

# PAVING RECTANGULAR REGIONS WITH RECTANGULAR TILES: TATAMI AND NON-TATAMI TILINGS.

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ABSTRACT. The number of complete tilings of  $m \times n$  floors for tiles of shape  $1 \times 2$ ,  $1 \times 3$ ,  $1 \times 4$  and  $2 \times 3$  is computed numerically for floors up to width  $m = 9$  and variable floor lengths  $n$ . Counts are obtained for two classes, for fixed tile stack orientation on one hand and for counts up to rotations and reflections on the other hand. Counts are refined by the number of points on the floor where 4 tiles meet, i.e., by the degree of violation of the requirement for Tatami tilings.

## 1. DEFINITIONS

**1.1. Rectangular floors and tiles.** We consider floors of rectangular dimension  $m \times n$  laid out by unit squares. We count full covers with tiles of width  $t_m$  and length  $t_n$ . The tiles may be of mixed orientation—of which there are two, aligning  $t_n$  with  $n$  or with  $m$ . Only one tile shape is considered at a time; hybrid covers by dominos ( $1 \times 2$  tiles) and unit-tiles, for example, will not be discussed.

An obvious requirement of each complete tiling of the floor is that the ratio of the floor area by the tile area is an integer:

**Definition 1.** (*Matching Condition*)

$$(1) \quad nm \equiv 0 \pmod{t_n t_m}.$$

Another appropriate constraint is that  $t_n$  and  $t_m$  are coprime, since otherwise a trivial congruent scaling of the floor geometry and dimensions by the largest common factor would generate essentially the same results.

We write  $T(n, m)$  for the counts of different tilings of the  $m \times n$  floor, but do not add labels that make the dependence on the side lengths  $t_n$  and  $t_m$  of the tile explicit.

**1.2. Tatami condition.** Four tiles may meet at between 0 and the maximum of  $\lfloor (n-1)/t_n \rfloor \lfloor (m-1)/t_m \rfloor$  and  $\lfloor (n-1)/t_m \rfloor \lfloor (m-1)/t_n \rfloor$  points at corners of the unit base squares. (This upper limit is poor in the sense that is an upper estimate assuming that all tiles are aligned vertically or horizontally.) At points of that type, the edges of the tiles form a 4-way crossing.

**Definition 2.** (*Counts by number of 4 tile meets*)  $T_t(n, m)$  denotes the number of tilings which contain  $t$  points where 4 tiles meet.

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If we call tilings where no four tiles meet “Tatami tilings,” the number of Tatami tilings is  $T_0(n, m)$ . The  $T_t(n, m)$  with  $t > 0$  count tilings that violate the Tatami property.

This work shows tables of  $T_t(n, m)$  where the floor width  $m$  and the tile shape  $t_n$  and  $t_m$  are fixed, where the floor length  $n$  increases along the table rows, and where  $t$  increases along the table columns (Tables 1–59). The leftmost column in the tables shows  $n$  whenever the matching condition is fulfilled. The second column in the tables are the row sums,

$$(2) \quad T(n, m) = \sum_{t \geq 0} T_t(n, m).$$

Ordinary generating functions downward columns are noted as follows:

**Definition 3.** (*Generating function for tilings where  $t$  points exists where 4 tiles meet*)

$$(3) \quad T_t(z, m) \equiv \sum_{n \geq 0} T_t(n, m)z^n.$$

**Definition 4.** (*Generating function for unrestricted tilings*)

$$(4) \quad T(z, m) = \sum_{n \geq 0} T(n, m)z^n.$$

Mutual insertion of the previous three equations shows that the generating function of the row sums is the sum over the generating functions of the columns,

$$(5) \quad T(z, m) = \sum_{t \geq 0} T_t(z, m).$$

Floor tilings may be equivalent (congruent) in the sense that reflections of the entire stack of tiles along the horizontal and/or vertical axis through the middle of the floor displays another tiling. This group of symmetry operations of the rectangle will be augmented by the 90-degree rotations if the floor is a square, i.e., if  $m = n$ . If only one representative of the set of 4 or 8 congruent tilings is counted, an overbar is added to the capital  $T$  to denote the number of incongruent tilings:

**Definition 5.** (*Counts by number of 4 tile meets*)  $\bar{T}_t(n, m)$  denotes the number of incongruent tilings which contain  $t$  points where 4 tiles meet.

Row sums and generating functions for counts of incongruent tilings are defined as for the full counts:

**Definition 6.** (*Number of incongruent tilings of the  $m \times n$  floor*)

$$(6) \quad \bar{T}(n, m) = \sum_{t \geq 0} \bar{T}_t(n, m).$$

**Definition 7.** (*Generating function of the incongruent tilings which contain  $t$  points where 4 tiles meet*)

$$(7) \quad \bar{T}_t(z, m) = \sum_{n \geq 0} \bar{T}_t(n, m)z^n.$$

**Definition 8.** (*Generating function for the number of incongruent tilings*)

$$(8) \quad \bar{T}(z, m) = \sum_{n \geq 0} \bar{T}(n, m)z^n.$$

The reader should keep in mind that these notations do not show the dependence on the tile's dimensions. We count coverage by  $1 \times 2$  tiles in Section 2, by  $1 \times 3$  tiles in Section 3, by  $1 \times 4$  tiles in Section 4, and by  $2 \times 3$  tiles in Section 5; each section defines a different set of  $T$ ,  $\bar{T}$  and associated generating functions.

**1.3. Classification by Slide Line Count.** Tilings of a  $m \times n$  floor may be stacks of tilings of the  $m_1 \times n$  floor and of the  $m_2 \times n$  floor where  $m = m_1 + m_2$  (or stacks of three or more such tilings). The resultant stack then has one or more “slide lines” that run parallel to the long edge of the floor and do not cut through any of the tiles.

Similar to the degree of violation of the Tatami property, the number of slides lines (between 0 and  $m - 1$ , inclusive) allows a (rough) classification of all tilings of the  $m \times n$  floor with tiles of shape  $t_m \times t_n$ .

**Definition 9.**  $\hat{T}_s(n, m)$  denotes the number of tilings of the  $m \times n$  floor with  $s$  slide lines.

**Definition 10.**  $\hat{\hat{T}}_s(n, m)$  denotes the number of incongruent tilings of the  $m \times n$  floor with  $s$  slide lines.

Equivalent to (2), tables of  $\hat{T}_s(n, m)$  have row sums  $T(n, m)$  and generating functions along the columns:

$$(9) \quad T(n, m) = \sum_{s=0}^{m-1} \hat{T}_s(n, m);$$

$$(10) \quad T(z, m) = \sum_{s=0}^{m-1} \hat{T}_s(z, m).$$

Section 6 is dedicated to tables of tilings refined according to their number of side lines.

## 2. DOMINO TILING

Tilings with  $1 \times 2$  tiles (also known as dominos) are represented by Tables 1–16.

**2.1. Results (full count).** The values for Tatami tilings (domino tilings)—represented by the columns  $T_0(n, m)$  in Tables 1–8 and  $\bar{T}_0(n, m)$  in 10–16 are well understood by a previous enumeration by Ruskey and Woodcock [6]. The row sums  $T(n, m)$  of Tables 2–8 have already been reported by Klarner and Pollack [5, 4, 8, 9].

We turn to a discussion of individual tables.

Row sums of Table 1 are the Fibonacci numbers, sequence A000045 in the Encyclopedia of Integer Sequences [7, A000045]. The generating function for column 0 is [7, A068921]

**Theorem 1.** (Table 1)

$$(11) \quad T_0(z, 2) = \frac{1 + z^2}{1 - z - z^3}.$$

**Conjecture 1.** (Table 1)

$$(12) \quad T_1(z, 2) = z^4 \frac{1}{(1 - z - z^3)^2}.$$

TABLE 1. Number  $T(n, 2)$  and  $T_t(n, 2)$  of domino tilings of  $2 \times n$  boards.

$n$		0	1	2	3	4	5	6	7	8	9	10
1	1	1	0	0	0	0	0	0	0	0	0	0
2	2	2	0	0	0	0	0	0	0	0	0	0
3	3	3	0	0	0	0	0	0	0	0	0	0
4	5	4	1	0	0	0	0	0	0	0	0	0
5	8	6	2	0	0	0	0	0	0	0	0	0
6	13	9	3	1	0	0	0	0	0	0	0	0
7	21	13	6	2	0	0	0	0	0	0	0	0
8	34	19	11	3	1	0	0	0	0	0	0	0
9	55	28	18	7	2	0	0	0	0	0	0	0
10	89	41	30	14	3	1	0	0	0	0	0	0
11	144	60	50	24	8	2	0	0	0	0	0	0
12	233	88	81	43	17	3	1	0	0	0	0	0
13	377	129	130	77	30	9	2	0	0	0	0	0
14	610	189	208	132	57	20	3	1	0	0	0	0
15	987	277	330	224	108	36	10	2	0	0	0	0
16	1597	406	520	379	193	72	23	3	1	0	0	0
17	2584	595	816	633	342	143	42	11	2	0	0	0
18	4181	872	1275	1047	605	264	88	26	3	1	0	0
19	6765	1278	1984	1722	1052	485	182	48	12	2	0	0
20	10946	1873	3077	2814	1808	891	345	105	29	3	1	0
21	17711	2745	4758	4570	3088	1602	654	225	54	13	2	0
22	28657	4023	7337	7385	5232	2843	1242	436	123	32	3	1
23	46368	5896	11286	11880	8796	5014	2298	850	272	60	14	2
24	75025	8641	17322	19029	14699	8760	4193	1663	537	142	35	3
25	121393	12664	26532	30363	24426	15167	7606	3155	1074	323	66	15
26	196418	18560	40563	48279	40371	26084	13650	5900	2159	648	162	38
27	317811	27201	61908	76518	66404	44571	24250	10976	4188	1327	378	72

**Conjecture 2.** (Table 1)

$$(13) \quad T_2(z, 2) = z^6 \frac{1-z}{(1-z-z^3)^3}.$$

Column  $T(n, m)$  of Table 2 is [7, A001835]:

**Theorem 2.** (Table 2)

$$(14) \quad T(z, 3) = \frac{1-z^2}{1-4z^2+z^4}.$$

Column  $T_0(n, m)$  is [7, A068922]:

**Theorem 3.** (Table 2)

$$(15) \quad T_0(z, 3) = z^2 - 1 + \frac{2}{1-z^2-z^4}.$$

For the next two columns we have:

**Conjecture 3.** (Table 2)

$$(16) \quad T_1(z, 3) = 2z^6 \frac{3+3z^2-3z^4-2z^6}{(1-z^2-z^4)^2}.$$

TABLE 2. Number  $T(n, 3)$  and  $T_t(n, 3)$  of domino tilings of  $3 \times n$  boards.

$n$		0	1	2	3	4	5	6	7	8
2	3	3	0	0	0	0	0	0	0	0
4	11	4	6	1	0	0	0	0	0	0
6	41	6	18	12	4	1	0	0	0	0
8	153	10	36	58	32	12	4	1	0	0
10	571	16	74	156	174	92	40	14	4	1
12	2131	26	142	384	578	518	284	128	50	16
14	7953	42	268	860	1646	1998	1578	886	422	170
16	29681	68	494	1838	4202	6408	6672	4912	2816	1374
18	110771	110	898	3780	10024	18238	23500	22004	15564	9010
20	413403	178	1612	7566	22732	47852	73190	83316	72300	49900
22	1542841	288	2866	14816	49638	118242	208586	279346	289268	237802
24	5757961	466	5054	28512	105190	279056	556128	854582	1030190	992446
26	21489003	754	8852	54080	217586	634978	1407596	2435918	3348484	3712826
28	80198051	1220	15414	101338	441146	1402550	3417114	6564072	10121734	12706318
30	299303201	1974	26706	187932	879436	3022324	8016016	16898784	28848726	40432874
32	1117014753	3194	46068	345410	1728056	6377980	18272816	41888806	78334170	121156904

**Conjecture 4.** (Table 2)

$$(17) \quad T_2(z, 3) = z^4 + 2z^6 \frac{6 + 11z^2 - 9z^4 - 12z^6 - z^8 + z^{10}}{(1 - z^2 - z^4)^3}.$$

Column  $T(n, m)$  of Table 3 is [7, A005178]:

**Theorem 4.** (Table 3)

$$(18) \quad T(z, 4) = \frac{1 - z^2}{1 - z - 5z^2 - z^3 + z^4}.$$

Column  $T_0(n, m)$  is [7, A068923]:

**Theorem 5.** (Table 3)

$$(19) \quad T_0(z, 4) = -1 + 3z^2 + 2z^3 + \frac{2 + z + z^2 + z^4}{1 - z^3 - z^5}.$$

Column  $T(n, m)$  of Table 4 is [7, A003775]; column  $T_0(n, m)$  is [7, A068924].

Column  $T(n, m)$  of Table 5 is [7, A028468] and column  $T_0(n, m)$  is [7, A068925], so the two generating functions are known:

**Theorem 6.** (Table 5)

$$(20) \quad T(z, 6) = \frac{1}{13} \left[ \frac{5 + 12z + 3z^2}{1 + 5z + 6z^2 + z^3} + \frac{6 - 10z + 3z^2}{1 - 6z + 5z^2 - z^3} + \frac{1}{1 + z} + \frac{1}{1 - z} \right].$$

**Theorem 7.** (Table 5)

$$(21) \quad T_0(z, 6) = -1 + 9z^2 + 6z^3 + 2z^4 + \frac{1}{13} \left[ \frac{12 - 9z}{1 - z + z^2} + \frac{14 + 24z + 19z^2 + 7z^3 + 4z^4}{1 + z - z^3 - z^4 - z^5} \right].$$

Column  $T(n, m)$  of Table 6 is [7, A028469].

Column  $T(n, m)$  of Table 7 is [7, A028470].

Column  $T(n, m)$  of Table 8 is [7, A028471].

TABLE 3. Number  $T(n, 4)$  and  $T_t(n, 4)$  of domino tilings of  $4 \times n$  boards.

$n$		0	1	2	3	4	5	6	7	8
1	1	1	0	0	0	0	0	0	0	0
2	5	4	1	0	0	0	0	0	0	0
3	11	4	6	1	0	0	0	0	0	0
4	36	2	14	18	2	0	0	0	0	0
5	95	3	20	43	26	3	0	0	0	0
6	281	3	24	74	113	56	10	1	0	0
7	781	3	30	123	248	272	88	15	2	0
8	2245	5	34	173	477	717	596	192	41	9
9	6336	5	38	252	792	1581	1962	1260	358	72
10	18061	6	53	318	1254	3036	4848	4906	2614	780
11	51205	8	64	404	1864	5361	10426	13826	11720	5479
12	145601	8	79	553	2600	8817	20258	32969	37265	26921
13	413351	11	98	717	3600	13656	36536	70299	97856	95707
14	1174500	13	116	937	4960	20364	61894	137367	227287	275616
15	3335651	14	150	1207	6760	29639	99486	251723	479984	694119
16	9475901	19	181	1532	9132	42467	154532	435514	944178	1576953
17	26915305	21	220	1989	12146	60088	233572	721548	1747606	3315355
18	76455961	25	277	2525	16126	83677	346240	1155602	3079704	6537383
19	217172736	32	330	3203	21298	115525	504476	1800597	5218642	12216799
20	616891945	35	413	4075	27895	158158	724090	2746127	8557790	21856138
21	1752296281	44	506	5126	36468	214484	1028182	4110548	13666350	37689958
22	4977472781	53	608	6491	47369	289373	1443894	6059733	21341722	63027592
23	14138673395	60	762	8157	61352	387604	2009478	8818495	32701156	102707437

TABLE 4. Number  $T(n, 5)$  and  $T_t(n, 5)$  of domino tilings of  $5 \times n$  boards.

$n$		0	1	2	3	4	5	6	7	8
2	8	6	2	0	0	0	0	0	0	0
4	95	3	20	43	26	3	0	0	0	0
6	1183	2	32	147	332	343	220	92	14	1
8	14824	2	38	271	1046	2695	3730	3370	2206	1061
10	185921	4	38	422	2302	8144	21058	35753	41254	35275
12	2332097	4	68	532	4074	19405	65662	169536	319122	439701
14	29253160	6	82	864	6206	37520	163410	538759	1390882	2792397
16	366944287	8	114	1224	10120	64464	339114	1382583	4458614	11576972

TABLE 5. Number  $T(n, 6)$  and  $T_t(n, 6)$  of domino tilings of  $6 \times n$  boards.

$n$		0	1	2	3	4	5	6	7	8	9	10
1	1	1	0	0	0	0	0	0	0	0	0	0
2	13	9	3	1	0	0	0	0	0	0	0	0
3	41	6	18	12	4	1	0	0	0	0	0	0
4	281	3	24	74	113	56	10	1	0	0	0	0
5	1183	2	32	147	332	343	220	92	14	1	0	0
6	6728	2	24	202	714	1326	1842	1508	782	288	38	2
7	31529	2	38	255	1214	3242	6126	7909	6784	3888	1576	438
8	167089	1	37	290	1665	6315	15372	27943	36693	35232	24777	12630
9	817991	1	42	350	2288	10342	32878	73852	126190	167827	167658	125352
10	4213133	2	39	429	2990	15183	57599	162389	350681	588547	783469	820190
11	21001799	3	24	470	3756	21304	93950	313656	817194	1673001	2739234	3644387
12	106912793	4	39	567	4624	28961	141741	543976	1654825	4022461	7941693	12855987
13	536948224	3	46	525	5408	37887	203462	880052	3045892	8594250	19871300	37953976

TABLE 6. Number  $T(n, 7)$  and  $T_t(n, 7)$  of domino tilings of  $7 \times n$  boards.

$n$		0	1	2	3	4	5	6	7	8	9	10
2	21	13	6	2	0	0	0	0	0	0	0	0
4	781	3	30	123	248	272	88	15	2	0	0	0
6	31529	2	38	255	1214	3242	6126	7909	6784	3888	1576	438
8	1292697	2	36	372	2606	11163	37162	90677	170694	244705	265598	220083
10	53175517	0	38	470	3802	24137	113316	404254	1148114	2630509	4931018	7521981
12	2188978117	2	28	528	5470	40512	239344	1130007	4329026	13626282	35621942	78462193

TABLE 7. Number  $T(n, 8)$  and  $T_t(n, 8)$  of domino tilings of  $8 \times n$  boards.

$n$		0	1	2	3	4	5	6	7	8	9	10
1	1	1	0	0	0	0	0	0	0	0	0	0
2	34	19	11	3	1	0	0	0	0	0	0	0
3	153	10	36	58	32	12	4	1	0	0	0	0
4	2245	5	34	173	477	717	596	192	41	9	1	0
5	14824	2	38	271	1046	2695	3730	3370	2206	1061	324	68
6	167089	1	37	290	1665	6315	15372	27943	36693	35232	24777	12630
7	1292697	2	36	372	2606	11163	37162	90677	170694	244705	265598	220083
8	12988816	2	34	310	3110	16712	68504	224484	569884	1127972	1798390	2307764
9	108435745	2	42	445	4058	24514	119062	453678	1403192	3503161	7121656	11916877
10	1031151241	1	29	460	4313	30449	176077	785735	2864328	8638123	21830440	46070190

TABLE 8. Number  $T(n, 9)$  and  $T_t(n, 9)$  of domino tilings of  $9 \times n$  boards.

$n$		0	1	2	3	4	5	6	7	8	9	10
2	55	28	18	7	2	0	0	0	0	0	0	0
4	6336	5	38	252	792	1581	1962	1260	358	72	14	2
6	817991	1	42	350	2288	10342	32878	73852	126190	167827	167658	125352
8	108435745	2	42	445	4058	24514	119062	453678	1403192	3503161	7121656	11916877

TABLE 9. Number  $\bar{T}(n, 2)$  and  $\bar{T}_t(n, 2)$  of incongruent domino tilings of  $2 \times n$  boards.

$n$		0	1	2	3	4	5	6	7	8	9	10	11
1	1	1	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0
3	2	2	0	0	0	0	0	0	0	0	0	0	0
4	4	3	1	0	0	0	0	0	0	0	0	0	0
5	5	4	1	0	0	0	0	0	0	0	0	0	0
6	9	6	2	1	0	0	0	0	0	0	0	0	0
7	12	8	3	1	0	0	0	0	0	0	0	0	0
8	21	12	6	2	1	0	0	0	0	0	0	0	0
9	30	16	9	4	1	0	0	0	0	0	0	0	0
10	51	24	16	8	2	1	0	0	0	0	0	0	0
11	76	33	25	13	4	1	0	0	0	0	0	0	0
12	127	49	42	24	9	2	1	0	0	0	0	0	0
13	195	69	65	40	15	5	1	0	0	0	0	0	0
14	322	102	106	70	30	11	2	1	0	0	0	0	0
15	504	145	165	115	54	19	5	1	0	0	0	0	0
16	826	214	263	196	99	39	12	2	1	0	0	0	0
17	1309	307	408	322	171	73	21	6	1	0	0	0	0
18	2135	452	642	535	306	137	46	14	2	1	0	0	0
19	3410	653	992	870	526	246	91	25	6	1	0	0	0
20	5545	960	1545	1426	910	454	176	56	15	2	1	0	0
21	8900	1393	2379	2300	1544	808	327	114	27	7	1	0	0
22	14445	2046	3678	3723	2626	1438	626	224	64	17	2	1	0
23	23256	2978	5643	5965	4398	2519	1149	429	136	31	7	1	0
24	37701	4371	8675	9564	7365	4409	2106	842	273	75	18	2	1
25	60813	6376	13266	15222	12213	7605	3803	1586	537	163	33	8	1
26	98514	9354	20302	24219	20210	13091	6842	2972	1086	331	84	20	2
27	159094	13665	30954	38324	33202	22324	12125	5503	2094	668	189	37	8
28	257608	20041	47198	60582	54412	37926	21414	10128	4017	1380	390	96	21
29	416325	29307	71770	95372	88678	63966	37455	18366	7621	2716	805	220	39

2.2. **Results (incongruent).** Counts of domino tilings where only one representative of the roto-reflected copies of each tiling is counted are shown in Tables 9–16.

Table 9 is characterized by:

**Theorem 8.** (Table 9 [7, A060312])

$$(22) \quad \bar{T}(z, 2) = -z^2 + \frac{1}{2} \left[ \frac{1}{1-z-z^2} + \frac{1+z+z^2}{1-z^2-z^4} \right].$$

**Theorem 9.** (Table 9 [7, A068927])

$$(23) \quad \bar{T}_0(z, 2) = -z^2 + \frac{1}{2} \left[ \frac{1+z+z^2+z^5}{1-z^2-z^6} + \frac{1+z^2}{1-z-z^3} \right].$$

**Conjecture 5.** (Table 9)

$$(24) \quad \bar{T}_1(z, 2) = z^4 \frac{(1+z+z^2)(1-z)^2}{(1-z^2-z^6)(1-z-z^3)^2}$$

Column  $\bar{T}_0(n, 3)$  of Table 10 is found in [7, A068928], which shows



TABLE 10. Number  $\bar{T}(n, 3)$  and  $\bar{T}_t(n, 3)$  of incongruent domino tilings of  $3 \times n$  boards.

$n$		0	1	2	3	4	5	6	7	8	9
2	2	2	0	0	0	0	0	0	0	0	0
4	5	2	2	1	0	0	0	0	0	0	0
6	14	2	5	5	1	1	0	0	0	0	0
8	46	4	10	17	9	4	1	1	0	0	0
10	156	5	19	43	45	27	10	5	1	1	0
12	561	9	37	102	148	137	73	35	13	5	1
14	2037	12	68	223	414	514	398	232	106	47	15
16	7525	21	126	473	1058	1626	1679	1248	709	353	148
18	27874	30	226	960	2512	4596	5885	5549	3899	2282	1131
20	103741	51	407	1919	5699	12027	18329	20908	18106	12535	7287
22	386386	76	719	3732	12421	29645	52174	69982	72350	59598	40361
24	1440946	127	1270	7182	26329	69916	139112	213893	257658	248374	196028
26	5374772	195	2217	13571	54419	158927	351963	609370	837226	928729	845193
28	20054945	322	3864	25437	110347	350976	854462	1641695	2530761	3177495	3292433
30	74835209	504	6683	47075	219901	755959	2004146	4225650	7212468	10109805	11772446

**Theorem 10.** (Table 10)

$$(25) \quad \bar{T}_0(z, 3) = z^2 + \frac{1}{2} \left[ \frac{1 + z^2 + z^4}{1 - z^4 - z^8} + \frac{1}{1 - z^2 - z^4} \right].$$

For the row sums we formulate

**Conjecture 6.** (Table 10)

$$(26) \quad \bar{T}(z, 3) = \frac{1}{8} \left[ \frac{1}{1+z} + \frac{1}{1-z} \right] + \frac{1 + 2z^2 - z^6}{2(1 - 4z^4 + z^8)} + \frac{1 - z^2}{4(1 - 4z^2 + z^4)}.$$

Column  $\bar{T}_0(n, 4)$  of Table 11 is found in [7, A068929], which provides

**Theorem 11.** (Table 11)

$$(27) \quad \bar{T}_0(z, 4) = 2z^2 + z^3 - z^4 + \frac{1}{2} \left[ z \frac{1 + z + 2z^3 + z^4 + z^5 + z^8}{1 - z^6 - z^{10}} + \frac{2 + z + z^2 + z^4}{1 - z^3 - z^5} \right].$$

Column  $\bar{T}_0(n, 5)$  of Table 12 is [7, A068930].

Column  $\bar{T}_0(n, 6)$  of Table 13 is [7, A068931].

TABLE 11. Number  $\bar{T}(n, 4)$  and  $\bar{T}_t(n, 4)$  of incongruent domino tilings of  $4 \times n$  boards.

$n$		0	1	2	3	4	5	6	7	8	9
1	1	1	0	0	0	0	0	0	0	0	0
2	4	3	1	0	0	0	0	0	0	0	0
3	5	2	2	1	0	0	0	0	0	0	0
4	9	1	3	4	1	0	0	0	0	0	0
5	33	2	6	15	8	2	0	0	0	0	0
6	98	2	9	25	34	22	5	1	0	0	0
7	230	2	10	38	67	80	26	6	1	0	0
8	658	3	12	57	134	203	166	62	16	4	1
9	1725	3	13	77	214	428	512	346	102	24	5
10	4876	4	19	100	347	819	1272	1300	702	230	61
11	13378	5	22	124	505	1421	2690	3577	3009	1451	428
12	37794	5	30	171	715	2347	5236	8493	9535	6947	3007
13	105761	6	35	221	983	3597	9383	17960	24818	24353	15439
14	299221	8	43	292	1369	5386	15924	35057	57610	69816	60038
15	844219	8	56	372	1857	7809	25514	64024	121268	175164	187455
16	2392040	11	68	480	2530	11214	39713	110741	238496	397460	508485
17	6773154	12	83	617	3359	15854	59923	183201	440817	834301	1236712
18	19211023	14	108	792	4489	22117	88979	293450	776790	1644337	2768166
19	54485124	17	125	1006	5930	30525	129574	456942	1315446	3070613	5787975
20	154636939	20	162	1284	7813	41885	186189	697095	2157279	5492248	11435056
21	438909205	24	196	1617	10217	56806	264434	1043191	3444055	9467905	21540497

TABLE 12. Number  $\bar{T}(n, 5)$  and  $\bar{T}_t(n, 5)$  of incongruent domino tilings of  $5 \times n$  boards.

$n$		0	1	2	3	4	5	6	7	8	9	10
2	5	4	1	0	0	0	0	0	0	0	0	0
4	33	2	6	15	8	2	0	0	0	0	0	0
6	329	1	8	43	88	96	58	30	4	1	0	0
8	3818	1	10	75	266	704	945	869	561	277	85	21
10	46878	1	10	113	579	2070	5288	9024	10354	8904	5815	3054
12	584386	2	17	141	1026	4903	16447	42564	79871	110221	114726	94591
14	7318152	2	21	221	1557	9441	40889	134928	347878	698731	1081974	1313568
16	91752831	3	29	318	2535	16202	84840	346053	1114901	2895523	6060041	10207144

TABLE 13. Number  $\bar{T}(n, 6)$  and  $\bar{T}_t(n, 6)$  of incongruent domino tilings of  $6 \times n$  boards.

$n$		0	1	2	3	4	5	6	7	8	9	10
1	1	1	0	0	0	0	0	0	0	0	0	0
2	9	6	2	1	0	0	0	0	0	0	0	0
3	14	2	5	5	1	1	0	0	0	0	0	0
4	98	2	9	25	34	22	5	1	0	0	0	0
5	329	1	8	43	88	96	58	30	4	1	0	0
6	930	1	4	29	96	181	247	211	105	48	7	1
7	8121	1	11	69	310	834	1558	2032	1733	1024	405	126
8	42873	1	13	82	438	1642	3926	7143	9339	9018	6335	3296
9	206420	1	13	97	587	2635	8279	18624	31728	42279	42204	31700
10	1060866	1	12	127	781	3903	14570	40942	88162	147973	196828	206307
11	5265647	2	7	130	970	5423	23645	78788	204821	419280	686092	913146
12	26782279	2	12	164	1207	7403	35771	136737	414961	1007932	1988590	3219095

TABLE 14. Number  $\bar{T}(n, 7)$  and  $\bar{T}_t(n, 7)$  of incongruent domino tilings of  $7 \times n$  boards.

$n$		0	1	2	3	4	5	6	7	8	9	10
2	12	8	3	1	0	0	0	0	0	0	0	0
4	230	2	10	38	67	80	26	6	1	0	0	0
6	8121	1	11	69	310	834	1558	2032	1733	1024	405	126
8	324617	1	9	98	661	2841	9347	22828	42804	61449	66562	55266
10	13303375	0	10	123	958	6083	28402	101293	287287	658240	1233331	1881672

TABLE 15. Number  $\bar{T}(n, 8)$  and  $\bar{T}_t(n, 8)$  of incongruent domino tilings of  $8 \times n$  boards.

$n$		0	1	2	3	4	5	6	7	8	9	10
1	1	1	0	0	0	0	0	0	0	0	0	0
2	21	12	6	2	1	0	0	0	0	0	0	0
3	46	4	10	17	9	4	1	1	0	0	0	0
4	658	3	12	57	134	203	166	62	16	4	1	0
5	3818	1	10	75	266	704	945	869	561	277	85	21
6	42873	1	13	82	438	1642	3926	7143	9339	9018	6335	3296
7	324617	1	9	98	661	2841	9347	22828	42804	61449	66562	55266
8	1629189	1	6	43	400	2133	8647	28309	71540	141600	225402	289306
9	27129182	1	12	119	1027	6191	29862	113815	351290	877086	1781753	2981718

TABLE 16. Number  $\bar{T}(n, 9)$  and  $\bar{T}_t(n, 9)$  of incongruent domino tilings of  $9 \times n$  boards.

$n$		0	1	2	3	4	5	6	7	8	9	10
2	30	16	9	4	1	0	0	0	0	0	0	0
4	1725	3	13	77	214	428	512	346	102	24	5	1
6	206420	1	13	97	587	2635	8279	18624	31728	42279	42204	31700
8	27129182	1	12	119	1027	6191	29862	113815	351290	877086	1781753	2981718

3. TILING WITH  $1 \times 3$  TILES

Tilings with tiles of shape  $1 \times 3$  are represented by Tables 17–31.

**3.1. Results (full count).** The row sums  $T(n, 3)$  in Table 17 are found in [7, A000930].

**Theorem 12.** (Table 17)

$$(28) \quad T(z, 3) = \frac{1}{1 - z - z^3}.$$

Column  $T_0(n, 3)$  appears to be [7, A003269]:

**Theorem 13.** (Table 17)

$$(29) \quad T_0(z, 3) = \frac{1 + z^3}{1 - z - z^4}.$$

**Conjecture 7.** (Table 17)

$$(30) \quad T_t(z, 3) = \begin{cases} z^{3(t/2+1)} \frac{(1-z)^{t/2-1}}{(1-z-z^4)^{t/2+1}}, & \text{even } t > 0 \\ 0, & \text{odd } t. \end{cases}$$

The previous three equations comply with the sum rule (5).

The row sums  $T(n, 4)$  of Table 19 are [7, A049086]:

**Theorem 14.** (Table 19)

$$(31) \quad T(z, 4) = \frac{1 - 2z^3 + z^6}{1 - 5z^3 + 3z^6 - z^9}.$$

For its column  $T_0$  we suspect

**Conjecture 8.** (Table 19)

$$(32) \quad T_0(z, 4) = -1 + z^3 + \frac{2}{1 - z^3 - 2z^6 - z^9},$$

which means these are essentially the sequence [7, A002478] multiplied by 2.

The denominator of the generating function of this conjecture claims the recurrence  $T_0(n, 4) = T_0(n - 3, 4) + 2T_0(n - 6, 4) + T_0(n - 9, 4)$ . This is interpreted with the aid of Table 18 as follows: To tile the  $4 \times n$  floor, take

- (1) a tiling of the  $4 \times (n - 3)$  floor and attach a  $4 \times 3$  super-tile with the one (out of 2) orientations that avoid a 4-crossing at the interface, or
- (2) take a tiling of the  $4 \times (n - 6)$  floor and attach a  $4 \times 6$  super-tile which come in two shapes, one with a block of 2 vertical tiles at the left and one with a block of 2 vertical tiles at the right, or
- (3) take a tiling of the  $4 \times (n - 9)$  floor and attach the  $4 \times 9$  super-tile with a slide line with the one (out of 2) orientations—second line in Table 18—that avoid a 4-crossing at the interface.

The claim is essentially that the final  $4 \times 9$  super-tile of the  $4 \times n$  Tatami tiling with  $1 \times 3$  tiles is always one of the shapes in Table 18.

The row sums in Table 20 are presumably generated by

**Conjecture 9.** (Table 20)

$$(33) \quad T(z, 3) = \frac{1 - 2z^3 + z^6}{1 - 6z^3 + 3z^6 - z^9}.$$

TABLE 17. Number  $T(n, 3)$  and  $T_t(n, 3)$  of tilings of  $3 \times n$  boards with  $1 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	6	5	0	1	0	0	0	0	0	0	0	0	0	0	0	0
7	9	7	0	2	0	0	0	0	0	0	0	0	0	0	0	0
8	13	10	0	3	0	0	0	0	0	0	0	0	0	0	0	0
9	19	14	0	4	0	1	0	0	0	0	0	0	0	0	0	0
10	28	19	0	7	0	2	0	0	0	0	0	0	0	0	0	0
11	41	26	0	12	0	3	0	0	0	0	0	0	0	0	0	0
12	60	36	0	19	0	4	0	1	0	0	0	0	0	0	0	0
13	88	50	0	28	0	8	0	2	0	0	0	0	0	0	0	0
14	129	69	0	42	0	15	0	3	0	0	0	0	0	0	0	0
15	189	95	0	64	0	25	0	4	0	1	0	0	0	0	0	0
16	277	131	0	97	0	38	0	9	0	2	0	0	0	0	0	0
17	406	181	0	144	0	60	0	18	0	3	0	0	0	0	0	0
18	595	250	0	212	0	97	0	31	0	4	0	1	0	0	0	0
19	872	345	0	312	0	155	0	48	0	10	0	2	0	0	0	0
20	1278	476	0	459	0	240	0	79	0	21	0	3	0	0	0	0
21	1873	657	0	672	0	368	0	134	0	37	0	4	0	1	0	0
22	2745	907	0	979	0	565	0	223	0	58	0	11	0	2	0	0
23	4023	1252	0	1422	0	867	0	356	0	99	0	24	0	3	0	0
24	5896	1728	0	2062	0	1320	0	563	0	175	0	43	0	4	0	1
25	8641	2385	0	2984	0	1995	0	894	0	301	0	68	0	12	0	2
26	12664	3292	0	4308	0	3003	0	1419	0	492	0	120	0	27	0	3
27	18560	4544	0	6206	0	4510	0	2228	0	798	0	220	0	49	0	4
28	27201	6272	0	8925	0	6752	0	3466	0	1304	0	389	0	78	0	13
29	39865	8657	0	12816	0	10071	0	5368	0	2130	0	648	0	142	0	30
30	58425	11949	0	18376	0	14972	0	8294	0	3431	0	1074	0	269	0	55
31	85626	16493	0	26310	0	22201	0	12764	0	5467	0	1800	0	487	0	88
32	125491	22765	0	37620	0	32844	0	19549	0	8673	0	3015	0	824	0	165
33	183916	31422	0	53728	0	48475	0	29818	0	13729	0	4964	0	1392	0	322
34	269542	43371	0	76648	0	71381	0	45341	0	21630	0	8074	0	2387	0	595
35	395033	59864	0	109230	0	104892	0	68748	0	33882	0	13080	0	4089	0	1020
36	578949	82629	0	155507	0	153844	0	103928	0	52824	0	21151	0	6862	0	1753
37	848491	114051	0	221184	0	225239	0	156652	0	82076	0	34016	0	11364	0	3070
38	1243524	157422	0	314325	0	329202	0	235504	0	127113	0	54342	0	18757	0	5367
39	1822473	217286	0	446320	0	480371	0	353204	0	196175	0	86364	0	30917	0	9160

TABLE 18. The  $T_0(9, 4) = 12$  Tatami tilings of row  $n = 9$  in Table 19 with  $\bar{T}_0(9, 4) = 4$  incongruent forms of row  $n = 9$  in Table 26.

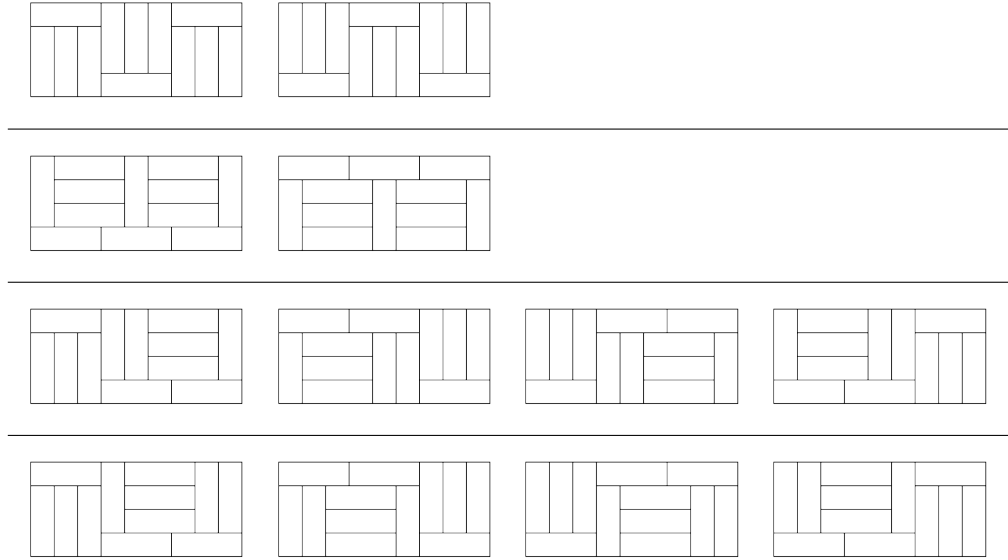


TABLE 19. Number  $T(n, 4)$  and  $T_t(n, 4)$  of tilings of  $4 \times n$  boards with  $1 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9
3	3	3	0	0	0	0	0	0	0	0	0
6	13	6	6	0	1	0	0	0	0	0	0
9	57	12	24	16	0	4	0	1	0	0	0
12	249	26	66	84	40	16	12	0	4	0	1
15	1087	56	176	306	264	134	76	44	12	14	0
18	4745	120	452	970	1170	892	504	316	160	78	50
21	20713	258	1128	2852	4324	4388	3152	1996	1232	642	368
24	90417	554	2762	7986	14414	18070	16298	11784	7808	4810	2694
27	394691	1190	6660	21590	44916	66492	72344	61720	45156	30510	18948
30	1722917	2556	15868	56862	133338	226324	287668	286806	237554	175262	119330
33	7520929	5490	37440	146714	381640	727634	1054508	1206850	1135412	925770	684908

TABLE 20. Number  $T(n, 5)$  and  $T_t(n, 5)$  of tilings of  $5 \times n$  boards with  $1 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11
3	4	4	0	0	0	0	0	0	0	0	0	0	0
6	22	4	8	9	0	1	0	0	0	0	0	0	0
9	121	3	20	34	32	25	0	6	0	1	0	0	0
12	664	2	28	90	140	160	124	72	20	21	0	6	0
15	3643	2	32	164	388	653	760	678	456	265	112	86	16
18	19987	2	36	245	832	1854	2908	3748	3548	2756	1816	1077	552
21	109657	2	40	330	1488	4254	8592	14014	17636	18502	15692	11626	7508
24	601624	2	44	422	2344	8378	21096	41721	65708	86523	93068	86956	68992
27	3300760	2	48	522	3408	14695	44912	105920	200464	317485	417696	475112	461888

TABLE 21. Number  $T(n, 6)$  and  $T_t(n, 6)$  of tilings of  $6 \times n$  boards with  $1 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9
1	1	1	0	0	0	0	0	0	0	0	0
2	1	0	1	0	0	0	0	0	0	0	0
3	6	5	0	1	0	0	0	0	0	0	0
4	13	6	6	0	1	0	0	0	0	0	0
5	22	4	8	9	0	1	0	0	0	0	0
6	64	0	18	20	24	0	2	0	0	0	0
7	155	6	32	41	40	33	0	3	0	0	0
8	321	8	46	78	81	60	44	0	4	0	0
9	783	6	40	134	184	194	120	91	0	13	0
10	1888	0	58	212	416	464	360	220	128	10	18
11	4233	10	88	356	676	983	924	651	316	183	20
12	9912	16	112	488	1154	1746	2164	1852	1305	588	360
13	23494	10	104	634	1844	3574	4576	4744	3808	2332	1032
14	54177	0	146	858	3086	6308	9650	10522	9680	6976	4009
15	126019	18	216	1280	4548	10800	17676	23085	22848	19265	13036
16	295681	32	260	1660	6572	16972	32846	46652	54195	49582	38418
17	687690	18	248	2044	9184	28006	57740	93112	116144	121213	101656
18	1600185	0	342	2670	13568	43180	100706	172550	240028	270006	257483
19	3738332	34	504	3778	18808	65924	162320	313866	471344	589001	609660
20	8712992	64	584	4800	25474	95010	264062	545618	911664	1225476	1394156
21	20293761	34	568	5754	33940	141048	414236	944758	1678144	2469202	3014824
22	47337405	0	778	7402	47404	202330	649428	1559924	3023840	4749534	6313134

TABLE 22. Number  $T(n, 7)$  and  $T_t(n, 7)$  of tilings of  $7 \times n$  boards with  $1 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9
3	9	7	0	2	0	0	0	0	0	0	0
6	155	6	32	41	40	33	0	3	0	0	0
9	2861	23	100	374	552	558	572	366	168	131	0
12	52817	28	288	1301	3884	7303	8976	9658	8516	5610	3764
15	972557	55	612	3900	16356	43495	86916	129728	154264	158222	133548
18	17892281	140	1388	11760	54828	196629	520436	1057125	1723036	2301996	2624904

TABLE 23. Number  $T(n, 8)$  and  $T_t(n, 8)$  of tilings of  $8 \times n$  boards with  $1 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9
3	13	10	0	3	0	0	0	0	0	0	0
6	321	8	46	78	81	60	44	0	4	0	0
9	8133	8	92	471	1052	1580	1552	1298	1008	623	236
12	204975	8	140	1026	4578	12340	22428	31240	34738	31358	25410
15	5158223	10	128	2030	12040	50140	139904	295867	488740	656582	751376

TABLE 24. Number  $T(n, 9)$  and  $T_t(n, 9)$  of tilings of  $9 \times n$  boards with  $1 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9
1	1	1	0	0	0	0	0	0	0	0	0
2	1	0	0	1	0	0	0	0	0	0	0
3	19	14	0	4	0	1	0	0	0	0	0
4	57	12	24	16	0	4	0	1	0	0	0
5	121	3	20	34	32	25	0	6	0	1	0
6	783	6	40	134	184	194	120	91	0	13	0
7	2861	23	100	374	552	558	572	366	168	131	0
8	8133	8	92	471	1052	1580	1552	1298	1008	623	236
9	37160	6	80	528	1832	4344	6432	7092	6016	4690	3040
10	143419	22	284	1441	4796	10973	17900	23948	25048	21910	16172
11	468816	46	408	2276	8040	21416	40512	60384	73908	77511	66884
12	1876855	12	220	2062	10084	34134	83864	157514	229952	281691	292240
13	7263468	35	444	3504	18432	69552	190316	402638	669380	917207	1058448
14	25496863	86	1048	7892	38008	137588	387664	879237	1602712	2461860	3207432
15	97187247	97	1100	8848	48832	204383	654700	1688610	3530980	6175615	9156644



**3.2. Results (incongruent).** Counts of tilings with  $1 \times 3$  tiles where only one representative of the roto-reflected copies of each tiling is counted are shown in Tables 25–31.

Row sums  $\bar{T}(n, 3)$  of Table 25 appear to be tabulated in [7, A102543] with row sums

**Conjecture 10.** (Table 25)

$$(34) \quad \bar{T}(z, 3) = -z^3 + \frac{1}{2} \left[ \frac{1}{1-z-z^3} + \frac{1+z+z^3}{1-z^2-z^6} \right].$$

Up to a shift in the indices these numbers appeared already in column 0 of Table 9.

Column  $\bar{T}_0(n, 3)$  seems to be [7, A192928]:

**Conjecture 11.** (Table 25)

$$(35) \quad \bar{T}_0(z, 3) = -z^3 + \frac{1}{2} \left[ \frac{1+z^3}{1-z-z^4} + \frac{1+z+z^3+z^7}{1-z^2-z^8} \right].$$

**Conjecture 12.** (Table 25)

$$(36) \quad \bar{T}_2(z, 3) = \frac{z^2}{2} \left[ \frac{1-z}{(1-z-z^4)^2} + \frac{1}{1-z-z^4} + \frac{z^4}{1-z^2-z^8} \right].$$

TABLE 25. Number  $\bar{T}(n, 3)$  and  $\bar{T}_t(n, 3)$  of incongruent tilings of  $3 \times n$  boards with  $1 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	4	3	0	1	0	0	0	0	0	0	0	0	0	0	0	0
7	6	5	0	1	0	0	0	0	0	0	0	0	0	0	0	0
8	8	6	0	2	0	0	0	0	0	0	0	0	0	0	0	0
9	12	9	0	2	0	1	0	0	0	0	0	0	0	0	0	0
10	16	11	0	4	0	1	0	0	0	0	0	0	0	0	0	0
11	24	16	0	6	0	2	0	0	0	0	0	0	0	0	0	0
12	33	20	0	10	0	2	0	1	0	0	0	0	0	0	0	0
13	49	29	0	14	0	5	0	1	0	0	0	0	0	0	0	0
14	69	37	0	22	0	8	0	2	0	0	0	0	0	0	0	0
15	102	53	0	32	0	14	0	2	0	1	0	0	0	0	0	0
16	145	69	0	50	0	20	0	5	0	1	0	0	0	0	0	0
17	214	98	0	72	0	33	0	9	0	2	0	0	0	0	0	0
18	307	130	0	108	0	50	0	16	0	2	0	1	0	0	0	0
19	452	183	0	156	0	82	0	24	0	6	0	1	0	0	0	0
20	653	245	0	232	0	122	0	41	0	11	0	2	0	0	0	0
21	960	343	0	336	0	191	0	67	0	20	0	2	0	1	0	0
22	1393	463	0	493	0	286	0	114	0	30	0	6	0	1	0	0
23	2046	646	0	711	0	444	0	178	0	53	0	12	0	2	0	0
24	2978	877	0	1036	0	666	0	285	0	89	0	22	0	2	0	1
25	4371	1220	0	1492	0	1014	0	447	0	156	0	34	0	7	0	1
26	6376	1664	0	2161	0	1511	0	714	0	248	0	62	0	14	0	2
27	9354	2310	0	3103	0	2280	0	1114	0	408	0	110	0	26	0	2
28	13665	3161	0	4472	0	3390	0	1740	0	656	0	198	0	40	0	7
29	20041	4381	0	6408	0	5073	0	2684	0	1079	0	324	0	75	0	15
30	29307	6009	0	9201	0	7507	0	4158	0	1723	0	542	0	136	0	28
31	42972	8319	0	13155	0	11156	0	6382	0	2757	0	900	0	250	0	44
32	62884	11430	0	18828	0	16454	0	9791	0	4349	0	1514	0	414	0	85
33	92191	15811	0	26864	0	24320	0	14909	0	6902	0	2482	0	707	0	161
34	134974	21751	0	38349	0	35739	0	22694	0	10834	0	4048	0	1198	0	302
35	197858	30070	0	54615	0	52568	0	34374	0	17000	0	6540	0	2062	0	510
36	289772	41405	0	77788	0	76994	0	51998	0	26442	0	10594	0	3440	0	883
37	424746	57216	0	110592	0	112799	0	78326	0	41129	0	17008	0	5713	0	1535
38	622198	78836	0	157210	0	164707	0	117802	0	63605	0	27200	0	9394	0	2692

TABLE 26. Number  $\bar{T}(n, 4)$  and  $\bar{T}_t(n, 4)$  of incongruent tilings of  $4 \times n$  boards with  $1 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10
3	2	2	0	0	0	0	0	0	0	0	0	0
6	5	2	2	0	1	0	0	0	0	0	0	0
9	18	4	6	6	0	1	0	1	0	0	0	0
12	69	8	18	22	11	4	4	0	1	0	1	0
15	287	16	44	83	66	38	19	11	3	5	0	1
18	1215	33	116	247	297	226	132	79	42	20	13	3
21	5244	69	282	729	1081	1119	788	511	308	165	92	54
24	22729	145	697	2010	3617	4534	4097	2953	1968	1205	682	394
27	98959	307	1665	5438	11229	16698	18086	15503	11289	7666	4737	2826
30	431273	653	3981	14253	33372	56644	71993	71751	59465	43839	29889	18918
33	1881481	1393	9360	36778	95410	182136	263627	302018	283853	231688	171227	117565
36	8210019	2978	21937	93182	265472	560002	908584	1173497	1244674	1127399	909441	673113

TABLE 27. Number  $\bar{T}(n, 5)$  and  $\bar{T}_t(n, 5)$  of incongruent tilings of  $5 \times n$  boards with  $1 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12
3	3	3	0	0	0	0	0	0	0	0	0	0	0	0
6	9	2	2	4	0	1	0	0	0	0	0	0	0	0
9	42	2	5	13	8	11	0	2	0	1	0	0	0	0
12	192	1	7	28	35	51	31	23	5	8	0	2	0	1
15	996	1	8	47	97	187	190	199	114	83	28	25	4	10
18	5206	1	9	66	208	485	727	1003	887	749	454	302	138	100
21	28091	1	10	89	372	1092	2148	3613	4409	4839	3923	3067	1877	1227
24	152212	1	11	111	586	2118	5274	10535	16427	21992	23267	22282	17248	12726
27	830974	1	12	138	852	3714	11228	26614	50116	79911	104424	120138	115472	101054
30	4543764	1	13	164	1174	5951	21405	59457	131820	245085	381516	515789	596135	615285

TABLE 28. Number  $\bar{T}(n, 6)$  and  $\bar{T}_t(n, 6)$  of incongruent tilings of  $6 \times n$  boards with  $1 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10
1	1	1	0	0	0	0	0	0	0	0	0	0
2	1	0	1	0	0	0	0	0	0	0	0	0
3	4	3	0	1	0	0	0	0	0	0	0	0
4	5	2	2	0	1	0	0	0	0	0	0	0
5	9	2	2	4	0	1	0	0	0	0	0	0
6	11	0	3	3	4	0	1	0	0	0	0	0
7	47	3	8	13	10	11	0	2	0	0	0	0
8	91	2	13	20	24	16	13	0	3	0	0	0
9	219	3	10	38	46	56	30	30	0	5	0	1
10	494	0	16	54	107	117	97	57	35	3	7	0
11	1106	4	22	99	169	256	231	177	79	54	5	8
12	2533	4	28	124	293	441	550	465	341	150	98	14
13	5978	4	26	170	461	916	1144	1211	952	607	258	158
14	13670	0	37	217	779	1583	2431	2638	2443	1747	1030	414
15	31765	7	54	335	1137	2734	4419	5824	5712	4878	3259	1922
16	74194	8	67	416	1656	4252	8239	11678	13590	12410	9650	6136
17	172508	7	62	529	2296	7071	14435	23380	29036	30431	25414	18417
18	400688	0	88	669	3403	10808	25223	43164	60073	67537	64465	50706
19	935891	11	126	973	4702	16574	40580	78666	117836	147498	152415	135777
20	2179732	16	146	1203	6382	23774	66072	136452	228070	306440	348724	333975
21	5076530	11	142	1471	8485	35410	103559	236504	419536	617812	753706	789385
22	11837691	0	195	1855	11871	50605	162461	390059	756195	1187512	1578687	1778376

TABLE 29. Number  $\bar{T}(n, 7)$  and  $\bar{T}_t(n, 7)$  of incongruent tilings of  $7 \times n$  boards with  $1 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10
3	6	5	0	1	0	0	0	0	0	0	0	0
6	47	3	8	13	10	11	0	2	0	0	0	0
9	769	10	25	105	138	155	143	103	42	42	0	5
12	13354	10	72	343	971	1856	2244	2447	2129	1431	941	535
15	244096	23	153	1020	4089	11002	21729	32621	38566	39749	33387	25076
18	4475868	45	347	2991	13707	49336	130109	264648	430759	575998	656226	647062

TABLE 30. Number  $\bar{T}(n, 8)$  and  $\bar{T}_t(n, 8)$  of incongruent tilings of  $8 \times n$  boards with  $1 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11
3	8	6	0	2	0	0	0	0	0	0	0	0	0
6	91	2	13	20	24	16	13	0	3	0	0	0	0
9	2126	4	23	130	263	417	388	354	252	170	59	58	0
12	51493	2	37	258	1153	3091	5637	7819	8734	7849	6398	4582	2831
15	1291743	4	32	529	3010	12637	34976	74236	122185	164546	187844	185619	160116

TABLE 31. Number  $\bar{T}(n, 9)$  and  $\bar{T}_t(n, 9)$  of incongruent tilings of  $9 \times n$  boards with  $1 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10
1	1	1	0	0	0	0	0	0	0	0	0	0
2	1	0	0	1	0	0	0	0	0	0	0	0
3	12	9	0	2	0	1	0	0	0	0	0	0
4	18	4	6	6	0	1	0	1	0	0	0	0
5	42	2	5	13	8	11	0	2	0	1	0	0
6	219	3	10	38	46	56	30	30	0	5	0	1
7	769	10	25	105	138	155	143	103	42	42	0	5
8	2126	4	23	130	263	417	388	354	252	170	59	58
9	4808	2	10	76	229	579	804	926	752	628	380	258
10	36250	7	71	375	1199	2791	4475	6083	6262	5571	4043	2648
11	118396	19	102	610	2010	5492	10128	15323	18477	19668	16721	12599
12	471234	5	55	542	2521	8640	20966	39641	57488	70824	73060	66147
13	1820940	14	111	915	4608	17634	47579	101220	167345	230236	264612	267931
14	6383748	28	262	2022	9502	34623	96916	220504	400678	616756	801858	904085

TABLE 32. Number  $T(n, 3)$  and  $T_t(n, 3)$  of tilings of  $3 \times n$  boards with  $1 \times 4$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
4	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
12	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
16	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
20	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
24	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
28	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
32	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

#### 4. TILING WITH $1 \times 4$ TILES

Tilings with  $1 \times 4$  tiles are represented by Tables 32–45.

**4.1. Results (full count).** Table 32 is simple because the floor width  $m = 3$  is too small to allow “vertical” placements of the tiles. So there is only one tiling with a “ferromagnetic” alignment of all tiles and a maximum number of points on the floor where 4 tiles meet.

Row sums  $T(n, 4)$  of Table 33 are [7, A003269] and column  $T_0(n, 4)$  is [7, A003520]. For two further columns we propose

**Conjecture 13.** (Table 33)

$$(37) \quad T_3(z, 4) = \frac{z^8}{(1 - z + z^2)^2(1 - z^2 - z^3)^2}.$$

**Conjecture 14.** (Table 33)

$$(38) \quad T_6(z, 4) = z^{12} \frac{1 - z}{(1 - z + z^2)^3(1 - z^2 - z^3)^3}.$$

For row sums in Table 34 we conjecture

**Conjecture 15.** (Table 34)

$$(39) \quad T(z, 5) = \frac{1 - 3z^4 + 3z^8 - z^{12}}{1 - 6z^4 + 6z^8 - 4z^{12} + z^{16}}.$$

**Conjecture 16.** (Table 34)

$$(40) \quad T_0(z, 5) = -1 + z^4 + \frac{2}{1 - z^4 - 3z^8 - 3z^{12} - z^{16}}.$$

For row sums in Table 35 we conjecture

**Conjecture 17.** (Table 35)

$$(41) \quad T(z, 6) = \frac{1 - 3z^4 + 3z^8 - z^{12}}{1 - 7z^4 + 6z^8 - 4z^{12} + z^{16}}.$$

For row sums in Table 36 we conjecture

**Conjecture 18.** (Table 36)

$$(42) \quad T(z, 7) = \frac{1 - z^4}{2} \left[ \frac{1}{1 - 2z^2 - 2z^4 + z^8} + \frac{1}{1 + 2z^2 - 2z^4 + z^8} \right].$$

TABLE 33. Number  $T(n, 4)$  and  $T_t(n, 4)$  of tilings of  $4 \times n$  boards with  $1 \times 4$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	5	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	7	6	0	0	1	0	0	0	0	0	0	0	0	0	0	0
9	10	8	0	0	2	0	0	0	0	0	0	0	0	0	0	0
10	14	11	0	0	3	0	0	0	0	0	0	0	0	0	0	0
11	19	15	0	0	4	0	0	0	0	0	0	0	0	0	0	0
12	26	20	0	0	5	0	0	1	0	0	0	0	0	0	0	0
13	36	26	0	0	8	0	0	2	0	0	0	0	0	0	0	0
14	50	34	0	0	13	0	0	3	0	0	0	0	0	0	0	0
15	69	45	0	0	20	0	0	4	0	0	0	0	0	0	0	0
16	95	60	0	0	29	0	0	5	0	0	1	0	0	0	0	0
17	131	80	0	0	40	0	0	9	0	0	2	0	0	0	0	0
18	181	106	0	0	56	0	0	16	0	0	3	0	0	0	0	0
19	250	140	0	0	80	0	0	26	0	0	4	0	0	0	0	0
20	345	185	0	0	115	0	0	39	0	0	5	0	0	1	0	0
21	476	245	0	0	164	0	0	55	0	0	10	0	0	2	0	0
22	657	325	0	0	230	0	0	80	0	0	19	0	0	3	0	0
23	907	431	0	0	320	0	0	120	0	0	32	0	0	4	0	0
24	1252	571	0	0	445	0	0	181	0	0	49	0	0	5	0	0
25	1728	756	0	0	620	0	0	269	0	0	70	0	0	11	0	0
26	2385	1001	0	0	864	0	0	390	0	0	105	0	0	22	0	0
27	3292	1326	0	0	1200	0	0	560	0	0	164	0	0	38	0	0
28	4544	1757	0	0	1660	0	0	805	0	0	257	0	0	59	0	0
29	6272	2328	0	0	2290	0	0	1161	0	0	394	0	0	85	0	0
30	8657	3084	0	0	3155	0	0	1674	0	0	585	0	0	131	0	0
31	11949	4085	0	0	4344	0	0	2400	0	0	860	0	0	212	0	0
32	16493	5411	0	0	5975	0	0	3420	0	0	1269	0	0	343	0	0
33	22765	7168	0	0	8206	0	0	4855	0	0	1882	0	0	539	0	0
34	31422	9496	0	0	11252	0	0	6881	0	0	2789	0	0	815	0	0
35	43371	12580	0	0	15408	0	0	9744	0	0	4100	0	0	1221	0	0
36	59864	16665	0	0	21078	0	0	13775	0	0	5980	0	0	1842	0	0
37	82629	22076	0	0	28810	0	0	19426	0	0	8684	0	0	2798	0	0
38	114051	29244	0	0	39344	0	0	27327	0	0	12592	0	0	4244	0	0
39	157422	38740	0	0	53680	0	0	38364	0	0	18244	0	0	6370	0	0
40	217286	51320	0	0	73173	0	0	53778	0	0	26375	0	0	9471	0	0
41	299915	67985	0	0	99662	0	0	75285	0	0	38006	0	0	14017	0	0
42	413966	90061	0	0	135640	0	0	105245	0	0	54591	0	0	20723	0	0
43	571388	119305	0	0	184480	0	0	146908	0	0	78220	0	0	30619	0	0
44	788674	158045	0	0	250740	0	0	204765	0	0	111878	0	0	45120	0	0
45	1088589	209365	0	0	340578	0	0	285032	0	0	159760	0	0	66222	0	0
46	1502555	277350	0	0	462316	0	0	396295	0	0	227726	0	0	96824	0	0
47	2073943	367411	0	0	627200	0	0	550380	0	0	323980	0	0	141176	0	0
48	2862617	486716	0	0	850420	0	0	763548	0	0	460057	0	0	205453	0	0
49	3951206	644761	0	0	1152480	0	0	1058151	0	0	652202	0	0	298455	0	0
50	5453761	854126	0	0	1561043	0	0	1464921	0	0	923225	0	0	432643	0	0
51	7527704	1131476	0	0	2113420	0	0	2026100	0	0	1305036	0	0	625721	0	0
52	10390321	1498887	0	0	2859925	0	0	2799700	0	0	1842184	0	0	902974	0	0

TABLE 34. Number  $T(n, 5)$  and  $T_t(n, 5)$  of tilings of  $5 \times n$  boards with  $1 \times 4$  tiles.

$n$		0	1	2	3	4	5	6	7	8
4	3	3	0	0	0	0	0	0	0	0
8	15	8	6	0	0	1	0	0	0	0
12	75	20	32	12	6	0	4	0	0	1
16	371	52	112	96	48	24	16	18	0	0
20	1833	138	368	464	340	192	112	110	56	12
24	9057	362	1168	1872	1856	1288	824	640	490	200
28	44753	952	3592	6928	8502	7384	5292	3992	3120	1966
32	221137	2504	10816	24248	35236	36772	30492	23778	18864	13864
36	1092699	6584	32048	81584	136456	165812	158516	133522	109200	86104
40	5399327	17314	93754	266592	502880	695648	755996	700938	603264	500044
44	26679563	45530	271468	851582	1784664	2763024	3366464	3444380	3169744	2758358
48	131831075	119728	779418	2671108	6147774	10513564	14189044	15958800	15823444	14535462

TABLE 35. Number  $T(n, 6)$  and  $T_t(n, 6)$  of tilings of  $6 \times n$  boards with  $1 \times 4$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	
2	0	0	0	0	0	0	0	0	0	0	0	0	
4	4	4	0	0	0	0	0	0	0	0	0	0	
6	0	0	0	0	0	0	0	0	0	0	0	0	
8	25	5	10	9	0	0	1	0	0	0	0	0	
10	0	0	0	0	0	0	0	0	0	0	0	0	
12	154	5	28	44	43	21	6	0	6	0	0	1	
14	0	0	0	0	0	0	0	0	0	0	0	0	
16	943	3	46	155	198	214	164	75	34	20	27	0	
18	0	0	0	0	0	0	0	0	0	0	0	0	
20	5773	2	56	342	695	1084	1120	1015	604	323	207	156	
22	0	0	0	0	0	0	0	0	0	0	0	0	
24	35344	2	62	565	1860	3795	5342	6427	5674	4360	2609	1763	
26	0	0	0	0	0	0	0	0	0	0	0	0	
28	216388	2	68	794	3892	10697	19470	29080	34291	33610	27010	19746	
30	0	0	0	0	0	0	0	0	0	0	0	0	
32	1324801	2	74	1027	6788	25048	58416	106132	154688	188377	191619	168341	
34	0	0	0	0	0	0	0	0	0	0	0	0	
36	8110882	2	80	1272	10492	50091	149232	328062	572908	839481	1031920	1094524	1018003
38	0	0	0	0	0	0	0	0	0	0	0	0	0
40	49657576	2	86	1534	15004	88602	331456	882617	1824950	3141422	4555427	5699764	6226786



TABLE 36. Number  $T(n, 7)$  and  $T_t(n, 7)$  of tilings of  $7 \times n$  boards with  $1 \times 4$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12
4	5	5	0	0	0	0	0	0	0	0	0	0	0	0
8	37	2	10	12	12	0	0	1	0	0	0	0	0	0
12	269	2	4	38	56	68	54	32	6	0	8	0	0	1
16	1949	2	4	40	98	258	386	356	348	228	114	44	24	38
20	14121	2	4	48	124	448	1036	1682	2294	2476	2150	1638	940	512
24	102313	2	4	56	152	666	1786	4154	8000	11640	14602	16012	14476	11104
28	741305	2	4	64	180	916	2696	7608	18120	34864	57616	80422	97260	104246
32	5371097	2	4	72	208	1198	3766	12238	33214	76820	157410	273472	412536	553940
36	38916077	2	4	80	236	1512	4996	18172	54156	143044	339912	700280	1277312	2062920

TABLE 37. Number  $T(n, 8)$  and  $T_t(n, 8)$  of tilings of  $8 \times n$  boards with  $1 \times 4$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9
1	1	1	0	0	0	0	0	0	0	0	0
2	1	0	1	0	0	0	0	0	0	0	0
3	1	0	0	1	0	0	0	0	0	0	0
4	7	6	0	0	1	0	0	0	0	0	0
5	15	8	6	0	0	1	0	0	0	0	0
6	25	5	10	9	0	0	1	0	0	0	0
7	37	2	10	12	12	0	0	1	0	0	0
8	100	0	10	30	28	30	0	0	2	0	0
9	229	13	30	53	40	52	36	2	0	3	0
10	454	20	61	87	95	62	74	47	4	0	4
11	811	11	80	147	168	154	84	98	58	6	0
12	1732	6	65	182	291	344	314	198	187	114	14
13	3777	2	56	302	534	741	760	573	296	310	138
14	7858	33	132	502	928	1274	1511	1347	938	470	430
15	15339	62	240	762	1588	2185	2616	2621	2198	1432	674
16	31273	31	342	1060	2354	3721	4467	4930	4637	3962	2541
17	65536	12	288	1286	3268	6265	8842	10044	10194	8781	7080
18	136600	10	224	1819	5076	10549	16351	20314	20956	19151	15498
19	276535	91	490	2715	8198	17134	28388	37836	42196	39726	33754
20	562728	182	895	3995	12713	27487	46732	65807	78456	80726	72305
21	1159942	95	1288	5404	17816	41810	74992	112822	144618	162166	158642
22	2400783	22	1107	6600	23252	62848	123834	198059	269447	316179	330597
23	4918159	30	808	8761	33098	95131	202666	344897	492644	607682	657528
24	10052140	267	1664	12278	50305	144614	325405	582052	874124	1127120	1277621
25	20627526	528	3178	17688	74316	219790	505436	949104	1490540	2029691	2427958
26	42480474	287	4568	23867	101648	318272	766146	1516726	2522492	3620544	4577179

TABLE 38. Number  $T(n, 9)$  and  $T_t(n, 9)$  of tilings of  $9 \times n$  boards with  $1 \times 4$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9
4	10	8	0	0	2	0	0	0	0	0	0
8	229	13	30	53	40	52	36	2	0	3	0
12	5764	63	314	534	870	973	870	751	616	350	240
16	143765	261	1020	4081	9208	14592	18820	21126	19374	17658	13658
20	3556413	668	4296	19034	61038	138550	239868	337187	420514	461175	449884

TABLE 39. Number of incongruent tilings of  $3 \times n$  boards with  $1 \times 4$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
4	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
12	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
16	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
20	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
24	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
28	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
32	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
36	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

4.2. **Results (incongruent).** Counts of tilings with  $1 \times 4$  tiles where only one representative of the roto-reflected copies of each tiling is counted are shown in Tables 39–45.

The row sums in Table 40 are found in [7, A192928], repeating those in Table 25.

**Conjecture 19.** (Table 40)

(43)

$$\bar{T}_0(z, 4) = -z^4 + \frac{1}{14} \left[ -\frac{1 - 3z - 3z^2 + 2z^3}{1 - z^2 + z^4} + \frac{3 - 2z}{1 - z + z^2} + \frac{8 + 4z + 5z^2 + 6z^3 + 3z^4 + 5z^5}{1 - z^4 - z^6} + \frac{4 + 6z + 5z^2}{1 - z^2 - z^3} \right].$$

**Conjecture 20.** (Table 40)

(44)

$$\bar{T}_3(z, 4) = z^8 \frac{(z^4 + z^3 + z^2 + z + 1)(z - 1)^2}{(z^4 - z^2 + 1)(1 - z^6 - z^4)(z^2 - z + 1)^2(z^3 + z^2 - 1)^2}.$$

TABLE 40. Number  $\bar{T}(n, 4)$  and  $\bar{T}_t(n, 4)$  of incongruent tilings of  $4 \times n$  boards with  $1 \times 4$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	5	4	0	0	1	0	0	0	0	0	0	0	0	0	0	0
9	6	5	0	0	1	0	0	0	0	0	0	0	0	0	0	0
10	9	7	0	0	2	0	0	0	0	0	0	0	0	0	0	0
11	11	9	0	0	2	0	0	0	0	0	0	0	0	0	0	0
12	16	12	0	0	3	0	0	1	0	0	0	0	0	0	0	0
13	20	15	0	0	4	0	0	1	0	0	0	0	0	0	0	0
14	29	20	0	0	7	0	0	2	0	0	0	0	0	0	0	0
15	37	25	0	0	10	0	0	2	0	0	0	0	0	0	0	0
16	53	34	0	0	15	0	0	3	0	0	1	0	0	0	0	0
17	69	43	0	0	20	0	0	5	0	0	1	0	0	0	0	0
18	98	58	0	0	29	0	0	9	0	0	2	0	0	0	0	0
19	130	74	0	0	40	0	0	14	0	0	2	0	0	0	0	0
20	183	99	0	0	59	0	0	21	0	0	3	0	0	1	0	0
21	245	128	0	0	82	0	0	29	0	0	5	0	0	1	0	0
22	343	171	0	0	117	0	0	43	0	0	10	0	0	2	0	0
23	463	223	0	0	160	0	0	62	0	0	16	0	0	2	0	0
24	646	297	0	0	225	0	0	95	0	0	25	0	0	3	0	0
25	877	388	0	0	310	0	0	137	0	0	35	0	0	6	0	0
26	1220	516	0	0	435	0	0	201	0	0	54	0	0	12	0	0
27	1664	676	0	0	600	0	0	284	0	0	82	0	0	20	0	0
28	2310	899	0	0	834	0	0	411	0	0	131	0	0	31	0	0
29	3161	1181	0	0	1145	0	0	587	0	0	197	0	0	44	0	0
30	4381	1569	0	0	1583	0	0	849	0	0	296	0	0	69	0	0
31	6009	2065	0	0	2172	0	0	1210	0	0	430	0	0	108	0	0
32	8319	2741	0	0	2995	0	0	1728	0	0	639	0	0	177	0	0
33	11430	3614	0	0	4103	0	0	2442	0	0	941	0	0	272	0	0
34	15811	4795	0	0	5636	0	0	3467	0	0	1400	0	0	415	0	0
35	21751	6330	0	0	7704	0	0	4892	0	0	2050	0	0	615	0	0
36	30070	8395	0	0	10552	0	0	6925	0	0	2998	0	0	932	0	0
37	41405	11091	0	0	14405	0	0	9741	0	0	4342	0	0	1407	0	0
38	57216	14705	0	0	19689	0	0	13716	0	0	6308	0	0	2138	0	0
39	78836	19440	0	0	26840	0	0	19222	0	0	9122	0	0	3198	0	0
40	108906	25770	0	0	36609	0	0	26962	0	0	13205	0	0	4761	0	0
41	150130	34085	0	0	49831	0	0	37700	0	0	19003	0	0	7028	0	0
42	207346	45176	0	0	67850	0	0	52725	0	0	27320	0	0	10401	0	0
43	285932	59775	0	0	92240	0	0	73536	0	0	39110	0	0	15337	0	0
44	394838	79215	0	0	125410	0	0	102527	0	0	55972	0	0	22618	0	0
45	544623	104845	0	0	170289	0	0	142631	0	0	79880	0	0	33151	0	0
46	751969	138930	0	0	231211	0	0	198350	0	0	113908	0	0	48496	0	0
47	1037425	183921	0	0	313600	0	0	275350	0	0	161990	0	0	70648	0	0
48	1432263	243696	0	0	425280	0	0	382056	0	0	230091	0	0	102847	0	0
49	1976229	322666	0	0	576240	0	0	529298	0	0	326101	0	0	149318	0	0

TABLE 41. Number  $\bar{T}(n, 5)$  and  $\bar{T}_t(n, 5)$  of incongruent tilings of  $5 \times n$  boards with  $1 \times 4$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9
4	2	2	0	0	0	0	0	0	0	0	0
8	6	3	2	0	0	1	0	0	0	0	0
12	23	6	8	5	2	0	1	0	0	1	0
16	103	16	30	26	13	6	4	6	0	0	1
20	478	37	92	124	86	52	29	28	14	3	6
24	2314	98	297	478	470	326	208	167	124	53	34
28	11285	245	898	1756	2128	1872	1327	1008	782	497	271
32	55529	646	2717	6099	8832	9225	7641	5973	4726	3489	2208
36	273652	1664	8012	20469	34121	41562	39643	33464	27314	21569	15483
40	1351040	4381	23473	66774	125799	174068	189092	175382	150883	125132	97403
44	6672248	11430	67867	213114	446184	691163	841663	861532	792499	689894	570063

TABLE 42. Number  $\bar{T}(n, 6)$  and  $\bar{T}_t(n, 6)$  of incongruent tilings of  $6 \times n$  boards with  $1 \times 4$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	3	3	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	11	3	3	4	0	0	1	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	53	3	7	16	12	10	2	0	2	0	0	1	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	278	2	12	49	52	66	45	24	10	5	10	0	0	2
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	1578	1	14	97	175	304	292	285	162	95	60	41	25	7
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24	9262	1	16	154	469	1006	1355	1696	1465	1156	690	464	314	205
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0
28	55530	1	17	209	973	2750	4885	7478	8692	8666	6905	5117	3426	2356
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0
32	336015	1	19	268	1704	6352	14625	26894	38865	47763	48399	42806	33568	24361
34	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	2044505	1	20	329	2623	12610	37312	82511	143431	211334	259060	275895	256334	214503

TABLE 43. Number  $\bar{T}(n, 7)$  and  $\bar{T}_t(n, 7)$  of incongruent tilings of  $7 \times n$  boards with  $1 \times 4$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12	13
4	3	3	0	0	0	0	0	0	0	0	0	0	0	0	0
8	13	1	3	4	4	0	0	1	0	0	0	0	0	0	0
12	77	1	1	11	14	19	14	12	2	0	2	0	0	1	0
16	513	1	1	11	26	69	100	92	91	59	31	11	6	12	0
20	3599	1	1	13	31	118	259	432	575	634	539	423	237	138	79
24	25763	1	1	15	39	173	449	1053	2015	2928	3671	4021	3638	2789	1902
28	185823	1	1	17	45	237	674	1922	4531	8760	14406	20186	24325	26147	24062

TABLE 44. Number  $\bar{T}(n, 8)$  and  $\bar{T}_t(n, 8)$  of incongruent tilings of  $8 \times n$  boards with  $1 \times 4$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10
1	1	1	0	0	0	0	0	0	0	0	0	0
2	1	0	1	0	0	0	0	0	0	0	0	0
3	1	0	0	1	0	0	0	0	0	0	0	0
4	5	4	0	0	1	0	0	0	0	0	0	0
5	6	3	2	0	0	1	0	0	0	0	0	0
6	11	3	3	4	0	0	1	0	0	0	0	0
7	13	1	3	4	4	0	0	1	0	0	0	0
8	19	0	2	6	4	6	0	0	1	0	0	0
9	70	5	8	17	11	15	11	1	0	2	0	0
10	138	6	18	25	28	20	21	16	1	0		
11	230	5	21	41	45	45	23	28	17	2	0	3
12	496	2	19	49	79	96	86	61	52	40	4	0
13	1014	1	14	84	137	197	197	159	80	85	39	8
14	2106	12	34	138	243	335	394	355	255	134	115	58
15	3993	16	61	199	405	568	668	680	567	385	178	152
16	8152	12	87	280	600	962	1146	1273	1191	1036	670	388
17	16803	4	73	330	824	1604	2235	2569	2589	2258	1810	1135
18	34946	3	57	477	1282	2693	4141	5169	5327	4889	3960	3034
19	70170	28	123	696	2063	4335	7150	9560	10646	10076	8548	6487
20	142629	47	230	1015	3210	6945	11769	16609	19765	20411	18272	14915
21	292627	29	324	1374	4472	10536	18822	28383	36316	40825	39922	34669
22	605021	7	283	1667	5846	15827	31088	49804	67646	79514	83092	76865
23	1236177	9	202	2223	8289	23894	50784	86534	123502	152467	164960	161521
24	2524938	77	418	3114	12625	36308	81557	145961	219059	282657	320298	323951
25	5173937	133	796	4450	18609	55108	126525	237764	373162	508477	608074	650492

TABLE 45. Number  $\bar{T}(n, 9)$  and  $\bar{T}_t(n, 9)$  of incongruent tilings of  $9 \times n$  boards with  $1 \times 4$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10
4	6	5	0	0	1	0	0	0	0	0	0	0
8	70	5	8	17	11	15	11	1	0	2	0	0
12	1505	19	80	144	219	254	223	196	157	94	62	45
16	36239	73	258	1048	2311	3697	4711	5325	4855	4454	3428	2431
20	890546	179	1074	4803	15272	34758	60008	84484	105184	115485	112524	100040

TABLE 46. Number  $T(n, 3)$  and  $T_t(n, 3)$  of tilings of  $3 \times n$  boards with  $2 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE 47. Number  $T(n, 4)$  and  $T_t(n, 4)$  of tilings of  $4 \times n$  boards with  $2 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
9	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
12	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
15	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
18	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
21	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
24	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
27	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
30	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
33	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
36	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0

5. TILING WITH  $2 \times 3$  TILES

Tilings with  $2 \times 3$  tiles are represented by Tables 46–59.

5.1. **Results (full count).** The pattern in Table 46 is obvious: all hexominoes have the same orientation along the strip.

The pattern in Table 47 is also obvious: all hexominoes have the same orientation along the strip, because there is only one composition of  $m = 4$  into parts of the side lengths  $t_n$  and  $t_m$ .

In Table 48 the row sums are powers of 2 because the width  $m = 5$  allows only tiling with super-tiles of shape  $6 \times 5$  containing 5 tiles of shape  $2 \times 3$  with two orientations. Each of these blocks may be inserted into the floor with two different orientations. The Tatami tilings are  $T_0(z, 5) = 2/(1 - z^6)$  because the two orientations must be filled in alternatingly to avoid 4-crossings. Distributing the places along the “long” edge of the floor where the Tatami violations may occur leads to  $T_t(6n, 5) = 2\binom{n-1}{t}$  [7, A028326].

In Table 49, the floor width  $m = 6$  allows only consecutive placements of super-tiles of  $6 \times 2$  (with 2 tiles) or super-tiles  $6 \times 3$  (with 3 tiles) along the long floor axis. Therefore the row sums  $T(n, 6)$  are basically the Padovan sequence [7, A000931].

TABLE 48. Number  $T(n, 5)$  and  $T_t(n, 5)$  of tilings of  $5 \times n$  boards with  $2 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
6	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	4	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0
18	8	2	4	2	0	0	0	0	0	0	0	0	0	0	0	0
24	16	2	6	6	2	0	0	0	0	0	0	0	0	0	0	0
30	32	2	8	12	8	2	0	0	0	0	0	0	0	0	0	0
36	64	2	10	20	20	10	2	0	0	0	0	0	0	0	0	0
42	128	2	12	30	40	30	12	2	0	0	0	0	0	0	0	0
48	256	2	14	42	70	70	42	14	2	0	0	0	0	0	0	0
54	512	2	16	56	112	140	112	56	16	2	0	0	0	0	0	0

**Theorem 15.** (Table 49)

$$(45) \quad T(z, 6) = \frac{1}{1 - z^2 - z^3}.$$

Column  $t = 0$  is periodic with period length 5 because the only way to avoid points where 4 tiles meet is to alternate the two forms of super-tiles:

**Theorem 16.** (Table 49)

$$(46) \quad T_0(z, 6) = \frac{(1+z)(1+z^2)(1-z+z^2)}{1-z^5}.$$

The other columns are basically given by considering the number of compositions of  $n$  into parts of 2 or 3, where pairs of adjacent 2 increase  $t$  by 1 and pairs of adjacent 3 increase  $t$  by 2.

**Conjecture 21.** (Table 49)

$$(47) \quad T_1(z, 6) = z^4 \frac{(1+z)^2(z^2-z+1)^2}{(1-z^5)^2}.$$

**Conjecture 22.** (Table 49)

$$(48) \quad T_2(z, 6) = z^6 \frac{2 + 2z^2 + 2z^3 + z^4 - z^5 + z^6 - 2z^7 - z^9}{(1-z^5)^3}.$$

The appearance of one more factor  $1-z^5$  in the denominator of the previous three equations each time  $t$  increases by 1 is a repeated convolution with  $1/(1-z^5) = 1 + z^5 + z^{10} + z^{15} + \dots$  and understood by repeated attachment of blocks two super-tiles with total dimension  $6 \times 5$  to the shorter floors of length  $n-5$ ,  $n-10$  and so on with adaptation of the up-down alignment of the new blocks to define one more point where 4 tiles meet at the interface.

The number of tilings in Table 50 are  $T(n, 3) = 3^{n/6}$ , powers of 3, because there are only 3 compositions of 7 in parts of the two side lengths  $t_n$  and  $t_m$  of the tile. The tiling consists of a linear package of super-tiles of shape  $7 \times 6$  that contain 7 tiles of shape  $2 \times 3$  each.

For the row sums in Table 51 we find:

**Conjecture 23.** (Table 51)

$$(49) \quad T(z, 8) = \frac{1 - 2z^3 - z^6}{1 - 3z^3 + z^9 - z^{12}}.$$



TABLE 49. Number  $T(n, 6)$  and  $T_t(n, 6)$  of tilings of  $6 \times n$  boards with  $2 \times 3$  tiles.

$n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0	0	0	0	0	0	0
4	1	0	1	0	0	0	0	0	0	0	0	0	0	0
5	2	2	0	0	0	0	0	0	0	0	0	0	0	0
6	2	0	0	2	0	0	0	0	0	0	0	0	0	0
7	3	1	2	0	0	0	0	0	0	0	0	0	0	0
8	4	1	0	2	1	0	0	0	0	0	0	0	0	0
9	5	0	2	2	0	1	0	0	0	0	0	0	0	0
10	7	2	1	1	2	1	0	0	0	0	0	0	0	0
11	9	0	0	5	2	2	0	0	0	0	0	0	0	0
12	12	1	4	1	2	2	1	1	0	0	0	0	0	0
13	16	1	0	4	6	3	2	0	0	0	0	0	0	0
14	21	0	3	6	1	6	2	3	0	0	0	0	0	0
15	28	2	2	2	8	7	4	2	0	1	0	0	0	0
16	37	0	0	9	8	7	7	3	3	0	0	0	0	0
17	49	1	6	3	6	12	8	9	2	2	0	0	0	0
18	65	1	0	6	16	13	13	8	4	3	0	1	0	0
19	86	0	4	12	4	18	16	17	10	3	2	0	0	0
20	114	2	3	3	18	24	21	19	9	10	3	2	0	0
21	151	0	0	14	20	17	32	24	26	11	4	2	0	1
22	200	1	8	6	12	36	33	42	25	20	11	4	2	0
23	265	1	0	8	32	36	42	48	36	35	12	11	2	2
24	351	0	5	20	10	40	60	63	66	36	31	12	5	2
25	465	2	4	4	32	59	66	79	66	67	44	25	12	3
26	616	0	0	20	40	35	90	100	114	93	52	42	13	13
27	816	1	10	10	20	80	96	136	128	116	102	59	38	13
28	1081	1	0	10	55	80	101	165	166	186	123	94	53	28
29	1432	0	6	30	20	75	160	184	252	204	198	141	80	51
30	1897	2	5	5	50	120	162	228	268	303	279	198	141	71
31	2513	0	0	27	70	65	200	300	364	420	322	312	184	135
32	3329	1	12	15	30	150	226	353	436	472	521	414	318	193
33	4410	1	0	12	86	155	206	430	548	677	646	553	458	287
34	5842	0	7	42	35	126	350	455	730	780	850	830	612	483
35	7739	2	6	6	72	216	342	534	804	1042	1164	1048	912	664
36	10252	0	0	35	112	112	385	735	960	1380	1350	1475	1240	978
37	13581	1	14	21	42	252	462	791	1170	1505	1940	1922	1744	1414
38	17991	1	0	14	126	273	378	945	1470	1965	2400	2397	2420	1929
39	23833	0	8	56	56	196	672	994	1780	2335	2860	3420	3104	2852
40	31572	2	7	7	98	357	651	1092	1995	2961	3765	4149	4207	3842
41	41824	0	0	44	168	182	672	1568	2220	3720	4420	5340	5680	5129
42	55405	1	16	28	56	392	854	1596	2682	4032	5890	6852	7248	7101
43	73396	1	0	16	176	448	644	1848	3416	4899	7170	8308	9560	9358
44	97229	0	9	72	84	288	1176	1974	3852	5913	8118	11250	12041	12580
45	128801	2	8	8	128	554	1148	2030	4340	7322	10254	13356	15444	16489
46	170625	0	0	54	240	282	1092	3024	4656	8748	12181	16137	20380	21058

TABLE 50. Number  $T(n, 7)$  and  $T_t(n, 7)$  of tilings of  $7 \times n$  boards with  $2 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
6	3	1	2	0	0	0	0	0	0	0	0	0	0	0	0	0
12	9	0	0	7	0	2	0	0	0	0	0	0	0	0	0	0
18	27	0	0	0	8	9	8	0	2	0	0	0	0	0	0	0
24	81	0	0	0	0	8	24	25	12	10	0	2	0	0	0	0
30	243	0	0	0	0	0	8	40	66	61	38	16	12	0	2	0
36	729	0	0	0	0	0	0	8	56	138	184	153	100	54	20	14
42	2187	0	0	0	0	0	0	0	8	72	242	436	496	409	262	148

TABLE 51. Number  $T(n, 8)$  and  $T_t(n, 8)$  of tilings of  $8 \times n$  boards with  $2 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12
3	1	1	0	0	0	0	0	0	0	0	0	0	0	0
6	4	1	0	2	1	0	0	0	0	0	0	0	0	0
9	11	2	4	0	4	0	0	1	0	0	0	0	0	0
12	33	3	0	9	10	4	0	6	0	0	1	0	0	0
15	96	1	4	14	22	12	24	6	8	0	4	0	0	1
18	281	2	4	31	21	57	38	51	36	18	6	12	0	4
21	821	1	16	14	68	106	118	145	124	66	84	36	20	6
24	2400	3	4	40	119	171	262	415	314	335	261	186	122	75
27	7015	2	8	58	170	263	632	772	926	1074	864	782	520	365
30	20505	1	12	95	162	526	1098	1558	2407	2551	2744	2671	2058	1576
33	59936	1	24	62	284	907	1646	3184	4960	6134	7694	7719	6772	6401
36	175193	4	8	99	447	1174	2698	6016	9313	13852	18880	19957	22022	19724
39	512089	1	16	132	584	1492	4608	9862	17156	29873	40004	51252	58818	59298
42	1496836	1	16	200	545	2322	7156	15128	31920	55890	84242	119677	144697	162460
45	4375251	2	32	136	796	3478	9568	24175	55092	100161	170268	255318	334008	420009

An appropriate fit to column  $T_0$  is

**Conjecture 24.** (Table 51)

$$(50) \quad T_0(z, 8) = \frac{1 + 2z^3 + 3z^6 + 4z^9 + 5z^{12} + 4z^{15} + 3z^{18}}{1 + z^3 + z^6 - z^{12} - z^{15} - z^{18}}.$$

TABLE 52. Number  $T(n, 9)$  and  $T_t(n, 9)$  of tilings of  $9 \times n$  boards with  $2 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
4	1	0	0	1	0	0	0	0	0	0	0	0	0	0
6	5	0	2	2	0	1	0	0	0	0	0	0	0	0
8	11	2	4	0	4	0	0	1	0	0	0	0	0	0
10	19	0	8	0	4	2	4	0	0	1	0	0	0	0
12	45	0	0	8	8	8	10	4	6	0	0	1	0	0
14	105	0	0	12	12	24	30	8	4	10	4	0	0	1
16	219	2	0	20	26	38	22	44	36	8	6	12	4	0
18	475	0	8	16	24	68	64	52	70	70	62	12	8	16
20	1061	0	0	16	36	72	120	178	172	144	114	98	68	8
22	2313	0	0	10	46	108	238	330	300	362	318	184	162	128
24	5027	2	0	12	92	158	288	512	684	662	582	644	524	312
26	11035	0	10	16	68	240	474	636	1002	1452	1542	1446	1188	1108
28	24173	0	0	26	68	206	584	1154	1828	2422	3032	3442	3122	2406
30	52793	0	0	12	66	260	752	1724	2990	4434	5782	6240	6448	6554
32	115499	2	0	12	106	376	924	2388	4590	7000	9774	12316	13694	13558
34	252827	0	12	20	80	492	1350	2902	5996	11076	16704	22040	27320	30388

TABLE 53. Number of  $\bar{T}(n, 3)$  and  $\bar{T}_t(n, 3)$  incongruent tilings of  $3 \times n$  boards with  $2 \times 3$  tiles. This is the same as Table 46 because there is only one tiling for each  $n$ .

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE 54. Number  $\bar{T}(n, 4)$  and  $\bar{T}_t(n, 4)$  of incongruent tilings of  $4 \times n$  boards with  $2 \times 3$  tiles. This is the same as Table 47 because there is only one tiling for each  $n$ .

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
9	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
12	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
15	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
18	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0

5.2. **Results (incongruent).** Counts of tilings with  $2 \times 3$  tiles where only one representative of the roto-reflected copies of each tiling is counted are shown in Tables 53–59.

The row sums of Table 55 are fitted by the following generating function:

**Conjecture 25.** (Table 55)

$$(51) \quad \bar{T}(z, 5) = \frac{1}{4} \left[ 1 + \frac{1}{1 - 2z^6} \right] + \frac{1 + z^6}{2(1 - 2z^{12})}.$$

Followup columns appear to have similarly simple forms:

**Conjecture 26.** (Table 55)

$$(52) \quad \bar{T}_0(z, 5) = \frac{1}{1 - z^6}.$$

**Conjecture 27.** (Table 55)

$$(53) \quad \bar{T}_1(z, 5) = z^{12} \frac{1}{(1 + z^6)(1 - z^6)^2}.$$

**Conjecture 28.** (Table 55)

$$(54) \quad \bar{T}_2(z, 5) = z^{18} \frac{1}{(1 + z^6)(1 - z^6)^3}.$$

**Conjecture 29.** (Table 55)

$$(55) \quad \bar{T}_3(z, 5) = z^{24} \frac{1 + z^{12}}{(1 + z^6)^2(1 - z^6)^4}.$$

TABLE 55. Number  $\bar{T}(n, 5)$  and  $\bar{T}_t(n, 5)$  of incongruent tilings of  $5 \times n$  boards with  $2 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
6	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
18	3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
24	6	1	2	2	1	0	0	0	0	0	0	0	0	0	0	0
30	10	1	2	4	2	1	0	0	0	0	0	0	0	0	0	0
36	20	1	3	6	6	3	1	0	0	0	0	0	0	0	0	0
42	36	1	3	9	10	9	3	1	0	0	0	0	0	0	0	0
48	72	1	4	12	19	19	12	4	1	0	0	0	0	0	0	0
54	136	1	4	16	28	38	28	16	4	1	0	0	0	0	0	0
60	272	1	5	20	44	66	66	44	20	5	1	0	0	0	0	0
66	528	1	5	25	60	110	126	110	60	25	5	1	0	0	0	0
72	1056	1	6	30	85	170	236	236	170	85	30	6	1	0	0	0
78	2080	1	6	36	110	255	396	472	396	255	110	36	6	1	0	0
84	4160	1	7	42	146	365	651	868	868	651	365	146	42	7	1	0
90	8256	1	7	49	182	511	1001	1519	1716	1519	1001	511	182	49	7	1
96	16512	1	8	56	231	693	1512	2520	3235	3235	2520	1512	693	231	56	8
102	32896	1	8	64	280	924	2184	4032	5720	6470	5720	4032	2184	924	280	64
108	65792	1	9	72	344	1204	3108	6216	9752	12190	12190	9752	6216	3108	1204	344

**Conjecture 30.** (Table 55)

$$(56) \quad \bar{T}_4(z, 5) = z^{30} \frac{1 + z^{12}}{(1 + z^6)^2 (1 - z^6)^5}.$$

In Table 56 we conjecture

**Conjecture 31.** (Table 56, row sum)

$$(57) \quad \bar{T}(z, 6) = 1 - z^6 + \frac{1}{2} \left[ \frac{1}{1 - z^2 - z^3} + \frac{1 + z^2 + z^3}{1 - z^4 - z^6} \right].$$

The column  $t = 0$  is periodic with period length 5 for the same reason as in Table 49, but entries of 2 are reduced to 1 because the two tilings counted in 49 are congruent to each other.

**Conjecture 32.** (Table 56)

$$(58) \quad \bar{T}_1(z, 6) = z^4 + z^7 \frac{1 + z^2 + z^3 + z^5 + z^7 - z^{12}}{(1 + z^5)(1 - z^5)^2}.$$

**Conjecture 33.** (Table 56)

$$(59) \quad \bar{T}_2(z, 6) = z^6 + z^8 \frac{1 + z + z^2 + 3z^3 + z^4 + z^6 - z^7 - z^{10} - 4z^{13} + 2z^{18}}{(1 + z^5)(1 - z^5)^3}.$$

The fit to row sums of Table 57 is:

**Conjecture 34.** (Table 57)

$$(60) \quad \bar{T}(z, 7) = \frac{1 - 2z^6 - 4z^{12} + 6z^{18}}{(1 - z^6)(1 - 3z^6)(1 - 3z^{12})}.$$

TABLE 56. Number  $\bar{T}(n, 6)$  and  $\bar{T}_t(n, 6)$  of incongruent tilings of  $6 \times n$  boards with  $2 \times 3$  tiles.

$n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
5	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
6	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0
7	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
8	3	1	0	1	1	0	0	0	0	0	0	0	0	0	0
9	3	0	1	1	0	1	0	0	0	0	0	0	0	0	0
10	5	1	1	1	1	1	0	0	0	0	0	0	0	0	0
11	5	0	0	3	1	1	0	0	0	0	0	0	0	0	0
12	8	1	2	1	1	1	1	1	0	0	0	0	0	0	0
13	9	1	0	2	3	2	1	0	0	0	0	0	0	0	0
14	13	0	2	3	1	4	1	2	0	0	0	0	0	0	0
15	15	1	1	1	4	4	2	1	0	1	0	0	0	0	0
16	22	0	0	6	4	4	4	2	2	0	0	0	0	0	0
17	26	1	3	2	3	6	4	5	1	1	0	0	0	0	0
18	37	1	0	3	9	7	7	5	2	2	0	1	0	0	0
19	45	0	2	6	2	10	8	9	5	2	1	0	0	0	0
20	63	1	2	2	9	14	11	10	5	6	2	1	0	0	0
21	78	0	0	8	10	9	16	12	13	6	2	1	0	1	0
22	108	1	4	4	6	18	18	23	13	11	6	3	1	0	0
23	136	1	0	4	16	19	21	25	18	18	6	6	1	1	0
24	186	0	3	10	6	22	30	34	34	19	16	7	3	1	0
25	237	1	2	2	16	31	33	40	33	35	22	13	6	2	1
26	322	0	0	12	20	19	46	51	59	49	27	22	7	8	1
27	414	1	5	6	10	40	48	70	64	59	51	30	19	7	2
28	559	1	0	5	29	41	52	86	83	96	63	50	27	15	8
29	724	0	3	15	10	39	80	94	126	104	99	72	40	26	7
30	973	1	3	3	25	64	82	116	136	155	142	102	72	37	22
31	1267	0	0	15	35	34	100	150	182	213	161	157	92	70	32
32	1697	1	6	9	15	75	116	181	220	241	262	212	161	100	49
33	2219	1	0	6	43	79	103	218	274	341	323	280	229	145	98
34	2964	0	4	21	19	66	175	234	368	394	428	421	309	245	127
35	3888	1	3	3	36	111	171	269	402	524	582	528	456	335	225
36	5183	0	0	20	56	59	194	369	485	699	679	744	623	498	349
37	6815	1	7	12	21	126	231	400	585	757	970	964	872	712	479
38	9071	1	0	7	65	138	192	480	735	992	1206	1209	1214	973	772
39	11949	0	4	28	28	100	336	502	890	1173	1430	1716	1552	1431	1037
40	15886	1	4	4	49	185	327	551	1002	1488	1890	2089	2111	1931	1542
41	20955	0	0	24	84	94	336	784	1110	1869	2210	2676	2840	2573	2245
42	27835	1	8	16	28	196	432	806	1346	2030	2948	3442	3634	3569	2998
43	36755	1	0	8	88	226	322	930	1708	2457	3585	4166	4780	4688	4236
44	48790	0	5	36	44	148	588	1000	1932	2967	4068	5643	6031	6311	5730
45	64476	1	4	4	64	282	574	1020	2170	3666	5127	6693	7722	8257	7766
46	85545	0	0	30	120	146	548	1514	2338	4395	6101	8091	10199	10560	10660
47	113115	1	9	20	36	288	732	1496	2748	4758	7734	10179	12345	13974	13830
48	150021	1	0	9	121	350	524	1672	3570	5493	9220	12225	15541	17995	18352
49	198460	0	5	45	60	205	960	1828	3816	6641	10146	15822	19323	22667	24235
50	263136	1	5	5	81	419	956	1768	4292	8158	12348	18501	24015	28850	31413

TABLE 57. Number  $\bar{T}(n, 7)$  and  $\bar{T}_t(n, 7)$  of incongruent tilings of  $7 \times n$  boards with  $2 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
6	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
12	4	0	0	3	0	1	0	0	0	0	0	0	0	0	0	0
18	10	0	0	0	3	4	2	0	1	0	0	0	0	0	0	0
24	25	0	0	0	0	3	6	9	3	3	0	1	0	0	0	0
30	70	0	0	0	0	0	3	12	18	17	11	5	3	0	1	0
36	196	0	0	0	0	0	0	3	14	39	46	43	25	16	5	4
42	574	0	0	0	0	0	0	0	3	20	64	114	128	107	68	38
48	1681	0	0	0	0	0	0	0	0	3	22	101	220	338	339	282
54	5002	0	0	0	0	0	0	0	0	0	3	28	142	404	741	961
60	14884	0	0	0	0	0	0	0	0	0	0	3	30	195	650	1485
66	44530	0	0	0	0	0	0	0	0	0	0	0	3	36	252	1014
72	133225	0	0	0	0	0	0	0	0	0	0	0	0	3	38	321

TABLE 58. Number  $\bar{T}(n, 8)$  and  $\bar{T}_t(n, 8)$  of incongruent tilings of  $8 \times n$  boards with  $2 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12	13
3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
6	3	1	0	1	1	0	0	0	0	0	0	0	0	0	0
9	4	1	1	0	1	0	0	1	0	0	0	0	0	0	0
12	13	2	0	3	3	2	0	2	0	0	1	0	0	0	0
15	28	1	1	4	6	3	6	3	2	0	1	0	0	1	0
18	84	1	1	10	6	17	10	16	9	5	3	4	0	1	0
21	216	1	4	4	18	29	30	37	31	18	22	9	5	3	3
24	639	2	1	12	31	49	66	110	81	92	66	48	32	22	11
27	1784	1	2	16	43	67	160	199	232	272	218	199	131	92	64
30	5238	1	3	29	42	138	277	404	603	656	693	682	518	407	303
33	15068	1	6	17	72	231	413	805	1243	1545	1925	1933	1697	1618	1142
36	44118	2	2	28	113	306	679	1534	2334	3499	4731	5032	5519	4973	4350
39	128257	1	4	35	148	379	1155	2481	4292	7486	10010	12840	14708	14853	14288
42	375126	1	4	56	138	597	1793	3828	7992	14044	21079	30027	36198	40734	42919
45	1094470	1	8	36	200	880	2400	6063	13779	25073	42582	63892	83522	105067	114960
48	3199848	2	4	53	279	1112	3295	9728	21969	44151	81627	127447	186220	246533	292094

TABLE 59. Number  $\bar{T}(n, 9)$  and  $\bar{T}_t(n, 9)$  of incongruent tilings of  $9 \times n$  boards with  $2 \times 3$  tiles.

$n$		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
6	3	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0
8	4	1	1	0	1	0	0	1	0	0	0	0	0	0	0	0
10	7	0	3	0	1	1	1	0	0	1	0	0	0	0	0	0
12	15	0	0	3	2	3	3	1	2	0	0	1	0	0	0	0
14	31	0	0	4	4	6	8	3	1	3	1	0	0	1	0	0
16	62	1	0	5	7	12	6	12	9	3	2	3	1	0	0	1
18	131	0	2	6	6	17	17	15	19	19	17	4	2	5	1	0
20	279	0	0	5	9	21	31	46	43	38	29	27	17	3	4	4
22	603	0	0	3	13	29	61	87	78	92	81	48	42	34	20	4
24	1289	1	0	3	23	42	72	133	172	172	146	163	133	82	57	50
26	2810	0	3	4	19	61	121	163	255	368	389	372	299	280	204	96
28	6115	0	0	8	17	55	146	293	460	615	760	874	784	613	481	413
30	13315	0	0	3	18	68	192	436	753	1117	1452	1572	1619	1652	1324	979
32	29026	1	0	3	27	98	231	610	1151	1765	2448	3095	3430	3411	3232	2981
34	63463	0	3	6	20	127	340	732	1507	2786	4191	5531	6851	7618	7535	6694



TABLE 60. Number  $T(n, 3)$  and  $\hat{T}_s(n, 3)$  of tilings of  $3 \times n$  boards with dominos.

$n$		0	1	2
2	3	0	2	1
4	11	2	8	1
6	41	16	24	1
8	153	86	66	1
10	571	394	176	1
12	2131	1666	464	1
14	7953	6734	1218	1
16	29681	26488	3192	1
18	110771	102410	8360	1
20	413403	391512	21890	1
22	1542841	1485528	57312	1
24	5757961	5607912	150048	1
26	21489003	21096168	392834	1

## 6. CLASSIFIED BY SLIDE LINES

Table 60–63 show the  $T(n, m)$  and the rows  $\hat{T}_s(n, m)$  for columns labeled by  $s$ .

**6.1. Results (full count).** Row sums of Table 60 are those of Table 2. The column with no slide lines is

**Conjecture 35.** (Table 60)

$$(61) \quad \hat{T}_0(z, 3) = 1 + 2z^4 \frac{1}{(1-z^2)(1-4z^2+z^4)(1-3z^2+z^4)}.$$

The column with  $s = 1$  slide lines is apparently

**Conjecture 36.** (Table 60)

$$(62) \quad \hat{T}_1(z, 3) = 2z^2 \frac{1}{(1-z^2)(1-3z^2+z^4)},$$

twice the sequence [7, A027941].

**Theorem 17.** (Table 60)  $\hat{T}_2(n, 3) = 1$  for even  $n$  and  $\hat{T}_2(z, 3) = z^2/(1-z^2)$ , because there is one tiling with 2 slide lines; this consists of a stack of 3 copies of the  $1 \times n$  floor tilings.

The previous equations for  $\hat{T}_s(z, 3)$  are compatible with the sum rule (10) and with Equation (14).

**Conjecture 37.** (Table 61)

$$(63) \quad \hat{T}_1(z, 4) = \frac{2(1-z^2)}{1-4z^2+z^4} + \frac{3-2z}{5(1-3z+z^2)} + \frac{17-2z}{5(1-z^2)} - \frac{3}{1-z-z^2} - \frac{3}{1+z-z^2}.$$

**Conjecture 38.** (Table 61)

$$(64) \quad \hat{T}_2(z, 4) = 3z^2 \frac{1}{(1-z^2)(1-3z^2+z^4)}.$$

Row sums of Table 61 are those in Table 3. The column with  $s = 0$  slide lines is deduced from the columns and the sum rule:

TABLE 61. Number  $T(n, 4)$  and  $\hat{T}_s(n, 4)$  of tilings of  $4 \times n$  boards with dominos.

$n$		0	1	2	3
1	1	0	1	0	0
2	5	0	1	3	1
3	11	2	9	0	0
4	36	3	20	12	1
5	95	31	64	0	0
6	281	68	176	36	1
7	781	340	441	0	0
8	2245	884	1261	99	1
9	6336	3311	3025	0	0
10	18061	9264	8532	264	1
11	51205	30469	20736	0	0
12	145601	87748	57156	696	1
13	413351	271222	142129	0	0
14	1174500	788323	384349	1827	1
15	3335651	2361482	974169	0	0
16	9475901	6870920	2600192	4788	1
17	26915305	20238249	6677056	0	0
18	76455961	58766200	17677220	12540	1
19	217172736	171407511	45765225	0	0
20	616891945	496283060	120576049	32835	1

TABLE 62. Number  $T(n, 5)$  and  $\hat{T}_s(n, 5)$  of tilings of  $5 \times n$  boards with dominos.

$n$		0	1	2	3	4
2	8	0	0	3	4	1
4	95	2	22	54	16	1
6	1183	134	520	480	48	1
8	14824	3722	7444	3525	132	1
10	185921	73282	87872	24414	352	1
12	2332097	1216178	948520	166470	928	1

**Conjecture 39.** (Table 61)

$$(65) \quad \hat{T}_0(z, 4) = 1 - \frac{2 - 2z^2}{1 - 4z^2 + z^4} - \frac{3 - 2z}{5(1 - 3z + z^2)} - \frac{7 - 2z}{5(1 - z^2)} + \frac{3}{2(1 - z - z^2)} \\ + \frac{3}{2(1 + z - z^2)} + \frac{1 - z^2}{1 - z - 5z^2 - z^3 + z^4}.$$

The main use of the classification of tilings by their number of slide lines is that the full number of tilings is computable from the tilings without slide lines by summing over all compositions of  $m$  into positive parts and building the product of all possibilities of the subfloor stacks, or summing over all partitions of  $m$  into

TABLE 63. Number  $T(n, 6)$  and  $\hat{T}_s(n, 6)$  of tilings of  $6 \times n$  boards with dominos.

$n$		0	1	2	3	4	5
1	1	0	0	1	0	0	0
2	13	0	0	1	6	5	1
3	41	2	12	27	0	0	0
4	281	3	32	121	104	20	1
5	1183	175	496	512	0	0	0
6	6728	499	2156	3084	928	60	1
7	31529	7988	14280	9261	0	0	0
8	167089	31244	73184	55617	6878	165	1
9	817991	287406	364210	166375	0	0	0
10	4213133	1315092	1932264	917296	48040	440	1

TABLE 64. Slide-line free  $\hat{T}_0(n, m)$  of tilings of  $m \times n$  boards with dominos.

$m$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
2	1	1	3	4	8	12	21	33	55	88	144	232	377	609
3	0	0	0	2	0	16	0	86	0	394	0	1666	0	6734
4	0	0	2	3	31	68	340	884	3311	9264	30469	87748	271222	788323
5	0	0	0	2	0	134	0	3722	0	73282	0	1216178	0	18339220

positive parts and inserting the multinomial coefficients as multiplicities:

$$\begin{aligned}
 (66) \quad T(n, m) &= \sum_{m=m_1+m_2+m_3+\dots} \hat{T}_0(n, m_1)\hat{T}_0(n, m_2)\dots \\
 &= \sum_{\pi(m)=\{m_1^{\alpha_1}; m_2^{\alpha_2}; m_3^{\alpha_3}; \dots\}} \frac{(\alpha_1 + \alpha_2 + \dots)!}{\alpha_1! \alpha_2! \dots} \hat{T}_0(n, m_1)^{\alpha_1} \hat{T}_0(n, m_2)^{\alpha_2} \dots
 \end{aligned}$$

This transformation of the sequence  $\hat{T}_0(n, m)$  into  $T(n, m)$  is known as the Invert Transform [1] and Cameron’s A-transform [3], see Deutsch’s comment in [7, A005178].

That same strategy delivers  $\hat{T}_s(n, m)$  for  $s > 0$  by restriction of the number of parts in these two equations whenever  $\hat{T}_0(n, m')$  are known for all  $1 \leq m' \leq m - s$ . In this respect, Table 64, extracted from Tables 60–63, contains all other columns of these tables.

**Example 1.** For fixed  $n = 12$  and the partitions  $4 = \{4^1; 1^13^1; 2^2; 1^22^1; 1^4\}$  the sum in column  $n$  of Table 64 is

$$\begin{aligned}
 (67) \quad T(12, 4) &= \hat{T}_0(12, 4) + 2\hat{T}_0(12, 1)T_0(12, 3) + \hat{T}_0(12, 2)^2 + 3\hat{T}_0(12, 1)^2T_0(12, 2) + \hat{T}_0(12, 1)^4 \\
 &= 87748 + 2 \times 1 \times 1666 + 232^2 + 3 \times 1 \times 232 + 1^4 = 145601
 \end{aligned}$$

as seen in row  $n = 12$  in Table 61.

**Remark 1.** A further reduction of the information may be defined if the tabulation is not only refined along the index  $s$  of the slide lines but along a pair of indices that test an orthogonal grid of crossed slide lines that cut through the floor.

TABLE 65. Number  $\bar{T}(n, 3)$  and  $\hat{T}_s(n, 3)$  of incongruent tilings of  $3 \times n$  boards with dominos.

$n$		0	1	2
2	2	0	1	1
4	5	1	3	1
6	14	5	8	1
8	46	25	20	1
10	156	105	50	1
12	561	434	126	1
14	2037	1715	321	1
16	7525	6699	825	1
18	27874	25739	2134	1
20	103741	98196	5544	1
22	386386	371941	14444	1
24	1440946	1403245	37700	1
26	5374772	5276258	98513	1

6.2. **Results (incongruent).** A fit to  $\hat{T}_1(n, 3)$  in Table 65 is

**Conjecture 40.** (Table 65)

$$(68) \quad \hat{T}_1(z, 3) = z^2 \frac{1 - 2z^2}{(1 - z^2)(1 - 3z^2 + z^4)(1 - z^2 - z^4)}.$$

The generating function for  $\hat{T}_0$  follows then from (26) with  $\hat{T}_2(z, 3) = z^2/(1 - z^2)$  and with the sum rule:

**Conjecture 41.** (Table 65)

$$(69) \quad \hat{T}_0(z, 3) = 1 + \frac{1 - z^2}{4(1 - 4z^2 + z^4)} - \frac{1 + z^2}{2(1 - z^2 - z^4)} + \frac{1}{4(1 - z^2)} - \frac{1}{4(1 - z - z^2)} - \frac{1}{4(1 + z - z^2)} + \frac{1 + 2z^2 - z^6}{2(1 - 4z^4 + z^8)}.$$

**Remark 2.** Definition 10 is poor if  $n = m$  and the matching condition is fulfilled, because we have defined slide lines to run parallel to the “long” edge. For square floors, a tiling rotation by 90 degrees may not conserve the slide line count as defined. So the  $\hat{T}_s(n, n)$  depend on which of the tilings of a congruent octet is chosen as the representative, and in that respect row  $n = 4$  in Table 66 is undefined. Consider the upper left configuration in Figure 1 for example: as shown it has 3 slide lines, but if rotated by 90 degrees only 1.

The tables are also defined for tiles of other shapes; one set for the full counts and one set for incongruent counts. We eventually list two examples of them.

The heuristics for  $1 \times 3$  tiles splits the row sums (28) as follows:  $\hat{T}_1(n, 3) = 0$ ,  $\hat{T}_2(z, 3) = z^3/(1 - z^3)$  and

**Conjecture 42.** (Tilings of  $3 \times n$  floors with  $1 \times 3$  tiles without slide lines [7, A099560])

$$(70) \quad \hat{T}_0(z, 3) = 1 + \frac{z}{(1 - z^3)(1 - z - z^3)}.$$

FIGURE 1. The  $\bar{T}(4, 4) = 9$  tilings recorded in Tables 11 and 66.

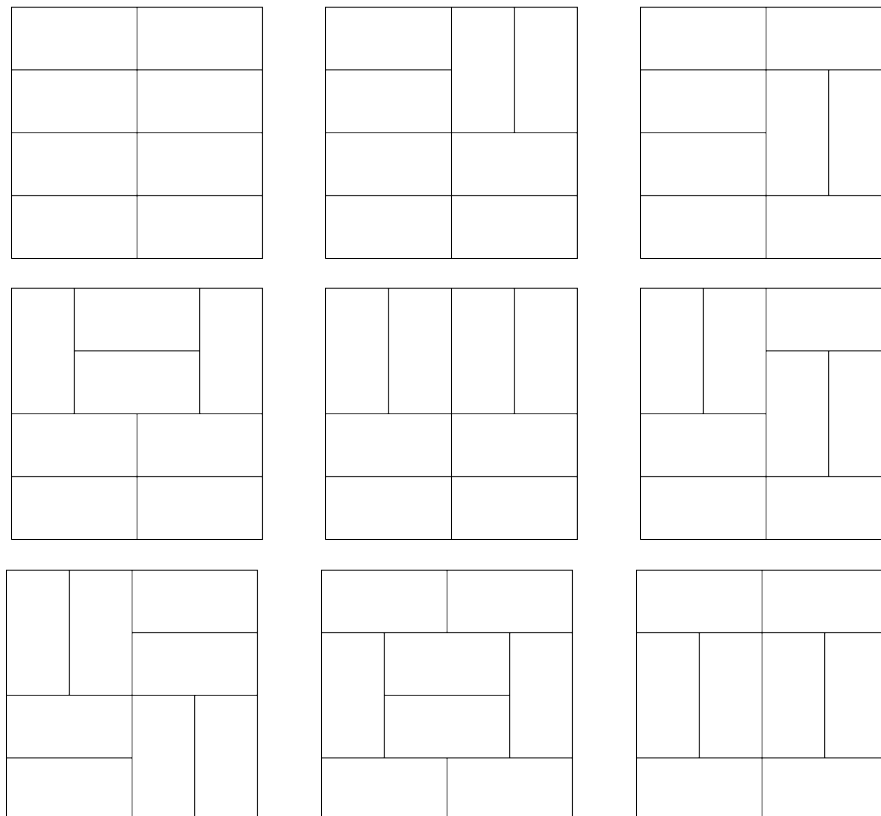


TABLE 66. Number  $\bar{T}(n, 4)$  and  $\hat{T}_s(n, 4)$  of incongruent tilings of  $4 \times n$  boards with dominos.

$n$		0	1	2	3
1	1	0	1	0	0
2	4	0	1	2	1
3	5	1	4	0	0
4	9	0	2	6	1
5	33	12	21	0	0
6	98	26	55	16	1
7	230	107	123	0	0
8	658	271	346	40	1
9	1725	935	790	0	0
10	4876	2554	2221	100	1
11	13378	8106	5272	0	0
12	37794	23021	14520	252	1
13	105761	69998	35763	0	0
14	299221	201871	96707	642	1
15	844219	600073	244146	0	0
16	2392040	1738736	651653	1650	1
17	6773154	5102309	1670845	0	0

The heuristics for  $1 \times 3$  tiles splits the row sums (31) as follows:  $\hat{T}_2(n, 4) = 0$ ,  $\hat{T}_3(z, 4) = z^3/(1 - z^3)$  and

**Conjecture 43.** (*Tilings of  $4 \times n$  floors with  $1 \times 3$  tiles with one slide line*)

$$(71) \quad \hat{T}_1(z, 4) = 2z^3 \frac{1}{(1 - z^3)(1 - 4z^3 + 3z^6 - z^9)}.$$

**Conjecture 44.** (*Tilings of  $4 \times n$  floors with  $1 \times 3$  tiles without slide lines*)

$$(72) \quad \hat{T}_0(z, 4) = 1 + 2z^6 \frac{1}{(1 - z^3)(1 - 5z^3 + 3z^6 - z^9)(1 - 4z^3 + 3z^6 - z^9)}.$$

#### APPENDIX A. C++ PROGRAM

The tables have been computed by a C++ program which is listed in the ancillary directory. Details of the implementation are documented in the source code. The code is compiled with

```
make
```

and a suite of tests is then run with

```
make test
```

A reminder of the options of the main program is shown by calling the main program without any arguments:

```
tatamiMain
```

The program generates all available tilings with an exhaustive recursive placement of new tiles into the current floor, starting with the empty floor. To support elimination of congruential tilings in case the `-i` option has been supplied, a ordering of tilings is defined by mapping each tiling onto a binary sequence, then admitting/counting only those tilings that are represented by the smallest binary number amongst the 4 or 8 roto-reflected congruential copies.

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