

Laurent Biorthogonal Polynomials and Riordan Arrays

Paul Barry
 School of Science
 Waterford Institute of Technology
 Ireland

Abstract

We show that Laurent biorthogonal polynomials whose defining three-term recurrence have constant coefficients have coefficient arrays that are Riordan arrays. For each such family of Laurent biorthogonal polynomials we associate in a natural way a family of orthogonal polynomials. We also extend these results to a notion of generalized orthogonal polynomials whose recurrence depends on three parameters.

1 Introduction

It is well known that orthogonal polynomials on the real line are characterized by their three-term recurrences [3, 6, 20]. Specifically, if $P_n(x)$ is a family of orthogonal polynomials, with $P_{-1}(x) = 0$ and $P_0(x) = 1$, then we can find two sequences α_n and β_n such that

$$P_n(x) = (x - \alpha_n)P_{n-1}(x) - \beta_n P_{n-2}(x).$$

In the case that $\alpha_n = \alpha$ and $\beta_n = \beta$ are constant sequences, then the coefficient array

$$\{d_{n,k}\}$$

of the polynomials is a Riordan array [1, 2, 16], where

$$P_n(x) = \sum_{k=0}^n d_{n,k} x^k.$$

This Riordan array will then be of the form

$$\left(\frac{1 - \delta x - \epsilon x^2}{1 + \alpha x + \beta x^2}, \frac{1}{1 + \alpha x + \beta x^2} \right).$$

In this note, we shall look at the situation of Laurent biorthogonal polynomials whose defining recurrences have constant coefficients. Based on the above results for orthogonal polynomials, it is natural to ask if there is a similar connection between Riordan arrays and Laurent biorthogonal polynomials whose defining recurrences have constant coefficients. We shall show that the answer is in the affirmative. We recall that a family of polynomials $P_n(x)$

is said to be a family of Laurent biorthogonal polynomials [8, 11, 10, 23] if there exist two sequences α_n and β_n such that

$$P_n(x) = (x - \alpha_n)P_{n-1}(x) - \beta_n x P_{n-2}(x).$$

The results in this note are typified by the following two propositions.

Proposition 1. *Let $P_n(x) = P_n(x; \alpha, \beta)$ be the family of Laurent biorthogonal polynomials defined by the recurrence*

$$P_n(x) = (x - \alpha)P_{n-1}(x) - \beta x P_{n-2}(x),$$

with $P_0(x) = 1$ and $P_1(x) = x - \alpha$. Then the coefficient array of the polynomials $P_n(x)$ is given by the Riordan array

$$\left(\frac{1}{1 + \alpha x}, \frac{x(1 - \beta x)}{1 + \alpha x} \right).$$

Similarly, we have

Proposition 2. *Let $P_n(x) = P_n(x; \alpha, \beta)$ be the family of Laurent biorthogonal polynomials defined by the recurrence*

$$P_n(x) = (x - \alpha)P_{n-1}(x) - \beta x P_{n-2}(x),$$

with $P_0(x) = 1$ and $P_1(x) = x - (\alpha + \beta)$. Then the coefficient array of the polynomials $P_n(x)$ is given by the Riordan array

$$\left(\frac{1 - \beta x}{1 + \alpha x}, \frac{x(1 - \beta x)}{1 + \alpha x} \right).$$

We will concentrate on this latter version, as we find it easier to work with the element $\left(\frac{1 - \beta x}{1 + \alpha x}, \frac{x(1 - \beta x)}{1 + \alpha x} \right)$ of the Bell subgroup of the Riordan group. Before we prove the above result, we recall some notation and elements of Riordan group theory. Readers familiar with Riordan groups may skip this section.

For an integer sequence a_n , that is, an element of $\mathbb{Z}^{\mathbb{N}}$, the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is called the *ordinary generating function* or g.f. of the sequence. a_n is thus the coefficient of x^n in this series. We denote this by $a_n = [x^n]f(x)$. For instance, $F_n = [x^n] \frac{x}{1 - x - x^2}$ is the n -th Fibonacci number [A000045](#), while $S_n = [x^n] \frac{1 - \sqrt{1 - 6x + x^2}}{2x}$ is the n -th (large) Schröder number [A006318](#).

For a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $f(0) = 0$ we define the reversion or compositional inverse of f to be the power series $\bar{f}(x)$ such that $f(\bar{f}(x)) = x$. We shall sometimes write this as $\bar{f} = \text{Rev}f$.

For a lower triangular matrix $(a_{n,k})_{n,k \geq 0}$ the row sums give the sequence with general term $\sum_{k=0}^n a_{n,k}$. The *Riordan group* [16, 19], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1 x + g_2 x^2 + \dots$ and $f(x) = f_1 x + f_2 x^2 + \dots$ where $f_1 \neq 0$ [19]. We assume in addition that $f_1 = 1$ in what follows. The associated matrix is the matrix whose i -th column is generated by $g(x)f(x)^i$

(the first column being indexed by 0). The matrix corresponding to the pair g, f is denoted by (g, f) or $\mathcal{R}(g, f)$. The group law is then given by

$$(g, f) \cdot (h, l) = (g, f)(h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where \bar{f} is the compositional inverse of f .

If \mathbf{M} is the matrix (g, f) , and $\mathbf{a} = (a_0, a_1, \dots)^T$ is an integer sequence with ordinary generating function $\mathcal{A}(x)$, then the sequence $\mathbf{M}\mathbf{a}$ has ordinary generating function $g(x)\mathcal{A}(f(x))$. The (infinite) matrix (g, f) can thus be considered to act on the ring of integer sequences $\mathbb{Z}^{\mathbb{N}}$ by multiplication, where a sequence is regarded as a (infinite) column vector. We can extend this action to the ring of power series $\mathbb{Z}[[x]]$ by

$$(g, f) : \mathcal{A}(x) \mapsto (g, f) \cdot \mathcal{A}(x) = g(x)\mathcal{A}(f(x)).$$

Example 3. The so-called *binomial matrix* \mathbf{B} is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group. It has general element $\binom{n}{k}$, and hence as an array coincides with Pascal's triangle. More generally, \mathbf{B}^m is the element $(\frac{1}{1-mx}, \frac{x}{1-mx})$ of the Riordan group, with general term $\binom{n}{k}m^{n-k}$. It is easy to show that the inverse \mathbf{B}^{-m} of \mathbf{B}^m is given by $(\frac{1}{1+mx}, \frac{x}{1+mx})$.

The proof of the propositions above is dependent on the sequence characterization of Riordan arrays [7, 14], and in particular on the A-sequence. One version of the sequence characterization of Riordan arrays is given below.

Proposition 4. [7] *Let $D = [d_{n,k}]$ be an infinite triangular matrix. Then D is a Riordan array if and only if there exist two sequences $A = [a_0, a_1, a_2, \dots]$ and $Z = [z_0, z_1, z_2, \dots]$ with $a_0 \neq 0$ such that*

- $d_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d_{n,k+j}, \quad (k, n = 0, 1, \dots)$
- $d_{n+1,0} = \sum_{j=0}^{\infty} z_j d_{n,j}, \quad (n = 0, 1, \dots).$

The coefficients a_0, a_1, a_2, \dots and z_0, z_1, z_2, \dots are called the *A-sequence* and the *Z-sequence* of the Riordan array $D = (g(x), f(x))$, respectively. Letting $A(x)$ and $Z(x)$ denote the generating functions of these sequences, respectively, we have [14] that

$$\frac{f(x)}{x} = A(f(x)), \quad g(x) = \frac{d_{0,0}}{1 - xZ(f(x))}.$$

We therefore deduce that

$$A(x) = \frac{x}{f(x)},$$

and

$$Z(x) = \frac{1}{f(x)} \left[1 - \frac{d_{0,0}}{g(f(x))} \right].$$

When $(g, f) = (g, xg)$, we then obtain that

$$Z(x) = \frac{A(x) - 1}{x}.$$

We now turn to the proof of the proposition.

Proof. We let $(d_{n,k})$ be the coefficient array of the polynomials $P_n(x)$ that are defined by the recurrence

$$P_n(x) = (x - \alpha)P_{n-1}(x) - \beta x P_{n-2}(x).$$

The recurrence implies that

$$\sum_{k=0}^n d_{n,k} x^k = (x - \alpha) \sum_{k=0}^{n-1} d_{n-1,k} x^k - \beta x \sum_{k=0}^{n-2} d_{n-2,k} x^k,$$

or

$$\sum_{k=0}^n d_{n,k} x^k = \sum_{k=0}^{n-1} d_{n-1,k} x^{k+1} - \alpha \sum_{k=0}^{n-1} d_{n-1,k} x^k - \beta \sum_{k=0}^{n-2} d_{n-2,k} x^{k+1},$$

or

$$\sum_{k=0}^n d_{n,k} x^k = \sum_{k=1}^n d_{n-1,k-1} x^k - \alpha \sum_{k=0}^{n-1} d_{n-1,k} x^k - \beta \sum_{k=1}^{n-1} d_{n-2,k-1} x^k.$$

Thus for $0 < k < n$, we require that

$$d_{n,k} = d_{n-1,k-1} - \alpha d_{n-1,k} - \beta d_{n-2,k-1}.$$

Now the A-sequence of the Riordan array $\left(\frac{1}{1+\alpha x}, \frac{x(1-\beta x)}{1+\alpha x}\right)$ (and that of $\left(\frac{1-\beta x}{1+\alpha x}, \frac{x(1-\beta x)}{1+\alpha x}\right)$) is generated by the power series

$$A(x) = \frac{x}{f(x)},$$

where $f(x) = \frac{x(1-\beta x)}{1+\alpha x}$. Thus

$$A(x) = \frac{2\beta x}{1 - \alpha x - \sqrt{1 - (\alpha + 2\beta)x + \alpha^2 x^2}},$$

and the A-sequence $[a_0, a_1, a_2, \dots]$ begins

$$1, -(\alpha + \beta), \beta(\alpha + \beta), -\beta(\alpha + \beta)(\alpha + 2\beta), -\beta(\alpha + \beta)((\alpha^2 + 5\alpha\beta + 5\beta^2)), \dots$$

By the sequence characterization of Riordan arrays, we have

$$d_{n,k+1} = a_0 d_{n-1,k} + a_1 d_{n-1,k+1} + a_2 d_{n-1,k+2} + \dots$$

We can reverse this relation [14] using the coefficients b_n obtained by

$$b_n = -\frac{1}{a_0} \sum_{j=1}^n a_j b_{n-j}, \quad b_0 = \frac{1}{a_0},$$

to obtain

$$d_{n-1,k} = a_0 d_{n,k+1} - a_1 d_{n,k+1} - (a_2 - a_1^2) d_{n,k+2} - (a_3 - 2a_1 a_2 + a_1^3) d_{n,k+3} + \dots$$

which in this case gives

$$d_{n-1,k} = d_{n,k+1} + (\alpha + \beta)d_{n,k+1} + (\alpha + \beta)(\alpha + 2\beta)d_{n,k+2} + (\alpha + \beta)(\alpha^2 + 5\alpha\beta + 5\beta^2)d_{n,k+3} + \dots$$

Thus we get

$$\begin{aligned} d_{n+1,k+1} &= d_{n,k} - (\alpha + \beta)d_{n,k+1} - \beta(\alpha + \beta)d_{n,k+2} - \beta(\alpha + \beta)(\alpha + 2\beta)d_{n,k+3} + \dots \\ &= d_{n,k} - \alpha d_{n,k+1} - \beta\{d_{n,k+1} + (\alpha + \beta)d_{n,k+2} + \beta(\alpha + \beta)(\alpha + 2\beta)d_{n,k+3} + \dots\}, \\ &= d_{n,k} - \alpha d_{n,k+1} - \beta d_{n-1,k}. \end{aligned}$$

□

We can derive an expression for $P_n(x)$ using the definition of Riordan arrays. The (n, k) -th element $d_{n,k}$ of the coefficient array $\left(\frac{1-\beta x}{1+\alpha x}, \frac{x(1-\beta x)}{1+\alpha x}\right)$ is given by

$$\begin{aligned} d_{n,k} &= [x^n] \frac{1-\beta x}{1+\alpha x} \left(\frac{x(1-\beta x)}{1+\alpha x}\right)^k \\ &= [x^{n-k}] \frac{(1-\beta x)^{k+1}}{(1+\alpha x)^{k+1}} \\ &= (-1)^{n-k} \sum_{j=0}^{k+1} \binom{k+1}{j} \binom{n-j}{n-k-j} \alpha^{n-k-j} \beta^j. \end{aligned}$$

Thus we get

$$P_n(x) = \sum_{k=0}^n (-1)^{n-k} \sum_{j=0}^{k+1} \binom{k+1}{j} \binom{n-j}{n-k-j} \alpha^{n-k-j} \beta^j x^k.$$

We note that an “ $\alpha_{i,j}$ ” matrix [14] for the Riordan array $\left(\frac{1-\beta x}{1+\alpha x}, \frac{x(1-\beta x)}{1+\alpha x}\right)$ is given by

$$\begin{bmatrix} -\beta & 0 \\ 1 & -\alpha \end{bmatrix}.$$

Example 5. We take the simple case of $\alpha = \beta = 1$. Thus we look at the Riordan array

$$\left(\frac{1-x}{1+x}, \frac{x(1-x)}{1+x}\right).$$

This matrix begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & -4 & 1 & 0 & 0 & 0 & \dots \\ -2 & 8 & -6 & 1 & 0 & 0 & \dots \\ 2 & -12 & 18 & -8 & 1 & 0 & \dots \\ -2 & 16 & -38 & 32 & -10 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which is the coefficient array of the polynomials $P_n(x) = P_n(x; 1, 1)$ which satisfy

$$P_n(x) = (x - 1)P_{n-1}(x) - xP_{n-2}.$$

The inverse array begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 22 & 16 & 6 & 1 & 0 & 0 & \cdots \\ 90 & 68 & 30 & 8 & 1 & 0 & \cdots \\ 394 & 304 & 146 & 48 & 10 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where we see that the moments are given by the large Schröder numbers 1, 2, 6, 22, 90, ..., [A06318](#).

Other examples of arrays of this type may be found for instance in the On-Line Encyclopedia of Integer Sequences [[17](#), [18](#)], where the last two number triangles are [A080246](#) and [A080247](#), respectively.

2 Associated orthogonal polynomials

We can associate a family of orthogonal polynomials to the family $P_n(x; \alpha, \beta)$ by taking a suitable β -fold inverse binomial transform.

Proposition 6. *The matrix*

$$\left(1, \frac{x}{1 - \beta x}\right)^{-1} \cdot \left(\frac{1 - \beta x}{1 + \alpha x}, \frac{x(1 - \beta x)}{1 + \alpha x}\right)$$

is the coefficient array of the family of orthogonal polynomials $\tilde{P}_n(x) = \tilde{P}_n(x; \alpha, \beta)$ that satisfies the three-term recurrence

$$\tilde{P}_n(x) = (x - (\alpha + 2\beta))\tilde{P}_{n-1}(x) - \beta(\alpha + \beta)\tilde{P}_{n-2}(x),$$

with $\tilde{P}_0(x) = 1$ and $\tilde{P}_1(x) = x - (\alpha + \beta)$.

Proof. We have

$$\left(1, \frac{x}{1 - \beta x}\right)^{-1} \cdot \left(\frac{1 - \beta x}{1 + \alpha x}, \frac{x(1 - \beta x)}{1 + \alpha x}\right) = \left(\frac{1}{1 + (\alpha + \beta)x}, \frac{x}{(1 + \beta x)(1 + (\alpha + \beta)x)}\right).$$

The inverse of this Riordan array has tri-diagonal production matrix which begins

$$\begin{pmatrix} \alpha + \beta & 1 & 0 & 0 & 0 & 0 & \cdots \\ \beta(\alpha + \beta) & \alpha + 2\beta & 1 & 0 & 0 & 0 & \cdots \\ 0 & \beta(\alpha + \beta) & \alpha + 2\beta & 1 & 0 & 0 & \cdots \\ 0 & 0 & \beta(\alpha + \beta) & \alpha + 2\beta & 1 & 0 & \cdots \\ 0 & 0 & 0 & \beta(\alpha + \beta) & \alpha + 2\beta & 1 & \cdots \\ 0 & 0 & 0 & 0 & \beta(\alpha + \beta) & \alpha + 2\beta & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

from which we deduce the three-term recurrence. □

Corollary 7.

$$\left(\frac{1 - \beta x}{1 + \alpha x}, \frac{x(1 - \beta x)}{1 + \alpha x} \right) = \left(1, \frac{x}{1 - \beta x} \right) \cdot \left(\frac{1}{1 + (\alpha + \beta)x}, \frac{x}{(1 + \beta x)(1 + (\alpha + \beta)x)} \right).$$

Corollary 8. *We have the relations*

$$P_n(x) = \sum_{k=0}^n \binom{n-1}{n-k} \beta^{n-k} \tilde{P}_k(x), \quad \tilde{P}_n(x) = \sum_{k=0}^n \binom{n-1}{n-k} (-\beta)^{n-k} P_k(x).$$

We finish this section by noting that

$$\left(\frac{1 - \beta x}{1 + \alpha x}, \frac{x(1 - \beta x)}{1 + \alpha x} \right) \cdot \frac{1}{1 - tx} = \frac{1 - \beta x}{1 + (\alpha - t)x + \beta tx^2},$$

and hence we have the generating function

$$\frac{1 - \beta x}{1 + (\alpha - t)x + \beta tx^2} = \sum_{n=0}^{\infty} P_n(t) x^n.$$

3 Moments and T-fractions

We identify the moments of $P_n(x)$ and $\tilde{P}_n(x)$ with the first column elements of the inverses of their coefficient arrays. The moments coincide, since they both have the generating function

$$\frac{1 - \alpha x - \sqrt{1 - (\alpha + 2\beta)x + \alpha^2 x^2}}{2\beta x}.$$

In order to find a closed expression for these moments, we calculate the inverse of $\left(\frac{1 - \beta x}{1 + \alpha x}, \frac{x(1 - \beta x)}{1 + \alpha x} \right)$.

We use Lagrange inversion [15] for this. Given that $\left(\frac{1 - \beta x}{1 + \alpha x}, \frac{x(1 - \beta x)}{1 + \alpha x} \right)$ is an element of the Bell subgroup of the Riordan group, its inverse will be of the form $\left(\frac{v(x)}{x}, v(x) \right)$, where

$$v(x) = \text{Rev} \left(\frac{x(1 - \beta x)}{1 + \alpha x} \right).$$

Then we have

$$\begin{aligned}
[x^n] \frac{v(x)}{x} v(x)^k &= [x^{n+1}] v(x)^{k+1} \\
&= [x^{n+1}] \left(\text{Rev} \left(\frac{x(1-\beta x)}{1+\alpha x} \right) \right)^{k+1} \\
&= \frac{1}{n+1} [x^n] (k+1)x^k \left(\frac{1+\alpha x}{1-\beta x} \right)^{n+1} \\
&= \frac{k+1}{n+1} [x^{n-k}] \left(\frac{1+\alpha x}{1-\beta x} \right)^{n+1} \\
&= \frac{k+1}{n+1} [x^{n-k}] \sum_{j=0}^{n+1} \alpha^j x^j \sum_{i=0}^{\infty} \binom{-(n+1)}{i} (-1)^i \beta^i x^i \\
&= \frac{k+1}{n+1} \sum_{j=0}^{n+1} \binom{n+1}{j} \binom{2n-k-j}{n-k-j} \alpha^j \beta^{n-k-j}.
\end{aligned}$$

Thus the moments μ_n of the family of Laurent biorthogonal polynomials $P_n(x)$ are given by

$$\mu_n = \frac{1}{n+1} \sum_{j=0}^{n+1} \binom{n+1}{j} \binom{2n-j}{n-j} \alpha^j \beta^{n-j}.$$

A more compact form is

$$\mu_n = \sum_{k=0}^n \binom{n+k}{2k} C_k \alpha^{n-k} \beta^k,$$

where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n -th Catalan number [A000108](#). The theory of T -fractions and Laurent biorthogonal polynomials now tells us that the moments μ_n are generated by the following continued fraction [\[21\]](#).

$$g_{\alpha,\beta}(x) = \frac{1}{1 - \alpha x - \frac{\beta x}{1 - \alpha x - \frac{\beta x}{1 - \alpha x - \dots}}}$$

Now the sequence μ_n is also the moment sequence for the orthogonal polynomials $\tilde{P}_n(x)$. Hence we also have the following Stieltjes continued fraction for $g_{\alpha,\beta}(x)$.

$$g_{\alpha,\beta}(x) = \frac{1}{1 - (\alpha + \beta)x - \frac{\beta(\alpha + \beta)x^2}{1 - (\alpha + 2\beta)x - \frac{\beta(\alpha + \beta)x^2}{1 - (\alpha + 2\beta)x - \dots}}}$$

From this we deduce the following result concerning the Hankel transform of the moments μ_n [\[12\]](#).

Proposition 9. *The Hankel transform of the moments μ_n of $P_n(x)$ is given by*

$$h_n = (\beta(\alpha + \beta))^{\binom{n+1}{2}}.$$

Example 10. The large Schröder numbers [A006318](#)

$$S_n = \sum_{k=0}^n \binom{n+k}{2k} C_k$$

are the case $\alpha = \beta = 1$. Thus the generating function for the large Schröder numbers may be written

$$\frac{1}{1 - x - \frac{x}{1 - x - \frac{x}{1 - x - \dots}}}$$

It is a classical result that the Hankel transform of the large Schröder numbers is $2^{\binom{n+1}{2}}$.

It is interesting to note that the row sums of the inverse matrix $\left(\frac{1-\beta x}{1+\alpha x}, \frac{x(1-\beta x)}{1+\alpha x}\right)^{-1}$ have a generating function that can be expressed as the continued fraction

$$g_{\alpha,\beta}(x) = \frac{1}{1 - (\alpha + \beta + 1)x - \frac{\beta(\alpha + \beta)x^2}{1 - (\alpha + 2\beta)x - \frac{\beta(\alpha + \beta)x^2}{1 - (\alpha + 2\beta)x - \dots}}}$$

Hence they too have a Hankel transform given by

$$h_n = (\beta(\alpha + \beta))^{\binom{n+1}{2}}.$$

The form of the continued fraction shows that the row sums are also moments for a family of orthogonal polynomials whose parameters can be read from the continued fraction.

Finally, the bi-variate generating function for the moment matrix $\left(\frac{1-\beta x}{1+\alpha x}, \frac{x(1-\beta x)}{1+\alpha x}\right)^{-1}$ is given by the generating function

$$\frac{1}{1 - (\alpha + \beta)x - xy - \frac{\beta(\alpha + \beta)x^2}{1 - (\alpha + 2\beta)x - \frac{\beta(\alpha + \beta)x^2}{1 - (\alpha + 2\beta)x - \dots}}}$$

Example 11. The orthogonal polynomials $\tilde{P}_n(x; 1, 1)$ associated with the Riordan array $\left(\frac{1-\beta x}{1+\alpha x}, \frac{x(1-\beta x)}{1+\alpha x}\right)$ have coefficient array given by

$$\left(\frac{1}{1+2x}, \frac{x}{1+3x+2x^2}\right).$$

The inverse of this matrix begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 5 & 1 & 0 & 0 & 0 & \cdots \\ 22 & 23 & 8 & 1 & 0 & 0 & \cdots \\ 90 & 107 & 49 & 11 & 1 & 0 & \cdots \\ 394 & 509 & 276 & 84 & 14 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

indicating that the large Schröder numbers are the moments for this family of orthogonal polynomials. This is borne out by the fact that the inverse is given by

$$\left(\frac{1-x-\sqrt{1-6x+x^2}}{2x}, \frac{1-3x-\sqrt{1-6x+x^2}}{4x} \right).$$

This is [A133367](#).

4 Derivatives

We consider the derivatives

$$\frac{d}{dx}P_n(x) = R_{n-1}(x), \quad n > 0.$$

Since $P_n(x)$ is precisely of degree n , $R_n(x)$ is also of degree n , with

$$R_0(x) = \frac{d}{dx}(x - (\alpha + \beta)) = 1.$$

We let $e_{n,k}$ be the (n, k) -th term of the coefficient array of $R_n(x)$. Thus we have

$$R_n(x) = \frac{d}{dx}P_{n+1}(x) = \sum_{k=0}^n e_{n,k}x^k.$$

Proposition 12. *The coefficient array whose (n, k) -th term is $\frac{1}{n+1}e_{n,k}$ is given by the Riordan array*

$$\left(\left(\frac{1-\beta x}{1+\alpha x} \right)^2, \frac{x(1-\beta x)}{1+\alpha x} \right).$$

This results from a general result concerning Riordan arrays. We first note that we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 4 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 5 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2x \\ 3x^2 \\ 4x^3 \\ 5x^4 \\ \vdots \end{pmatrix}.$$

Thus we have

$$\begin{pmatrix} d_{0,0} & 0 & 0 & 0 & 0 & 0 & \cdots \\ d_{1,0} & d_{1,1} & 0 & 0 & 0 & 0 & \cdots \\ d_{2,0} & d_{2,1} & d_{2,2} & 0 & 0 & 0 & \cdots \\ d_{3,0} & d_{3,1} & d_{3,2} & d_{3,3} & 0 & 0 & \cdots \\ d_{4,0} & d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & 0 & \cdots \\ d_{5,0} & d_{5,1} & d_{5,2} & d_{5,3} & d_{5,4} & d_{5,5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 4 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 5 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ R_0(x) \\ R_1(x) \\ R_2(x) \\ R_3(x) \\ R_4(x) \\ \vdots \end{pmatrix},$$

where in this case $(d_{n,k})$ represents the Riordan array $(g, f) = \left(\frac{1-\beta x}{1+\alpha x}, \frac{x(1-\beta x)}{1+\alpha x}\right)$. We then have

$$\begin{pmatrix} d_{1,1} & 0 & 0 & 0 & 0 & 0 & \cdots \\ d_{2,1} & d_{2,2} & 0 & 0 & 0 & 0 & \cdots \\ d_{3,1} & d_{3,2} & d_{3,3} & 0 & 0 & 0 & \cdots \\ d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & 0 & 0 & \cdots \\ d_{5,1} & d_{5,2} & d_{5,3} & d_{5,4} & d_{5,5} & 0 & \cdots \\ d_{6,1} & d_{6,2} & d_{6,3} & d_{6,4} & d_{6,5} & d_{6,6} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 4 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 5 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ \vdots \end{pmatrix} = \begin{pmatrix} R_0(x) \\ R_1(x) \\ R_2(x) \\ R_3(x) \\ R_4(x) \\ R_5(x) \\ \vdots \end{pmatrix}.$$

By the Riordan array structure of (g, f) , the leftmost matrix is given by (gf, f) . In other words, we have

$$\left(\left(\frac{1-\beta x}{1+\alpha x}\right)^2, \frac{x(1-\beta x)}{1+\alpha x}\right) \cdot \text{Diag}(1, 2, 3, \dots) = \begin{pmatrix} R_0(x) \\ R_1(x) \\ R_2(x) \\ R_3(x) \\ R_4(x) \\ R_5(x) \\ \vdots \end{pmatrix}.$$

5 Generalized orthogonality

Ismail and Masson [9] have studied a notion of generalized orthogonality, associated with polynomials that satisfy recurrences of the type

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n(x - a_n)P_{n-2}.$$

We specialize this to the constant coefficient case. Thus in this section we let $P_n(x) = P_n(x; \alpha, \beta, \gamma)$ be a family of polynomials that obeys the recurrence

$$P_n(x) = (x - \alpha)P_{n-1}(x) - \beta(x - \gamma)P_{n-2}(x),$$

with $P_0(x) = 1$ and $P_1(x) = x - \alpha - \beta$.

Following a similar development to that in the first section, we arrive at the following results.

Proposition 13. *The coefficient array of the polynomials $P_n(x)$ is given by the Riordan array*

$$\left(\frac{1 - \beta x}{1 + \alpha x - \beta \gamma x^2}, \frac{x(1 - \beta x)}{1 + \alpha x - \beta \gamma x^2} \right).$$

Note that when $\gamma = 0$, we retrieve our original case. When $\alpha + \beta \neq \gamma$, we have

Proposition 14. *The moments μ_n of the family of polynomials $P_n(x)$ are also the moments of the family of associated orthogonal polynomials $\tilde{P}_n(x)$ whose coefficient array is given by the Riordan array*

$$\left(\frac{1 + \beta x}{1 + (\alpha + 2\beta)x + \beta(\alpha + \beta - \gamma)x^2}, \frac{x}{1 + (\alpha + 2\beta)x + \beta(\alpha + \beta - \gamma)x^2} \right).$$

Corollary 15. *The Hankel transform of the moments μ_n is given by*

$$h_n = (\beta(\alpha + \beta - \gamma))^{\binom{n+1}{2}}.$$

The relationship between the generalized orthogonal polynomials $P_n(x)$ of this section and their associated orthogonal polynomials $\tilde{P}_n(x)$ is given by the following proposition, which can be verified by applying the multiplication rule for Riordan arrays.

Proposition 16.

$$\left(\frac{1 - \beta x}{1 + \alpha x - \beta \gamma x^2}, \frac{x(1 - \beta x)}{1 + \alpha x - \beta \gamma x^2} \right) = \tilde{B} \cdot \left(\frac{1 + \beta x}{1 + (\alpha + 2\beta)x + \beta(\alpha + \beta - \gamma)x^2}, \frac{x}{1 + (\alpha + 2\beta)x + \beta(\alpha + \beta - \gamma)x^2} \right),$$

where

$$\tilde{B} = \left(1, \frac{x}{1 - \beta x} \right).$$

Example 17. We consider the polynomials $P_n(x) = P_n(x; 2, 1, 1)$ with coefficient array

$$\left(\frac{1 - x}{1 + 2x - x^2}, \frac{x(1 - x)}{1 + 2x - x^2} \right).$$

This begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 7 & -6 & 1 & 0 & 0 & 0 & \cdots \\ -17 & 23 & -9 & 1 & 0 & 0 & \cdots \\ 41 & -76 & 48 & -12 & 1 & 0 & \cdots \\ -99 & 233 & -204 & 82 & -15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$P_n(x)$ satisfies the recurrence

$$P_n(x) = (x - 2)P_{n-1} - (x - 1)P_{n-2}$$

with $P_0(x) = 1$ and $P_1(x) = x - 3$. The moment array is given by $\left(\frac{1-x}{1+2x-x^2}, \frac{x(1-x)}{1+2x-x^2}\right)^{-1}$ which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 11 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 47 & 31 & 9 & 1 & 0 & 0 & \cdots \\ 223 & 160 & 60 & 12 & 1 & 0 & \cdots \\ 1135 & 849 & 366 & 98 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus the moment sequence μ_n starts

$$1, 3, 11, 47, 223, 1135, \dots$$

This is the binomial transform [A174347](#) of the large Schröder numbers.

$$\mu_n = \sum_{k=0}^n \binom{n}{k} S_k.$$

This is a consequence of the fact that

$$\left(\frac{1-x}{1+2x-x^2}, \frac{x(1-x)}{1+2x-x^2}\right) \cdot B = \left(\frac{1-x}{1+x}, \frac{x(1-x)}{1+x}\right). \quad (1)$$

We note that $\left(\frac{1-x}{1+x}, \frac{x(1-x)}{1+x}\right)$ is the coefficient array of the Laurent biorthogonal polynomials

$$Q_n(x) = (x-1)Q_{n-1} - xQ_{n-2}.$$

Equation (1) is equivalent to

$$P_n(x+1) = Q_n(x).$$

We can also transform the coefficient array $\left(\frac{1-\beta x}{1+\alpha x-\beta\gamma x^2}, \frac{x(1-\beta x)}{1+\alpha x-\beta\gamma x^2}\right)$ of $P_n(x; \alpha, \beta, \gamma)$ by multiplication on the right to obtain the coefficient array of an associated family of Laurent biorthogonal polynomials. This is the content of the next result, which is verifiable by straight-forward Riordan array multiplication.

Proposition 18. *We have*

$$\left(\frac{1-\beta x}{1+\alpha x-\beta\gamma x^2}, \frac{x(1-\beta x)}{1+\alpha x-\beta\gamma x^2}\right) \cdot B_\gamma = \left(\frac{1-\beta x}{1+(\alpha-\gamma)x}, \frac{x(1-\beta x)}{1+(\alpha-\gamma)x}\right),$$

where

$$B_\gamma = \left(\frac{1}{1-\gamma x}, \frac{x}{1-\gamma x}\right).$$

Corollary 19. *Using an obvious notation, we have*

$$P_n(x+\gamma; \alpha, \beta, \gamma) = P_n(x; \alpha-\gamma, \beta).$$

Proof. This follows since

$$B_\gamma \cdot (1, x, x^2, \dots)^T = (1, x+\gamma, (x+\gamma)^2, \dots)^T.$$

□

6 Determinant representations

It is possible to give determinant representations for the polynomial sequences [22] in this note.

Taking the polynomials defined by the Riordan array $M = \left(\frac{1-\beta x}{1+\alpha x}, \frac{x(1-\beta x)}{1+\alpha x} \right)$, we calculate $A(x)$ and $Z(x)$ for the inverse $\left(\frac{1-\beta x}{1+\alpha x}, \frac{x(1-\beta x)}{1+\alpha x} \right)^{-1}$ to get

$$A(x) = \frac{1 + \alpha x}{1 - \beta x}, \quad Z(x) = \frac{\alpha + \beta}{1 - \beta x}.$$

Thus the production matrix of the inverse M^{-1} begins

$$\begin{pmatrix} \alpha + \beta & 1 & 0 & 0 & 0 & 0 & \dots \\ \beta(\alpha + \beta) & \alpha + \beta & 1 & 0 & 0 & 0 & \dots \\ \beta^2(\alpha + \beta) & \beta(\alpha + \beta) & \alpha + \beta & 1 & 0 & 0 & \dots \\ \beta^3(\alpha + \beta) & \beta^2(\alpha + \beta) & \beta(\alpha + \beta) & \alpha + \beta & 1 & 0 & \dots \\ \beta^4(\alpha + \beta) & \beta^3(\alpha + \beta) & \beta^2(\alpha + \beta) & \beta(\alpha + \beta) & \alpha + \beta & 1 & \dots \\ \beta^5(\alpha + \beta) & \beta^4(\alpha + \beta) & \beta^3(\alpha + \beta) & \beta^2(\alpha + \beta) & \beta(\alpha + \beta) & \alpha + \beta & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

Letting \mathbf{P}_n denote the principal submatrix of order n of this production matrix, we have

$$P_n(x) = P_n(x; \alpha, \beta) = \det(xI_n - \mathbf{P}_n),$$

where I_n is the identity matrix of order n .

Similar remarks hold for the other polynomials defined in this article.

References

- [1] P. Barry, Riordan arrays, orthogonal polynomials as moments, and Hankel transforms, *J. Integer Seq.*, **14** (2011), [Article 11.2.2](#).
- [2] P. Barry and A. Hennessy, Meixner-type results for Riordan arrays and associated integer sequences, *J. Integer Seq.*, **13** (2010), [Article 10.9.4](#).
- [3] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.
- [4] E. Deutsch, L. Ferrari, and S. Rinaldi, Production matrices, *Adv. in Appl. Math.* **34** (2005), 101–122.
- [5] E. Deutsch, L. Ferrari, and S. Rinaldi, Production matrices and Riordan arrays, <http://arxiv.org/abs/math/0702638v1>, February 22 2007.
- [6] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation*, Clarendon Press - Oxford, 2003.

- [7] Tian-Xiao He and R. Sprugnoli, Sequence characterization of Riordan arrays, *Discrete Math.* **309** (2009), 3962–3974.
- [8] E. Hendriksen and O. Njåstad, Biorthogonal Laurent polynomials with biorthogonal derivatives, *Rocky Mountain J. Math.*, **21** (1991), 301–317.
- [9] M. E. H Ismail and D. R. Masson, Generalized orthogonality and continued fractions, *J. Approx. Theory*, **83** (1995), 1–40.
- [10] S. Kamioka, A combinatorial derivation with Schröder paths of a determinant representation of Laurent biorthogonal polynomials, *Electron. J. Combin.*, **15** (2008), R76.
- [11] S. Kamioka, A combinatorial representation with Schröder paths of biorthogonality of Laurent biorthogonal polynomials, *Electron. J. Combin.*, **14** (2007), R37.
- [12] J. W. Layman, The Hankel transform and some of its properties, *J. Integer Seq.* **4** (2001), [Article 01.1.5](#).
- [13] A. Luzón and M. A. Morón, Recurrence relations for polynomial sequences via Riordan matrices, *Linear Alg. Appl.*, **433** (2010), 1422–1446.
- [14] D. Merlini, D. G. Rogers, R. Sprugnoli, and M. C. Verri, On some alternative characterizations of Riordan arrays, *Canad. J. Math.*, **49** (1997), 301–320.
- [15] D. Merlini, R. Sprugnoli, and M. C. Verri, Lagrange inversion: when and how, *Acta Appl. Math.*, **94** (2006), 233–249.
- [16] L. W. Shapiro, S. Getu, W-J. Woan, and L.C. Woodson, The Riordan Group, *Discr. Appl. Math.*, **34** (1991), 229–239.
- [17] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*. Published electronically at <http://oeis.org>, 2013.
- [18] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, *Notices Amer. Math. Soc.*, **50** (2003), 912–915.
- [19] R. Sprugnoli, Riordan arrays and combinatorial sums, *Discrete Math.* **132** (1994), 267–290.
- [20] G. Szegő, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc., (1975).
- [21] H. S. Wall, *Analytic Theory of Continued Fractions*, AMS Chelsea Publishing.
- [22] S-L Yang and S-N Zheng, Determinant representation of polynomial sequences of Riordan type, *Journal of Discrete Mathematics*, **2013** (2013), Article ID 734836, 6 pages.
- [23] A. Zhedanov, The “classical” Laurent biorthogonal polynomials, *J. Comput. Appl. Math.*, **98** (1998), 121–147.

2010 *Mathematics Subject Classification*: Primary 11C20; Secondary 11B83, 15B36, 33C45.
Keywords: Riordan array, Laurent biorthogonal polynomials, orthogonal polynomials, generalized orthogonal polynomial, A sequence, Z sequence, production matrix.

Concerns sequences

[A000045](#), [A006318](#), [A080246](#), [A080247](#), [A133367](#), [A174347](#).