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AN EXPLICIT FORMULA FOR THE PRIME COUNTING FUNCTION

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ABSTRACT. This paper studies the behaviour of the prime counting function at some certain points.
We show that there is an exact formula for $\pi(n)$ which is valid for infinitely many naturals numbers n.

1. Introduction

The prime counting function is at the center of mathematical research for centuries and many asymptotic distributions of $\pi(n)$ are well known.
Many formulas have been discovered by mathematicians [1] but almost all of them are using all the prime numbers not greater than n to calculate $\pi(n)$. In this paper we give an exact formula which holds when a standard condition is satisfied.

2. Some basic theorems

Theorem 2.1. *Let $\pi(n)$ be the number of primes not greater than n and $n \geq 1$. Then $\frac{n}{\pi(n)}$ takes on every integer value greater than* 1.

Proof. The proof is presented in [2]. It uses only the fact that $\pi(N) = o(N)$ and $\pi(N+1)-\pi(N)$ is 0 or 1.
We can conclude from this theorem that $\frac{n}{\pi(n)}$ is an integer infinitelly often and we will use this fact in order to prove the existence of our formula. □

Theorem 2.2. $\frac{n}{lnn-\frac{1}{2}} < \pi(n) < \frac{n}{lnn-\frac{3}{2}}$ *for every $n \geq 67$.*

Proof. This is a theorem proved by J. Barkley Rosser and Lowell Schoenfeld and the proof is presented at [3].
□

3. The formula for $\pi(n)$

Theorem 3.1. *For infinitely many natural numbers n the following formula is valid:*

$$\pi(n) = \frac{n}{\left\lfloor lnn-\frac{1}{2} \right\rfloor}$$

Proof. We will make use of the above mentioned inequality in order to prove our formula.

We have $\frac{n}{lnn-\frac{1}{2}} < \pi(n) < \frac{n}{lnn-\frac{3}{2}}$ for every $n \geq 67$.
Inverse the inequality and multiply by n. We can see that the inequality now has the form:

$$lnn - \tfrac{3}{2} < \tfrac{n}{\pi(n)} < lnn - \tfrac{1}{2}\ .$$

So $\frac{n}{\pi(n)}$ lies between two real numbers $a-1$ and a with $a = lnn - \frac{1}{2}$.

This means that for every $n \geq 67$ when $\frac{n}{\pi(n)}$ is an integer we must have:

$$\frac{n}{\pi(n)} = \left\lfloor lnn - \frac{1}{2} \right\rfloor \Leftrightarrow \pi(n) = \frac{n}{\left\lfloor lnn - \frac{1}{2} \right\rfloor}.$$

This completes the proof. □

We can see below at Table 1 that the formula $\left\lfloor \frac{n}{lnn - \frac{1}{2}} \right\rfloor$ gives exactly the value of $\pi(n)$ for every natural number $67 \leq n < 4000$ with $\frac{n}{\pi(n)}$ being an integer[4].

n	$\pi(n)$	$\left\lfloor \frac{n}{lnn - \frac{1}{2}} \right\rfloor$
96	24	24
100	25	25
120	30	30
330	66	66
335	67	67
340	68	68
350	70	70
355	71	71
360	72	72
1008	168	168
1080	180	180
1092	182	182
1116	186	186
1122	187	187
1128	188	188
1134	189	189
3059	437	437
3066	438	438
3073	439	439
3080	440	440
3087	441	441
3094	442	442

TABLE 1

References

[1] Hardy, G. H., E. M. Wright An Introduction to the Theory of Numbers (5th Edition), Oxford, England, Clarendon Press, 1979.

[2] On the Ratio of N to $\pi(N)$ Solomon W. Golomb The American Mathematical Monthly Vol. 69, No. 1 (Jan., 1962), pp. 36-37

[3] J. Barkley Rosser and Lowell Schoenfeld, Approximate formulas for some functions of prime numbers, Ill. Journ. Math. 6 (1962) 64-94.

[4] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.

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