# Connectivity for bridge-alterable graph classes

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**Abstract**

A collection $\mathcal{A}$ of graphs is called bridge-alterable if, for each graph $G$ with a bridge $e$, $G$ is in $\mathcal{A}$ if and only if $G - e$ is. For example the class $\mathcal{F}$ of forests is bridge-alterable. For a random forest $F_n$ sampled uniformly from the set $\mathcal{F}_n$ of forests on vertex set $\{1, \ldots, n\}$, a classical result of Rényi (1959) shows that the probability that $F_n$ is connected is $e^{-\frac{1}{2}+o(1)}$.

Recently Addario-Berry, McDiarmid and Reed (2012) and Kang and Panagiotou (2013) independently proved that, given a bridge-alterable class $\mathcal{A}$, for a random graph $R_n$ sampled uniformly from the graphs in $\mathcal{A}$ on $\{1, \ldots, n\}$, the probability that $R_n$ is connected is at least $e^{-\frac{1}{2}+o(1)}$. Here we give a stronger non-asymptotic form of this result, with a more straightforward proof. We see that the probability that $R_n$ is connected is at least the minimum over $n/3 < t \leq n$ of the probability that $F_t$ is connected.

**Keywords:** random graph, connectivity, bridge-addable, bridge-alterable

## 1 Introduction

A collection $\mathcal{A}$ of graphs is *bridge-addable* if for each graph $G$ in $\mathcal{A}$ and pair of vertices $u$ and $v$ in different components, the graph $G + uv$ obtained by adding the edge (bridge) $uv$ is also in $\mathcal{A}$; that is, if $\mathcal{A}$ is closed under adding bridges. This property was introduced in [8] (under the name 'weakly addable'). If also $\mathcal{A}$ is closed under deleting bridges we call $\mathcal{A}$ *bridge-alterable*. Thus $\mathcal{A}$ is bridge-alterable exactly when, for each graph $G$ with a bridge $e$, $G$ is in $\mathcal{A}$ if and only if $G - e$ is in $\mathcal{A}$. The class $\mathcal{F}$ of forests is bridge-alterable, as for example is the class of series-parallel graphs, the class of planar graphs, and indeed the class of graphs embeddable in any given surface. All natural examples of bridge-addable classes seem to satisfy the stronger condition of being bridge-alterable.

Given a class $\mathcal{A}$ of graphs we let $\mathcal{A}_n$ denote the set of graphs in $\mathcal{A}$ on vertex set $[n] := \{1, \ldots, n\}$. Also, we use the notation $R_n \in_u \mathcal{A}$ to mean that $R_n$ is a random graph sampled uniformly from $\mathcal{A}_n$ (where we assume implicitly that $\mathcal{A}_n \neq \emptyset$). For a random forest $F_n \in_u \mathcal{F}$, a classical result of Rényi (1959) shows that, as $n \to \infty$

$$\mathbb{P}(F_n \text{ is connected}) = e^{-\frac{1}{2}+o(1)}. \tag{1}$$

It was conjectured [9] in 2006 that, when $\mathcal{A}$ is bridge-addable, for $R_n \in_u \mathcal{A}$

$$\mathbb{P}(R_n \text{ is connected}) \geq e^{-\frac{1}{2}+o(1)}\,? \tag{2}$$

and this conjecture was strengthened (see Conjecture 1.2 of [2], Conjecture 5.1 of [1], or Conjecture 6.2 of [7]) to the non-asymptotic form

$$\mathbb{P}(R_n \text{ is connected}) \geq \mathbb{P}(F_n \text{ is connected})\,? \tag{3}$$

Early progress was made on conjecture (2) by Balister, Bollobás and Gerke [2, 3], and very recently Norin [11] has made significant further progress, showing that $\mathbb{P}(R_n \text{ is connected}) \geq e^{-\frac{2}{3}+o(1)}$; but the full conjecture remains open.

Recently Addario-Berry, McDiarmid and Reed (2012) and Kang and Panagiotou (2013) independently proved the special case of conjecture (2) when $\mathcal{A}$ is bridge-alterable.

**Theorem 1.1.** *[1, 5] Let $\mathcal{A}$ be a bridge-alterable class of graphs, and let $R_n \in_u \mathcal{A}$. Then*

$$\mathbb{P}(R_n \textit{ is connected}) \geq e^{-\frac{1}{2}+o(1)}.$$

Here we give a short and reasonably straightforward proof of the following non-asymptotic form of this result, which together with (1) gives Theorem 1.1. This is a first step towards conjecture (3) for a bridge-alterable class.

**Theorem 1.2.** *Let $\mathcal{A}$ be a bridge-alterable class of graphs, let $n$ be a positive integer, let $R_n \in_u \mathcal{A}$, and let $F_t \in_u \mathcal{F}$ for $t = 1, 2, \ldots$. Then*

$$\mathbb{P}(R_n \textit{ is connected}) \geq \min_{n/3 < t \leq n} \mathbb{P}(F_t \textit{ is connected}). \tag{4}$$

The value $n/3$ can be increased towards $n/2$, see the final section of the paper.

## 2 Proof of Theorem 1.2

We use two lemmas in the proof.

**Lemma 2.1.** *Let $\mathcal{A}$ be a bridge-alterable class of graphs, let $n$ be a positive integer, let $R_n \in_u \mathcal{A}$, and let $F_t \in_u \mathcal{F}$ for $t = 1, 2, \ldots$. Then*

$$\mathbb{P}(R_n \textit{ is connected}) \geq \min_{t=1,\ldots,n} \max\{e^{-\frac{t}{n}}, \mathbb{P}(F_t \textit{ is connected})\}. \tag{5}$$

**Lemma 2.2.** *For each $n = 2, 3, \ldots$*

$$\mathbb{P}(F_n \textit{ is connected}) < e^{-\frac{1}{3}}.$$

To deduce Theorem 1.2 from these lemmas, observe that by Lemma 3.1, for each $2 \leq t \leq n/3$

$$e^{-\frac{t}{n}} \geq e^{-\frac{1}{3}} \geq \mathbb{P}(F_n \text{ is connected}),$$

and so the right side in (5) is at least the right side in (4).

**Proof of Lemma 2.1** Our proof initially follows the lines of the proofs in [1] and [5] of Theorem 1.1, in that we aim to lower bound the probability of connectedness for the random graph $F^{\mathbf{n}}$ introduced below. Consider a fixed $n \geq 2$.

Given a graph $G$, let $b(G)$ be the graph obtained by removing all bridges from $G$. We say $G$ and $G'$ are *equivalent* if $b(G) = b(G')$. This is an equivalence relation on graphs, and if a graph $G$ is in $\mathcal{A}_n$ then so is the whole equivalence class $[G]$. Thus $\mathcal{A}_n$ is a union of disjoint equivalence classes. To prove the theorem we consider an arbitrary (fixed) equivalence class.

Fix a bridgeless graph $G$ on vertex set $[n]$ and let $\mathcal{B} = [G]$. Let $G$ have $t$ components, with $n_1, \ldots, n_t$ vertices, where $n = \sum_{i=1}^{t} n_i$. We use $\mathbf{n} = (n_1, \ldots, n_t)$ to define probabilities. First, given a forest $F \in \mathcal{F}_t$, let

$$\text{mass}\,(F) = \prod_{i=1}^{t} n_i^{d_F(i)},$$

where $d_F(i)$ denotes the degree of vertex $i$ in $F$. For $\mathcal{F}' \subseteq \mathcal{F}_t$ let $\text{mass}\,(\mathcal{F}') = \sum_{F \in \mathcal{F}'} \text{mass}\,(F)$. Now let

$$\mathbb{P}(F^{\mathbf{n}} = F) = \frac{\text{mass}\,(F)}{\text{mass}\,(\mathcal{F}_t)} \quad \text{for each } F \in \mathcal{F}_t.$$

By Lemma 2.3 of [1], for a uniformly random element $R^{\mathcal{B}}$ of $\mathcal{B}$,

$$\mathbb{P}(R^{\mathcal{B}} \text{ is connected}) = \mathbb{P}(F^{\mathbf{n}} \text{ is connected}).$$

Hence to prove the theorem it suffices to consider $F^{\mathbf{n}}$, and show that

$$\mathbb{P}(F^{\mathbf{n}} \text{ is connected}) \geq \max\{e^{-\frac{t}{n}}, \mathbb{P}(F_t \text{ is connected})\}. \tag{6}$$

To see this, observe that then the probability that $R_n$ is connected is an average of values each at least the right side of (6) for some $t$, and so it is at least the right side in (5).

The proof of (6) breaks into two parts, and the first is standard. By Lemma 3.2 of [1], for $i = 1, \ldots, t-1$

$$\mathbb{P}(\kappa(F^{\mathbf{n}}) = i+1) \leq \frac{1}{i}\frac{t}{n}\mathbb{P}(\kappa(F^{\mathbf{n}}) = i),$$

and thus

$$\mathbb{P}(\kappa(F^{\mathbf{n}}) = i+1) \leq \frac{1}{i!}\left(\frac{t}{n}\right)^i \mathbb{P}(\kappa(F^{\mathbf{n}}) = 1).$$

Hence

$$1 = \sum_{i=0}^{t-1} \mathbb{P}(\kappa(F^{\mathbf{n}}){=}i{+}1) \quad \leq \quad \sum_{i=0}^{t-1} \frac{1}{i!}\left(\frac{t}{n}\right)^i \mathbb{P}(\kappa(F^{\mathbf{n}}){=}1) < e^{\frac{t}{n}} \cdot \mathbb{P}(\kappa(F^{\mathbf{n}}){=}1)$$

and so $\mathbb{P}(F^{\mathbf{n}} \text{ is connected}) > e^{-\frac{t}{n}}$ (as noted at the end of Section 3 of [1]).

It remains to show that

$$\mathbb{P}(F^{\mathbf{n}} \text{ is connected}) \geq \mathbb{P}(F_t \text{ is connected}). \tag{7}$$

We may assume that $t \geq 2$. Let $\mathcal{T}$ be the class of trees. Then

$$\text{mass}\,(\mathcal{T}_t) = \prod_{i=1}^{t} n_i \cdot n^{t-2}. \tag{8}$$

This result is proved for example in [1] (see the proof of Lemma 4.2) and in [5], though in fact it has long been known, see Theorem 6.1 of Moon [10] (1970), and see also Problems 5.3 and 5.4 of Lovász [6]. We let $N = \prod_{i=1}^{t} n_i$ and rewrite (8) as

$$\text{mass}\,(\mathcal{T}_t) = N(\frac{n}{t})^{t-2} \cdot |\mathcal{T}_t|. \tag{9}$$

For the case $t = 2$, $\text{mass}\,(\mathcal{T}_2) = n_1 n_2$ and $\text{mass}\,(\mathcal{F}_2) = \text{mass}\,(\mathcal{T}_2) + 1$, so

$$\mathbb{P}(F^{\mathbf{n}} \text{ is connected}) = \frac{n_1 n_2}{n_1 n_2 + 1} \geq \frac{1}{2} = \mathbb{P}(F_2 \text{ is connected}).$$

Thus we may assume from now on that $t \geq 3$.

For each integer $k$ with $1 \leq k \leq t$ let $\mathcal{F}_t^k$ be the set of forests in $\mathcal{F}_t$ with $k$ components. We shall show that for each such $k$

$$\text{mass}\,(\mathcal{F}_t^k) \leq N(\frac{n}{t})^{t-2} \cdot |\mathcal{F}_t^k|. \tag{10}$$

Summing over $k$ will then give

$$\text{mass}\,(\mathcal{F}_t) \leq N(\frac{n}{t})^{t-2} \cdot |\mathcal{F}_t|,$$

and so, using also (9)

$$\mathbb{P}(F^{\mathbf{n}} \text{ is connected}) = \frac{\text{mass}\,(\mathcal{T}_t)}{\text{mass}\,(\mathcal{F}_t)} \geq \frac{|\mathcal{T}_t|}{|\mathcal{F}_t|} = \mathbb{P}(F_t \text{ is connected}).$$

This will complete the proof of (7) and thus of the lemma. Hence it remains now to prove (10).

Fix an integer $k$ with $1 \leq k \leq t$. Given a partition $\mathbf{U} = (U_1, \ldots, U_k)$ of $[t]$ into $k$ unordered sets, let $J = J(\mathbf{U}) = \{i : |U_i| \geq 2\}$, and let $\mathcal{F}(\mathbf{U})$ be the set of forests in $\mathcal{F}_t^k$ such that the $U_i$ are the vertex sets of the $k$ component trees. For non-empty sets $U \subseteq [t]$, let $p(U) = \prod_{i \in U} n_i$ and $s(U) = \sum_{i \in U} n_i$. Observe that the mass of a forest is the product of the masses of its component trees, and a singleton component just gives a factor 1. Now fix a partition $\mathbf{U} = (U_1, \ldots, U_k)$ as above.

If $J = \emptyset$ than $\text{mass}\,(\mathcal{F}(\mathbf{U})) = 1 = |\mathcal{F}(\mathbf{U})|$. Now suppose that $J \neq \emptyset$. Then by (9)

$$\begin{aligned} \text{mass}\,(\mathcal{F}(\mathbf{U})) &= \prod_{i \in J} p(U_i) \left( \frac{s(U_i)}{|U_i|} \right)^{|U_i|-2} |U_i|^{|U_i|-2} \\ &\leq N \cdot \prod_{i \in J} \left( \frac{s(U_i)}{|U_i|} \right)^{|U_i|-2} \cdot \prod_{i \in J} |U_i|^{|U_i|-2} \\ &= N \cdot \prod_{i \in J} \left( \frac{s(U_i)}{|U_i|} \right)^{|U_i|-2} \cdot |\mathcal{F}(\mathbf{U})|. \end{aligned}$$

To handle the middle factor here, we can use Jensen's inequality, since $\log(x)$ is concave: we have

$$\begin{aligned}
& \log \prod_{i\in J} \left(\frac{s(U_i)}{|U_i|}\right)^{|U_i|-2} \\
= & \ (t-2)\sum_{i\in J} \frac{|U_i|-2}{t-2} \log \frac{s(U_i)}{|U_i|} \\
\le & \ (t-2)\sum_{i\in J} \frac{|U_i|}{t} \log \frac{s(U_i)}{|U_i|} \qquad \text{since } |U_i| \le t \\
\le & \ (t-2)\sum_{i=1}^{k} \frac{|U_i|}{t} \log \frac{s(U_i)}{|U_i|} \\
\le & \ (t-2)\log \left(\sum_{i=1}^{k} \frac{|U_i|}{t} \frac{s(U_i)}{|U_i|}\right) \qquad \text{since log is concave} \\
= & \ (t-2)\log \frac{n}{t}.
\end{aligned}$$

Hence in each case

$$\text{mass}\,(\mathcal{F}(\mathbf{U})) \le N \left(\frac{n}{t}\right)^{t-2} |\mathcal{F}(\mathbf{U})|.$$

So, summing over partitions $\mathbf{U} = (U_1, \ldots, U_k)$ of $[t]$,

$$\begin{aligned}
\text{mass}\,(\mathcal{F}_t^k) &= \sum_{\mathbf{U}=(U_1,\ldots,U_k)} \text{mass}\,(\mathcal{F}(\mathbf{U})) \\
&\le \sum_{\mathbf{U}=(U_1,\ldots,U_k)} N \left(\frac{n}{t}\right)^{t-2} |\mathcal{F}(\mathbf{U})| \\
&= N \left(\frac{n}{t}\right)^{t-2} |\mathcal{F}_t^k|.
\end{aligned}$$

This completes the proof of (10), and thus the proof of Lemma 2.1. □

To prove Lemma 2.2 we will use the standard inequality

$$(1-\frac{j}{n})^{n-j} \ge e^{-j} \qquad \text{for } 1 \le j < n. \tag{11}$$

[To see this, fix $j$ and let $g(x) = (x-j)\log(1-\frac{j}{x})$ for $x > j$. Then

$$g'(x) = (x-j)(\frac{1}{x-j} - \frac{1}{x}) + \log(1-\frac{j}{x}) = \frac{j}{x} + \log(1-\frac{j}{x}) < 0,$$

and so $g(n)$ is decreasing for $n > j$. But $g(n) \to e^{-j}$ as $n \to \infty$, so $g(n) > e^{-j}$ for each $n > j$.]

**Proof of Lemma 2.2** For a graph $G$ let frag$(G)$ be the number of vertices in $G$ less the number of vertices in a largest component; and for integers $n$ and $j$ with $1 \leq j < n$ let $f(n,j)$ be the number of forests $F$ on $[n]$ with frag$(F) = j$. By (11), for $1 \leq j < n/2$

$$\begin{aligned} f(n,j) &= \binom{n}{j} |\mathcal{F}_j| (n-j)^{n-j-2} \\ &= n^{n-2} \cdot \frac{|\mathcal{F}_j|}{j!} \cdot \frac{(n)_j}{n^j} (1 - \frac{j}{n})^{n-j-2} \\ &\geq n^{n-2} \cdot \frac{|\mathcal{F}_j|}{j!\, e^j} \cdot \frac{(n)_j}{n^j} (1 - \frac{j}{n})^{-2}. \end{aligned}$$

Now consider just $j \leq 2$ and let $n \geq 5$. Then $\frac{(n)_j}{n^j}(1 - \frac{j}{n})^{-2} \geq 1$, so

$$\frac{|\mathcal{F}_n|}{n^{n-2}} > \sum_{j=0}^{2} \frac{|\mathcal{F}_j|}{j!\, e^j} = 1 + \frac{1}{e} + \frac{2}{2!\, e^2} \approx 1.5032 \approx e^{0.4076}.$$

It is easy to check that this holds also for $n = 2, 3$ and 4; so

$$\mathbb{P}(F_n \text{ is connected}) < e^{-2/5} < e^{-1/3} \quad \text{for each } n \geq 2$$

as required. □

# 3 Concluding Remarks

We can improve on Lemma 2.2 by pushing the proof further.

**Lemma 3.1.** *If we set $\alpha = 0.48$ then for each $n = 2, 3, \ldots$*

$$\mathbb{P}(F_n \textit{ is connected}) < e^{-\alpha}.$$

*Proof.* It is easy to check that $\frac{(n)_j}{n^j}(1 - \frac{j}{n})^{-2} \geq 1$ for each $j \leq 6$ and $n > 12$. Hence, from the proof of Lemma 2.2, for $n > 12$

$$\frac{|\mathcal{F}_n|}{n^{n-2}} > \sum_{j=0}^{6} \frac{|\mathcal{F}_j|}{j!\, e^j} \approx 1.6167 \approx e^{0.4804} > e^{0.48},$$

and this holds also for $2 \leq n \leq 12$. To check this we may for example use [13] for $|\mathcal{F}_j|$ for $j \leq 12$. □

Lemma 3.1 allows us to strengthen Theorem 1.2 as follows: with the same premises, if we set $\alpha = 0.48$ then

$$\mathbb{P}(R_n \text{ is connected}) \geq \min_{\alpha n < t \leq n} \mathbb{P}(F_t \text{ is connected}). \tag{12}$$

It is well known (see for example Flajolet and Sedgewick [4] Section II.5.3) that $\sum_{j \geq 1} \frac{|\mathcal{T}_j|}{j!\, e^j} = \frac{1}{2}$ and so by the exponential formula $\sum_{j \geq 0} \frac{|\mathcal{F}_j|}{j!\, e^j} = e^{\frac{1}{2}}$. We could expect with more work to increase the value $\alpha = 0.48$ in (12) to nearer $\frac{1}{2}$ – but can we can go further?

(a) Perhaps $\mathbb{P}(F_n \text{ is connected}) \leq e^{-\frac{1}{2}}$ for each $n \geq 2$? In that case we could replace the value $\alpha = 0.48$ in (12) by the more satisfactory value $\frac{1}{2}$.

(b) Indeed, perhaps $\mathbb{P}(F_n \text{ is connected})$ is increasing for $n \geq 5$? In that case we could improve the bounds in Theorem 1.2 and in (12) to

$$\mathbb{P}(R_n \text{ is connected}) \geq \mathbb{P}(F_{\lceil n/2 \rceil} \text{ is connected}), \tag{13}$$

which is getting closer to Conjecture (3).

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