Chapter 1

On Landau's Function $g(n)$

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1.1 Introduction

Let S_n be the symmetric group of n letters. Landau considered the function $g(n)$ defined as the maximal order of an element of S_n; Landau observed that (cf. [9])

$$g(n) = \max \ \operatorname{lcm}(m_1, \ldots, m_k) \tag{1.1}$$

where the maximum is taken on all the partitions $n = m_1 + m_2 + \ldots + m_k$ of n and proved that, when n tends to infinity

$$\log \ g(n) \sim \sqrt{n \log n}. \tag{1.2}$$

More precise asymptotic estimates have been given in [22, 25, 11]. In [25] and [11] one also can find asymptotic estimates for the number of prime factors of $g(n)$. In [8] and [3], the largest prime factor $P^+(g(n))$ of $g(n)$ is investigated. In [10] and [12], effective upper and lower bounds of $g(n)$ are given. In [17], it is proved that $\lim_{n\to\infty} g(n+1)/g(n) = 1$. An algorithm able to calculate $g(n)$ up to 10^{15} is given in [2] (see also [26]). The sequence of distinct values of $g(n)$ is entry A002809 of [24]. A nice survey paper was written by W. Miller in 1987 (cf. [13]).

My very first mathematical paper [15] was about Landau's function, and the main result was that $g(n)$, which is obviously non decreasing, is constant on arbitrarily long intervals (cf. also [16]). First time I met A. Schinzel in Paris in May 1967. He told me that he was interested in my results, but that P. Erdős would be more interested than himself. Then I wrote my first letter to Paul with a copy of my work. I received an answer dated of June 12 1967 saying " I sometimes thought about $g(n)$ but my results were very much less complete than yours". Afterwards, I met my advisor, the late Professor Pisot, who, in view of this letter, told me that my work was good for a thesis.

The main idea of my work about $g(n)$ was to use the tools introduced by S. Ramanujan to study highly composite numbers (cf. [19, 20]). P. Erdős was very well aware of this paper of Ramanujan (cf. [1, 4, 6, 5]) as well as of the symmetric group and the order of its elements, (cf. [7]) and I think that he enjoyed the connection between these two areas of mathematics. Anyway, since these first letters, we had many occasions to discuss Landau's function.

Let us define $n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4, n_5 = 5, n_6 = 7$, etc $\ldots, n_k$ (see a table of $g(n)$ in [16, p. 187]), such that

$$g(n_k) > g(n_k - 1). \tag{1.3}$$

The above mentioned result can be read:

$$\overline{\lim}\,(n_{k+1} - n_k) = +\infty. \tag{1.4}$$

Here, I shall prove the following result:

Theorem 1.

$$\underline{\lim}\,(n_{k+1} - n_k) < +\infty. \tag{1.5}$$

Let us set $p_1 = 2, p_2 = 3, p_3 = 5, \ldots, p_k =$ the k-th prime. It is easy to deduce Theorem 1 from the twin prime conjecture (i.e. $\underline{\lim}\,(p_{k+1} - p_k) = 2$) or even from the weaker conjecture $\underline{\lim}\,(p_{k+1} - p_k) < +\infty$. (cf. §1.4 below). But I shall prove Theorem 1 independently of these deep conjectures. Moreover I shall explain below why it is reasonable to conjecture that the mean value of $n_{k+1} - n_k$ is 2; in other terms one may conjecture that

$$n_k \sim 2k \tag{1.6}$$

and that $n_{k+1} - n_k = 2$ has infinitely many solutions. Due to a parity phenomenon, $n_{k+1} - n_k$ seems to be much more often even than odd; nevertheless, I conjecture that:

$$\underline{\lim}\,(n_{k+1} - n_k) = 1. \tag{1.7}$$

The steps of the proof of Theorem 1 are first to construct the set G of values of $g(n)$ corresponding to the so called superior highly composite numbers introduced by S. Ramanujan, and then, when $g(n) \in G$, to build the table of $g(n + d)$ when d is small. This will be done in §1.4 and §1.5. Such values of $g(n+d)$ will be linked with the number of distinct differences of the form $P - Q$ where P and Q are primes satisfying $x - x^\alpha \le Q \le x < P \le x + x^\alpha$, where x goes to infinity and $0 < \alpha < 1$. Our guess is that these differences $P - Q$ represent almost all even numbers between 0 and $2x^\alpha$, but we shall only prove in §1.3 that the number of these differences is of the order of magnitude of x^α, under certain strong hypothesis on x and α, and for that a result due to Selberg about the primes between x and $x + x^\alpha$ will be needed (cf. §1.2).

To support conjecture (1.6), I think that what has been done here with $g(n) \in G$ can also be done for many more values of $g(n)$, but, unfortunately, even assuming strong hypotheses, I do not see for the moment how to manage it.

I thank very much E. Fouvry who gave me the proof of Proposition 2.

1.1.1 Notation

p will denote a generic prime, p_k the k-th prime; P, Q, P_i, Q_j will also denote primes. As usual $\pi(x) = \sum_{p \le x} 1$ is the number of primes up to x.

$|S|$ will denote the number of elements of the set S. The sequence n_k is defined by (1.3).

1.2 About the distribution of primes

Proposition 1. *Let us define $\pi(x) = \sum_{p \le x} 1$, and let α be such that $\frac{1}{6} < \alpha < 1$, and $\varepsilon > 0$. When ξ goes to infinity, and $\xi' = \xi + \xi/\log \xi$, then for all x in the interval $[\xi, \xi']$ but a subset of measure $O((\xi' - \xi)/\log^3 \xi)$ we have:*

$$\left| \pi(x + x^\alpha) - \pi(x) - \frac{x^\alpha}{\log x} \right| \le \varepsilon \frac{x^\alpha}{\log x} \tag{1.8}$$

$$\left| \pi(x) - \pi(x - x^\alpha) - \frac{x^\alpha}{\log x} \right| \le \varepsilon \frac{x^\alpha}{\log x} \tag{1.9}$$

$$\left| \frac{x}{\log x} - \frac{Q^k - Q^{k-1}}{\log Q} \right| \ge \frac{\sqrt{x}}{\log^4 x} \text{ for all primes } Q, \text{ and } k \ge 2. \tag{1.10}$$

Proof. This proposition is an easy extension of a result of Selberg (cf. [21]) who proved that (1.8) holds for most x in (ξ, ξ'). In [18], I gave a first extension of Selberg's result by proving that (1.8) and (1.9) hold simultaneously for all x in (ξ, ξ') but for a subset of measure $O((\xi' - \xi)/\log^3 \xi)$. So, it suffices to prove that the measure of the set of values of x in (ξ, ξ') for which (1.10) does not hold is $O((\xi' - \xi)/\log^3 \xi)$.

We first count the number of primes Q such that for one k we have:

$$\frac{\xi}{\log \xi} \le \frac{Q^k - Q^{k-1}}{\log Q} \le \frac{\xi'}{\log \xi'}. \tag{1.11}$$

If Q satisfies (1.11), then $k \le \frac{\log \xi'}{\log 2}$ for ξ' large enough. Further, for k fixed, (1.11) implies that $Q \le (\xi')^{1/k}$, and the total number of solutions of (1.11) is

$$\le \sum_{k=2}^{\log \xi' / \log 2} (\xi')^{1/k} = O(\sqrt{\xi'}) = O(\sqrt{\xi}).$$

With a more careful estimation, this upper bound could be improved, but this crude result is enough for our purpose. Now, for all values of $y = \frac{Q^k - Q^{k-1}}{\log Q}$ satisfying (1.11), we cross out the interval $\left(y - \frac{\sqrt{\xi'}}{\log^4 \xi'}, y + \frac{\sqrt{\xi'}}{\log^4 \xi'}\right)$. We also cross out this interval whenever $y = \frac{\xi}{\log \xi}$ and $y = \frac{\xi'}{\log \xi'}$. The total sum of the lengths of the crossed out intervals is $O\left(\frac{\xi}{\log^4 \xi}\right)$, which is smaller than the length of the interval $\left(\frac{\xi}{\log \xi}, \frac{\xi'}{\log \xi'}\right)$ and if $\frac{x}{\log x}$ does not fall into one of these forbidden intervals, (1.10) will certainly hold. Since the derivative of the function $\varphi(x) = x/\log x$ is $\varphi'(x) = \frac{1}{\log x} - \frac{1}{\log^2 x}$ and satisfies $\varphi'(x) \sim \frac{1}{\log \xi}$ for all $x \in (\xi, \xi')$, the measure of the set of values of $x \in (\xi, \xi')$ such that $\varphi(x)$ falls into one of the above forbidden intervals is, by the mean value theorem $O\left(\frac{\xi}{\log^3 \xi}\right)$, and the proof of Proposition 1 is completed.

1.3 About the differences between primes

Proposition 2. *Suppose that there exists $\alpha, 0 < \alpha < 1$, and x large enough such that the inequalities*

$$\pi(x + x^\alpha) - \pi(x) \geq (1 - \varepsilon) x^\alpha / \log x \tag{1.12}$$

$$\pi(x) - \pi(x - x^\alpha) \geq (1 - \varepsilon) x^\alpha / \log x \tag{1.13}$$

hold. Then the set

$$E = E(x, \alpha) = \{P - Q; P, Q \text{ primes},\ x - x^\alpha < Q \leq x < P \leq x + x^\alpha\}$$

satisfies:

$$|E| \geq C_2 x^\alpha$$

where $C_2 = C_1 \alpha^4 (1 - \varepsilon)^4$ and C_1 is an absolute constant ($C_1 = 0.00164$ works).

Proof. The proof is a classical application of the sieve method that Paul Erdős enjoys very much. Let us set, for $d \leq 2x^\alpha$,

$$r(d) = |\{(P, Q); x - x^\alpha < Q \leq x < P \leq x + x^\alpha, P - Q = d\}|.$$

Clearly we have

$$|E| = \sum_{\substack{0 < d \leq 2x^\alpha \\ r(d) \neq 0}} 1 \tag{1.14}$$

and

$$\sum_{0 < d \leq 2x^\alpha} r(d) = (\pi(x + x^\alpha) - \pi(x))(\pi(x) - \pi(x - x^\alpha)) \geq (1 - \varepsilon)^2 x^{2\alpha} / \log^2 x. \tag{1.15}$$

Now to get an upper bound for $r(d)$, we sift the set

$$A = \{n; x - x^\alpha < n \le x\}$$

with the primes $p \le z$. If p divides d, we cross out the $n's$ satisfying $n \equiv 0 \pmod p$, and if p does not divide d, the $n's$ satisfying

$$n \equiv 0 \pmod p \quad \text{or} \quad n \equiv -d \pmod p$$

so that we set for $p \le z$:

$$w(p) = \begin{cases} 1 & \text{if } p \text{ divides } d \\ 2 & \text{if } p \text{ does not divide } d. \end{cases}$$

By applying the large sieve (cf. [14, Corollary 1]), we have

$$r(d) \le \frac{|A|}{L(z)}$$

with

$$L(z) = \sum_{n \le z} \left(1 + \frac{3}{2} n |A|^{-1} z\right)^{-1} \mu(n)^2 \left(\prod_{p|n} \frac{w(p)}{p - w(p)}\right)$$

(μ is the Möbius function), and with the choice $z = (\frac{2}{3}|A|)^{1/2}$, it is proved in [23] that

$$\frac{|A|}{L(z)} \le 16 \prod_{p \ge 3} \left(1 - \frac{1}{(p-1)^2}\right) \frac{|A|}{\log^2(|A|)} \prod_{\substack{p|d \\ p>2}} \frac{p-1}{p-2}.$$

The value of the above infinite product is $0.6602\ldots < 2/3$. We set $f(d) = \prod_{\substack{p|d \\ p>2}} \frac{p-1}{p-2}$, and we observe that $|A| \ge x^\alpha - 1$, so that for x large enough

$$r(d) \le \frac{32}{3\alpha^2} \frac{|A|}{\log^2 x} f(d). \tag{1.16}$$

Now, for the next step, we shall need an upper bound for $\sum_{n \le x} f^2(n)$. By using the convolution method and defining

$$h(n) = \sum_{a|n} \mu(a) f^2(n/a)$$

one gets $h(2) = h(2^2) = h(2^3) = \ldots = 0$ and, for $p \ge 3$, $h(p) = \frac{2p-3}{(p-2)^2}$, $h(p^2) = h(p^3) = \ldots = 0$, so that

$$\begin{aligned} \sum_{n \le x} f^2(n) &= \sum_{n \le x} \sum_{a|n} h(a) = \sum_{a \le x} h(a) \left\lfloor \frac{x}{a} \right\rfloor \\ &\le x \sum_{a=1}^{\infty} \frac{h(a)}{a} = x \prod_{p \ge 3} \left(1 + \frac{2p-3}{p(p-2)^2}\right) \qquad (1.17) \\ &= 2.63985\ldots x \le \frac{8}{3} x. \end{aligned}$$

From (1.15) and (1.16), one can deduce

$$\frac{(1-\varepsilon)^2 x^{2\alpha}}{\log^2 x} \le \sum_{\substack{0<d\le 2x^\alpha\\ r(d)\neq 0}} r(d) \le \frac{32}{3\alpha^2}\frac{|A|}{\log^2 x}\sum_{\substack{0<d\le 2x^\alpha\\ r(d)\neq 0}} f(d).$$

which implies

$$\sum_{\substack{0<d\le 2x^\alpha\\ r(d)\neq 0}} f(d) \ge \frac{3\alpha^2 x^{2\alpha}(1-\varepsilon)^2}{32|A|}\cdot$$

By Cauchy-Schwarz's inequality, one has

$$\left(\sum_{\substack{0<d\le 2x^\alpha\\ r(d)\neq 0}} 1\right)\left(\sum_{\substack{0<d\le 2x^\alpha\\ r(d)\neq 0}} f^2(d)\right) \ge \frac{9\alpha^4 x^{4\alpha}(1-\varepsilon)^4}{1024|A|^2}$$

and, by (1.14) and (1.17)

$$|E| \ge \frac{9\alpha^4 x^{4\alpha}(1-\varepsilon)^4}{1024|A|^2}\Big/\frac{8}{3}(2x^\alpha) = \frac{27}{16384}\frac{x^{3\alpha}(1-\varepsilon)^4}{|A|^2}\cdot$$

Since $|A| \le x^\alpha + 1$, and x has been supposed large enough, proposition 2 is proved.

1.4 Some properties of $g(n)$

Here, we recall some known properties of $g(n)$ which can be found for instance in [16]. Let us define the arithmetic function ℓ in the following way: ℓ is additive, and, if p is a prime and $k \ge 1$, then $\ell(p^k) = p^k$. It is not difficult to deduce from (1.1) (cf. [13] or [16]) that

$$g(n) = \max_{\ell(M)\le n} M. \tag{1.18}$$

Now the relation (cf. [16], p. 139)

$$M \in g(\mathbb{N}) \quad\Longleftrightarrow\quad (M' > M \implies \ell(M') > \ell(M)) \tag{1.19}$$

easily follows from (1.18), and shows that the values of the Landau function g are the "champions" for the small values of ℓ. So the methods introduced by Ramanujan (cf. [19]) to study highly composite numbers can also be used for $g(n)$. Indeed M is highly composite, if it is a "champion" for the divisor function d, that is to say if

$$M' < M \quad\Longrightarrow\quad d(M') < d(M).$$

Corresponding to the so-called superior highly composite numbers, one introduces the set $G : N \in G$ if there exists $\rho > 0$ such that

$$\forall M \geq 1, \quad \ell(M) - \rho \log M \geq \ell(N) - \rho \log N. \tag{1.20}$$

(1.19) and (1.20) easily imply that $G \subset g(\mathbb{N})$. Moreover, if $\rho > 2/\log 2$, let us define $x > 4$ such that $\rho = x/\log x$ and

$$N_\rho = \prod_{p \leq x} p^{\alpha_p} = \prod_p p^{\alpha_p} \tag{1.21}$$

with

$$\alpha_p = \begin{cases} 0 & \text{if} \quad p > x \\ 1 & \text{if} \quad \frac{p}{\log p} \leq \rho < \frac{p^2 - p}{\log p} \\ k \geq 2 & \text{if} \quad \frac{p^k - p^{k-1}}{\log p} \leq \rho < \frac{p^{k-1} - p^k}{\log p} \end{cases}$$

then $N_\rho \in G$. With the above definition, since $x \geq 4$, it is not difficult to show that (cf. [11, (5)])

$$p^{\alpha_p} \leq x \tag{1.22}$$

holds for $p \leq x$, whence N_ρ is a divisor of the l.c.m. of the integers $\leq x$. Here we can prove

Proposition 3. *For every prime p, there exists n such that the largest prime factor of $g(n)$ is equal to p.*

Proof. We have $g(2) = 2, g(3) = 3$. If $p \geq 5$, let us choose $\rho = p/\log p > 2/\log 2$. N_ρ defined by (1.21) belongs to $G \subset g(\mathbb{N})$, and its largest prime factor is p, which proves Proposition (3).

From Proposition 3, it is easy to deduce a proof of Theorem 1, under the twin prime conjecture. Let $P = p + 2$ be twin primes, and n such that the largest prime factor of $g(n)$ is p. The sequence n_k being defined by (1.3), we define k in terms of n by $n_k \leq n < n_{k+1}$, so that $g(n_k) = g(n)$ has its largest prime factor equal to p. Now, from (1.18) and (1.19),

$$\ell(g(n_k)) = n_k$$

and $g(n_k+2) > g(n_k)$ since $M = \frac{P}{p} g(n_k)$ satisfies $M > g(n_k)$ and $\ell(M) = n_k+2$. So $n_{k+1} \leq n_k + 2$, and Theorem 1 is proved under this strong hypothesis.

Let us introduce now the so-called benefit method. For a fixed $\rho > 2/\log 2$, $N = N_\rho$ is defined by (1.21), and for any integer M,

$$M = \prod_p p^{\beta_p},$$

one defines the benefit of M:

$$\text{ben}(M) = \ell(M) - \ell(N) - \rho \log M/N. \tag{1.23}$$

Clearly, from (1.20), $\mathrm{ben}(M) \geq 0$ holds, and from the additivity of ℓ one has

$$\mathrm{ben}(M) = \sum_p \left(\ell(p^{\beta_p}) - \ell(p^{\alpha_p}) - \rho(\beta_p - \alpha_p)\log p\right). \tag{1.24}$$

In the above formula, let us observe that $\ell(p^\beta) = p^\beta$ if $\beta \geq 1$, but that $\ell(p^\beta) = 0 \neq p^\beta = 1$ if $\beta = 0$, and, due to the choice of α_p in (1.21), that, in the sum (1.24), all the terms are non negative: for all p and for $\beta \geq 0$, we have

$$\ell(p^\beta) - \ell(p^{\alpha_p}) - \rho(\beta - \alpha_p)\log p \geq 0 \tag{1.25}$$

Indeed, let us consider the set of points (0,0) and $(\beta, p^\beta/\log p)$ for β integer ≥ 1. For all p, the piecewise linear curve going through these points is convex, and for a given ρ, α_p is chosen so that the straight line L of slope ρ going through $\left(\alpha_p, \frac{p^{\alpha_p}}{\log p}\right)$ does not cut that curve. The left-hand side of (1.25), (which is $\mathrm{ben}(Np^{\beta-\alpha_p})$) can be seen as the product of log p by the vertical distance of the point $\left(\beta, \frac{p^\beta}{\log p}\right)$ to the straight line L, and because of convexity, we shall have for all p,

$$\mathrm{ben}(Np^t) \geq t\ \mathrm{ben}(Np), \quad t \geq 1 \tag{1.26}$$

and for $p \leq x$,

$$\mathrm{ben}(Np^{-t}) \geq t\ \mathrm{ben}(Np^{-1}), \quad 1 \leq t \leq \alpha_p. \tag{1.27}$$

1.5 Proof of Theorem 1

First the following proposition will be proved:

Proposition 4. *Let $\alpha < 1/2$, and x large enough such that (1.10) holds. Let us denote the primes surrounding x by:*

$$\ldots < Q_j < \ldots < Q_2 < Q_1 \leq x < P_1 < P_2 < \ldots < P_i < \ldots$$

Let us define $\rho = x/\log x, N = N_\rho$ by (1.21), $n = \ell(N)$. Then for $n \leq m \leq n + 2x^\alpha, g(m)$ can be written

$$g(m) = N\frac{P_{i_1}P_{i_2}\ldots P_{i_r}}{Q_{j_1}Q_{j_2}\ldots Q_{j_r}} \tag{1.28}$$

with $r \geq 0$ and $i_1 < \ldots < i_r, j_1 < \ldots < j_r, P_{i_r} \leq x + 4x^\alpha, Q_{j_r} \geq x - 4x^\alpha$.

Proof. First, from (1.18), one has $\ell(g(m)) \leq m$, and from (1.23) and (1.18)

$$\mathrm{ben}(g(m)) = \ell(g(m)) - \ell(N) - \rho\log\frac{g(m)}{N} \leq m - n \leq 2x^\alpha \tag{1.29}$$

for $n \leq m \leq 2x^\alpha$.

Further, let $Q \leq x$ be a prime, and $k = \alpha_Q \geq 1$ the exponent of Q in the standard factorization of N. Let us suppose that for a fixed m, Q divides $g(m)$

with the exponent $\beta_Q = k+t, t>0$. Then, from (1.24), (1.25), and (1.26), one gets

$$\mathrm{ben}(g(m)) \geq \mathrm{ben}(NQ^t) \geq \mathrm{ben}(NQ) \tag{1.30}$$

and

$$\begin{aligned} \mathrm{ben}(NQ) &= Q^{k+1} - Q^k - \rho \log Q \\ &= \log Q \left(\frac{Q^{k+1} - Q^k}{\log Q} - \rho \right). \end{aligned}$$

From (1.21), the above parenthesis is non negative, and from (1.10), one gets:

$$\mathrm{ben}(NQ) \geq \log 2 \frac{\sqrt{x}}{\log^4 x} \cdot \tag{1.31}$$

For x large enough, there is a contradiction between (1.29), (1.30) and (1.31), and so, $\beta_Q \leq \alpha_Q$.

Similarly, let us suppose $Q \leq x, k = \alpha_Q \geq 2$ and $\beta_Q = k-t, 1 \leq t \leq k$. One has, from (1.24), (1.25) and (1.27),

$$\mathrm{ben}(g(m)) \geq \mathrm{ben}(NQ^{-t}) \geq \mathrm{ben}(NQ^{-1})$$

and

$$\begin{aligned} \mathrm{ben}(NQ^{-1}) &= Q^{k-1} - Q^k + \rho \log Q \\ &= \log Q \left(\rho - \frac{Q^k - Q^{k-1}}{\log Q} \right) \geq \log 2 \frac{\sqrt{x}}{\log^4 x} \end{aligned}$$

which contradicts (1.29), and so, for such a Q, $\beta_Q = \alpha_Q$.

Now, let us suppose $Q \leq x, \alpha_Q = 1$, and $\beta_Q = 0$ for some $m, n \leq m \leq n + 2x^\alpha$. Then

$$\mathrm{ben}(g(m)) \geq \mathrm{ben}(NQ^{-1}) = -Q + \rho \log Q = y(Q)$$

by setting $y(t) = \rho \log t - t$. From the concavity of $y(t)$ for $t > 0$, for $x \geq e^2$, we get

$$\begin{aligned} y(Q) \geq y(x) + (Q-x)y'(x) &= (Q-x)\left(\frac{\rho}{x} - 1\right) \\ &= (x-Q)\left(1 - \frac{1}{\log x}\right) \geq \frac{1}{2}(x-Q) \end{aligned}$$

and so,

$$\mathrm{ben}(g(m)) \geq \frac{1}{2}(x-Q)$$

which, from (1.29) yields

$$x - Q \leq 4x^\alpha.$$

In conclusion, the only prime factors allowed in the denominator of $\frac{g(m)}{N}$ are the $Q's$, with $x - 4x^\alpha \leq Q \leq x$, and $\alpha_Q = 1$.

What about the numerator? Let $P > x$ be a prime number and suppose that P^t divides $g(m)$ with $t \geq 2$. Then, from (1.26) and (1.23),

$$\mathrm{ben}(Np^t) \geq \mathrm{ben}(Np^2) = P^2 - 2\rho \log P.$$

But the function $t \mapsto t^2 - 2\rho \log t$ is increasing for $t \geq \sqrt{\rho}$, so that,

$$\mathrm{ben}(NP^t) \geq x^2 - 2x > 2x^\alpha$$

for x large enough, which contradicts (1.29). The only possibility is that P divides $g(m)$ with exponent 1. In that case, from the convexity of the function $z(t) = t - \rho \log t$, inequality (1.26) yields

$$\begin{aligned} \mathrm{ben}(g(m)) \;\geq\; \mathrm{ben}(NP) \;=\; z(P) \;&\geq\; z(x) + (P - x)z'(x) \\ &= (P - x)\left(1 - \frac{1}{\log x}\right) \;\geq\; \frac{1}{2}(P - x) \end{aligned}$$

for $x \geq e^2$, which, with (1.29), implies

$$P - x \leq 4x^\alpha.$$

Up to now, we have shown that

$$g(m) = N \frac{P_{i_1} \dots P_{i_r}}{Q_{j_1} \dots Q_{j_s}}$$

with $P_{i_r} \leq x + 4x^\alpha, Q_{j_s} \geq x - 4x^\alpha$. It remains to show that $r = s$. First, since $n \leq m \leq n + 2x^\alpha$, and N belongs to G, we have from (1.18) and (1.19)

$$n \leq \ell(g(m)) \leq n + 2x^\alpha. \tag{1.32}$$

Further,

$$\ell(g(m)) - n = \sum_{t=1}^{r} P_{i_t} - \sum_{t=1}^{s} Q_{j_t}$$

and since $r \leq 4x^\alpha$, and $s \leq 4x^\alpha$,

$$\begin{aligned} \ell(g(m)) - n \;&\leq\; r(x + 4x^\alpha) - s(x - 4x^\alpha) \\ &\leq\; (r - s)x + 32x^{2\alpha}. \end{aligned}$$

From (1.32), $\ell(g(m)) - n \geq 0$ holds and as $\alpha < 1/2$, this implies that $r \geq s$ for x large enough. Similarly,

$$\ell(g(m)) - n \geq (r - s)x,$$

so, from (1.32), $(r - s)x$ must be $\leq 2x^\alpha$, which, for x large enough, implies $r \leq s$; finally $r = s$, and the proof of Proposition 4 is completed.

Lemma 1. *Let x be a positive real number, $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k$ be real number such that*

$$b_k \le b_{k-1} \le \dots \le b_1 \le x < a_1 \le a_2 \le \dots \le a_k$$

and Δ be defined by $\Delta = \sum_{i=1}^{k}(a_i - b_i)$. Then the following inequalities

$$\frac{x+\Delta}{x} \le \prod_{i=1}^{k} \frac{a_i}{b_i} \le \exp\left(\frac{\Delta}{x}\right)$$

hold.

Proof. It is easy, and can be found in [16], p. 159.

Now it is time to prove Theorem 1. With the notation and hypothesis of Proposition 4, let us denote by B the set of integers M of the form

$$M = N \frac{P_{i_1} P_{i_2} \dots P_{i_r}}{Q_{j_1} Q_{j_2} \dots Q_{j_r}}$$

satisfying

$$\ell(M) - \ell(N) = \sum_{t=1}^{r} (P_{i_t} - Q_{j_t}) \le 2x^\alpha.$$

From Proposition 4, for $n \le m \le 2x^\alpha, g(m) \in B$, and thus, from (1.18),

$$g(m) = \max_{\substack{\ell(M) \le m \\ M \in B}} M. \tag{1.33}$$

Further, for $0 \le d \le 2x^\alpha$, define

$$B_d = \{M \in B; \ell(M) - \ell(N) = d\}.$$

I claim that, if $d < d'$ (which implies $d \le d' - 2$), any element of B_d is smaller than any element of $B_{d'}$. Indeed, let $M \in B_d$, and $M' \in B_{d'}$. From Lemma 1, one has

$$\frac{M}{N} \le \exp\left(\frac{d}{x}\right) \quad \text{and} \quad \frac{M'}{N} \ge \frac{x + d'}{x} \ge \frac{x+d+2}{x}.$$

Since $d < 2x^\alpha < x$, and $e^t \le \frac{1}{1-t}$ for $0 \le t < 1$, one gets

$$\frac{M}{N} \le \frac{1}{1 - d/x} = \frac{x}{x-d}.$$

This last quantity is smaller than $\frac{x+d+2}{x}$ if $(d+1)^2 < 2x + 1$, which is true for x large enough, because $d \le 2x^\alpha$ and $\alpha < 1/2$.

From the preceding claim, and from (1.33), it follows that, if B_d is non empty, then

$$g(n+d) = \max B_d.$$

Further, since $N \in G$, we know that $n = \ell(N)$ belongs to the sequence (n_k) where g is increasing, and so, $n = n_{k_0}$. If $0 < d_1 < d_2 < \ldots < d_s \leq 2x^\alpha$ denote the values of d for which B_d is non empty, then one has

$$n_{k_{0+i}} = n + d_i, 1 \leq i \leq s. \tag{1.34}$$

Suppose now that $\alpha < 1/2$ and x have been chosen in such a way that (1.12) and (1.13) hold. With the notation of Proposition 2, the set $E(x, \alpha)$ is certainly included in the set $\{d_1, d_2, \ldots, d_s\}$, and from Proposition 2,

$$s \geq C_2 x^\alpha \tag{1.35}$$

which implies that for at least one $i, d_{i+1} - d_i \leq \frac{2}{C_2}$, and thus

$$n_{k_0+i+1} - n_{k_0+i} \leq \frac{2}{C_2}.$$

Finally, for $\frac{1}{6} < \alpha < \frac{1}{2}$, Proposition 1 allows us to choose x as wished, and thus, the proof of Theorem 1 is completed. With ε very small, and α close to 1/2, the values of C_1 and C_2 given in Proposition 2 yield that for infinitely many $k's$,

$$n_{k+1} - n_k \leq 20000.$$

To count how many such differences we get, we define

$$\gamma(n) = \mathrm{Card}\{m \leq n; g(m) > g(m-1)\}.$$

Therefore, with the notation (1.3), we have $n_{\gamma(n)} = n$.

In [16, 162–164], it is proved that

$$n^{1-\tau/2} \ll \gamma(n) \leq n - c\frac{n^{3/4}}{\sqrt{\log n}}$$

where τ is such that the sequence of consecutive primes satisfies $p_{i+1} - p_i \ll p_i^\tau$. Without any hypothesis, the best known τ is $> 1/2$.

Proposition 5. *We have $\gamma(n) \geq n^{3/4-\varepsilon}$ for all $\varepsilon > 0$, and n large enough.*

Proof. With the definition of $\gamma(n)$, (1.34) and (1.35) give

$$\gamma(n + 2x^\alpha) - \gamma(n) \geq s \gg x^\alpha \tag{1.36}$$

whenever $n = \ell(N), N = N_\rho, \rho = x/\log x$, and x satisfies Proposition 1. But, from (1.21), two close enough distinct values of x can yield the same N.

I now claim that, with the notation of Proposition 1, the number of primes p_i between ξ and ξ' such that there is at least one $x \in [p_i, p_{i+1})$ satisfying (1.8), (1.9) and (1.10) is bigger than $\frac{1}{2}(\pi(\xi') - \pi(\xi))$. Indeed, for each i for which $[p_i, p_{i+1})$ does not contain any such x, we get a measure $p_{i+1} - p_i \geq 2$, and if there are more than $\frac{1}{2}(\pi(\xi') - \pi(\xi))$ such $i's$, the total measure will be greater than $\pi(\xi') - \pi(\xi) \sim \xi/\log^2 \xi$, which contradicts Proposition 1.

From the above claim, there will be at least $\frac{1}{2}(\pi(\xi') - \pi(\xi))$ distinct $N's$, with $N = N_\rho, \rho = x/\log x$, and $\xi \le x \le \xi'$. Moreover, for two such distinct N, say $N' < N$", we have from (1.21), $\ell(N'') - \ell(N') \ge \xi$.

Let $N^{(1)}$ and $N^{(0)}$ the biggest and the smallest of these $N's$, and $n^{(1)} = \ell(N^{(1)}), n^{(0)} = \ell(N^{(0)})$, then from (1.36),

$$\gamma(n^{(1)}) \ge \gamma(n^{(1)}) - \gamma(n^{(0)}) \ge \frac{1}{2}\left(\pi(\xi') - \pi(\xi)\right)\xi^\alpha \gg \frac{\xi^{1+\alpha}}{\log^2 \xi}. \tag{1.37}$$

But from (1.21) and (1.22), $x \sim \log N_\rho$, and from (1.2),

$$x \sim \log N_\rho \sim \sqrt{n \log n} \quad \text{with} \quad n = \ell(N_p)$$

so

$$\xi \sim \sqrt{n^{(1)} \log n^{(1)}}$$

and since α can be choosen in (1.37) as close as wished of 1/2, this completes the proof of Proposition 5.

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