ENUMERATION OF k-FIBONACCI PATHS USING INFINITE WEIGHTED AUTOMATA

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Abstract. In this paper, we introduce a new family of generalized colored Motzkin paths, where horizontal steps are colored by means of $F_{k,l}$ colors, where $F_{k,l}$ is the l-th k-Fibonacci number. We study the enumeration of this family according to the length. For this, we use infinite weighted automata.

1. Introduction

A lattice path of length n is a sequence of points $P_1, P_2, \dots, P_n$ with $n \geqslant 1$ such that each point P_i belongs to the plane integer lattice and each two consecutive points P_i and P_{i+1} connect by a line segment. We will consider lattice paths in $\mathbb{Z} \times \mathbb{Z}$ using three step types: a rise step $U = (1,1)$, a fall step $D = (1,-1)$ and a $F_{k,l}$–colored length horizontal step $H_l = (l,0)$ for every positive integer l, such that H_l is colored by means of $F_{k,l}$ colors, where $F_{k,l}$ is the l-th k-Fibonacci number.

Many kinds of generalizations of the Fibonacci Numbers have been presented in the literature [10, 11] and the corresponding references. One of them is the *k-Fibonacci Numbers*. For any positive integer number k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbb{N}}$, is defined recurrently by

$$F_{k,0} = 0, \quad F_{k,1} = 1, \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \text{ for } n \geqslant 1.$$

The generating function of the k-Fibonacci numbers is $f_k(x) = \frac{x}{1-kx-x^2}$, [4, 6]. This sequence was studied by Horadam in [9]. Recently, Falcón and Plaza [6] found the k-Fibonacci numbers by studying the recursive application of two geometrical transformations used in the four-triangle longest-edge (4TLE) partition. The interested reader is also referred to [1, 3, 4, 5, 6, 12, 13, 16], for further information about this.

A *generalized $F_{k,l}$-colored Motzkin path* or simply *k-Fibonacci path* is a sequence of rise, fall and $F_{k,l}$–colored length horizontal steps ($l = 1, 2, \dots$) running from $(0,0)$ to $(n,0)$ that never pass below the x-axis. We denote by $\mathcal{M}_{F_{k,n}}$ the set of all k-Fibonacci paths of length n and $\mathcal{M}_k = \bigcup_{n=0}^{\infty} \mathcal{M}_{F_{k,n}}$. In Figure 1 we show the set $\mathcal{M}_{F_{2,3}}$.

A *grand k-Fibonacci path* is a k-Fibonacci path without the condition that never going below the x-axis. We denote by $\mathcal{M}^*_{F_{k,n}}$ the set of all grand k-Fibonacci paths of length n and $\mathcal{M}^*_k = \bigcup_{n=0}^{\infty} \mathcal{M}^*_{F_{k,n}}$. A *prefix k-Fibonacci path* is a k-Fibonacci path without the

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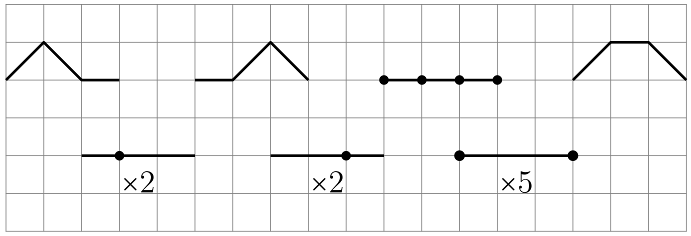


FIGURE 1. k-Fibonacci Paths of length 3, $|\mathcal{M}_{F_{2,3}}| = 13$.

condition that ending on the x-axis. We denote by $\mathcal{PM}_{F_{k,n}}$ the set of all prefix k-Fibonacci paths of length n and $\mathcal{PM}_k = \bigcup_{n=0}^{\infty} \mathcal{PM}_{F_{k,n}}$. Analogously, we have the family of *prefix grand k-Fibonacci paths*. We denote by $\mathcal{PM}^*_{F_{k,n}}$ the set of all prefix grand k-Fibonacci paths of length n and $\mathcal{PM}^*_k = \bigcup_{n=0}^{\infty} \mathcal{PM}^*_{F_{k,n}}$.

In this paper, we study the generating function for the k-Fibonacci paths, grand k-Fibonacci paths, prefix k-Fibonacci paths, and prefix grand k-Fibonacci paths, according to the length. We use Counting Automata Methodology (CAM) [2], which is a variation of the methodology developed by Rutten [14] called Coinductive Counting. Counting Automata Methodology uses infinite weighted automata, weighted graphs and continued fractions. The main idea of this methodology is find a counting automaton such that there exist a bijection between all words recognized by an automaton $\mathcal{M}$ and the family of combinatorial objects. From the counting automaton $\mathcal{M}$ is possible find the ordinary generating function (GF) of the family of combinatorial objects [2].

2. Counting Automata Methodology

The terminology and notation are mainly those of Sakarovitch [15]. An *automaton* $\mathcal{M}$ is a 5-tuple $\mathcal{M} = (\Sigma, Q, q_0, F, E)$, where Σ is a nonempty input alphabet, Q is a nonempty set of states of $\mathcal{M}$, $q_0 \in Q$ is the initial state of $\mathcal{M}$, $\varnothing \neq F \subseteq Q$ is the set of final states of $\mathcal{M}$ and $E \subseteq Q \times \Sigma \times Q$ is the set of transitions of $\mathcal{M}$. The language recognized by an automaton $\mathcal{M}$ is denoted by $L(\mathcal{M})$. If Q, Σ and E are finite sets, we say that $\mathcal{M}$ is a finite automaton [15].

Example 1. Consider the finite automaton $\mathcal{M} = (\Sigma, Q, q_0, F, E)$ where $\Sigma = \{a, b\}$, $Q = \{q_0, q_1\}$, $F = \{q_0\}$ and $E = \{(q_0, a, q_1), (q_0, b, q_0), (q_1, a, q_0)\}$. The transition diagram of $\mathcal{M}$ is as shown in Figure 2. It is easy to verify that $L(\mathcal{M}) = (b \cup aa)^*$.

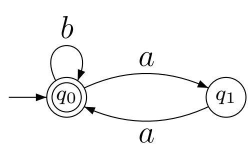


FIGURE 2. Transition diagram of $\mathcal{M}$, Example 1.

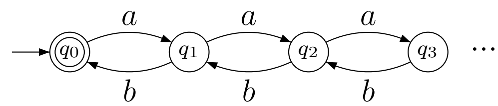


FIGURE 3. Transition diagram of $\mathcal{M}_{\mathcal{D}}$, Example 2.

Example 2. Consider the infinite automaton $\mathcal{M}_{\mathcal{D}} = (\Sigma, Q, q_0, F, E)$, where $\Sigma = \{a, b\}$, $Q = \{q_0, q_1, \dots\}$, $F = \{q_0\}$ and $E = \{(q_i, a, q_{i+1}), (q_{i+1}, b, q_i) : i \in \mathbb{N}\}$. The transition diagram of $\mathcal{M}_{\mathcal{D}}$ is as shown in Figure 3.
The language accepted by $\mathcal{M}_{\mathcal{D}}$ is

$$L(\mathcal{M}_{\mathcal{D}}) = \{w \in \Sigma^* : |w|_a = |w|_b \text{ and for all prefix } v \text{ of } w, |v|_b \leq |v|_a\}.$$

An ordinary generating function $F = \sum_{n=0}^{\infty} f_n z^n$ corresponds to a formal language L if $f_n = |\{w \in L : |w| = n\}|$, i.e., if the n-th coefficient f_n gives the number of words in L with length n.
Given an alphabet Σ and a semiring $\mathbb{K}$. A *formal power series* or *formal series* S is a function $S : \Sigma^* \to \mathbb{K}$. The image of a word w under S is called the *coefficient* of w in S and is denoted by s_w. The series S is written as a formal sum $S = \sum_{w \in \Sigma^*} s_w w$. The set of formal power series over Σ with coefficients in $\mathbb{K}$ is denoted by $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle$.

An automaton over Σ^* with weights in $\mathbb{K}$, or $\mathbb{K}$-*automaton* over Σ^* is a graph labelled with elements of $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle$, associated with two maps from the set of vertices to $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle$. Specifically, a *weighted automaton* $\mathcal{M}$ over Σ^* with weights in $\mathbb{K}$ is a 4-tuple $\mathcal{M} = (Q, I, E, F)$ where Q is a nonempty set of *states* of $\mathcal{M}$, E is an element of $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle^{Q\times Q}$ called *transition matrix*. I is an element of $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle^{Q}$, i.e., I is a function from Q to $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle$. I is the *initial function* of $\mathcal{M}$ and can also be seen as a row vector of dimension Q, called *initial vector* of $\mathcal{M}$ and F is an element of $\mathbb{K}\langle\langle\Sigma^*\rangle\rangle^{Q}$. F is the *final function* of $\mathcal{M}$ and can also be seen as a column vector of dimension Q, called *final vector* of $\mathcal{M}$.
We say that $\mathcal{M}$ is a *counting automaton* if $\mathbb{K} = \mathbb{Z}$ and $\Sigma^* = \{z\}^*$. With each automaton, we can associate a counting automaton. It can be obtained from a given automaton replacing every transition labelled with a symbol a, $a \in \Sigma$, by a transition labelled with z. This transition is called a *counting transition* and the graph is called a *counting automaton* of $\mathcal{M}$. Each transition from p to q yields an equation

$$L(p)(z) = zL(q)(z) + [p \in F] + \cdots.$$

We use L_p to denote $L(p)(z)$. We also use Iverson's notation, $[P] = 1$ if the proposition P is true and $[P] = 0$ if P is false.

2.1. **Convergent Automata and Convergent Theorems.** We denote by $L^{(n)}(\mathcal{M})$ the number of words of length n recognized by the automaton $\mathcal{M}$, including repetitions.

Definition 3. We say that an automaton $\mathcal{M}$ is convergent if for all integer $n \geqslant 0$, $L^{(n)}(\mathcal{M})$ is finite.

The proof of following theorems and propositions can be found in [2].

Theorem 4 (First Convergence Theorem)**.** *Let $\mathcal{M}$ be an automaton such that each vertex (state) of the counting automaton of $\mathcal{M}$ has finite degree. Then $\mathcal{M}$ is convergent.*

Example 5. The counting automaton of the automaton $\mathcal{M}_{\mathcal{D}}$ in Example 2 is convergent.

The following definition plays an important role in the development of applications because it allows to simplify counting automata whose transitions are formal series.

Definition 6. Let $\mathcal{M}$ be an automaton, and let $f(z) = \sum_{n=0}^{\infty} f_n z^n$ be a formal power series with $f_n \in \mathbb{N}$ for all $n \geqslant 0$ and $f_0 = 0$. In a counting automaton of $\mathcal{M}$ the set of counting transitions from state p to state q, without intermediate final states, see Figure 4 (left), is represented by a graph with a single edge labeled by $f(z)$, see Figure 4 (right).

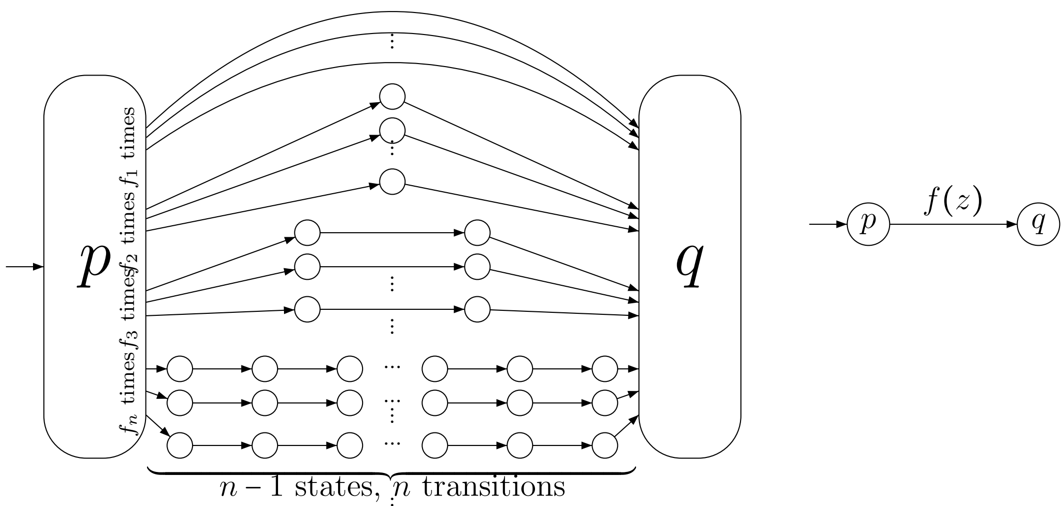


FIGURE 4. Transitions from the state p to q and its transition in parallel.

This kind of transition is called a transition in parallel. The states p and q are called visible states and the intermediate states are called hidden states.

Example 7. In Figure 5 (left) we display a counting automaton $\mathcal{M}_1$ without transitions in parallel, i.e., every transition is label by z. The transitions from state q_1 to q_2 correspond to the series $\frac{1-\sqrt{1-4z}}{2} = z + z^2 + 2z^3 + 5z^4 + 14z^5 + \cdots$. However, this automaton can also be represented using transitions in parallel. Figure 5 (right) displays two examples.

Theorem 8 (Second Convergence Theorem)**.** *Let $\mathcal{M}$ be an automaton, and let $f_1^q(z), f_2^q(z), \ldots,$ be transitions in parallel from state $q \in Q$ in a counting automaton of $\mathcal{M}$. Then $\mathcal{M}$ is convergent if the series*

$$F^q(z) = \sum_{k=1}^{\infty} f_k^q(z)$$

is a convergent series for each visible state $q \in Q$ of the counting automaton.

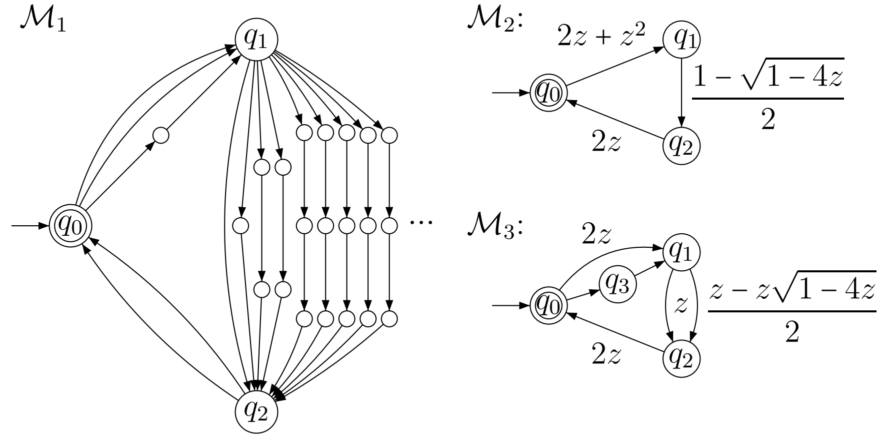


FIGURE 5. Counting automata with transitions in parallel, Example 7.

Proposition 9. *If $f(z)$ is a polynomial transition in parallel from state p to q in a finite counting automaton $\mathcal{M}$, then this gives rise to an equation in the system of GFs equations of $\mathcal{M}$*

$$L_p = f(z)L_q + [p \in F] + \cdots.$$

Proposition 10. *Let $\mathcal{M}$ be a convergent automaton such that a counting automaton of $\mathcal{M}$ has a finite number of visible states $q_0, q_1, \ldots, q_r$, in which the number of transitions in parallel starting from each state is finite. Let $f_1^{q_t}(z), f_2^{q_t}(z), \ldots, f_{s(t)}^{q_t}(z)$ be the transitions in parallel from the state $q_t \in Q$. Then the GF for the language $L(\mathcal{M})$ is $L_{q_0}(z)$. It is obtained by solving the system of $r+1$ GFs equations*

$$L(q_t)(z) = f_1^{q_t}(z)L(q_{t_1})(z) + f_2^{q_t}(z)L(q_{t_2})(z) + \cdots + f_{s(t)}^{q_t}(z)L(q_{t_{s(t)}})(z) + [q_t \in F],$$

with $0 \le t \le r$, where q_{t_k} is the visible state joined with q_t through the transition in parallel $f_k^{q_t}$, and $L(q_{t_k})$ is the GF for the language accepted by $\mathcal{M}$ if q_{t_k} is the initial state.

Example 11. The system of GFs equations associated with $\mathcal{M}_2$, see Example 7, is

$$\begin{cases} L_0 &= (2z+z^2)L_1 + 1 \\ L_1 &= \dfrac{1-\sqrt{1-4z}}{2} L_2 \\ L_2 &= 2zL_0. \end{cases}$$

Solving the system for L_0, we find the GF for the language $\mathcal{M}_2$ and therefore of $\mathcal{M}_1$ and $\mathcal{M}_3$

$$L_0 = \frac{1}{1-(2z^2+z^3)(1-\sqrt{1-4z})} = 1 + 4z^3 + 6z^4 + 10z^5 + 40z^6 + 114z^7 + \cdots.$$

2.2. **An Example of the Counting Automata Methodology (CAM).** A counting automaton associated with an automaton $\mathcal{M}$ can be used to model combinatorial objects

if there is a bijection between all words recognized by the automaton $\mathcal{M}$ and the combinatorial objects. Such method, along with the previous theorems and propositions constitute the **Counting Automata Methodology (CAM)**, see [2].
We distinguish three phases in the CAM:

(1) Given a problem of enumerative combinatorics, we have to find a convergent automaton $\mathcal{M}$ (see Theorems 4 and 8) whose GF is the solution of the problem.
(2) Find a general formula for the GF of $\mathcal{M}'$, where $\mathcal{M}'$ is an automaton obtained from $\mathcal{M}$ truncating a set of states or edges see Propositions 9 and 10. Sometimes we find a relation of iterative type, such as a continued fraction.
(3) Find the GF $f(z)$ to which converge the GFs associated to each $\mathcal{M}'$, which is guaranteed by the Convergences theorems.

Example 12. A *Motzkin path* of length n is a lattice path of $\mathbb{Z}\times\mathbb{Z}$ running from $(0,0)$ to $(n,0)$ that never passes below the x-axis and whose permitted steps are the up diagonal step $U=(1,1)$, the down diagonal step $D=(1,-1)$ and the horizontal step $H=(1,0)$. The number of Motzkin paths of length n is the *n-th Motzkin number* m_n, sequence A001006[1]. The number of words of length n recognized by the convergent automaton $\mathcal{M}_{\text{Mot}}$, see Figure 6, is the nth Motzkin number and its GF is

$$M(z)=\sum_{i=0}^{\infty} m_i z^i=\frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}.$$

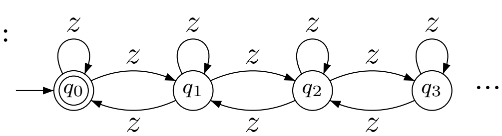
$\mathcal{M}_{\text{Mot}}$:


FIGURE 6. Convergent automaton associated with Motzkin paths.

In this case the edge from state q_i to state q_{i+1} represents a rise, the edge from the state q_{i+1} to q_i represents a fall and the loops represent the level steps, see Table 1.

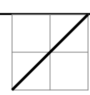
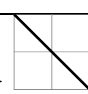
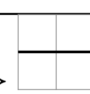
$(q_i,z,q_{i+1})\in E \Leftrightarrow$ 	$(q_{i+1},z,q_i)\in E \Leftrightarrow$ 	$(q_i,z,q_i)\in E \Leftrightarrow$ 

TABLE 1. Bijection between $\mathcal{M}_{\text{Mot}}$ and Motzkin paths.

Moreover, it is clear that a word is recognized by $\mathcal{M}_{\text{Mot}}$ if and only if the number of steps to the right and to the left coincide, which ensures that the path is well formed. Then

$$m_n=\left|\{w\in L(\mathcal{M}_{\text{Mot}}):|w|=n\}\right|=L^{(n)}(\mathcal{M}_{\text{Mot}}).$$

[1]Many integer sequences and their properties are found electronically on the On-Line Encyclopedia of Sequences [17].

Let $\mathcal{M}_{\text{Mot}s}$, $s \geq 1$ be the automaton obtained from $\mathcal{M}_{\text{Mot}}$, by deleting the states $q_{s+1}, q_{s+2}, \dots$. Therefore the system of GFs equations of $\mathcal{M}_{\text{Mot}s}$ is

$$\begin{cases} L_0 = zL_0 + zL_1 + 1, \\ L_i = zL_{i-1} + zL_i + zL_{i+1}, \quad 1 \leq i \leq s-1, \\ L_s = zL_{s-1} + zL_s. \end{cases}$$

Substituting repeatedly into each equation L_i, we have

$$L_0 = \cfrac{H}{1 - \cfrac{F^2}{1 - \cfrac{F^2}{\cfrac{\vdots}{1 - F^2}}}} \quad s \text{ times},$$

where $F = \frac{z}{1-z}$ and $H = \frac{1}{1-z}$. Since $\mathcal{M}_{\text{Mot}}$ is convergent, then as $s \to \infty$ we obtain a convergent continued fraction M of the GF of $\mathcal{M}_{\text{Mot}}$. Moreover,

$$M = \frac{H}{1 - F^2\left(\frac{M}{H}\right)}.$$

Hence $z^2M^2 - (1-z)M + 1 = 0$ and

$$M(z) = \frac{1 - z \pm \sqrt{1 - 2z - 3z^2}}{2z^2}.$$

Since $\epsilon \in L(\mathcal{M}_{\text{Mot}})$, $M \to 0$ as $z \to 0$. Hence, we take the negative sign for the radical in $M(z)$.

3. Generating Function for the k-Fibonacci Paths

In this section we find the generating function for k-Fibonacci paths, grand k-Fibonacci paths, prefix k-Fibonacci paths and prefix grand k-Fibonacci paths, according to the length.

Lemma 13 ([2])**.** *The GF of the automaton $\mathcal{M}_{\text{Lin}}$, see Figure 7, is*

$$E(z) = \cfrac{1}{1 - h_0(z) - \cfrac{f_0(z)\,g_0(z)}{1 - h_1(z) - \cfrac{f_1(z)\,g_1(z)}{\ddots}}},$$

where $f_i(z), g_i(z)$ and $h_i(z)$ are transitions in parallel for all integer $i \geqslant 0$.

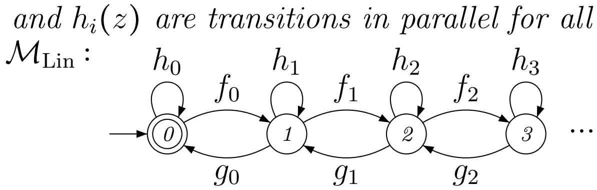


Figure 7. Linear infinite counting automaton $\mathcal{M}_{Lin}$

The last lemma coincides with Theorem 1 in [7] and Theorem 9.1 in [14]. However, this presentation extends their applications, taking into account that $f_i(z), g_i(z)$ and $h_i(z)$ are GFs, which can be GFs of several variables.

Corollary 14. *If for all integers $i \geq 0$, $f_i(z) = f(z), g_i(z) = g(z)$ and $h_i(z) = h(z)$ in $\mathcal{M}_{\text{Lin}}$, then the GF is*

$$B(z) = \frac{1 - h(z) - \sqrt{(1 - h(z))^2 - 4f(z)g(z)}}{2f(z)g(z)} \tag{1}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_n \binom{m+2n}{m} (f(z)\, g(z))^n \, (h(z))^m \tag{2}$$

$$= \cfrac{1}{1 - h(z) - \cfrac{f(z)\, g(z)}{1 - h(z) - \cfrac{f(z)\, g(z)}{1 - h(z) - \cfrac{f(z)\, g(z)}{\ddots}}}}, \tag{3}$$

where C_n is the nth Catalan number, sequence A000108.

Theorem 15. *The generating function for the k-Fibonacci paths according to the their length is*

$$T_k(z) = \sum_{i=0}^{\infty} |\mathcal{M}_{F_{k,i}}| z^i \tag{4}$$

$$= \frac{1 - (k+1)z - z^2 - \sqrt{(1 - (k+1)z - z^2)^2 - 4z^2(1 - kz - z^2)^2}}{2z^2(1 - kz - z^2)} \tag{5}$$

$$= \cfrac{1}{1 - \frac{z}{1-kz-z^2} - \cfrac{z^2}{1 - \frac{z}{1-kz-z^2} - \cfrac{z^2}{1 - \frac{z}{1-kz-z^2} - \cfrac{z^2}{\ddots}}}} \tag{6}$$

and

$$[z^t]\, T_k(z) = \sum_{n=0}^{t} \sum_{m=0}^{t-2n} \binom{m+2n}{m} C_n F_{k,t-2n-m+1}^{(m)},$$

where C_n is the n-th Catalan number and $F_{k,j}^{(r)}$ is a convolved k-Fibonacci number.

Convolved k-Fibonacci numbers $F_{k,j}^{(r)}$ are defined by

$$f_k^{(r)}(x) = (1 - kx - x^2)^{-r} = \sum_{j=0}^{\infty} F_{k,j+1}^{(r)} x^j, \quad r \in \mathbb{Z}^+.$$

Note that

$$F^{(r)}_{k,m+1} = \sum_{j_1+j_2+\cdots+j_r=m} F_{k,j_1+1}F_{k,j_2+1}\cdots F_{k,j_r+1}.$$

Moreover, using a result of Gould [8, p. 699] on Humbert polynomials (with $n = j, m = 2, x = k/2, y = -1, p = -r$ and $C = 1$), we have

$$F^{(r)}_{k,j+1} = \sum_{l=0}^{\lfloor j/2 \rfloor} \binom{j+r-l-1}{j-l}\binom{j-l}{l} k^{j-2l}.$$

Ramírez [13] studied some properties of convolved k-Fibonacci numbers.

Proof. Equations (5) and (6) are clear from Corollary 14 taking $f(z) = z = g(z)$ and $h(z) = \frac{z}{1-kz-z^2}$. Note that $h(z)$ is the GF of k-Fibonacci numbers. In this case the edge from state q_i to state q_{i+1} represents a rise, the edge from the state q_{i+1} to q_i represents a fall and the loops represent the $F_{k,l}$–colored length horizontal steps ($l = 1, 2, \dots$). Moreover, from Equation (2), we obtain

$$\begin{aligned} T_k(z) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_n \binom{m+2n}{m} z^{2n}\left(\frac{z}{1-kz-z^2}\right)^m \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_n \binom{m+2n}{m} z^{2n+m}\left(\frac{1}{1-kz-z^2}\right)^m \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_n \binom{m+2n}{m} z^{2n+m}\sum_{i=0}^{\infty} F^{(m)}_{k,i+1} z^i \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty} C_n F^{(m)}_{k,i+1}\binom{m+2n}{m} z^{2n+m+i}, \end{aligned}$$

taking $s = 2n + m + i$

$$T_k(z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=2n+m}^{\infty} C_n F^{(m)}_{k,s-2n-m+1}\binom{m+2n}{m} z^s.$$

Hence

$$[z^t]\, T_k(z) = \sum_{n=0}^{t}\sum_{m=0}^{t-2m} C_n F^{(m)}_{k,t-2n-m+1}\binom{m+2n}{m}.$$

□

In Table 2 we show the first terms of the sequence $|\mathcal{M}_{F_{k,i}}|$ for $k = 1, 2, 3, 4$.

k	Sequence
1	1, 1, 3, 8, 23, 67, 199, 600, 1834, 5674, 17743, ...
2	1, 1, 4, 13, 47, 168, 610, 2226, 8185, 30283, 112736, ...
3	1, 1, 5, 20, 89, 391, 1735, 7712, 34402, 153898, 690499, ...
4	1, 1, 6, 29, 155, 820, 4366, 23262, 124153, 663523, 3551158, ...

Table 2. Sequences $|\mathcal{M}_{F_{k,i}}|$ for $k = 1, 2, 3, 4$.

Definition 16. For all integers $i \geq 0$ we define the continued fraction $E_i(z)$ by:

$$E_i(z) = \cfrac{1}{1 - h_i(z) - \cfrac{f_i(z)\, g_i(z)}{1 - h_{i+1}(z) - \cfrac{f_{i+1}(z)\, g_{i+1}(z)}{\ddots}}},$$

where $f_i(z), g_i(z), h_i(z)$ are transitions in parallel for all integers positive i.

Lemma 17 ([2])**.** *The GF of the automaton $\mathcal{M}_{\text{BLin}}$, see Figure 8, is*

$$E_b(z) = \frac{1}{1 - h_0(z) - f_0(z)g_0(z)E_1(z) - f'_0(z)g'_0(z)E'_1(z)},$$

where $f_i(z), f'_i(z), g_i(z), g'_i(z), h_i(z)$ and $h'_i(z)$ are transitions in parallel for all $i \in \mathbb{Z}$.

$\mathcal{M}_{\text{BLin}}$:

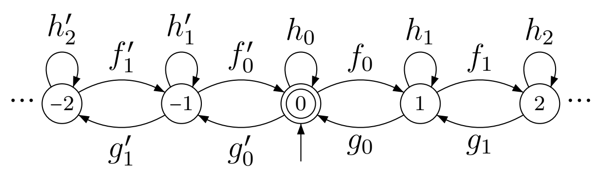


Figure 8. Linear infinite counting automaton $\mathcal{M}_{BLin}$.

Corollary 18. *If for all integers i, $f_i(z) = f(z) = f'_i(z), g_i(z) = g(z) = g'_i(z)$ and $h_i(z) = h(z) = h'_i(z)$ in $\mathcal{M}_{\text{BLin}}$, then we have the GF*

$$B_b(z) = \frac{1}{\sqrt{(1 - h(z))^2 - 4f(z)g(z)}} \tag{7}$$

$$= \cfrac{1}{1 - h(z) - \cfrac{2f(z)g(z)}{1 - h(z) - \cfrac{f(z)g(z)}{1 - h(z) - \cfrac{f(z)g(z)}{\ddots}}}}, \tag{8}$$

where $f(z), g(z)$ and $h(z)$ are transitions in parallel. Moreover, if $f(z) = g(z)$, then we have the GF

$$B_b(z) = \frac{1}{1 - h(z)} + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 2^n \frac{n}{n + 2k}\binom{n + 2k}{k}\binom{l + 2n + 2k}{l} f(z)^{2n+2k} h(z)^l. \tag{9}$$

Theorem 19. *The generating function for the grand k-Fibonacci paths according to the their length is*

$$T_k^*(z)=\sum_{i=0}^{\infty}|\mathcal{M}^*_{F_{k,i}}|z^i=\frac{1-kz-z^2}{\sqrt{(1-(k+1)z-z^2)^2-4z^2(1-kz-z^2)^2}} \tag{10}$$

$$=\cfrac{1}{1-\frac{z}{1-kz-z^2}-\cfrac{2z^2}{1-\frac{z}{1-kz-z^2}-\cfrac{z^2}{1-\frac{z}{1-kz-z^2}-\cfrac{z^2}{\ddots}}}} \tag{11}$$

and

$$[z^t]\,T_k^*(z)=F^{(1)}_{k+1,t}+\sum_{n=1}^{t}\sum_{m=0}^{t}\sum_{l=0}^{t-2n-2m}2^n\frac{n}{n+2m}\binom{n+2m}{m}\binom{l+2n+2m}{l}F^{(l)}_{k,t-2n-2m-l+1}, \tag{12}$$

with $t\geqslant 1$.

Proof. Equations (10) and (11) are clear from Corollary 18, taking $f(z)=z=g(z)$ and $h(z)=\frac{z}{1-kz-z^2}$. Moreover, from Equation (9), we obtain

$$\begin{aligned}T_k^*(z)&=\frac{1}{1-\frac{z}{1-kz-z^2}}+\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}2^n\frac{n}{n+2m}\binom{n+2m}{m}\binom{l+2n+2m}{l}z^{2n+2m}\left(\frac{z}{1-kz-z^2}\right)^l\\&=1+\sum_{j=0}^{\infty}F^{(1)}_{k+1,j}z^j+\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{u=0}^{\infty}2^n\frac{n}{n+2m}\binom{n+2m}{m}\binom{l+2n+2m}{l}F^{(l)}_{k,u}z^{2n+2m+u+1},\end{aligned}$$

taking $s=2n+2m+l+u$

$$T_k^*(z)=1+\sum_{j=0}^{\infty}F^{(1)}_{k+1,j}z^j+\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{s=2n+2m+l}^{\infty}2^n\frac{n}{n+2m}\binom{n+2m}{m}\binom{l+2n+2m}{l}F^{(l)}_{k,s-2n-2m-l}z^s.$$

Therefore Equation (12) is clear. □

In Table 3 we show the first terms of the sequence $|\mathcal{M}^*_{F_{k,i}}|$ for $k=1,2,3,4$.

k	Sequence
1	1, 4, 11, 36, 115, 378, 1251, 4182, 14073, 47634, ...
2	1, 5, 16, 63, 237, 920, 3573, 14005, 55156, 218359, ...
3	1, 6, 23, 108, 487, 2248, 10371, 48122, 223977, 1046120, ...
4	1, 7, 32, 177, 949, 5172, 28173, 153963, 842940, 4624581, ...

TABLE 3. Sequences $|\mathcal{M}^*_{F_{k,i}}|$ for $k=1,2,3,4$ and $i\geqslant 1$.

In Figure 9 we show the set $\mathcal{M}^*_{F_{2,3}}$.

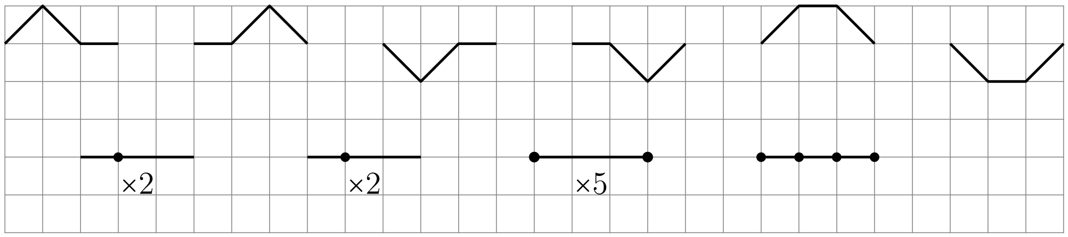


FIGURE 9. grand k-Fibonacci Paths of length 3, $|\mathcal{M}^*_{F_{2,3}}| = 16$.

Lemma 20 ([2]). *The GF of the automaton* $\text{FIN}_{\mathbb{N}}(\mathcal{M}_{Lin})$*, see Figure 10, is*

$$G(z) = E(z) + \sum_{j=1}^{\infty}\left(\prod_{i=0}^{j-1}(f_i(z)E_i(z))E_j(z)\right),$$

where $E(z)$ *is the GF in Lemma 13.*

$\text{FIN}_{\mathbb{N}}(\mathcal{M}_{Lin})$:

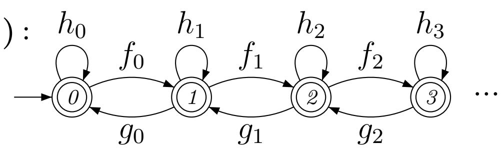


FIGURE 10. Linear infinite counting automaton $\text{FIN}_{\mathbb{N}}(\mathcal{M}_{Lin})$.

Corollary 21. *If for all integer* $i \geqslant 0$, $f_i(z) = f(z), g_i(z) = g(z)$ *and* $h_i(z) = h(z)$ *in* $\text{FIN}_{\mathbb{N}}(\mathcal{M}_{Lin})$*, then the GF is:*

$$G(z) = \frac{1 - 2f(z) - h(z) - \sqrt{(1-h(z))^2 - 4f(z)g(z)}}{2f(z)\left(f(z) + g(z) + h(z) - 1\right)} \tag{13}$$

$$= \cfrac{1}{1 - f(z) - h(z) - \cfrac{f(z)g(z)}{1 - h(z) - \cfrac{f(z)g(z)}{1 - h(z) - \cfrac{f(z)g(z)}{\ddots}}}}, \tag{14}$$

where $f(z), g(z)$ *and* $h(z)$ *are transitions in parallel and* $B(z)$ *is the GF in Corollary 14. Moreover, if* $f(z) = g(z)$ *and* $h(z) \neq 0$*, then we obtain the GF*

$$G(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{n+1}{n+k+1}\binom{n+2k+l}{k, l, k+n} f^{2k+n}(z)h^l(z). \tag{15}$$

Theorem 22. *The generating function for the prefix k-Fibonacci paths according to the their length is*

$$PT_k(z) = \sum_{i=0}^{\infty} |\mathcal{PM}_{F_{k,i}}| z^i$$
$$= \frac{(1-2z)(1-kz-z^2) - z - \sqrt{(1-z(k+1)-z^2)^2 + 4z^2(1-kz-z^2)^2}}{2z((1-kz-z^2)(2z-1)+z)}$$

and

$$[z^t]\, PT_k(z) = \sum_{n=0}^{t}\sum_{m=0}^{t}\sum_{l=0}^{t-2m-n} \frac{n+1}{n+m+1}\binom{n+2m+l}{m, l, m+n} F^{(l)}_{k,t-2m-n-l+1}, \quad t \geqslant 0.$$

Proof. The proof is analogous to the proof of Theorem 15 and 19. □

In Table 4 we show the first terms of the sequence $|\mathcal{PM}_{F_{k,i}}|$ for $k = 1, 2, 3, 4$.

k	Sequence
1	1, 2, 6, 19, 62, 205, 684, 2298, 7764, 26355, 89820, ...
2	1, 2, 7, 26, 101, 396, 1564, 6203, 24693, 98605, 394853, ...
3	1, 2, 8, 35, 162, 757, 3558, 16766, 79176, 374579, 1775082, ...
4	1, 2, 9, 46, 251, 1384, 7668, 42555, 236463, 1315281, 7322967, ...

TABLE 4. Sequences $|\mathcal{PM}_{F_{k,i}}|$ for $k = 1, 2, 3, 4$.

In Figure 11 we show the set $\mathcal{MP}_{F_{2,3}}$.

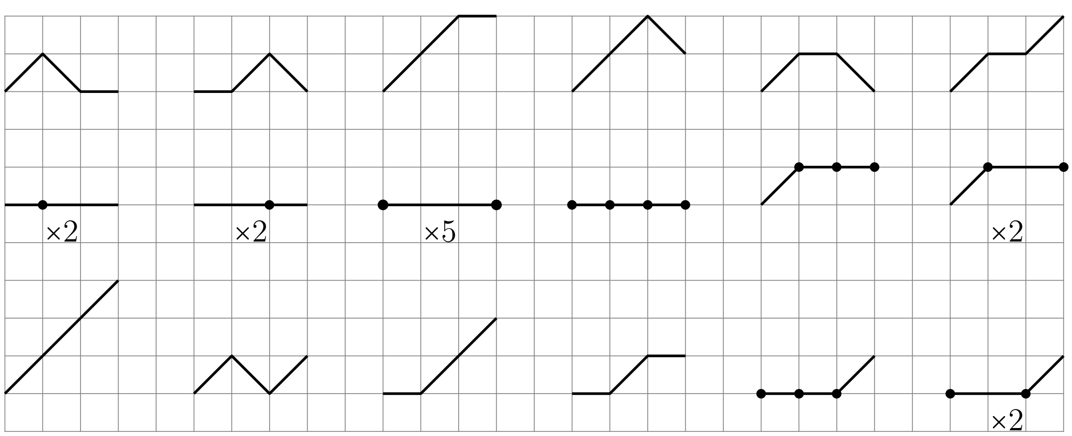


FIGURE 11. prefix k-Fibonacci paths of length 3, $|\mathcal{PM}_{F_{2,3}}| = 26$.

Lemma 23. *The GF of the automaton* $\text{FIN}_{\mathbb{Z}}(\mathcal{M}_{BLin})$*, see Figure 12, is*

$$H(z) = \frac{EE'}{E + E' - EE'(1 - h_0)}\left(1 + \sum_{j=1}^{\infty}\prod_{k=1}^{j-1} f_k E_k f_0 E_j + \sum_{j=1}^{\infty}\prod_{k=1}^{j-1} g'_k E'_k g'_0 E'_j\right)$$
$$= \frac{E'(z)G(z) + E(z)G'(z) - E(z)E'(z)}{E(z) + E'(z) - E(z)E'(z)(1 - h_0(z))},$$

where $G(z)$ *is the GF in Lemma 20 and* $G'(z), E'(z)$ *are the GFs obtained from* $G(z)$ *and* $E(z)$ *changing* $f(z)$ *to* $g'(z)$ *and* $g(z)$ *to* $f'(z)$.

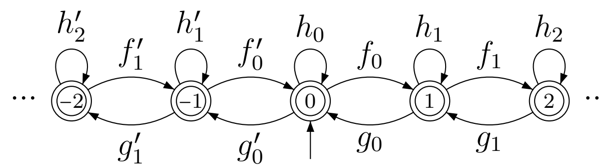


FIGURE 12. Linear infinite counting automaton $\text{FIN}_{\mathbb{Z}}(\mathcal{M}_{BLin})$.

Moreover, if for all integer $i \geqslant 0$, $f_i(z) = f(z) = f'_i(z), g_i(z) = g(z) = g'_i(z)$ *and* $h_i(z) = h(z) = h'_i(z)$ *in* $\text{FIN}_{\mathbb{Z}}(\mathcal{M}_{BLin})$*, then the GF is*

$$H(z) = \frac{1}{1 - f(z) - g(z) - h(z)}. \tag{16}$$

Theorem 24. *The generating function for the prefix grand k-Fibonacci paths according to the their length is*

$$PT^*_k(z) = \sum_{i=0}^{\infty} |\mathcal{PM}_{F^*_{k,i}}| z^i = \frac{1 - kz - z^2}{1 - (k+3)z - (1-2k)z^2 + 2z^3}.$$

Proof. The proof is analogous to the proof of Theorem 15 and 19. □

In Table 5 we show the first terms of the sequence $|\mathcal{PM}^*_{F_{k,i}}|$ for $k = 1, 2, 3, 4$.

k	Sequence
1	1, 3, 10, 35, 124, 441, 1570, 5591, 19912, 70917, 252574, ...
2	1, 3, 11, 44, 181, 751, 3124, 13005, 54151, 225492, 938997, ...
3	1, 3, 12, 55, 264, 1285, 6280, 30727, 150392, 736157, 3603528, ...
4	1, 3, 13, 68, 379, 2151, 12268, 70061, 400249, 2286780, 13065595 ...

TABLE 5. Sequences $|\mathcal{PM}^*_{F_{k,i}}|$ for $k = 1, 2, 3, 4$.

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