

COUNTING SMALLER ELEMENTS IN THE TAMARI AND m -TAMARI LATTICES

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ABSTRACT. We introduce new combinatorial objects, the interval-posets, that encode intervals of the Tamari lattice. We then find a combinatorial interpretation of the bilinear operator that appears in the functional equation of Tamari intervals described by Chapoton. Thus, we retrieve this functional equation and prove that the polynomial recursively computed from the bilinear operator on each tree T counts the number of trees smaller than T in the Tamari order.

Then we show that a similar $(m+1)$ -linear operator is also used in the functional equation of m -Tamari intervals. We explain how the m -Tamari lattices can be interpreted in terms of $(m+1)$ -ary trees or a certain class of binary trees. We then use the interval-posets to recover the functional equation of m -Tamari intervals and to prove a generalized formula that counts the number of elements smaller than or equal to a given tree in the m -Tamari lattice.

1. INTRODUCTION

The combinatorics of planar binary trees is known to have very interesting algebraic properties. Loday and Ronco first introduced the Hopf Algebra **PBT** based on these objects [12]. It was re-built by Hivert, Novelli and Thibon [10] through the introduction of the sylvester monoid. The structure of **PBT** involves a very nice object which is connected to both algebra and classical algorithmic: the *Tamari lattice*.

It was introduced by Tamari himself in 1962 as an order on formal bracketings [18] and was proved later to be a lattice [11]. It can be realized as a polytope called the associahedron. On binary trees, it can be described by a very common operation in algorithmic: the *right rotation* (see Figure 5). More generally, the cover relations of the Tamari order can be translated to many other combinatorial objects counted by Catalan numbers [16], like Dyck paths.

In this paper, we study the enumeration of the intervals of the Tamari lattice. Surprisingly, the number of intervals is given by a very beautiful

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formula

$$(1.1) \quad I_n = \frac{2}{n(n+1)} \binom{4n+1}{n-1},$$

where I_n is the number of intervals of the Tamari lattice of binary trees of size n . It was proven by Chapoton [8] using a functional equation on the generating function of the intervals. Very recently, Bergeron and Préville-Ratelle introduced a new set of lattices generalizing the Tamari lattice [5]. They are called the m -Tamari lattices and their elements are counted by the m -Catalan numbers. In this case also, one can obtain a formula counting the number intervals

$$(1.2) \quad I_{n,m} = \frac{m+1}{n(mn+1)} \binom{(m+1)^2n+m}{n-1}.$$

This was conjectured in [5] and proved in [7]. The proof also uses a functional equation that generalizes the classical case studied by Chapoton.

Here, we propose refined versions of both results by studying a new object that we call *interval-poset*. Each interval-poset corresponds to an interval of the Tamari lattice. To construct these objects, we use the strong relations between the Tamari order and the weak order on permutations. It has been known since Björner and Wachs [6] that linear extensions of a certain labelling of binary trees correspond to intervals of the weak order on permutations. This was more explicitly described in [10] with sylvester classes. The elements of the basis \mathbf{P} of \mathbf{PBT} are indexed by binary trees and defined as a sum on a sylvester class of elements of \mathbf{FQSym} . The \mathbf{PBT} algebra also admits two other bases \mathbf{H} and \mathbf{E} . An element of \mathbf{H} (resp. \mathbf{E}) is a sum of elements \mathbf{P}_T over an initial (resp. final) interval of Tamari lattice. They can be indexed by planar forests and, with a well chosen labelling, their linear extensions are intervals of the weak order on permutations corresponding to a reunion of sylvester classes. By combining the forests of the initials and finals intervals of two comparable trees in one single poset, we obtain what we call an interval-poset. Its linear extensions are exactly the sylvester classes corresponding to the interval in the weak order. This new object has nice combinatorial properties and allows to perform computations on Tamari intervals.

Thereby, we give a new proof of the formula of Chapoton (1.1). This proof is based on the study of a bilinear operator that already appeared in [8] but was not explored yet. It leads to the definition of a new family of polynomials:

Definition 1.1. Let T be a binary tree, the polynomial $\mathcal{B}_T(x)$ is recursively defined by

$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x \mathcal{B}_L(x) \frac{x \mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$

where L and R are respectively the left and right subtrees of T . We call $\mathcal{B}_T(x)$ the Tamari polynomial of T and the set of Tamari polynomials is the image of the map $T \mapsto \mathcal{B}_T(x)$.

This family of polynomials is yet unexplored in this context but seems to appear in a different computation made by Chapoton on rooted trees [9]. We give all polynomials for binary trees of size $n \leq 4$ in Figure 1. Our approach on Tamari interval-posets allows us to prove the following theorem in Section 3.3:

Theorem 1.2. Let T be a binary tree. Its Tamari polynomial $\mathcal{B}_T(x)$ counts the trees smaller than or equal to T in the Tamari order according to the number of nodes on their leftmost branch. In particular, $\mathcal{B}_T(1)$ is the number of trees smaller than or equal to T .

Symmetrically, if $\tilde{\mathcal{B}}_T$ is defined by exchanging the role of left and right children in Definition 1.1, then it counts the number of trees greater than or equal to T according to the number of nodes on their right border.

It was shown in [7] that the m -Tamari lattices can be seen as ideals of the Tamari lattice of size $n \times m$. Therefore, an interval of the m -Tamari lattice is an interval of Tamari which satisfies some conditions. This can be expressed in terms of interval-posets. Thus, it allows us to easily generalize our results to the m -Tamari case. We re-obtain the functional equation on the generating function described in [7] along with a generalization of Theorem 1.2 to count smaller elements in the m -Tamari lattices.

We first recall in Section 2 some definitions and properties of the Tamari lattice. We then introduce the notion of *interval-poset* to encode a Tamari interval. In Section 3, we show the implicit bilinear operator that appears in the functional equation of the generating functions of Tamari intervals. We then explain how interval-posets can be used to give a combinatorial interpretation of this bilinear operator and thereby give a new proof of the functional equation. Theorem 1.2 follows naturally. In Section 3.4, we discuss the similarity between Tamari polynomials and some bivariate polynomials that appeared in the context of flows of rooted trees [9].

Section 4 is dedicated to the study of the m -Tamari lattices defined in [5]. A functional equation for the intervals of these lattices is shown in [7] and contains a $m+1$ -linear operator that generalizes the binary case. In Section 4.2, we explain how the m -Tamari lattice can be seen on a certain class of binary trees which are in bijection with $(m+1)$ -ary trees. Thus, we are able to use again the interval-posets with a generalized combinatorial $m+1$ -operator to reobtain the functional equation of intervals of the m -Tamari order. We then prove Theorem 4.13, the generalization of Theorem 1.2 for the m -Tamari order.

$$\begin{array}{lll}
\mathcal{B}_{\nearrow}(x) = x^2 & \mathcal{B}_{\swarrow}(x) = x^4 & \mathcal{B}_{\searrow}(x) = x^4 + x^3 + x^2 \\
\mathcal{B}_{\setminus}(x) = x^2 + x & \mathcal{B}_{\swarrow}(x) = x^4 + x^3 & \mathcal{B}_{\wedge}(x) = x^4 + 2x^3 + 2x^2 \\
\mathcal{B}_{\nearrow}(x) = x^3 & \mathcal{B}_{\swarrow}(x) = x^4 + x^3 & \mathcal{B}_{\searrow}(x) = x^4 + x^3 + x^2 + x \\
\mathcal{B}_{\swarrow}(x) = x^3 + x^2 & \mathcal{B}_{\searrow}(x) = x^4 + x^3 + x^2 & \mathcal{B}_{\wedge}(x) = x^4 + 2x^3 + 2x^2 + 2x \\
\mathcal{B}_{\searrow}(x) = x^3 + x^2 + x & \mathcal{B}_{\swarrow}(x) = x^4 + 2x^3 + 2x^2 & \mathcal{B}_{\searrow}(x) = x^4 + 2x^3 + 2x^2 + 2x \\
\mathcal{B}_{\wedge}(x) = x^3 + x^2 & \mathcal{B}_{\swarrow}(x) = x^4 + x^3 & \mathcal{B}_{\searrow}(x) = x^4 + 2x^3 + 3x^2 + 3x \\
\mathcal{B}_{\setminus}(x) = x^3 + 2x^2 + 2x & \mathcal{B}_{\swarrow}(x) = x^4 + 2x^3 + x^2 & \mathcal{B}_{\setminus}(x) = x^4 + 3x^3 + 5x^2 + 5x
\end{array}$$

FIGURE 1. Tamari polynomials for binary trees of size $n \leq 4$.

2. INTERVAL-POSETS OF TAMARI LATTICE

2.1. The Tamari order on paths and binary trees. Originally, the Tamari lattice has been described on bracketing [18] but it is also commonly defined on Dyck paths.

Definition 2.1. *A Dyck path of size n is a lattice path from the origin $(0, 0)$ to the point $(2n, 0)$ made from a sequence of up steps $(1, 1)$ and down steps $(1, -1)$ such that the path stays above the line $y = 0$.*

A Dyck path can also be considered as a binary word by replacing up steps by the letter 1 and down steps by 0. We call a Dyck path *primitive* if it only touches the line $y = 0$ on its end points. A *rotation* consists of switching a down step d with the primitive Dyck path starting right after d , see Figure 2.

The Tamari order on Dyck paths is defined as the transitive and reflexive closure of the rotation operation: a path D' is greater than a path D if it can be obtained by applying a sequence of right rotation on D . It is indeed an order and even a lattice [11, 18]. See Figure 3 for the lattices on Dyck paths of sizes 3 and 4.

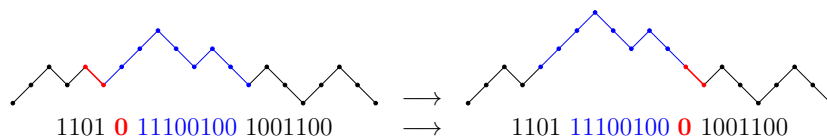


FIGURE 2. Rotation on Dyck Paths.

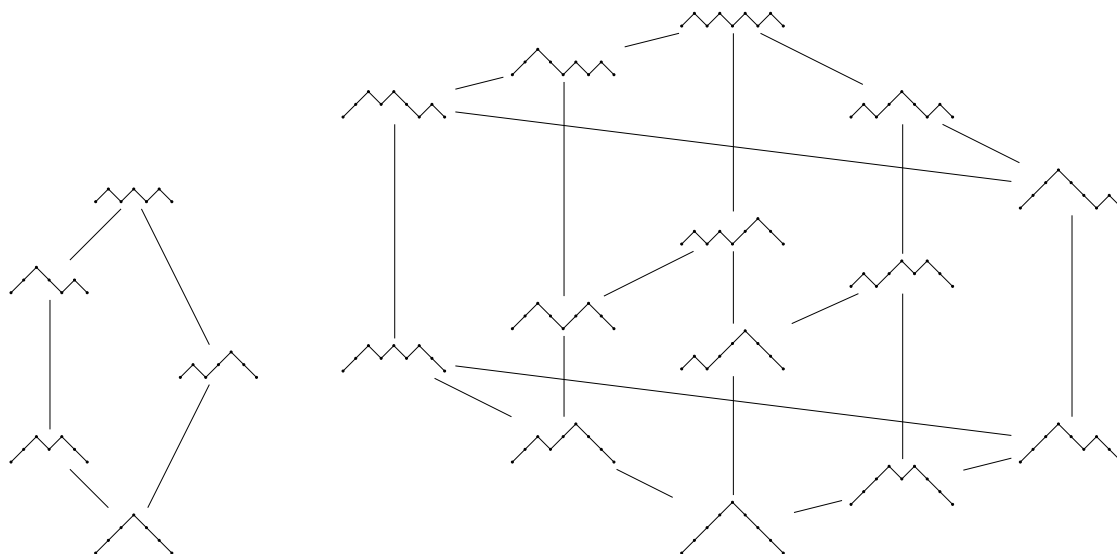


FIGURE 3. Tamari lattices of sizes 3 and 4 on Dyck paths.

A binary tree is recursively defined by being either the empty tree (\emptyset) or a pair of binary trees, respectively called *left* and *right* subtrees, grafted on an internal node¹. If a tree T is composed of a root node x with A and B as respectively left and right subtrees, we write $T = x(A, B)$. The number of nodes of a tree T is called the size of T .

There are many ways to define a bijection between Dyck paths and binary trees. The one we use here is the only one which is consistent with the usual definition of the Tamari order on binary trees through the right rotation (see Definition 2.2 later). Similarly to a binary tree, a Dyck path can be seen as a recursive binary object: it is either an empty path or a word $D_1 1 D_2 0$ where D_1 and D_2 are two Dyck paths (potentially empty ones). The subpath D_1 corresponds to the left factor of D up to the last touching point of D before the end. Consequently, if D is primitive, D_1 is empty. If both D_1 and D_2 are empty, then D is the only dyck path of size 1: the word 10. We define recursively

¹Note that what we call binary tree is actually a planar binary tree. All binary trees consider in this paper are planar, *i.e.*, the subtrees are ordered.

the binary tree T corresponding to D . If D is the empty word, then T is the empty tree. Otherwise, T is a binary tree whose left subtree (resp. right subtree) corresponds to D_1 (resp. D_2). See Figure 4 for an example of the bijection.

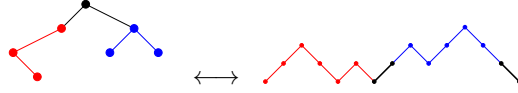


FIGURE 4. Bijection between Dyck paths and binary trees.

Through this bijection, the rotation on Dyck paths can be interpreted directly in terms of binary trees through an operation called the *right rotation*. This is a well known operation on binary trees, used in many different contexts, especially sorting algorithms [4].

Definition 2.2. Let y be a node of T with a non-empty left subtree x . The right rotation of T on y is a local rewriting which follows Figure 5, that is replacing $y(x(A, B), C)$ by $x(A, y(B, C))$ (note that A , B , or C might be empty).

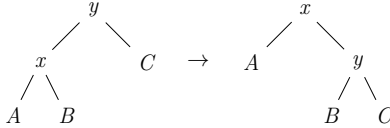


FIGURE 5. Right rotation on a binary tree.

The right rotation is then the cover relation of the Tamari order on binary trees, as illustrated in Figure 6.

2.2. Relation with the weak order. An interesting property is the relation between the Tamari lattice and the weak order on permutations. Indeed, the Tamari lattice is a sublattice of the right weak order: it can be induced from it by choosing the proper subset of permutations. It is also a quotient lattice: one can define a relation on permutations given by a surjective map to binary trees. The quotient lattice of the right weak order by this relation gives the Tamari lattice. These results are originally due to Tonks [19]. They are also explained in [10]. In this paper, we use the combinatorial constructions of the latter that we recall now. They are based on a very classical object in computer science: *binary search trees*.

Definition 2.3. A binary search tree is a labelled binary tree where for each node of label k , any label in its left (resp. right) subtree is smaller than or equal to (resp. larger than) k .

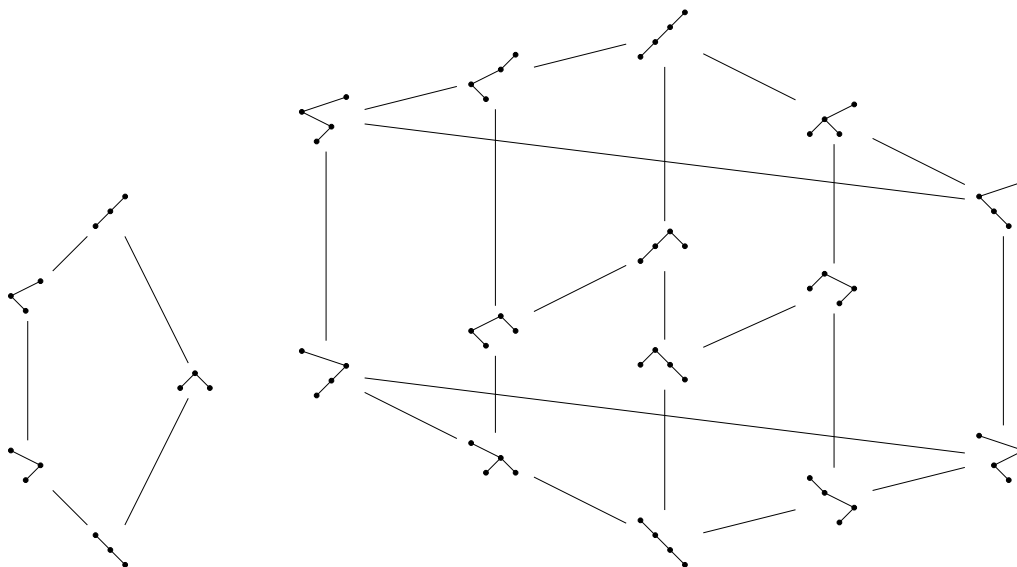
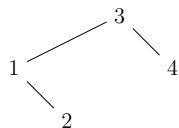


FIGURE 6. Tamari lattice of sizes 3 and 4 on binary trees.

Note that, in general, binary search trees are labelled by any set of numbers, allowing repetitions. However, we will only consider binary search trees with distinct labels. Figure 8 shows an example of such a tree. There is only one way to label a binary tree of size n with distinct labels $1, \dots, n$ to make it a binary search tree. We call this the *binary search tree labelling* of the tree and often identify the two objects. Such a labelled tree can be interpreted as a poset. The order relation, denoted \triangleleft^2 , is defined by $x \triangleleft y$ if and only if x is in the subtree whose root is y . For example, the tree



is the poset where $2 \triangleleft 1 \triangleleft 3$ and $4 \triangleleft 3$. A linear extension of this poset is a permutation of the labels of the tree where for all labels a and b , if $a \triangleleft b$ in the poset, then the number a is before b in the permutation. For example, 4213, 2413, and 2143 are the three linear extensions of the above tree. The permutation 1423 is not because 1 appears before 2 whereas $2 \triangleleft 1$. The set of linear extensions of a given tree is called the *sylvester class* of the tree: it forms an interval of the right weak

²We use the notation \triangleleft for all posets of integers to differentiate with the natural order on integers. When necessary, we index the notation by the name of the object. If T is a tree (or a forest or a poset), \triangleleft_T is the order relation given by the tree T .

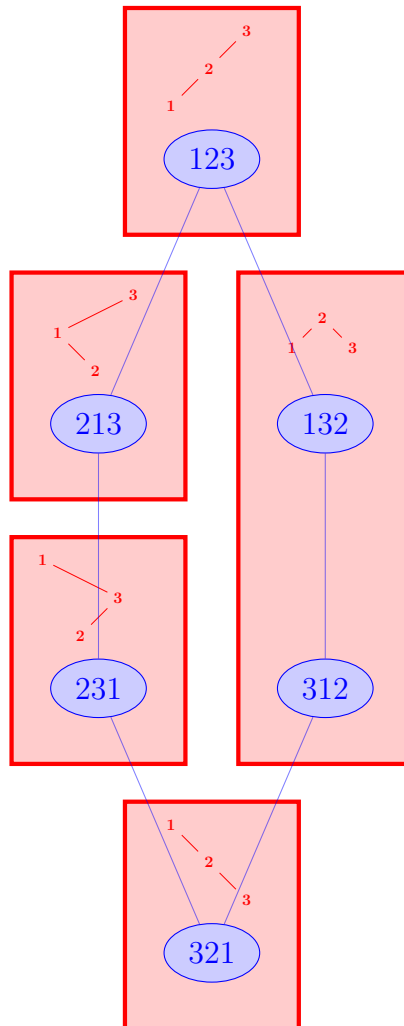
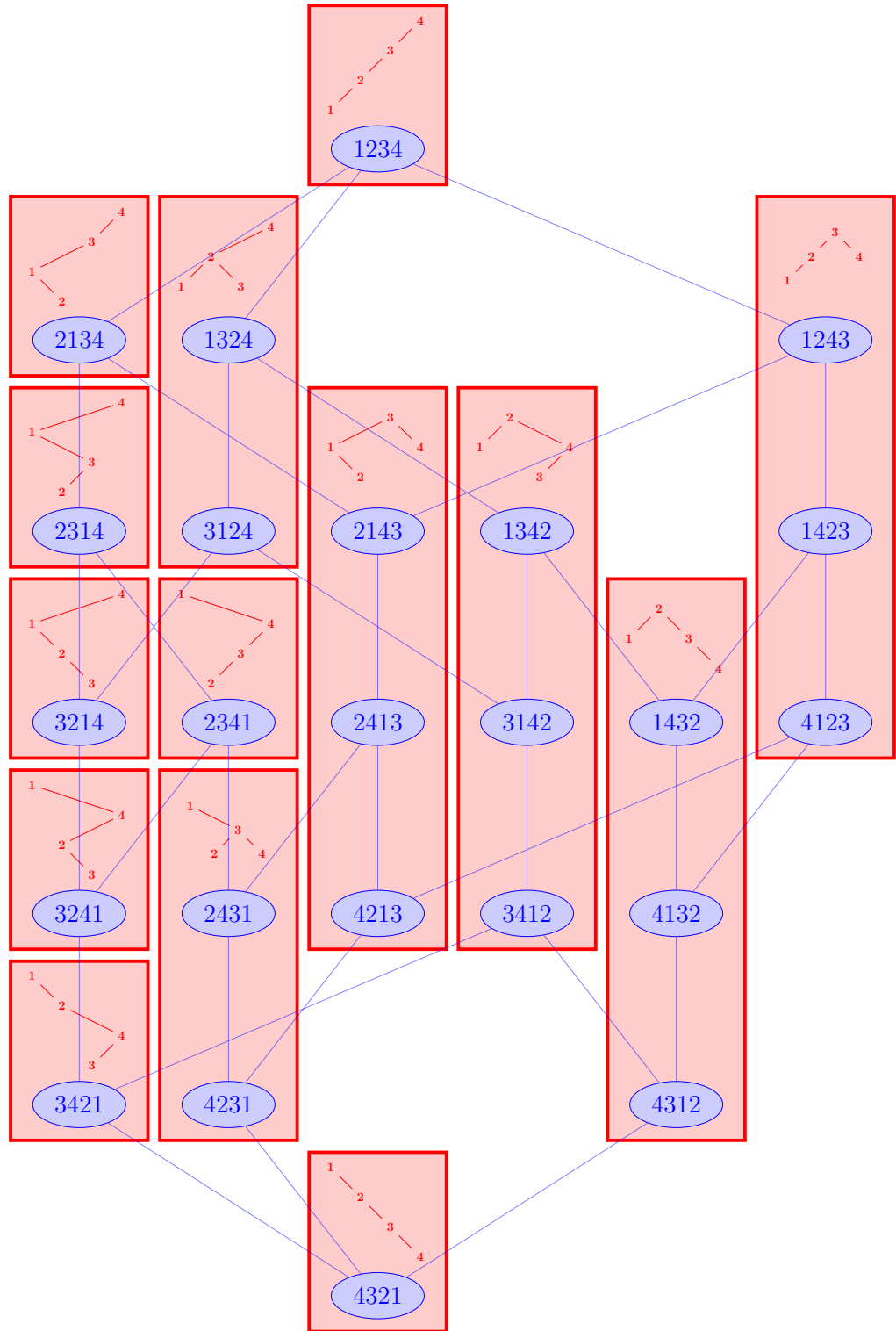


FIGURE 7. The Tamari order of size 3 as $\text{ad } 4$ a quotient of the weak order.



order as illustrated in Figure 8. Indeed, the set of linear extensions of a tree can be computed recursively using the concatenation and shuffle product which are stable operations on intervals of the weak order, for example, in Figure 8 the set of linear extensions is given by $(21 \sqcup 4).3$. The sylvester classes of the binary trees of size n form a partition of \mathfrak{S}_n . The ordering between classes is well defined as a quotient order of the weak order and it corresponds to the Tamari order on binary trees [10, 19]. This is illustrated by Figure 7: two binary trees T_1 and T_2 are such that $T_1 \leq T_2$ if and only if there exists two linear extensions σ_1 and σ_2 of respectively T_1 and T_2 such that $\sigma_1 \leq \sigma_2$ for the right weak order.

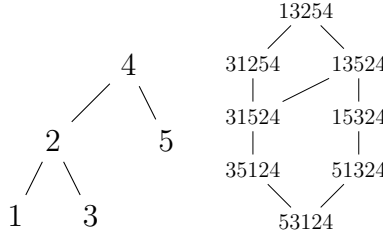


FIGURE 8. A binary search tree and its corresponding sylvester class.

2.3. Construction of interval-posets. We now introduce more general objects: *interval-posets* in bijection with the intervals of the Tamari order. Let us first recall two bijections between binary search trees and forests of planar trees. We say that a binary search tree has an *increasing* relation between a and b if $a < b$ and $a \triangleleft b$, which means a is in the left subtree of b . Symmetrically, a binary search tree has a *decreasing* relation if $a < b$ and $b \triangleleft a$, i.e., b is in the right subtree of a . From a binary search tree T , one can construct a poset containing only increasing (resp. decreasing) relations of T . These posets are actually forests, we call them the *initial* and *final* forest of the binary tree.

Definition 2.4. *The initial forest of a binary search tree T , denoted $F_{\leq}(T)$ or simply F_{\leq} when there is no ambiguity, is a forest poset on the nodes of T constructed by keeping only increasing relations of T , i.e.:*

$$a \triangleleft_{F_{\leq}} b \text{ if, and only if } a < b \text{ and } a \triangleleft_T b.$$

It is equivalent to the following construction, for all nodes x of T :

- if y is the left son of x in T , then y is the first son of x in $F_{\leq}(T)$,

- if y is the right son of x in T , then y is the right brother of x in $F_{\leq}(T)$, i.e. if x is the i^{th} son of its parent node in F then y is the $(i + 1)^{\text{th}}$ son (we consider that all root nodes of F have a common parent node).

In the same way, one can define the final forest (denoted F_{\geq}) by reversing the roles of the right and left son in the previous construction or, in terms of posets:

$$b \triangleleft_{F_{\geq}} a \text{ if, and only if } b < a \text{ and } b \triangleleft_T a.$$

In our definition, F_{\leq} and F_{\geq} are directly defined as posets. They actually correspond to some labelled planar forests. Indeed, our constructions are the translation of some well-known bijections between unlabelled binary trees and unlabelled planar forest. The labellings we obtain on F_{\geq} and F_{\leq} are canonical. For example, for F_{\leq} it corresponds to the recursive traversal of each root from left to right: recursively label the subtrees from left to right then label the root. As both are bijections, the tree T can be recursively retrieved from one forest among $F_{\leq}(T)$ and $F_{\geq}(T)$. On the initial forest, the root of the corresponding binary tree is the left-most (minimal) root of the forest trees. The left subtree is obtained recursively from the subtrees of the root and the right subtree from the remaining trees of the forest. The construction is symmetric for the final forest. Both bijections are illustrated in Figure 9. As a convention, initial forest are always written in blue with trees oriented from left to right and final forests are in red with trees oriented from bottom to top.

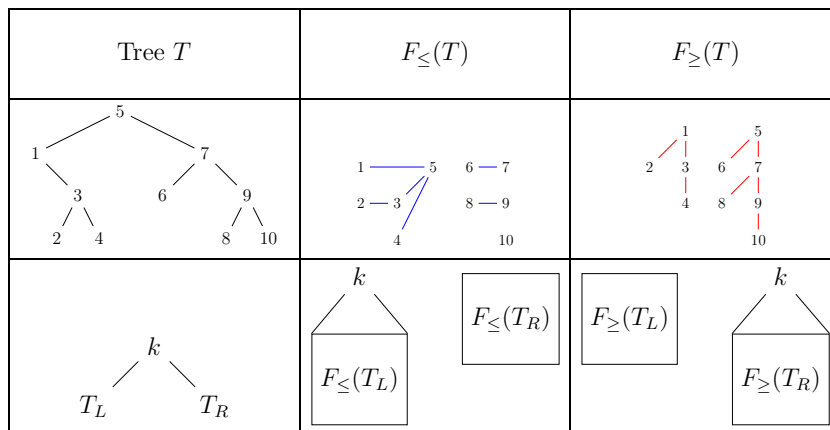


FIGURE 9. A tree with its corresponding initial and final forests.

Now, the question we want to answer is: if P is a poset labelled with distinct integers $1, \dots, n$, is it the initial (or final) forest of some

binary tree? The following lemma gives the the sufficient and necessary conditions for this to happen.

Lemma 2.5. *Let F be a labelled poset. Then F is the initial forest of a binary tree T if for every $a \triangleleft_F c$, we have $a < c$ and $b \triangleleft_F c$ for every b such that $a < b < c$. Symmetrically, F is the final forest of a binary tree T if for every $c \triangleleft_F a$ we have $c > a$ and $b \triangleleft_F a$ for every b such that $a < b < c$.*

Proof. The final and initial forests cases are symmetric, we only give the proof of the final forest case.

First let us proof that the final forest $F := F_{\geq}(T)$ of a binary tree T satisfies the necessary condition. Let $c > a$ be such that $c \triangleleft_F a$. By construction of F , we also have $c \triangleleft_T a$ which means that c is in the right subtree of a . Let b be such that $a < b < c$. Only three configurations are possible: either a is in the left subtree of b , or a and b are not comparable in T , or b is in the right subtree of a . The first configuration never happens because it implies that c is in the left subtree of b which contradicts the binary search tree condition. Then, if a and b are not comparable, it means they have a common root b' with $a < b' < b$ and a is in the left subtree of b' . The situation is then similar to the previous one and leads to a contradiction as c is also in the left subtree of b' . Only the third configuration is possible which makes $b \triangleleft_T a$ and by construction $b \triangleleft_F a$.

Now, let F be a labelled poset satisfying the condition of the final forest. The poset F is made of r connected components F_1, \dots, F_r . For each F_i , there is a unique minimal poset element x_i : $y \triangleleft_F x_i$ for all $y \in F_i$. We call it the root of F_i . Indeed, if x, x' , and y are in F_i with $y \triangleleft_F x$ and $y \triangleleft_F x'$ then either $x < x' < y$ and $x' \triangleleft_F x$ or $x' < x < y$ and $x \triangleleft_F x'$. As all relations of F are decreasing relations, x_i is also the minimal label of F_i : $y > x_i$ for all $y \in F_i$. Furthermore, if x_i and x_j are the roots of two different components F_i and F_j then $x_i < x_j$ implies $y < z$ for all $y \in F_i$ and $z \in F_j$. Now, following the construction described by Figure 9, we set k to be the maximal label among the roots x_1, \dots, x_r . If we cut out the root k from its connected component, the remaining poset F_L still satisfies the condition and all its labels are bigger than k . The poset F_R made from the other remaining connected components also satisfy the condition and all its label are smaller than k . Then we can recursively construct the binary tree $T := k(T_L, T_R)$ where T_L and T_R are obtained from respectively F_L and F_R . By construction, T is a binary search tree and $F = F_{\geq}(T)$. \square

We have seen that the linear extensions of a binary tree T form an interval of the right weak order. The linear extensions of the initial and final forests of T correspond to initial and final intervals [6] and can be interpreted in terms of the Tamari order.

Proposition 2.6. *The linear extensions of the initial forest $F_{\leq}(T)$ of a binary tree T are the sylvester classes of all trees $T' \leq T$ in the Tamari order (initial interval) and the linear extensions of the final forest $F_{\geq}(T)$ of T are the sylvester classes of all trees $T' \geq T$ (final interval).*

Proof. We only give the proof for $F_{\geq}(T)$. By symmetry of the right weak order and the Tamari order, it also proves the result for $F_{\leq}(T)$. Let α_T be the minimal element of the sylvester class of T . We want to prove that the linear extensions of $F_{\geq}(T)$ are exactly the interval $[\alpha_T, \omega]$ where ω is the maximal element of the right weak order. Since the Tamari order is a quotient of the right weak order, the Proposition is entirely proved by this result.

Let us recall that a coinversion (a, b) of a permutation σ is couple of numbers such that $a < b$ and b appears before a in σ . As an example, $(1, 4)$ is a coinversion of 2431 as well as $(1, 2)$, $(3, 4)$ and $(1, 3)$. We have that $\mu \leq \sigma$ in the right weak order if and only if the coinversions of μ are contained in the coinversions of σ . For the previous example, the permutation $\mu = 2314$ is smaller than σ because its coinversions $(1, 2)$ and $(1, 3)$ are also coinversions of σ .

The linear extensions of $F_{\geq}(T)$ are the permutations containing all coinversions (a, b) where $b \triangleleft_{F_{\geq}} a$. It is clear by construction that a linear extension of $F_{\geq}(T)$ contains these coinversions. It is also a sufficient condition. Indeed let σ be a permutation that is not a linear extension of $F_{\geq}(T)$. Then there is (a, b) with $b \triangleleft_{F_{\geq}} a$ and a before b in σ . The permutation σ does not contain the coinversion (a, b) .

Finally, the permutation α_T contains exactly the coinversions given by the $F_{\geq}(T)$ relations (it does not contain other coinversions). Indeed, it is known [10] that α_T is read on the binary search tree by a recursive printing: left subtree, right subtree, root. Let $b > a$ be such that $F_{\geq}(T)$ does not contain the relation $b \triangleleft_{F_{\geq}} a$. It means b is not on the right subtree of a . There are only two possible configurations: either a is on the left subtree of b , either they have a common root b' and a is on the left subtree of b' and b on the right subtree of b' . In both cases, a is read before b in α_T and then α_T does not contain the coinversion (a, b) .

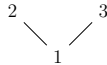
To conclude, the linear extensions of $F_{\geq}(T)$ are the permutations whose coinversions contain the coinversions of α_T . In other words, they are the permutations greater than or equal to α_T . \square

If two trees T and T' are such that $T \leq T'$, then $F_{\geq}(T)$ and $F_{\leq}(T')$ share some linear extensions (by Proposition 2.6). More precisely, we denote by $\text{ExtL}(F)$ the set of linear extensions of a poset F . Then we have $\text{ExtL}(F_{\geq}(T)) \cap \text{ExtL}(F_{\leq}(T')) = [\alpha_T, \omega_{T'}]$ where α_T (resp. $\omega_{T'}$) is the minimal permutation (resp. maximal permutation) of the sylvester class of T (resp. T'). This set corresponds exactly to the linear extensions of the trees of the interval $[T, T']$ in the Tamari order. It is then natural to construct a poset that would contain relations of both $F_{\geq}(T)$ and $F_{\leq}(T')$. That is what we call an *interval-poset*. We give a first example in Figure 10. Note that unlike $F_{\geq}(T)$ and $F_{\leq}(T')$, the interval-poset formed by the reunion of their relations is not necessary a forest itself. The characterisation of interval-posets follows naturally from Lemma 2.5.

Definition 2.7. *An interval-poset P is a poset labelled with distinct integers $\{1, \dots, n\}$ such that the following conditions hold:*

- $a \triangleleft_P c$ and $a < c$ implies that for all $a < b < c$, we have $b \triangleleft_P c$,
- $c \triangleleft_P a$ and $a < c$ implies that for all $a < b < c$, we have $b \triangleleft_P a$.

As an example, poset



is not an interval-poset. Indeed, we have $1 \triangleleft 3$ without $2 \triangleleft 3$ so it does not satisfy the second condition of the definition. An example of an interval-poset is given in Figure 10. By convention, even though an interval-poset is by definition a poset we do not represent it by its Hasse diagram. For clarity, we draw the union of the Hasse diagrams formed respectively by its increasing relations (in blue) and decreasing relations (in red).

Theorem 2.8. *Interval-posets are in bijection with intervals of the Tamari order.*

More precisely, to each interval-poset corresponds a couple of binary trees $T_1 \leq T_2$ such that the linear extensions of the interval-poset are exactly the linear extensions of the binary trees $T' \in [T_1, T_2]$.

And conversely, interval-posets are the only labelled posets whose linear extensions are intervals of the right weak order $[\alpha_{T_1}, \omega_{T_2}]$ with α_{T_1} (resp. ω_{T_2}) the minimal permutation (resp. maximal permutation) of a sylvester class.

Proof. Let $[T_1, T_2]$ be an interval of the Tamari order. We build a poset containing all the relations from both $F_{\geq}(T_1)$ and $F_{\leq}(T_2)$. Note that relations from $F_{\geq}(T_1)$ and $F_{\leq}(T_2)$ together can never produce a

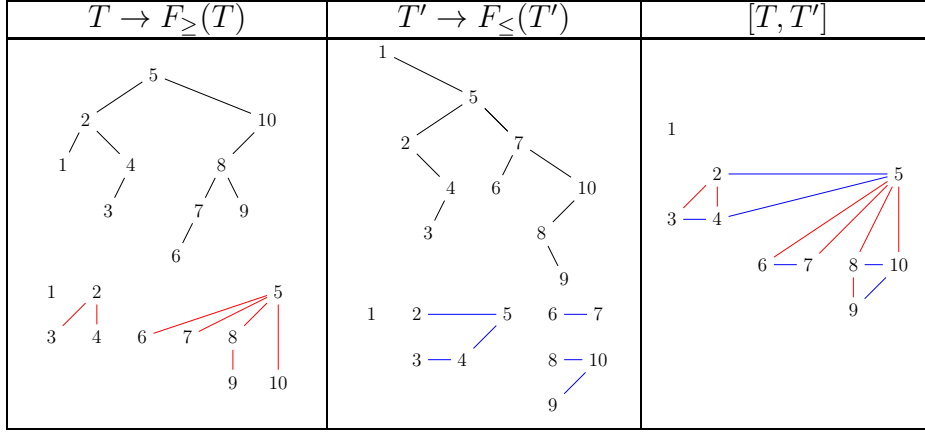


FIGURE 10. Two trees T and T' with $T < T'$, their final and initial forest and the interval-poset $[T, T']$. This Tamari interval is shown in Figure 11.

cycle. Indeed any linear extension of T_1 for example satisfies both by Proposition 2.6. It is clear by Lemma 2.5 that the resulting poset is an interval-poset.

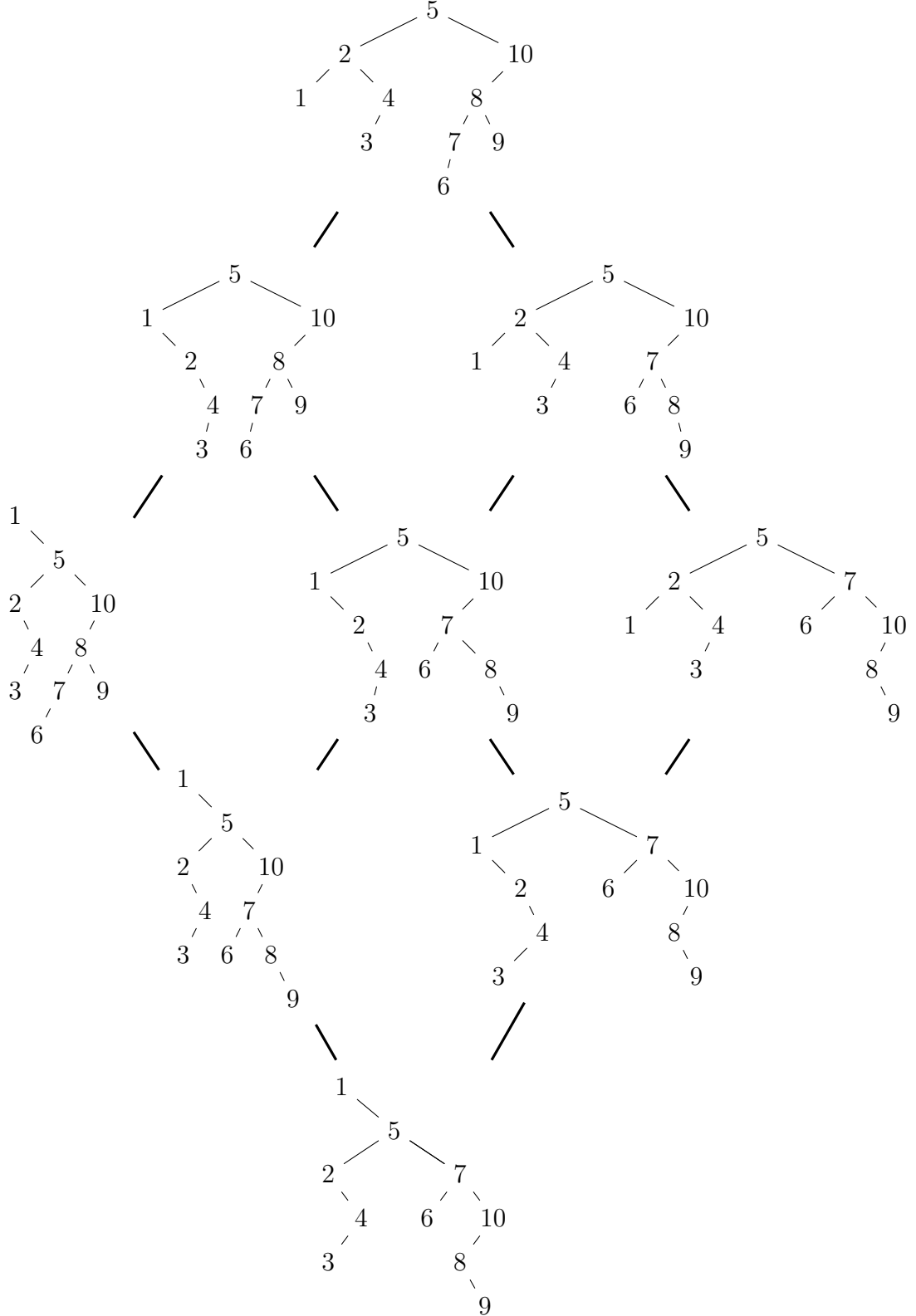
Conversely, from an interval-poset P , we build F_{\geq} and F_{\leq} by keeping respectively decreasing and increasing relations of P . By Lemma 2.5, the two resulting posets are respectively a final forest of a binary tree T_1 and an initial forest of a binary tree T_2 . Let σ be a linear extension of P whose sylvester class corresponds to a binary tree T' . By definition, the permutation σ is also a linear extension of F_{\geq} and F_{\leq} and we have by Proposition 2.6 that $T_1 \leq T' \leq T_2$. As $T_1 \leq T_2$, the interval $[T_1, T_2]$ is well defined. \square

Many operations on intervals can be easily adapted on interval-posets, all with trivial proofs.

Proposition 2.9. (i) *The intersection between two intervals I_1 and I_2 is given by the union of their relations I_3 . If I_3 is a valid poset, i.e., there is contradictions between the relations of I_1 and I_2 , then I_3 is an interval-poset, otherwise, the intersection is empty.*

(ii) *An interval $I_1 := [T_1, T'_1]$ is contained in an interval $I_2 := [T_2, T'_2]$, i.e., $T_1 \geq T_2$ and $T'_1 \leq T'_2$, if and only if all relations of the interval-poset I_1 are satisfied by the interval-poset I_2 .*

(iii) *If $I_1 := [T_1, T'_1]$ is an interval, then $I_2 = [T_2, T'_1]$, $T_2 \geq T_1$, if and only if all relations of the interval-poset I_1 are satisfied by I_2 and all new relations of I_2 are decreasing. Symmetrically, $I_3 =$*

FIGURE 11. The interval between the trees T and T' of Figure 10.

$[T_1, T_3], T_3 \leq T'_1$, if and only if all relations of the interval-poset I_1 are satisfied by I_3 and all new relations of I_3 are increasing.

The interval-poset consisting of the set of points $1, \dots, n$ without any relations corresponds to the whole Tamari lattice and the linear extensions are all permutations of size n . More generally, let I be the interval-poset of an interval $[T, T']$. If \tilde{T} is the binary tree T where a right rotation has been applied on a couple of nodes $x < y$, then \tilde{I} is the interval-poset I where the decreasing relation $y \triangleleft x$ has been added. Symmetrically, applying a left rotation on a couple of nodes $y > x$ on T' corresponds to adding the increasing relation $x \triangleleft y$. This is illustrated in Figure 12.

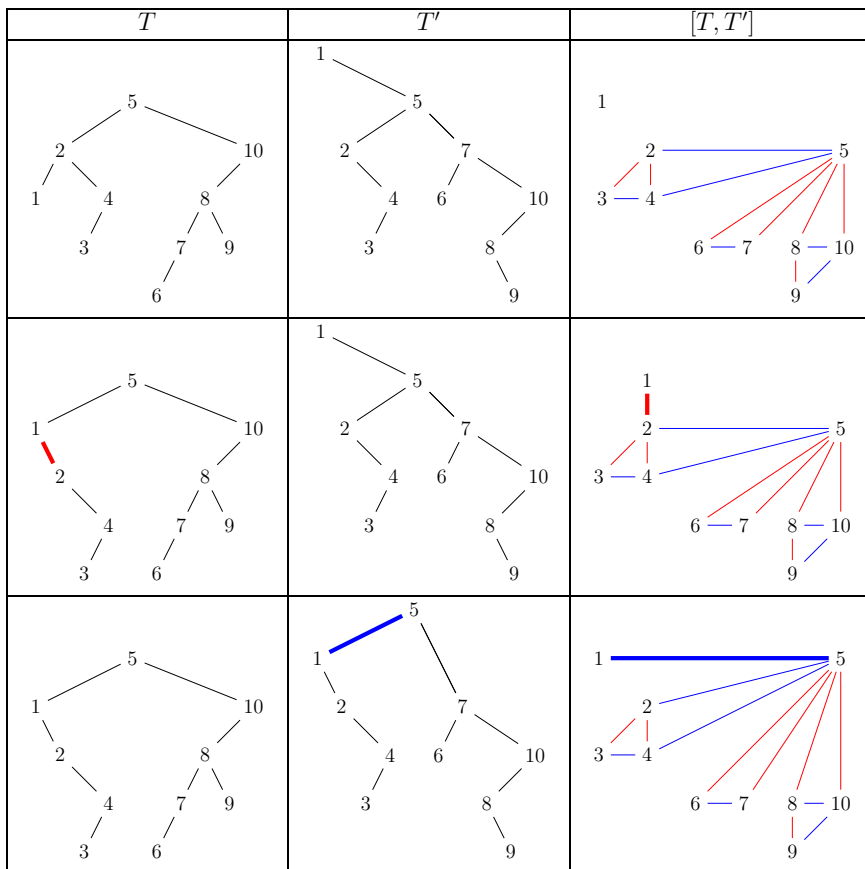


FIGURE 12. Adding relations on interval-posets: adding a decreasing relation to the poset makes a right rotation on T and adding an increasing relation makes a left rotation on T' .

As posets satisfying simple properties, interval-posets are easy to implement in any computer algebra system. Thereby they allow for computer exploration on the combinatorics of Tamari intervals. In this purpose, we decided to integrate them into the mathematical software Sage [17]. They were developed in [14] and are available since Sage 6.3. Especially, all basic operations such as the ones of Proposition 2.9 are implemented. The details of the different functionalities are given in the documentation [14], we give some basic examples in Appendix A.1.

3. TAMARI POLYNOMIALS

3.1. Composition of interval-posets. Let $\phi(y)$ be the generating function of Tamari intervals,

$$(3.1) \quad \phi(y) := \sum_{n \geq 0} I_n y^n$$

where I_n is the number of intervals of trees of size n , equivalently this is the number of interval-posets with n vertices. The first values of I_n are given in [3]

$$(3.2) \quad \phi(y) = 1 + y + 3y^2 + 13y^3 + 68y^4 + \dots$$

In [8], Chapoton gives a refined version of ϕ ,

$$(3.3) \quad \Phi(x, y) := \sum_{n, m \geq 0} I_{n, m} x^m y^n$$

where $I_{n, m}$ is the number of intervals $[T_1, T_2]$ of trees of size n such that T_1 has exactly m nodes on its leftmost branch. This gives

$$(3.4) \quad \Phi(x, y) = 1 + xy + (x + 2x^2)y^2 + (3x + 5x^2 + 5x^3)y^3 + \dots$$

The statistic of the number of nodes on the leftmost branch is well known [1]. On Dyck paths, it corresponds to the number of touch points: the number of contacts between the path and the bottom line [2]. It can also be read on $F_{\geq}(T)$: it is the number of connected components.

Definition 3.1. Let $[T_1, T_2]$ be an interval and I its interval-poset, we denote by

- (1) $\text{size}(I)$ the number of vertices of I , i.e., the size of the trees T_1 and T_2 .
- (2) $\text{trees}(I)$ the number of connected components (or trees) of $F_{\geq}(I)$, the poset obtained by keeping only decreasing relations of I .

We then define $\mathcal{P}(I) := x^{\text{trees}(I)} y^{\text{size}(I)}$ which we extend by linearity to linear combinations of interval-posets.

The refined generating functions Φ can be expressed as

$$(3.5) \quad \Phi(x, y) = \sum_I \mathcal{P}(I)$$

summed on all interval-posets. We prove the following theorem.

Theorem 3.2. *The generating function $\Phi(x, y)$ satisfies the functional equation*

$$(3.6) \quad \Phi(x, y) = \mathbb{B}(\Phi, \Phi) + 1$$

where

$$(3.7) \quad \mathbb{B}(f, g) := xyf(x, y) \frac{xg(x, y) - g(1, y)}{x - 1}.$$

This theorem was already proven by Chapoton in [8]³. In Section 3.2, we give a new proof of the theorem based on a combinatorial interpretation of the operator \mathbb{B} . Let us define now what we call the *composition* of interval-posets.

Definition 3.3. *Let I_1 and I_2 be two interval-posets of size respectively k_1 and k_2 . Then $\mathbb{B}(I_1, I_2)$ is the formal sum of all interval-posets of size $k_1 + k_2 + 1$ where,*

- (i) *the relations between vertices $1, \dots, k_1$ are exactly the ones from I_1 ,*
- (ii) *the relations between $k_1 + 2, \dots, k_1 + k_2 + 1$ are exactly the ones from I_2 shifted by $k_1 + 1$,*
- (iii) *we have $i \triangleleft k_1 + 1$ for all $i \leq k_1$,*
- (iv) *there is no relation $k_1 + 1 \triangleleft j$ for all $j > k_1 + 1$.*

We call this operation the composition of intervals and extend it by bilinearity to all linear sums of intervals.

The sum we obtain by composing interval-posets actually corresponds to all possible ways of adding decreasing relations between the second poset and the new vertex $k_1 + 1$, as seen in Figure 13. Especially, there is no relations between vertices $1, \dots, k_1$ and $k_1 + 2, \dots, k_1 + k_2 + 1$. Indeed, condition (iii) makes it impossible to have any relation $j \triangleleft i$ with $i < k_1 + 1 < j$ as this would imply by Definition 2.7 that $k_1 + 1 \triangleleft i$. And condition (iv) makes it impossible to have $i \triangleleft j$ as this would imply $k_1 + 1 \triangleleft j$.

The number of elements in the sum is given by $\text{trees}(I_2) + 1$. Indeed, if $x_1 < x_2 < \dots < x_m$ are the tree roots of $F_{\geq}(I_2)$, a decreasing relation

³Our equation is slightly different from the one of [8, formula (6)]. Indeed, the definition of the degree of x differs by one and in our case Φ also counts the interval of size 0.

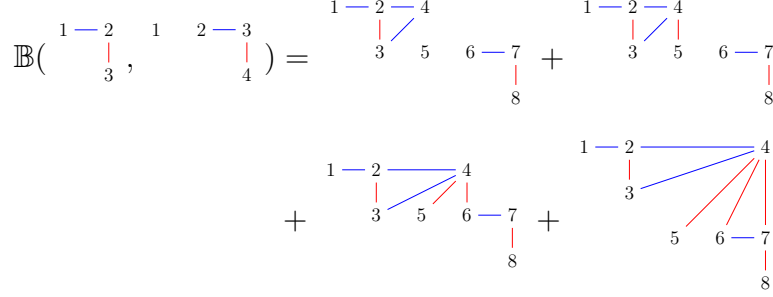


FIGURE 13. Composition of interval-posets: the four terms of the sum are obtained by adding respectively no, 1, 2, and 3 decreasing relations between the second poset and the vertex 4.

$x_i \triangleleft k_1 + 1$ can be added only if all relations $x_j \triangleleft k_1 + 1$ for $j < i$ have already been added. We then obtain

$$(3.8) \quad \mathbb{B}(I_1, I_2) = \sum_{0 \leq i \leq m} P_i$$

where P_i is the interval-poset where exactly i relations have been added: $x_j \triangleleft k_1 + 1$ for $j \leq i$.

Proposition 3.4. *Let I_1 be an interval-poset of size k_1 with $[T_1, T'_1]$ as corresponding interval and I_2 an interval-poset of size k_2 with $[T_2, T'_2]$ as corresponding interval. We set $k := k_1 + 1$ and*

- (1) Q_α , the binary tree obtained by grafting $k(T_1, \emptyset)$ on the left of the leftmost node of T_2 ,
- (2) Q_ω , the binary tree $k(T_1, T_2)$,
- (3) Q' , the binary tree $k(T'_1, T'_2)$.

Then we have

$$(3.9) \quad \mathbb{B}(I_1, I_2) = \sum_{Q \in [Q_\alpha, Q_\omega]} P_{[Q, Q']}$$

where $P_{[Q, Q']}$ is the interval-poset of $[Q, Q']$.

Proof. The composition of I_1 and I_2 is a sum of interval-posets P_0, \dots, P_m where $m = \text{trees}(I_2)$ and where P_i is the interval-poset where exactly i decreasing relations have been added. The maximal tree of all intervals is always the same as they all share the same increasing relations. This maximal tree is $Q' := k(T'_1, T'_2)$. The final forest $F_{\geq}(P_0)$ of P_0 contains $\text{trees}(I_1) + \text{trees}(I_2) + 1$ connected components. The nodes on the leftmost branch of its minimal tree are given by those of T_1 , then k , then

Interval-poset	Corresponding interval

FIGURE 14. Interval interpretation of the composition of interval-posets.

those of T_2 , *i.e.* this is exactly Q_α . Let Q_i be the minimal tree of P_i . To go from P_i to P_{i+1} , a decreasing relation is added to the vertex k : this corresponds to a rotation between the node k of Q_i and its parent node. This process ends when T_2 has been completely switched to the right side of the node k . We then obtain the tree $Q_m = Q_\omega$.

The interval between Q_α and Q_ω is actually a saturated chain: $Q_\alpha = Q_0 \leq Q_1 \leq \dots \leq Q_m = Q_\omega$. \square

As an example, the interpretation of the computation in Figure 13 in terms of intervals is given in Figure 14.

The composition of intervals-posets can be decomposed into two different operations: a left product $\vec{\bullet}$ and a right product $\overleftarrow{\delta}$.

Definition 3.5. Let I_1 and I_2 be two interval-posets such that $\text{trees}(I_2) = m$ with $x_1 < x_2 < \dots < x_m$ the roots of the trees of $F_{\geq}(I_2)$. Let α and ω be respectively the label of minimal value of I_2 and the label of maximal value of I_1 . Then

- (1) $I_1 \vec{\bullet} I_2$ is the interval-poset obtained by a shifted concatenation of I_1 and I_2 with the increasing relations $y \triangleleft \alpha$ for all $y \in I_1$.
- (2) $I_1 \overleftarrow{\delta} I_2$ is the sum of the $m + 1$ interval-posets P_0, P_1, \dots, P_m where P_i is the shifted concatenation of I_1 and I_2 with exactly i added decreasing relations: $x_j \triangleleft \omega$ for $j \leq i$.

As an example,

$$\begin{array}{c} 1 \\ | \\ 2-3 \end{array} \vec{\bullet} \begin{array}{c} 1 \ 2 \\ | \\ 3 \end{array} = \begin{array}{c} 1 \text{---} 4 \ 5 \\ | \quad / \\ 2-3 \quad 6 \end{array}$$

$$\begin{array}{c} 1 \\ | \\ 2-3 \end{array} \overleftarrow{\delta} \begin{array}{c} 1 \ 2 \\ | \\ 3 \end{array} = \begin{array}{c} 1 \\ | \\ 2-3 \\ 4 \end{array} \begin{array}{c} 5 \\ | \\ 6 \end{array} + \begin{array}{c} 1 \\ | \\ 2-3 \\ 4 \end{array} \begin{array}{c} 5 \\ | \\ 6 \end{array} + \begin{array}{c} 1 \text{---} 3 \\ | \quad / \\ 4 \quad 5 \\ | \\ 6 \end{array}.$$

From the description of the composition given by (3.8), it is clear that

$$(3.10) \quad \mathbb{B}(I_1, I_2) = I_1 \vec{\bullet} u \overleftarrow{\delta} I_2$$

where u is the interval-poset with a single vertex. For example, the composition of Figure 13 reads

$$\mathbb{B}\left(\begin{array}{c} 1-2 \\ | \\ 3 \end{array}, \begin{array}{c} 1 \ 2-3 \\ | \\ 4 \end{array}\right) = \begin{array}{c} 1-2 \\ | \\ 3 \end{array} \vec{\bullet}_1 \overleftarrow{\delta} \begin{array}{c} 1 \ 2-3 \\ | \\ 4 \end{array}.$$

The order on the two operations is not important: $(I_1 \vec{\bullet} u) \overleftarrow{\delta} I_2 = I_1 \vec{\bullet} (u \overleftarrow{\delta} I_2)$. In Appendix A.2, we give the Sage code of the composition using left and right products.

3.2. Enumeration of interval-posets. The B operator can also be decomposed into two operations,

$$(3.11) \quad f \succ g := fg, \quad f \prec_{\delta} g := f\Delta(g),$$

where

$$(3.12) \quad \Delta(g) := \frac{xg(x, y) - g(1, y)}{x - 1}.$$

And we have

$$(3.13) \quad \mathbb{B}(f, g) = f \succ xy \prec_{\delta} g.$$

The composition of interval-posets is a combinatorial interpretation of the B operator as stated in the following Proposition.

Proposition 3.6. *Let I_1 and I_2 be two interval-posets and \mathcal{P} the linear map from Definition 3.1. Then*

$$(3.14) \quad \mathcal{P}(I_1 \bullet I_2) = \mathcal{P}(I_1) \succ \mathcal{P}(I_2),$$

$$(3.15) \quad \mathcal{P}(I_1 \overset{\leftarrow}{\delta} I_2) = \mathcal{P}(I_1) \prec_{\delta} \mathcal{P}(I_2),$$

and consequently

$$(3.16) \quad \mathcal{P}(\mathbb{B}(I_1, I_2)) = \mathbb{B}(\mathcal{P}(I_1), \mathcal{P}(I_2)).$$

As an example, in Figure 13, $\mathcal{P}(I_1) = x^2y^3$ and $\mathcal{P}(I_2) = x^3y^4$. And we have $\mathcal{P}(\mathbb{B}(I_1, I_2)) = y^8(x^6 + x^5 + x^4 + x^3) = \mathbb{B}(x^2y^3, x^3y^4)$.

Proof. Let I_1 and I_2 be two interval-posets. The left product $I_1 \bullet I_2$ is the shifted concatenation of I_1 and I_2 on which only increasing relations have been added. Clearly,

$$\mathcal{P}(I_1 \bullet I_2) = y^{\text{size}(I_1)+\text{size}(I_2)} x^{\text{trees}(I_1)+\text{trees}(I_2)} = \mathcal{P}(I_1) \mathcal{P}(I_2)$$

which proves (3.14).

Now, let I_2 be such that $\text{trees}(I_2) = m$ and let $x_1 < x_2 < \dots < x_m$ be the roots of $F_{\geq}(I_2)$. By definition,

$$I_1 \overset{\leftarrow}{\delta} I_2 = \sum_{0 \leq i \leq m} P_i$$

where exactly i decreasing relations have been added between roots x_1, \dots, x_i of $F_{\geq}(I_2)$ and the vertex with maximal label of I_1 . We have $\text{trees}(P_i) = \text{trees}(I_1) + \text{trees}(I_2) - i$ because each added decreasing relation reduces the number of trees by one. Then

$$\begin{aligned} \mathcal{P}(I_1 \overset{\leftarrow}{\delta} I_2) &= y^{\text{size}(I_1)+\text{size}(I_2)} x^{\text{trees}(I_1)} (1 + x + x^2 + \dots + x^m) \\ &= y^{\text{size}(I_1)+\text{size}(I_2)} x^{\text{trees}(I_1)} \frac{x^{m+1} - 1}{x - 1} \\ &= \mathcal{P}(I_1) \prec_{\delta} \mathcal{P}(I_2). \quad \square \end{aligned}$$

Now, to prove Theorem 3.2, we only need the following proposition.

Proposition 3.7. *Let I be an interval-poset, then, there is exactly one pair of intervals I_1 and I_2 such that I appears in the composition $\mathbb{B}(I_1, I_2)$.*

Proof. Let I be an interval-poset of size n and let k be the vertex of I with maximal label such that $i \triangleleft k$ for all $i < k$. The vertex 1 satisfies this property, so one can always find such a vertex. We prove that I only appears in the composition of I_1 by I_2 , where I_1 is formed

by the vertices and relations of $1, \dots, k-1$ and I_2 is formed by the re-normalized vertices and relations of $k+1, \dots, n$. Note that one or both of these intervals can be of size 0.

Conditions (i), (ii), and (iii) of Definition 3.3 are clearly satisfied by construction. If condition (iv) is not satisfied, it means that we have a relation $k \triangleleft j$ with $j > k$. Then, by definition of an interval-poset, we also have $\ell \triangleleft j$ for all $k < \ell < j$ and by definition of k , we have $i \triangleleft k \triangleleft j$ for all $i < k$, so for all $i < j$, we have $i \triangleleft j$. This is not possible as k has been chosen to be maximal among vertices with this property.

This proves that I appears in the composition of I_1 by I_2 . Now, if I appears in $\mathbb{B}(I'_1, I'_2)$, the vertex $k' := |I'_1| + 1$ is by definition the vertex where for all $i < k'$, we have $i \triangleleft k'$ and for all $j > k'$, we have $k' \not\triangleleft j$, this is exactly the definition of k . So $k' = k$ which makes $I'_1 = I_1$ and $I'_2 = I_2$. \square

Proof of Theorem 3.2. Let $\mathbb{S} := \sum_{T_1 \leq T_2} P_{[T_1, T_2]}$ be the formal power series of interval-posets. By Proposition 3.7, we have

$$\mathbb{S} = \mathbb{B}(\mathbb{S}, \mathbb{S}) + \emptyset.$$

And by Proposition 3.6, we have

$$\begin{aligned} \Phi &= \mathcal{P}(\mathbb{S}) = \mathcal{P}(\mathbb{B}(\mathbb{S}, \mathbb{S})) + 1 \\ &= \mathbb{B}(\Phi, \Phi) + 1. \end{aligned} \quad \square$$

3.3. Counting smaller elements in Tamari. By developing (3.6), we obtain

$$\begin{aligned} \Phi &= 1 + \mathbb{B}(1, 1) + \mathbb{B}(\mathbb{B}(1, 1), 1) + \mathbb{B}(1, \mathbb{B}(1, 1)) + \dots \\ &= \sum_T y^{|T|} \mathcal{B}_T, \end{aligned}$$

where \mathcal{B}_T is the Tamari polynomial of Definition 1.1. Theorem 1.2 is proved by the following proposition.

Proposition 3.8. *Let $T := k(T_L, T_R)$ be a binary tree and $S_T := \sum_{T' \leq T} P_{[T', T]}$ the sum of all interval-posets whose maximal tree is T . Then $S_T = \mathbb{B}(S_{T_L}, S_{T_R})$.*

Proof. Let T be a binary tree of size n such that $T = k(T_L, T_R)$. The interval-poset of the initial interval $[T_0, T]$ is $F_{\leq}(T)$, the initial forest of T . From Proposition 2.9 (iii), the sum S_T is the sum of all interval-posets I which extends $F_{\leq}(T)$ by adding only decreasing relations.

Let I be an interval of S_T . Let I_L and I_R be the sub-posets obtained by restricting I to respectively $1, \dots, k-1$ and $k+1, \dots, n$. By the

recursive definition of initial forests given by Figure 9, I_L and I_R are poset extensions of respectively $F_{\leq}(T_L)$ and $F_{\leq}(T_R)$ where only decreasing relations have been added. And then $I_L \in S_{T_L}$ and $I_R \in S_{T_R}$. Finally, it is clear that $I \in \mathbb{B}(I_L, I_R)$. Indeed, I is a poset extension of $F_{\leq}(T)$ and so $i \triangleleft k$ for $i < k$ and $k \not\triangleleft j$ for $j > k$.

Conversely, if I_L and I_R are elements of respectively S_{T_L} and S_{T_R} , then any interval I of $\mathbb{B}(I_L, I_R)$ is in S_T . Indeed, by construction, I is an extension of $F_{\leq}(T)$ where only decreasing relations have been added. \square

For a given tree T , the coefficient of the monomial with maximal degree in x in \mathcal{B}_T is always 1. It corresponds to the initial interval $F_{\leq}(T)$. The interval with the maximal number of decreasing relations corresponds to $[T, T]$. An example of \mathcal{B}_T and of the computation of smaller trees is presented in Figure 15.

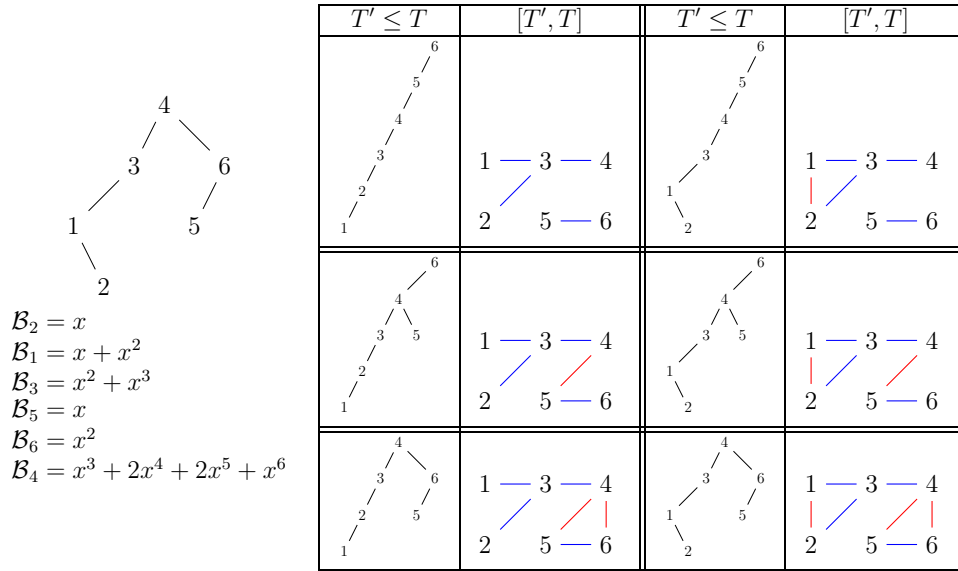


FIGURE 15. Example of the computation of \mathcal{B}_T and list of all smaller trees with associated intervals.

Proof of Theorem 1.2. Counting the number of trees $T' \leq T$ refined by the number of nodes on their leftmost branch can be done by counting the number of intervals $I = [T', T]$ refined by $\text{trees}(T')$. We then want to prove that $\mathcal{B}_T = \mathcal{P}(S_T)$ where $S_T = \sum_{T' \leq T} P_{[T', T]}$. It can be done by induction on the size of T . The initial case is trivial. And if we set $T = k(T_L, T_R)$, by the induction hypothesis, we have that $\mathcal{B}_{T_L} = \mathcal{P}(S_{T_L})$ and $\mathcal{B}_{T_R} = \mathcal{P}(S_{T_R})$. The result is then a direct

consequence of Propositions 3.6 and 3.8,

$$\begin{aligned} \mathcal{B}_T &= \mathbb{B}(\mathcal{P}(S_{T_L}), \mathcal{P}(S_{T_R})) = \mathcal{P}(\mathbb{B}(S_{T_L}, S_{T_R})) \\ &= \mathcal{P}(S_T). \end{aligned} \quad \square$$

3.4. Bivariate polynomials. In some recent work [9], Chapoton computed some bivariate polynomials that seem to be similar to the ones we study. The computation of the first examples of [9, formula (7)] with $b = 1$ and $t = 1 - 1/x$ leads us to conjecture that each polynomial is equal to some $\mathcal{B}_T(x)$ where T is a binary tree with no left subtree. More precisely, the non planar rooted tree corresponding to T is the non planar version of the planar forest $F_{\geq}(T)$.

A b parameter can be also be added to our formula. For an interval $[T', T]$, it is either the number of nodes in T' which have a right subtree, or in the interval-poset the number of nodes x with a relation $y \triangleleft x$ and $y > x$. By a generalization of the linear function \mathcal{P} , one can associate a monomial in b , x , and y with each interval-poset. The bilinear form now reads:

$$(3.17) \quad \mathbb{B}(f, g) = y \left(xbf \frac{xg - g_{x=1}}{x-1} - bxf g + xfg \right),$$

where f and g are polynomials in x , b , and y . Proposition 3.6 still holds, since a node with a decreasing relation is added in all terms of the composition but one. As an example, in Figure 13, one has $\mathbb{B}(y^3x^2b, y^4x^3b) = y^8(x^6b^2 + x^5b^3 + x^4b^3 + x^3b^3)$.

With this definition of the parameter b , the bivariate polynomials $\mathcal{B}_T(x, b)$ where T has no left subtree seem to be exactly the ones computed by Chapoton in [9] when taken on $t = 1 - 1/x$. This seems to indicate some combinatorial and algebraic links between structures from very different mathematical contexts.

4. m -TAMARI LATTICES

4.1. Definition. The m -Tamari lattices are a generalization of the Tamari lattice where objects have a $(m + 1)$ -ary structure instead of binary. They were introduced in [5] and can be described in terms of m -ballot paths. A m -ballot path is a lattice path from $(0, 0)$ to (nm, n) made from horizontal steps $(1, 0)$ and vertical steps $(0, 1)$ which always stays above the line $y = \frac{x}{m}$. When $m = 1$, a m -ballot path is just a Dyck path where up steps and down steps have been replaced by respectively vertical steps and horizontal steps. They are well known

combinatorial objects counted by the m -Catalan numbers

$$(4.1) \quad \frac{1}{mn + 1} \binom{(m + 1)n}{n}.$$

They can also be interpreted as words on a binary alphabet and the notion of *primitive path* still holds. Indeed, a primitive path is a m -ballot path which does not touch the line $y = \frac{x}{m}$ outside its extremal points. From this, the definition of the rotation on Dyck path given in Section 2 can be naturally extended to m -ballot-paths, see Figure 16.

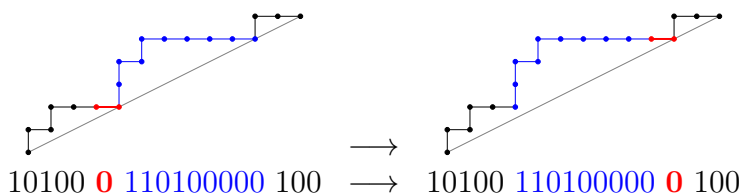


FIGURE 16. Rotation on m -ballot paths.

When interpreted as a cover relation, the rotation on m -ballot paths induces a well defined order, and even a lattice [5]. This is what we call the m -Tamari lattice or $\mathcal{T}_n^{(m)}$, see Figure 17 for an example.

A formula counting the number of intervals in $\mathcal{T}_n^{(m)}$ was conjectured in [5] and was proven in [7]. The authors use a functional equation that is a direct generalization of (3.6). Let $\Phi^{(m)}(x, y)$ be the generating function of intervals of the m -Tamari lattice where y is the size n and x a statistic called number of contacts, then [7, formula (3)] reads⁴

$$(4.2) \quad \Phi^{(m)}(x, y) = 1 + B^{(m)}(\Phi, \Phi, \dots, \Phi),$$

where $B^{(m)}$ is a $(m + 1)$ -linear form defined by

$$(4.3) \quad B^{(m)}(f, g_1, \dots, g_m) := xyf\Delta(g_1\Delta(\dots\Delta(g_m))\dots),$$

$$(4.4) \quad \Delta(g) := \frac{xg(x, y) - g(1, y)}{x - 1}.$$

Expanding (3.6), we obtain a sum of $(m + 1)$ -ary trees. This leads to conjecture that the formula of Theorem 1.2 for counting smaller elements in the lattice generalizes in the m -Tamari case, this is indeed true and we prove it in this section.

⁴for consistency with the first part of the article, the x parameter counts the number of contacts minus 1 and so the formula of [7] has been divided by x .

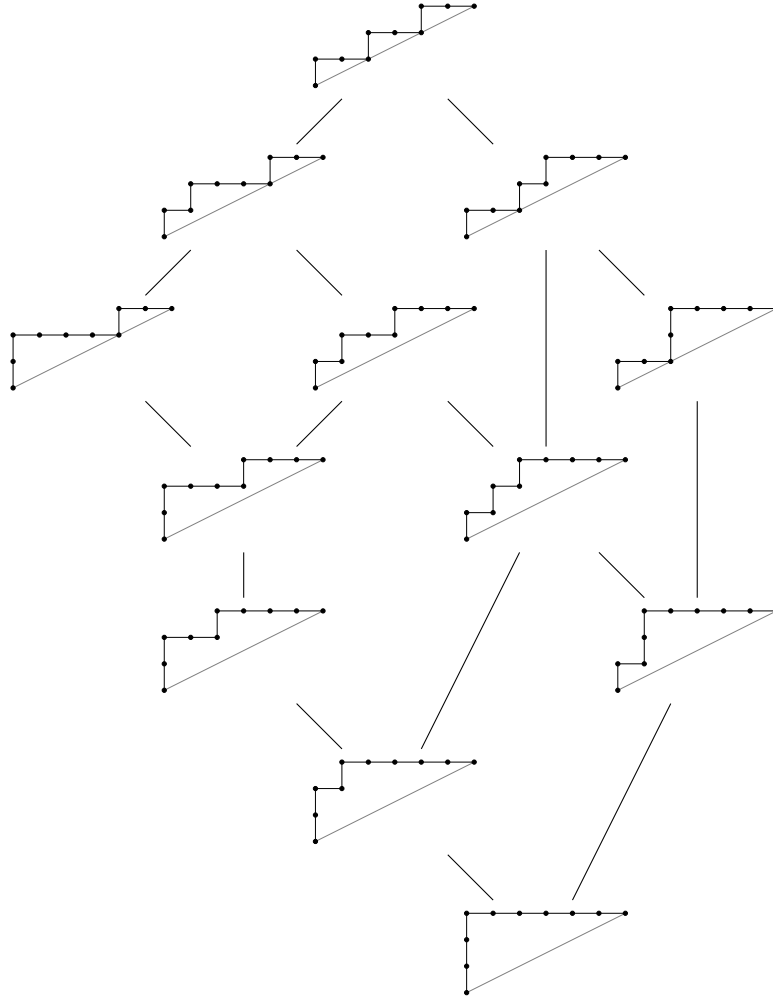


FIGURE 17. m -Tamari on m -ballot paths: $\mathcal{T}_3^{(2)}$.

4.2. Interpretation in terms of trees. It was proven in [7] that $\mathcal{T}_n^{(m)}$ is actually an upper ideal of $\mathcal{T}_{n \times m}^{(1)}$. Indeed, there is a natural injection from m -ballot paths of size n to Dyck paths of size mn by replacing each vertical step of a m -ballot path by m adjacent up steps, see Figure 18. The result set is made of all Dyck paths whose numbers of adjacent up steps are divisible by m . We call these paths m -Dyck paths and they are in clear bijection with m -ballot paths. The set of m -Dyck paths is stable by the rotation operation. It is the upper ideal generated by the Dyck word $(1^m . 0^m)^n$ which is the image of the m -ballot path $(1.0^m)^n$, see Figure 19.

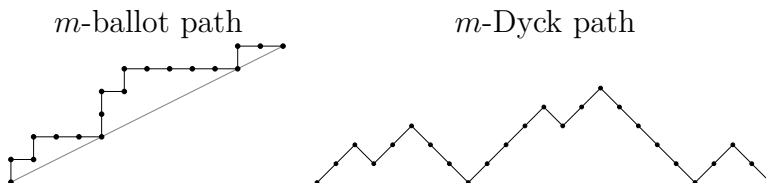


FIGURE 18. A 2-ballot path and its corresponding 2-Dyck path.

It is then possible to compute the binary tree image of the minimal m -Dyck path by the bijection described in Section 2. We call this tree the (n, m) -comb: it is a left-comb of n right-combs of size m , as illustrated in Figure 19.

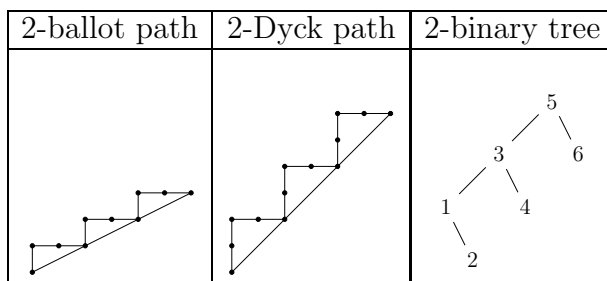


FIGURE 19. Minimal element of $\mathcal{T}_3^{(2)}$.

Because the m -Tamari lattice corresponds to the upper ideal of the Tamari lattice generated by the word $(1^m.0^m)^n$, it is now clear that it also corresponds to the upper ideal generated by the (n, m) -comb, which we give in Figure 21. To fully understand the lattice in terms of trees, two questions remains:

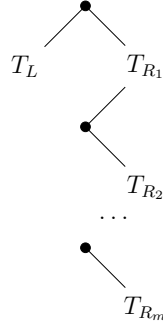
- How to characterize a binary tree belonging to the upper ideal of the (n, m) -comb, *i.e.*, the images of the m -Dyck paths?
- What is the bijection between those trees and the m -ballot paths?

We can answer both questions by defining a new class of binary trees which we call the m -binary trees. This definition is crucial to our work on the m -Tamari lattice, especially to generalize our results using interval-posets. In this Section, we first give the definition of m -binary trees, then prove that they correspond to the upper ideal generated by the (n, m) -comb. Finally, we give the explicit bijection between m -ballot paths, m -binary trees and $(m+1)$ -ary trees using the $(m+1)$ -ary structure of m -binary trees.

Definition 4.1. We define m -binary trees recursively by being either the empty binary tree or a binary tree T of size $m \times n$ constructed from $m + 1$ subtrees $T_L, T_{R_1}, \dots, T_{R_m}$ such that

- the sum of the sizes of $T_L, T_{R_1}, \dots, T_{R_m}$ is $m \times (k - 1)$,
- each subtree $T_L, T_{R_1}, \dots, T_{R_m}$ is itself a m -binary tree.

And T follows the structure bellow,



which we now describe. The left subtree of T is T_L . To construct the right subtree T_R of T from T_{R_1}, \dots, T_{R_m} , we follow this algorithm:

```

 $T_R \leftarrow T_{R_1}$ 
for  $i = 2$  to  $m$  do
    graft a single node  $x$  on left of the leftmost node of  $T_{R_{i-1}}$ 
    graft  $T_{R_i}$  on the right of  $x$ 
end for

```

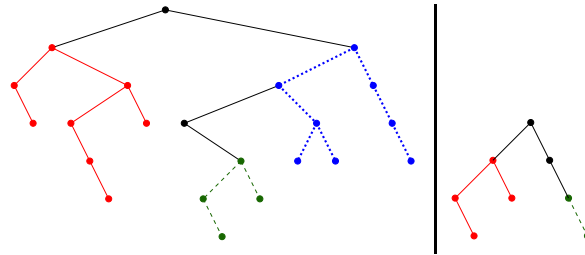


FIGURE 20. Examples of m -binary trees for $m = 2$: T_L is in red, T_{R_1} is in dotted blue and T_{R_2} is in dashed green. In the second example, T_{R_1} is empty.

Note that T contains the nodes of $T_L, T_{R_1}, \dots, T_{R_m}$ plus exactly m extra nodes. Those nodes are called the *root nodes* of the m -binary tree T . The actual root of the binary tree T is called the *main root* and the other ones, the *secondary roots*. If T_{R_i} is empty, then the i^{th} root node is directly on the right of the $(i - 1)^{\text{th}}$ root node. Figure 20 shows two examples of m -binary trees for $m = 2$ and the second one

illustrates the case where T_{R_i} is empty. The whole set of m -binary trees for $m = 2$ and $n = 3$ is given in Figure 21.

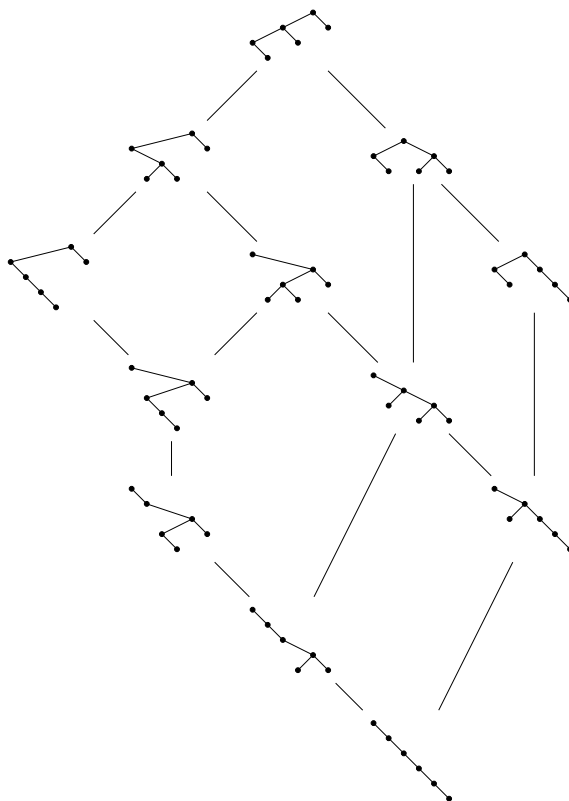


FIGURE 21. The lattice $\mathcal{T}_3^{(2)}$ on 2-binary tree.

From the definition, a m -binary tree is a special kind of binary tree, just as a m -Dyck path is a special kind of Dyck path. The following proposition gives a useful criteria to decide if a given binary tree is indeed a m -binary tree. We then use this criteria to prove that m -binary trees are the elements of the upper ideal generated by the (n, m) -comb.

Proposition 4.2. *A binary tree T is a m -binary tree if and only if it is of size $n \times m$ and its binary search tree satisfies:*

$$(4.5) \quad i \cdot m \triangleleft i \cdot m - 1 \triangleleft \dots \triangleleft i \cdot m - (m - 1)$$

for all $1 \leq i \leq n$.

This property can be checked in Figure 22 which is the binary search tree of the m -binary tree of Figure 20.

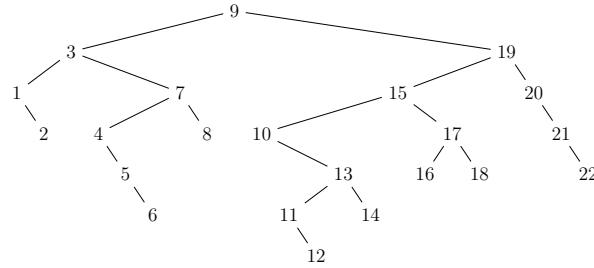


FIGURE 22. Binary search tree of a 2-binary tree. We always have $2k \triangleleft 2k - 1$.

Proof. The property is proved by induction on n . Let T be a m -binary tree composed of the m -binary trees $T_L, T_{R_1}, \dots, T_{R_m}$ which satisfy (4.5) by induction hypothesis. We prove that T also satisfies (4.5). The main root of T is labelled by $x = |T_L| + 1$. Because T_L is a m -binary tree, we have that $|T_L| = km$ for some $k \in \mathbb{N}$, and so $x = km + 1$. The m -binary tree structure makes it clear that $x + m - 1 \triangleleft x + m - 2 \triangleleft \dots \triangleleft x$. Besides, the labelling of T_L in T has not been changed, the one of T_{R_m} has been shifted by $|T_L| + m$, the one of $T_{R_{m-1}}$ has been shifted by $|T_L| + |T_{R_m}| + m$ and so on. The labels are only shifted by multiples of m which means that (4.5) still holds on T .

Now let T be a binary search tree which satisfies (4.5), we have to prove that T satisfies the recursive structure of m -binary trees. Let x be the root of T . The node x does not precede any element of T so it has to be of the form $x = km + 1$ for some $0 \leq k < n$. Let T_L be the left subtree of T , then $|T_L| = km$ and by induction, T_L is a m -binary tree. We have $x + 1 \triangleleft x$, i.e., $x + 1$ is in the right subtree of x . More precisely, it is the leftmost node of the right subtree. Let T_{R_1} be the binary tree in-between x and $x + 1$. For $0 \leq a < n$ and $1 \leq b \leq m$ we have that $y = am + b$ is in T_{R_1} if and only if all nodes $am + m \triangleleft_T am + m - 1 \triangleleft_T \dots \triangleleft_T am + 1$ are also in T_{R_1} . It means T_{R_1} satisfies (4.5) and is a m -binary tree by induction. The same holds for $T_{R_2}, T_{R_3}, \dots, T_{R_m}$ which gives T the recursive structure of a m -binary tree. \square

Proposition 4.3. *The upper ideal generated by the (n, m) -comb is the set of all m -binary trees.*

Proof. The final forest of the (n, m) -comb is exactly the poset given by (4.5). We proved by Proposition 4.2 that m -binary trees are the binary trees whose final forests are poset extensions of (4.5). This proves the result by using the properties of interval-posets (Proposition 2.9). \square

By this last proposition, the sub-lattice of the Tamari lattice of size $m \times n$ on m -binary trees is indeed the m -Tamari lattice. To obtain a m -ballot path W from a m -binary tree T , we first get the Dyck path D corresponding to T . This algorithm is just the well known bijection between binary trees and Dyck paths that we gave in Figure 4. Proposition 4.3 assures us that D is actually a m -Dyck path, *i.e.*, the numbers of adjacent up steps are always divisible by m . This last property can also be proved directly from the bijection and the structure of m -binary trees. Because D is a m -Dyck path, one can obtain the m -ballot path W by replacing each sequence of two consecutive up steps by one vertical step and each down step by an horizontal step. This way, one obtains a m -ballot path from a m -binary tree. An example of the bijection is given in Figure 23.

There is also another, equivalent, way to go from a m -binary tree to a m -ballot path using both $(m + 1)$ -ary structures of the objects. Indeed, m -binary trees are in clear bijection with $(m + 1)$ -ary trees: the m -binary tree T , composed from $T_L, T_{R_1}, \dots, T_{R_m}$, is associated to the $(m + 1)$ -ary tree T' with one root and $m + 1$ subtrees $T'_L, T'_{R_1}, \dots, T'_{R_m}$ respective images of $T_L, T_{R_1}, \dots, T_{R_m}$. The m -ballot paths can also be described by a recursive $(m + 1)$ -ary structure. A m -ballot path W is either the empty path or given by the word

$$W = W_L \ 1 \ W_{R_m} \ 0 \ W_{R_{m-1}} \ \dots \ 0 \ W_{R_1} \ 0$$

where $W_1, W_{R_1}, W_{R_2}, \dots, W_{R_m}$ are themselves m -ballot paths. We obtain the bijection with m -binary trees (and $(m + 1)$ -ary trees) by setting that the empty path is the image of the empty tree and a non-empty m -ballot path W is the image of T if and only if $W_1, W_{R_1}, W_{R_2}, \dots, W_{R_m}$ are the images of respectively $T_L, T_{R_1}, \dots, T_{R_m}$. Note that the order of W_{R_1}, \dots, W_{R_m} has to be reversed in W to match the subtrees of T and the first description of the bijection. This is illustrated in Figure 23.

The bijection between m -binary trees and $(m + 1)$ -ary trees also allows us to answer a question asked in [7]: what is the description of the m -Tamari lattice in terms of $(m + 1)$ -ary trees? The cover relation on m -binary trees is the usual cover relation of the Tamari lattice on binary trees: the right rotation. By Proposition 4.3, we know that it preserves the m -binary structure. We can then observe what becomes of $T_L, T_{R_1}, \dots, T_{R_m}$ when applying a rotation on one of the root nodes of T . It appears that two different cases arise depending on whether the rotation is made on the main root or on one of the secondary roots. These two cases and their translations in terms of $(m + 1)$ -ary trees are drawn in Figure 25 (for $m = 2$). We also give the m -Tamari lattices of sizes 2 and 3 for $m = 2$ in terms of ternary trees in Figure 24. Note

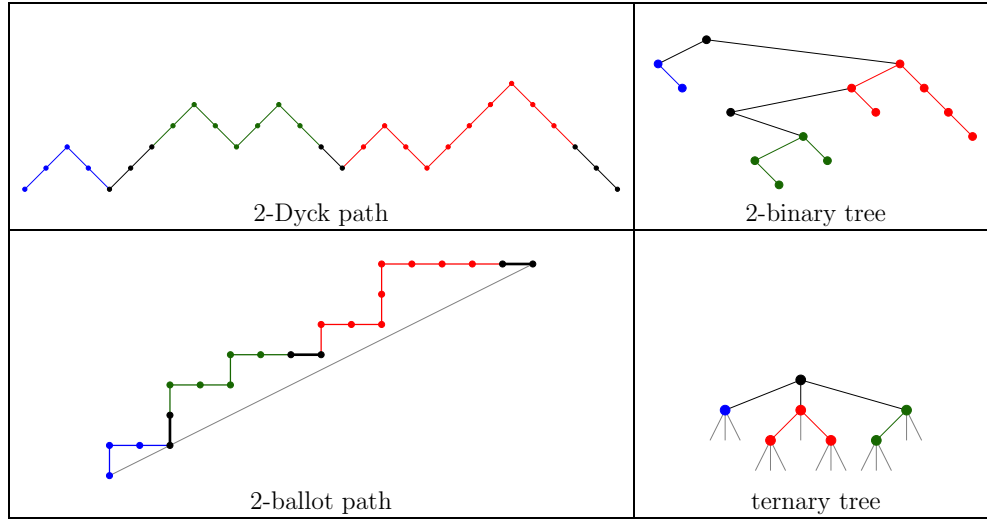


FIGURE 23. Bijection between ternary trees and 2-ballot paths: the 2-binary tree is sent to a Dyck path by the usual bijection, the Dyck path can then be interpreted as a 2-ballot path. Both the 2-binary tree and 2-ballot paths have a ternary structure and are bijectively sent to a ternary tree.

that the description of the lattice in terms of $(m + 1)$ -ary trees is not needed for the rest of the paper, we will be using only the m -binary structure.

4.3. m -Composition and enumeration of intervals.

Theorem 4.4. *Let $\Phi^{(m)}(x, y)$ be the generating function of intervals of m -Tamari where y counts the size of the objects and x the number of touching points of the lowest path (number of contacts with $y = \frac{x}{m}$ after the starting point). Then*

$$(4.6) \quad \Phi^{(m)} = B^{(m)}(\Phi^{(m)}, \dots, \Phi^{(m)}) + 1$$

where $B^{(m)}$ is the $(m + 1)$ -linear operator defined by

$$(4.7) \quad B^{(m)}(f, g_1, \dots, g_m) = f \succ xy \prec_{\delta} (g_1 \prec_{\delta} (\dots \prec_{\delta} (g_{m-1} \prec_{\delta} g_m)) \dots)$$

where \succ and \prec_{δ} are the left and right products defined in (3.11).

This new definition of $B^{(m)}$ (4.7) is equivalent to the previous one (4.3) and this theorem is just a reformulation of Proposition 8 of [7]. In this section, we propose a new proof by generalizing the concept of interval-poset.

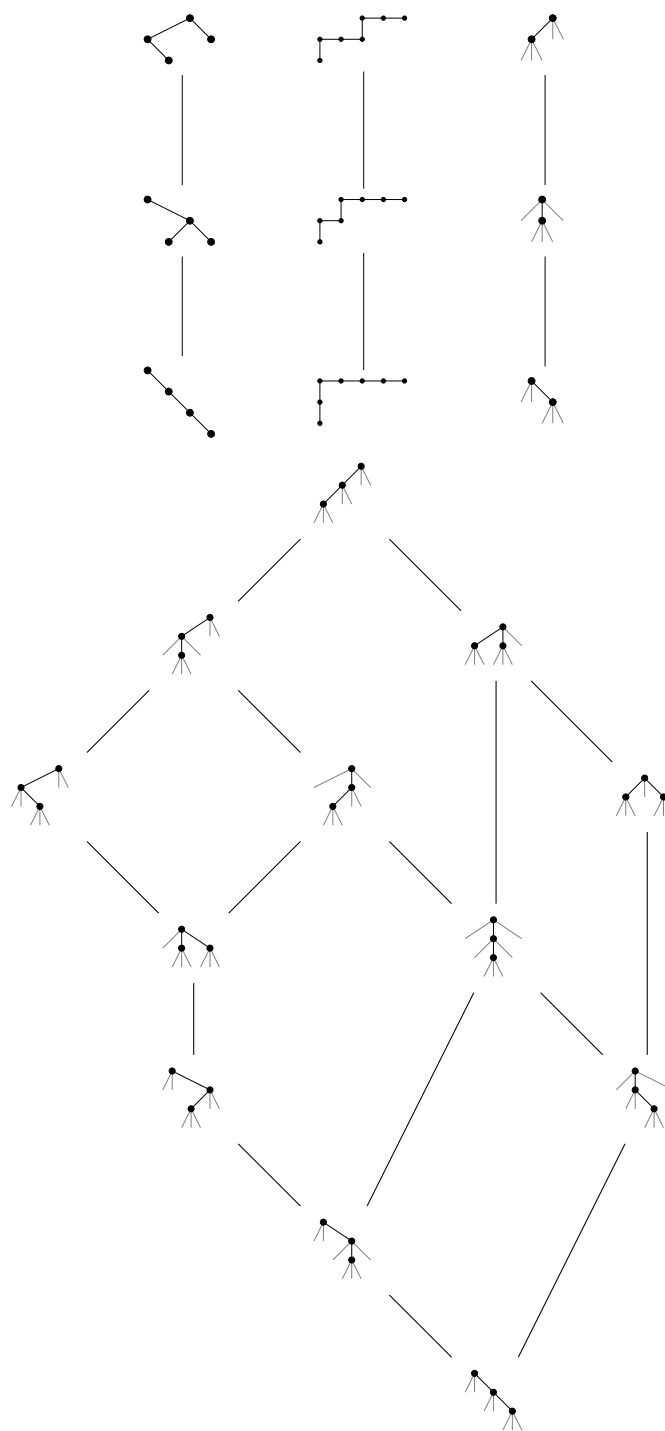


FIGURE 24. $\mathcal{T}_2^{(2)}$ and $\mathcal{T}_3^{(2)}$ on ternary trees.

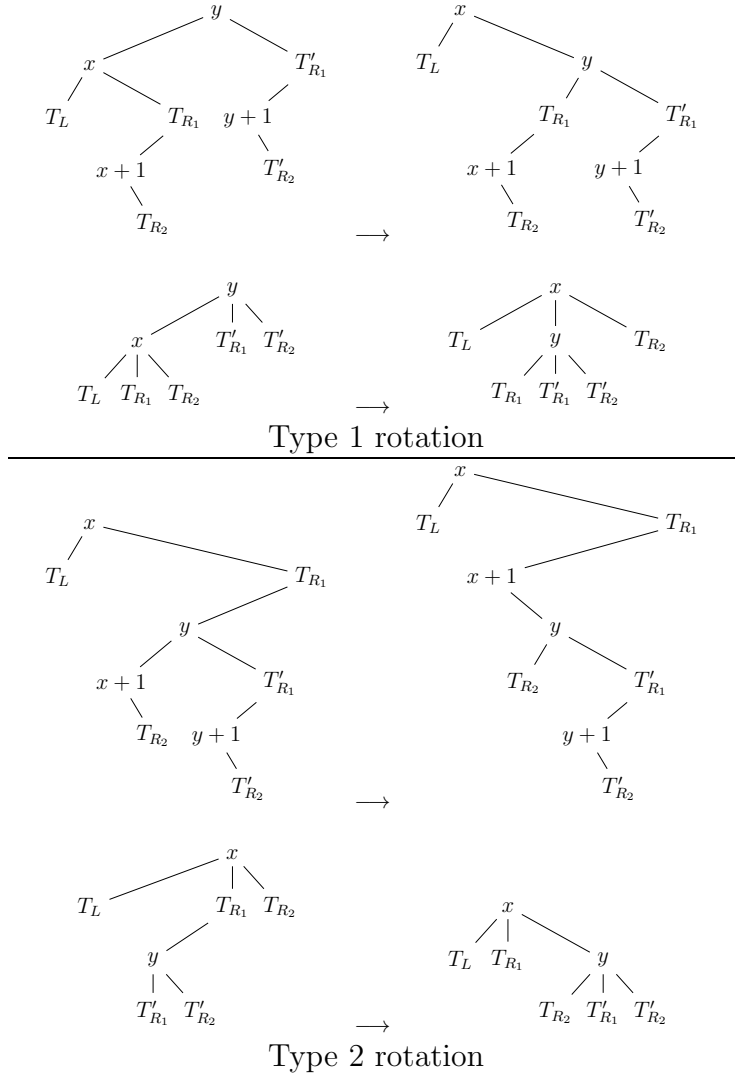


FIGURE 25. Rotations of type 1 and 2 in m -binary trees and $(m+1)$ -ary trees.

Definition 4.5. A m -interval-poset is an interval-poset of size $n \times m$ with

$$(4.8) \quad i \cdot m \triangleleft i \cdot m - 1 \triangleleft \dots \triangleleft i \cdot m - (m - 1)$$

for all $1 \leq i \leq n$.

Theorem 4.6. The m -interval-posets of size n are in bijection with intervals of $\mathcal{T}_n^{(m)}$.

Proof. A m -interval-poset I corresponds to an interval $[T_1, T_2]$ of the Tamari order of size $n \times m$. By Proposition 4.2, the binary tree T_1 is a m -binary tree. As $T_2 \geq T_1$, then T_2 is also a m -binary tree and I is an interval of $\mathcal{T}_n^{(m)}$. \square

The number of nodes on the border of a m -binary tree is the same as on its associated $(m + 1)$ -ary tree and still corresponds to the number of touch points of the m -ballot path. We then define

$$(4.9) \quad \mathcal{P}^{(m)}(I) := x^{\text{trees}(I)} y^{\frac{\text{size}(I)}{m}},$$

and we have

$$(4.10) \quad \Phi^{(m)}(x, y) = \sum_I \mathcal{P}^{(m)}(I)$$

summed on all m -interval-posets.

By composing two m -interval-posets, one does not obtain a sum on m -interval-posets: the sizes are not multiples of m any more. We have to generalize the \mathbb{B} composition to a m -composition which has to be a $(m + 1)$ -linear operator. A simple translation of (4.7) in terms of $\vec{\bullet}$ and $\overleftarrow{\delta}$ is not enough. Indeed, it wouldn't generate all m -interval-posets. However, the following expression

$$(4.11) \quad g_1 \prec_{\delta} g_2 = \frac{(xy \prec_{\delta} g_2) \succ g_1}{xy}.$$

reflects the m -binary structure given by Figure 20. We then rewrite (4.7) by using this observation. As an example, for $m = 3$, one obtains

$$\begin{aligned} \mathbb{B}^{(3)}(f, g_1, g_2, g_3) &= f \succ xy \prec_{\delta} (g_1 \prec_{\delta} (g_2 \prec_{\delta} g_3)) \\ &= f \succ xy \prec_{\delta} \frac{1}{xy} \left(\left(xy \prec_{\delta} \frac{1}{xy} ((xy \prec_{\delta} g_3) \succ g_2) \right) \succ g_1 \right) \\ &= \frac{1}{y^2} \left(f \succ xy \prec_{\frac{\delta}{x}} \left(\left(xy \prec_{\frac{\delta}{x}} ((xy \prec_{\delta} g_3) \succ g_2) \right) \succ g_1 \right) \right) \end{aligned}$$

where

$$(4.12) \quad f \prec_{\frac{\delta}{x}} g := f \prec_{\delta} \left(\frac{g}{x} \right) = f \Delta \left(\frac{g}{x} \right).$$

The $\prec_{\frac{\delta}{x}}$ operation can be interpreted on interval-posets.

Definition 4.7. *Let I_1 and I_2 be two interval-posets such that $\text{trees}(I_2) = k$. Let y be the maximal label of I_1 and x_1, \dots, x_k be the roots of $F_{\geq}(I_2)$. Then $I_1 \overset{\leftarrow{\delta}}{x} I_2$ is the sum of the k interval-posets P_1, \dots, P_k where P_i is the shifted concatenation of I_1 and I_2 with the i added decreasing relations: $x_j \triangleleft y$ for $j \leq i$.*

The sum $I_1 \overset{\leftarrow{\delta}}{x} I_2$ is just the sum $I_1 \overset{\leftarrow{\delta}}{\delta} I_2$ of Definition 3.5 minus the P_0 poset (the shifted concatenation with no extra decreasing relation). In particular, this means that the obtained interval-posets all have the relation $2 \triangleleft 1$ because 1 is always the minimal root of $F_{\geq}(I_2)$.

Proposition 4.8. *The $(m + 1)$ -linear operator $\mathbb{B}^{(m)}$ on m -interval-posets is defined by*

$$\mathbb{B}^{(m)}(I_L, I_{R_1}, I_{R_2}, \dots, I_{R_m}) := I_L \overset{\rightarrow}{\bullet} u \overset{\leftarrow{\delta}}{x} \left((u \overset{\leftarrow{\delta}}{x} (u \overset{\leftarrow{\delta}}{x} \dots ((u \overset{\leftarrow{\delta}}{\delta} I_{R_m}) \overset{\rightarrow}{\bullet} I_{R_{m-1}}) \overset{\rightarrow}{\bullet} \dots) \overset{\rightarrow}{\bullet} I_{R_1}) \right)$$

where u the interval-poset containing a single vertex. Recursively, the definition reads

$$\mathbb{B}^{(m)}(I_L, I_{R_1}, \dots, I_{R_m}) := I_L \overset{\rightarrow}{\bullet} \mathbb{R}(I_{R_1}, \dots, I_{R_m})$$

with

$$\begin{aligned} \mathbb{R}(I) &:= u \overset{\leftarrow{\delta}}{\delta} I, \\ \mathbb{R}(I_1, \dots, I_k) &:= u \overset{\leftarrow{\delta}}{x} (\mathbb{R}(I_2, \dots, I_k) \overset{\rightarrow}{\bullet} I_1). \end{aligned}$$

The result is a sum of m -interval-posets. The $\mathbb{B}^{(m)}$ operator is called the m -composition.

Note that we give the Sage code corresponding to this definition in Appendix 1.3.

Proof. Let us first notice that we compose with the interval-poset u exactly m times. It means that m vertices have been added to $I_L, I_{R_1}, I_{R_2}, \dots, I_{R_m}$: the size of the obtained intervals are multiples of m .

The first operation is $\mathbb{R}(I_{R_m}) = u \overset{\leftarrow{\delta}}{\delta} I_{R_m}$ which is a sum of interval-posets of size $1 + |I_{R_m}|$. The labels of I_{R_m} have been shifted by 1. The next operation is

$$(4.13) \quad \mathbb{R}(I_{R_{m-1}}, I_{R_m}) = u \overset{\leftarrow{\delta}}{x} (\mathbb{R}(I_{R_m}) \overset{\rightarrow}{\bullet} I_{R_{m-1}}).$$

The computation $\mathbb{R}(I_{R_m}) \overset{\rightarrow}{\bullet} I_{R_{m-1}}$ consists of attaching $I_{R_{m-1}}$ to the interval-posets of $u \overset{\leftarrow{\delta}}{\delta} I_{R_m}$ without adding any decreasing relations. The labels of $I_{R_{m-1}}$ are shifted by $1 + |I_{R_m}|$. By doing $u \overset{\leftarrow{\delta}}{x} (\mathbb{R}(I_{R_m}) \overset{\rightarrow}{\bullet} I_{R_{m-1}})$, we obtain a sum of interval-posets which all have the relation $2 \triangleleft 1$.

$2 \triangleleft \dots \triangleleft k$. Indeed, because of the increasing relations, there can not be any decreasing relations in-between intervals $I_L, I_{R_1}, \dots, I_{R_m}$. By definition of $\overleftarrow{\delta}$ and $\frac{\overleftarrow{\delta}}{x}$ it is then clear that the intervals of $\mathbb{B}^{(m)}(I_L, I_{R_1}, \dots, I_{R_m})$ are exactly the extensions of I_0 as defined by Proposition 4.9. \square

As explained in the proof, the intervals resulting of a m -compositions all share the same maximal tree given by the structure of the m -binary tree. The minimal trees range from a tree where all minimal trees of composed intervals have been grafted at the left of one another to the m -binary tree formed by all minimal trees. This illustrated in the case where $m = 2$ in Figure 26.

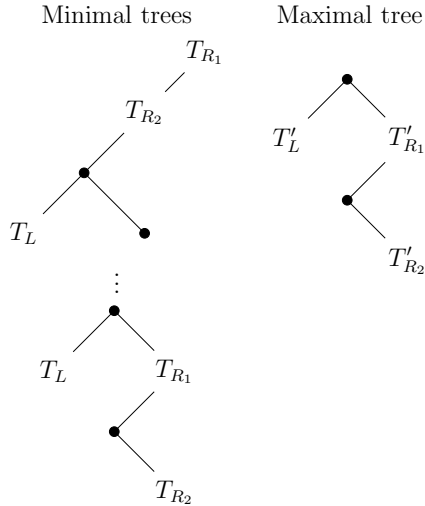


FIGURE 26. Minimal and maximal trees of the intervals of a m -composition.

Proposition 4.10. *Let $I_L, I_{R_1}, \dots, I_{R_m}$ be m -interval-posets. Then*

$$\mathcal{P}^{(m)}(\mathbb{B}^{(m)}(I_L, I_{R_1}, \dots, I_{R_m})) = \mathbb{B}^{(m)}(\mathcal{P}^{(m)}(I_L), \mathcal{P}^{(m)}(I_{R_1}), \dots, \mathcal{P}^{(m)}(I_{R_m}))$$

Proof. The only thing to prove is

$$(4.17) \quad \mathcal{P}(I_1 \frac{\overleftarrow{\delta}}{x} I_2) = \mathcal{P}(I_1) \prec_{\frac{\delta}{x}} \mathcal{P}(I_2)$$

for I_1 and I_2 two interval-posets. Indeed, let $Y = y^{\frac{1}{m}}$ and I be a m -interval-poset of size nm , then

$$\mathcal{P}^{(m)}(I)(x, y) = \mathcal{P}(I)(x, Y).$$

And so if (4.17) is satisfied, so are (3.14) and (3.15) and we have

$$\begin{aligned}
\mathcal{P}^{(m)}(\mathbb{B}^{(m)}(I_L, I_{R_1}, \dots, I_{R_m})) &= \mathcal{P}(\mathbb{B}^{(m)}(I_L, I_{R_1}, \dots, I_{R_m}))(x, Y) \\
&= \mathcal{P}\left(I_L \overset{\leftarrow}{\bullet} u \frac{\overset{\leftarrow}{\delta}}{x} \left((u \frac{\overset{\leftarrow}{\delta}}{x} \dots ((u \overset{\leftarrow}{\delta} I_{R_m}) \overset{\leftarrow}{\bullet} I_{R_{m-1}}) \overset{\leftarrow}{\bullet} \dots) \overset{\leftarrow}{\bullet} I_{R_1} \right)\right)(x, Y) \\
&= \mathcal{P}(I_L) \succ x.Y \prec_{\frac{\delta}{x}} \left((x.Y \prec_{\frac{\delta}{x}} \dots ((x.Y \prec_{\delta} \mathcal{P}(I_{R_m})) \succ \mathcal{P}(I_{R_{m-1}})) \succ \dots) \succ \mathcal{P}(I_{R_1}) \right) \\
&= Y^{m-1} \mathcal{P}(I_L) \succ x.Y \prec_{\delta} (\mathcal{P}(I_{R_1}) \prec_{\delta} \dots \prec_{\delta} (\mathcal{P}(I_{R_{m-1}}) \prec_{\delta} \mathcal{P}(I_{R_m}))) \dots \\
&= \mathbb{B}^{(m)}(\mathcal{P}(I_L), \mathcal{P}(I_{R_1}), \dots, \mathcal{P}(I_{R_m}))(x, Y) \\
&= \mathbb{B}^{(m)}(\mathcal{P}^{(m)}(I_L), \mathcal{P}^{(m)}(I_{R_1}), \dots, \mathcal{P}^{(m)}(I_{R_m})).
\end{aligned}$$

We then prove (4.17). We set $k := \text{trees}(I_2)$, we have

$$\begin{aligned}
\Delta\left(\frac{\mathcal{P}(I_2)}{x}\right) &= \Delta(y^{\text{size}(I_2)} x^{k-1}) \\
&= y^{\text{size}(I_2)} (1 + x + x^2 + \dots + x^{k-1}), \\
\mathcal{P}(I_1) \prec_{\frac{\delta}{x}} \mathcal{P}(I_2) &= y^{\text{size}(I_1) + \text{size}(I_2)} x^{\text{trees}(I_1)} (1 + x + x^2 + \dots + x^{k-1})
\end{aligned}$$

Besides, $I_1 \overset{\leftarrow}{x} I_2$ is the sum of interval-posets P_i , $1 \leq i \leq k$ where $\text{size}(P_i) = \text{size}(I_1) + \text{size}(I_2)$ and $\text{trees}(P_i) = \text{trees}(I_1) + k - i$ which proves the result. \square

We can check (4.17) on (4.15).

$$\begin{aligned}
xy \prec_{\frac{\delta}{x}} y^7 (x^4 + x^3) &= y^8 x (1 + x + x^2 + x^3 + 1 + x + x^2) \\
&= y^8 (2x + 2x^2 + 2x^3 + x^4)
\end{aligned}$$

Besides, by computing

$$\begin{aligned}
\mathbb{B}^{(m)}(xy, x^2 y^2, xy) &= xy \succ (xy \prec_{\delta} (x^2 y^2 \prec_{\delta} xy)) \\
&= y^5 x (x \prec_{\delta} x^2 (1 + x)) \\
&= y^5 x^2 (1 + x + x^2 + 1 + x + x^2 + x^3) \\
&= y^5 (2x^2 + 2x^3 + 2x^4 + x^5),
\end{aligned}$$

we check the result on (4.16).

Proposition 4.11. *Let I be a m -interval-poset, then there is exactly one list I_1, \dots, I_{m+1} of m -interval-posets such that I appears in the m -composition $\mathbb{B}^{(m)}(I_1, \dots, I_{m+1})$.*

Proof. We define k the same way as in the proof of Proposition 3.7: k is the maximal label such that $i \triangleleft k$ for all $i < k$. And for the same reasons, k is unique and I_L is made of vertices $i < k$. For $1 \leq j < m$, let a_j be the minimal label such that $k + j + 1 \triangleleft a_j$ and $k + j \not\triangleleft a_j$.

If there is no such label, we set $a_j := \emptyset$. And let $a_m := k + m$ if $k + m - 1 \not\prec k + m$ or \emptyset otherwise.

The vertices a_1, \dots, a_m satisfy Condition (iii) of Proposition 4.9. They allow us to cut I into $m + 1$ subposets. If $a_j = \emptyset$ then $I_{R_j} = \emptyset$, otherwise I_{R_j} is the subposet of I of which a_j is the minimal label.

All conditions of Proposition 4.9 are satisfied and so $I \in \mathbb{B}^{(m)}(I_L, I_{R_1}, \dots, I_{R_m})$. Besides, the vertices a_1, \dots, a_m are the only one to satisfy Condition (iii) of Proposition 4.9 without adding any increasing relations to I_0 : they give the only way to cut the poset I . \square

Proof of Theorem 4.4. The proof is direct by Propositions 4.10 and 4.11 by the same reasoning as for $m = 1$ of Theorem 3.2. \square

With Propositions 4.10 and 4.11, we now have a new proof of the functional equation (4.2) already described in [7]. We can go further and give a generalized version of Theorem 1.2.

4.4. Counting smaller elements in m -Tamari.

Proposition 4.12. *Let T be a m -binary tree and $S_T := \sum_{T' \leq T} P_{[T', T]}$, the sum of all m -interval-posets with maximal tree T . If \bar{T} is composed of the m -binary trees $T_L, T_{R_1}, \dots, T_{R_m}$, then $S_T = \mathbb{B}^{(m)}(S_{T_L}, S_{T_{R_1}}, \dots, S_{T_{R_m}})$.*

Proof. Let I_0 be the interval $[T_0, T]$ where T_0 is the (n, m) -comb, i.e., the minimal m -binary tree. The increasing relations of I_0 are the ones of T and the decreasing relations are (4.5). We cut I_0 into the subposets $I_L, I_{R_1}, I_{R_2}, \dots, I_{R_m}$ following the cutting of T (the labels of I_{R_j} in I_0 are the ones of T_{R_j} in T). By construction, the m -interval-posets $I_L, I_{R_1}, \dots, I_{R_m}$ are the initial m -Tamari intervals of respectively $T_L, T_{R_1}, \dots, T_{R_m}$.

Let P be an interval-poset of the sum S_T , i.e. an poset extension of I_0 where only decreasing relations have been added. By cutting P in the same way than I_0 , then $P_L, P_{R_1}, \dots, P_{R_m}$ are extensions of respectively $I_L, I_{R_1}, \dots, I_{R_m}$ and so appear respectively in $S_{T_L}, S_{T_{R_1}}, \dots, S_{T_{R_m}}$. And by Proposition 4.9, because the increasing relations of P are those of I_0 , then $P \in \mathbb{B}^{(m)}(P_L, P_{R_1}, \dots, P_{R_m})$.

Conversely, if $P_L, P_{R_1}, \dots, P_{R_m}$ are elements of respectively $S_L, S_{R_1}, \dots, S_{R_m}$, then the increasing relations of the elements of $\mathbb{B}^{(m)}(P_L, P_{R_1}, \dots, P_{R_m})$ are those of T which make them elements of S_T . \square

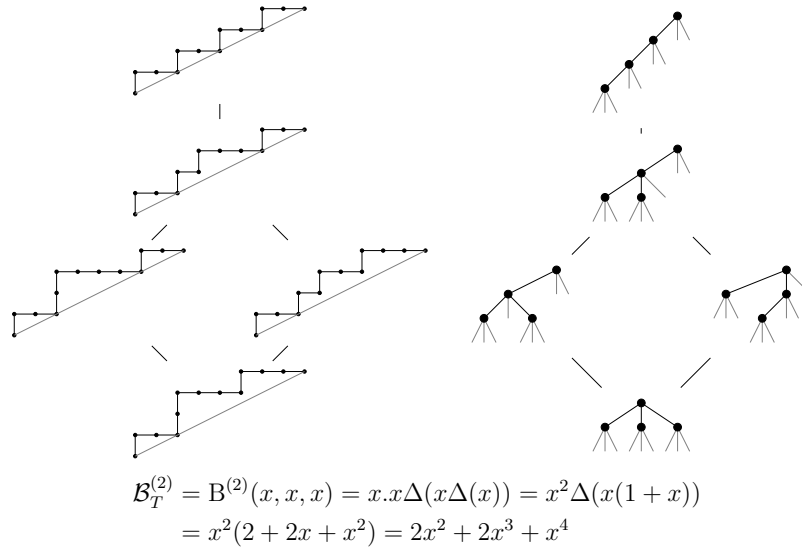


FIGURE 27. Example of $\mathcal{B}_T^{(m)}$ computation. We compute $\mathcal{B}_T^{(2)}$ for the tree at the bottom of the graph and obtain $\mathcal{B}_T^{(2)}(1) = 5$ which corresponds to the number of elements smaller than or equal to T .

Theorem 4.13. *Let T be a $(m+1)$ -ary tree, we define recursively $\mathcal{B}_T^{(m)}(x)$ by:*

$$\begin{aligned} \mathcal{B}_\emptyset^{(m)} &:= 1, \text{ and} \\ \mathcal{B}_T^{(m)} &:= \mathcal{B}^{(m)}_{y=1}(\mathcal{B}_{T_L}^{(m)}, \mathcal{B}_{T_{R_1}}^{(m)}, \dots, \mathcal{B}_{T_{R_m}}^{(m)}) \end{aligned}$$

where $T_L, T_{R_1}, \dots, T_{R_m}$ are the subtrees of T . Then $\mathcal{B}_T^{(m)}(x)$ counts the number of elements smaller than T in $\mathcal{T}_n^{(m)}$ according to the number of nodes on their leftmost branch (or the number of contacts on their ballot-path). In particular, $\mathcal{B}_T^{(m)}(1)$ is the number of elements smaller than T in $\mathcal{T}_n^{(m)}$.

See an example of this computation of Figure 27: one can check that the power of x corresponds either to the number of nodes on the leftmost branch of the tree or to the number of contacts minus 1 on ballot paths.

Proof. As in Theorem 1.2, we want to prove

$$\mathcal{B}_T^{(m)} = \mathcal{P}^{(m)}(S_T).$$

The result is obtained by an induction on n by Propositions 4.10 and 4.12

$$\begin{aligned}
 \mathcal{B}_T^{(m)} &= \mathbb{B}^{(m)}(\mathcal{B}_{T_L}^{(m)}, \mathcal{B}_{T_{R_1}}^{(m)}, \dots, \mathcal{B}_{T_{R_m}}^{(m)}) \\
 &= \mathbb{B}^{(m)}(\mathcal{P}^{(m)}(S_{T_L}), \mathcal{P}^{(m)}(S_{T_{R_1}}), \dots, \mathcal{P}^{(m)}(S_{T_{R_m}})) \\
 &= \mathcal{P}^{(m)}(\mathbb{B}^{(m)}(S_{T_L}, S_{T_{R_1}}, \dots, S_{T_{R_m}})) \\
 &= \mathcal{P}^{(m)}(S_T). \quad \square
 \end{aligned}$$

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APPENDIX A. SAGE IMPLEMENTATION OF INTERVAL-POSETS

This code is also available from the author’s webpage as a Demo sage worksheet on SageMathCloud [13].

A.1. Basic example. Below is the sage code to create an interval-poset, compute its endpoints as binary trees and the list of Dyck paths in the interval.

```
sage: ip = TamariIntervalPoset(4,[(2,1),(3,1),(2,4),(3,4)])
      ; ip
The tamari interval of size 4 induced by relations [(2, 4),
      (3, 4), (3, 1), (2, 1)]
sage: view(ip)
sage: ip.lower_binary_tree()
[[., [[., .], .], .]
sage: ip.upper_binary_tree()
[., [[., [., .]], .]]
sage: list(ip.dyck_words())
[[1, 1, 1, 0, 0, 1, 0, 0],
 [1, 1, 1, 0, 0, 0, 1, 0],
 [1, 1, 0, 1, 0, 1, 0, 0],
 [1, 1, 0, 1, 0, 0, 1, 0]]
sage: IP4 = TamariIntervalPosets(4); IP4
Interval-posets of size 4
sage: IP4.cardinality()
68
sage: ip in IP4
True
```

A.2. Composition. The composition function is not yet included in the `TamariIntervalPoset` package. But it can be easily coded using left and right products.

```
def left_product(ip1, ip2):
    size = ip1.size() + ip2.size()
    # Juxtaposition of ip1 and shifted ip2
    relations = list(ip1._cover_relations) + [(i+ip1.size(),
        j+ip1.size()) for (i,j) in ip2._cover_relations]
    # Increasing relations between ip1 and the first vertex
    # of ip2
    relations += [(i, ip1.size()+1) for i in ip1.
        increasing_roots()]
    return TamariIntervalPoset(size, relations)

def right_product(ip1, ip2):
    size = ip1.size() + ip2.size()
    # Juxtaposition of ip1 and shifted ip2
    relations = list(ip1._cover_relations) + [(i+ip1.size(),
        j+ip1.size()) for (i,j) in ip2._cover_relations]
    # First element: no extra decreasing relation
    yield TamariIntervalPoset(size, relations)
    for j in ip2.decreasing_roots():
        # Adding decreasing relations 1 by 1
        relations.append((j+ip1.size(), ip1.size()))
        yield TamariIntervalPoset(size, relations)

def composition(ip1, ip2):
    u = TamariIntervalPoset(1, [])
    left = left_product(ip1, u)
    for r in right_product(left, ip2):
        yield r
```

Here is how we now obtain the computation of Figure 13.

```
sage: ip1 = TamariIntervalPoset(3, [(1, 2), (3, 2)])
sage: ip2 = TamariIntervalPoset(4, [(2, 3), (4, 3)])
sage: list(composition(ip1, ip2))
[The tamari interval of size 8 induced by relations [(1, 2),
, (2, 4), (3, 4), (6, 7), (8, 7), (3, 2)],
The tamari interval of size 8 induced by relations [(1, 2),
, (2, 4), (3, 4), (6, 7), (8, 7), (5, 4), (3, 2)],
The tamari interval of size 8 induced by relations [(1, 2),
, (2, 4), (3, 4), (6, 7), (8, 7), (6, 4), (5, 4), (3,
2)],
```

The tamari interval of size 8 induced by relations $[(1, 2), (2, 4), (3, 4), (6, 7), (8, 7), (7, 4), (6, 4), (5, 4), (3, 2)]$

1.3. **m -Composition.** To obtain the m -composition, we add a function corresponding to $\frac{\delta}{x}$ and then follow the definition of Proposition 4.8.

```
def right_product_dim(ip1, ip2):
    size = ip1.size() + ip2.size()
    # Juxtaposition of ip1 and shifted ip2
    relations = list(ip1._cover_relations) + [(i+ip1.size(),
        j+ip1.size()) for (i,j) in ip2._cover_relations]
    for j in ip2.decreasing_roots():
        # Adding decreasing relations 1 by 1
        relations.append((j+ip1.size(), ip1.size()))
    yield TamariIntervalPoset(size, relations)

def mcomposition(ips):
    rights = ips[1:] # we take the list of intervals
    except the first one
    def compute_rights(rights): # we define a recursive
    method to compute the right part
        u = TamariIntervalPoset(1, [])
        if len(rights)==1:
            for r in right_product(u, rights[0]):
                yield r
        else:
            for r1 in compute_rights(rights[1:]):
                r1 = left_product(r1, rights[0])
                for r2 in right_product_dim(u, r1):
                    yield r2
    for r in compute_rights(rights):
        yield left_product(ips[0], r)
```

We can now compute the example of (4.16).

```
sage: ip1 = TamariIntervalPoset(2, [(2, 1)])
sage: ip2 = TamariIntervalPoset(4, [(2, 1), (4, 3), (2, 3)])
sage: ip3 = ip1
sage: list(mcomposition([ip1, ip2, ip3]))
[The tamari interval of size 10 induced by relations [(1,
3), (2, 3), (4, 7), (5, 7), (6, 7), (8, 9), (10, 9), (8,
7), (6, 5), (4, 3), (2, 1)],
```


- The tamari interval of size 10 induced by relations $[(1, 3), (2, 3), (4, 7), (5, 7), (6, 7), (8, 9), (10, 9), (8, 7), (6, 5), (5, 3), (4, 3), (2, 1)]$,
- The tamari interval of size 10 induced by relations $[(1, 3), (2, 3), (4, 7), (5, 7), (6, 7), (8, 9), (10, 9), (8, 7), (7, 3), (6, 5), (5, 3), (4, 3), (2, 1)]$,
- The tamari interval of size 10 induced by relations $[(1, 3), (2, 3), (4, 7), (5, 7), (6, 7), (8, 9), (10, 9), (9, 3), (8, 7), (7, 3), (6, 5), (5, 3), (4, 3), (2, 1)]$,
- The tamari interval of size 10 induced by relations $[(1, 3), (2, 3), (4, 7), (5, 7), (6, 7), (8, 9), (10, 9), (8, 7), (6, 5), (5, 4), (4, 3), (2, 1)]$,
- The tamari interval of size 10 induced by relations $[(1, 3), (2, 3), (4, 7), (5, 7), (6, 7), (8, 9), (10, 9), (8, 7), (7, 3), (6, 5), (5, 4), (4, 3), (2, 1)]$,
- The tamari interval of size 10 induced by relations $[(1, 3), (2, 3), (4, 7), (5, 7), (6, 7), (8, 9), (10, 9), (9, 3), (8, 7), (7, 3), (6, 5), (5, 4), (4, 3), (2, 1)]$