignement Math.* **(2) 16** (1970), 219–246 (1971), Errata: **(2) 17** (1971), 186.
- [17] J. E. DELANY, Groups of arithmetical functions, *Math. Mag.*, **78** (2005), 83–97.
- [18] P. DIACONIS, P. ERDŐS, On the distribution of the greatest common divisor, in *A festschrift for Herman Rubin, IMS Lecture Notes Monogr. Ser., Inst. Math. Statist.*, **45**, (2004), 56-61 (original version: Technical Report No. 12, Department of Statistics, Stanford University, Stanford, 1977).
- [19] J. L. FERNÁNDEZ, P. FERNÁNDEZ, Asymptotic normality and greatest common divisors, Preprint, 2013, arXiv:1302.2357 [math.PR].
- [20] J. L. FERNÁNDEZ, P. FERNÁNDEZ, On the probability distribution of the gcd and lcm of r -tuples of integers, Preprint, 2013, arXiv:1305.0536 [math.NT].

- [21] J. L. FERNÁNDEZ, P. FERNÁNDEZ, Equidistribution and coprimality, Preprint, 2013, arXiv:1310.3802 [math.NT].
- [22] J. L. FERNÁNDEZ, P. FERNÁNDEZ, Random index of codivisibility, Preprint, 2013, arXiv:1310.4681v1[math.NT].
- [23] S. FINCH, Modular forms on $SL_2(\mathbb{Z})$, manuscript, 2005, <http://www.people.fas.harvard.edu/~sfinch/resolve/frs.pdf>
- [24] G. HALÁSZ, Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen, *Acta Math. Acad. Sci. Hung.*, **19** (1968), 365–403.
- [25] M. HAMPEJS, N. HOLIGHAUS, L. TÓTH, C. WIESMEYR, On the subgroups of the group $\mathbb{Z}_m \times \mathbb{Z}_n$, Preprint, 2012, arXiv:1211.1797 [math.GR].
- [26] M. HAMPEJS, L. TÓTH, On the subgroups of finite Abelian groups of rank three, *Annales Univ. Sci. Budapest., Sect. Comp.*, **39** (2013), 111–124.
- [27] G. H. HARDY, E. M. WRIGHT, *An Introduction to the Theory of Numbers*, Sixth Edition, Edited and revised by D. R. Heath-Brown and J. H. Silverman, Oxford University Press, 2008.
- [28] P. HAUKKANEN, Derivation of arithmetical functions under the Dirichlet convolution, manuscript.
- [29] E. HEPPNER, Über benachbarte multiplikative zahlentheoretische Funktionen mehrerer Variablen, *Arch. Math. (Basel)*, **35** (1980), 454–460.
- [30] E. HEPPNER, Über Mittelwerte multiplikativer zahlentheoretischer Funktionen mehrerer Variablen, *Monatsh. Math.*, **91** (1981), 1–9.
- [31] J. HU, The probability that random positive integers are k -wise relatively prime, *Int. J. Number Theory*, **9** (2013), 1263–1271.
- [32] M. N. HUXLEY, Exponential sums and lattice points III., *Proc. London Math. Soc.* **87** (2003), 591–609.
- [33] K-H. INDLEKOFER, Multiplikative Funktionen mehrerer Variablen, *J. Reine Angew. Math.*, **256** (1972), 180–184.
- [34] K. R. JOHNSON, Reciprocity in Ramanujan’s sum, *Math. Mag.*, **59** (1986), 216–222.
- [35] E. KRÄTZEL, W. G. NOWAK, L. TÓTH, On certain arithmetic functions involving the greatest common divisor, *Cent. Eur. J. Math.*, **10** (2012), 761–774.
- [36] M. KÜHLEITNER, W. G. NOWAK, On a question of A. Schinzel: Omega estimates for a special type of arithmetic functions, *Cent. Eur. J. Math.*, **11** (2013), 477–486.
- [37] D. H. LEHMER, A new calculus of numerical functions, *Amer. J. Math.*, **53** (1931), no. 4, 843–854.
- [38] D. H. LEHMER, On a theorem of von Sterneck, *Bull. Amer. Math. Soc.*, **37** (1931), no. 10, 723–726.
- [39] D. H. LEHMER, Arithmetic of double series, *Trans. Amer. Math. Soc.*, **33** (1931), 945–957.
- [40] V. A. LISKOVETS, A multivariate arithmetic function of combinatorial and topological significance, *Integers* **10** (2010), 155–177.
- [41] T. MACHENRY, A subgroup of the group of units in the ring of arithmetic functions, *Rocky Mountain J. Math.*, **29** (1999), 1055–1065.
- [42] P. J. MCCARTHY, *Introduction to Arithmetical Functions*, Springer, 1986.
- [43] W. NARKIEWICZ, On a summation formula of E. Cohen, *Colloq. Math.*, **11** (1963), 85–86.
- [44] W. G. NOWAK, L. TÓTH, On the average number of subgroups of the group $\mathbb{Z}_m \times \mathbb{Z}_n$, *Int. J. Number Theory*, **10** (2014), 363–374, arXiv:1307.1414 [math.NT].

- [45] T. ONOZUKA, The multiple Dirichlet product and the multiple Dirichlet series, manuscript, 2013.
- [46] A. PAKAPONGPUN, T. WARD, Functorial orbit counting, *J. Integer Sequences*, **12** (2009), Article 09.2.4, 20 pp.
- [47] M. PETER, Rekurrente zahlentheoretische Funktionen in mehreren Variablen, *Arch. Math. (Basel)*, **68** (1997), 202–213.
- [48] R. W. RYDEN, Groups of arithmetic functions under Dirichlet convolution, *Pacific J. Math.*, **44** (1973), 355–360.
- [49] J. SÁNDOR, A. BEGE, The Möbius function: generalizations and extensions, *Adv. Stud. Contemp. Math. (Kyungshang)*, **6** (2003), no. 2, 77–128.
- [50] J. SÁNDOR, B. CRSTICI, *Handbook of Number Theory II*, Kluwer Academic Publishers, 2004.
- [51] H. SCHEID, Einige Ringe zahlentheoretischer Funktionen, *J. Reine Angew. Math.*, **237** (1969), 1–11.
- [52] E. D. SCHWAB, Generalized arithmetical functions of three variables, *Int. J. Number Theory*, **6** (2010), 1689–1699.
- [53] W. SCHWARZ, J. SPILKER, *Arithmetical functions*, An introduction to elementary and analytic properties of arithmetic functions and to some of their almost-periodic properties, London Mathematical Society Lecture Note Series, 184. Cambridge University Press, Cambridge, 1994.
- [54] H. N. SHAPIRO, On the convolution ring of arithmetic functions, *Comm. Pure Appl. Math.*, **25** (1972), 287–336.
- [55] R. SIVARAMAKRISHNAN, *Classical Theory of Arithmetic Functions*, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 126, Marcel Dekker, 1989.
- [56] N. J. A. SLOANE, The On-Line Encyclopedia of Integer Sequences, <http://oeis.org>.
- [57] D. A. SMITH, *Generalized arithmetic function algebras*, The theory of arithmetic functions (Proc. Conf., Western Michigan Univ., Kalamazoo, Mich., 1971), pp. 205–245, Lecture Notes in Math., Vol. 251, Springer, Berlin, 1972.
- [58] J. SNELLMAN, Truncations of the ring of arithmetical functions with unitary convolution, *Int. J. Math. Game Theory Algebra*, **13** (2003), 485–519.
- [59] J. SNELLMAN, The ring of arithmetical functions with unitary convolution: divisorial and topological properties, *Arch. Math. (Brno)*, **40** (2004), 161–179.
- [60] R. D. VON STERNECK, Ableitung zahlentheoretischer Relationen mit Hilfe eines mehrdimensionalen Systemes von Gitterpunkten, *Monatsh. Math. Phys.*, **5** (1894), 255–266.
- [61] M. V. SUBBARAO, On some arithmetic convolutions, in *The Theory of Arithmetic Functions*, Lecture Notes in Mathematics No. **251**, 247–271, Springer, 1972.
- [62] L. TÓTH, On the asymptotic densities of certain subsets of \mathbb{N}^k , *Riv. Mat. Univ. Parma*, **(6) 4** (2001), 121–131.
- [63] L. TÓTH, The probability that k positive integers are pairwise relatively prime, *Fibonacci Quart.*, **40** (2002), 13–18.
- [64] L. TÓTH, A survey of gcd-sum functions, *J. Integer Sequences*, **13** (2010), Article 10.8.1, 23 pp.
- [65] L. TÓTH, Menon’s identity and arithmetical sums representing functions of several variables, *Rend. Sem. Mat. Univ. Politec. Torino*, **69** (2011), 97–110.
- [66] L. TÓTH, On the number of cyclic subgroups of a finite Abelian group, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)*, **55 (103)** (2012), 423–428.

- [67] L. TÓTH, Some remarks on a paper of V. A. Liskovets, *Integers*, **12** (2012), 97–111.
- [68] L. TÓTH, Sums of products of Ramanujan sums, *Ann. Univ. Ferrara*, **58** (2012), 183–197.
- [69] L. TÓTH, Two generalizations of the Busche-Ramanujan identities, *Int. J. Number Theory*, **9** (2013), 1301–1311.
- [70] L. TÓTH, A survey of the alternating sum-of-divisors function, *Acta Univ. Sapientiae, Math.*, **5** (2013), 93–107, arXiv:1111.4842 [math.NT].
- [71] L. TÓTH, Averages of Ramanujan sums: Note on two papers by E. Alkan, *Ramanujan J.*, accepted, arXiv:1305.6018 [math.NT].
- [72] L. TÓTH, P. HAUKKANEN, On the binomial convolution of arithmetical functions, *J. Comb. Number Theory*, **1** (2009), 31–47.
- [73] N. USHIROYA, Mean-value theorems for multiplicative arithmetic functions of several variables, *Integers*, **12** (2012), 989–1002.
- [74] R. VAIDYANATHASWAMY, The theory of multiplicative arithmetic functions, *Trans. Amer. Math. Soc.*, **33** (1931), 579–662.
- [75] A. ZAHARESCU, M. ZAKI, Derivations and generating degrees in the ring of arithmetical functions, *Proc. Indian Acad. Sci. (Math. Sci.)*, **117** (2007), No. 2, 167–175.

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