

# Quandle Colorings of Knots and Applications

W. Edwin Clark, Mohamed Elhamdadi, Masahico Saito, Timothy Yeatman

Department of Mathematics and Statistics  
University of South Florida

## Abstract

We present a set of 26 finite quandles that distinguish (up to reversal and mirror image) by number of colorings, all of the 2977 prime oriented knots with up to 12 crossings. We also show that 1058 of these knots can be distinguished from their mirror images by the number of colorings by quandles from a certain set of 23 finite quandles. We study the colorings of these 2977 knots by all of the 431 connected quandles of order at most 35 found by L. Vendramin. Among other things, we collect information about quandles that have the same number of colorings for all of the 2977 knots. For example, we prove that if  $Q$  is a simple quandle of prime power order then  $Q$  and the dual quandle  $Q^*$  of  $Q$  have the same number of colorings for all knots and conjecture that this holds for all Alexander quandles  $Q$ . We study a knot invariant based on a quandle homomorphism  $f : Q_1 \rightarrow Q_0$ . We also apply the quandle colorings we have computed to obtain some new results for the bridge index, the Nakanishi index, the tunnel number, and the unknotting number. In an appendix we discuss various properties of the quandles in Vendramin's list. Links to the data computed and various programs in C, GAP and Maple are provided.

*Key words:* Quandles, knot colorings, mirror images of knots, knot invariants.

## 1 Introduction

Sets with certain self-distributive operations called *quandles* have been studied since the 1940s in various areas with different names. They have been studied, for example, as algebraic systems for symmetries [45], as quasi-groups [18], and in relation to modules [21, 37]. Typical examples of quandles arise from conjugacy classes of groups and from modules over the integral Laurent polynomial ring  $\mathbb{Z}[t, t^{-1}]$ , called Alexander quandles. The *fundamental quandle* of a knot was defined in a manner similar to the fundamental group [25, 32] of a knot, which made quandles an important tool in knot theory. The number of homomorphisms from the fundamental quandle to a fixed finite quandle has an interpretation as colorings of knot diagrams by quandle elements, and has been widely used as a knot invariant. Algebraic homology theories for quandles were defined [6, 17], and investigated in [29, 34, 38, 39, 40]. Extensions of quandles by cocycles have been studied [1, 5, 14], and invariants derived thereof are applied to various properties of knots and knotted surfaces (see [7] and references therein).

Tables of small quandles have been made previously (e.g., [7, 15, 37]). Computations using GAP by L. Vendramin [47] significantly expanded the list for connected quandles. He found all connected quandles of order up to 35. There are 431 of them. These quandles may be found in the

GAP package RIG [46]. We refer to these quandles as *RIG* quandles, and use the notation  $C[n, i]$  for the  $i$ -th quandle of order  $n$  in his list. As a matrix  $C[n, i]$  is the transpose of the quandle matrix *SmallQuandle*( $n, i$ ) in [46].

In this paper, we investigate to what extent the number of quandle colorings of a knot by a finite quandle can distinguish the oriented knots with at most 12 crossings from the knot table at KnotInfo [8].

In Section 2, we provide definitions and conventions. The coloring of knots by quandles and the relationship to symmetry of knots are discussed in Section 3 and Section 4. It turns out that *RIG* quandles do not suffice to distinguish all of the prime, oriented knots with at most 12 crossings. To distinguish all of these knots we generated several thousand conjugation quandles and generalized Alexander quandles. Eventually we found a set of 26 quandles that distinguish, up to orientation and mirror image, all knots with up to 12 crossings. These computations extend the results by Dionisio and Lopes [12] for 10 Alexander quandles and 249 prime knots with at most 10 crossings.

We write  $m(K)$  for the mirror image of knot  $K$  and  $r(K)$  for  $K$  with the orientation reversed. The 2977 knots given at KnotInfo [8] are representatives up to mirrors and reverses of the prime, oriented knots with at most 12 crossings (see [9]). It is known [30, 32] that quandle colorings do not distinguish  $K$  from  $rm(K)$  for any knot  $K$ . Thus it is of interest which finite quandles, if any, can distinguish  $K$  from  $m(K)$ . This is only possible for a knot which is chiral or negative amphicheiral. There are 1366 chiral or negative amphicheiral knots among the 2977 knots with up to 12 crossings at KnotInfo [8]. We have found a set of 23 quandles that distinguish  $K$  from  $m(K)$  for 1058 knots out of these 1366 knots. It remains an open question whether the remaining 308 of these knots can be so distinguished.

Many sets of RIG quandles share the same number of colorings for all knots in the KnotInfo table. We explore this phenomenon in Section 5. For example, we conjecture that a connected Alexander quandle and its dual quandle always give the same number of colorings for any knot, and we prove this for a special class of Alexander quandles. In Section 6, we introduce an invariant of knots based on quandle homomorphisms and mention a relation to the quandle cocycle invariant. The computational results are applied to other knot invariants in Section 7, such as the bridge index, the Nakanishi index, the tunnel number and the unknotting number. Various properties of quandles are further discussed in Appendix A, and a summary of computational results is given as to these properties for RIG quandles. Descriptions of quandles used for distinguishing knots and their mirror images can be found in Appendix B. Files containing the quandle matrices, programs and the results of computations can be found at [48, 49]. Appendix C contains the list of 12 crossing knots discussed in Section 7. Appendix D contains some information on programs used for colorings.

## 2 Preliminaries

We briefly review some definitions and examples of quandles. More details can be found, for example, in [1, 7, 17].

A *quandle*  $X$  is a set with a binary operation  $(a, b) \mapsto a * b$  satisfying the following conditions.

- (1) For any  $a \in X$ ,  $a * a = a$ .
- (2) For any  $b, c \in X$ , there is a unique  $a \in X$  such that  $a * b = c$ .
- (3) For any  $a, b, c \in X$ , we have  $(a * b) * c = (a * c) * (b * c)$ .

A *quandle homomorphism* between two quandles  $X, Y$  is a map  $f : X \rightarrow Y$  such that  $f(a *_X b) = f(a) *_Y f(b)$ , where  $*_X$  and  $*_Y$  denote the quandle operations of  $X$  and  $Y$ , respectively. A *quandle isomorphism* is a bijective quandle homomorphism, and two quandles are *isomorphic* if there is a quandle isomorphism between them.

Typical examples of quandles include the following.

- Any non-empty set  $X$  with the operation  $a * b = a$  for any  $a, b \in X$  is a quandle called a *trivial quandle*.
- A conjugacy class  $X$  of a group  $G$  with the quandle operation  $a * b = b^{-1}ab$ . We call this a *conjugation quandle*.
- A *generalized Alexander quandle* is defined by a pair  $(G, f)$  where  $G$  is a group and  $f \in \text{Aut}(G)$ , and the quandle operation is defined by  $x * y = f(xy^{-1})y$ . If  $G$  is abelian, this is called an *Alexander quandle*.
- A *Galkin quandle* is defined as follows. Let  $A$  be an abelian group, also regarded naturally as a  $\mathbb{Z}$ -module. Let  $\mu : \mathbb{Z}_3 \rightarrow \mathbb{Z}$ ,  $\tau : \mathbb{Z}_3 \rightarrow A$  be functions. These functions  $\mu$  and  $\tau$  need not be homomorphisms. Define a binary operation on  $\mathbb{Z}_3 \times A$  by

$$(x, a) * (y, b) = (2y - x, -a + \mu(x - y)b + \tau(x - y)) \quad x, y \in \mathbb{Z}_3, a, b \in A.$$

Then for any abelian group  $A$ , the above operation  $*$  defines a quandle structure on  $\mathbb{Z}_3 \times A$  if  $\mu(0) = 2$ ,  $\mu(1) = \mu(2) = -1$ , and  $\tau(0) = 0$ . Galkin gave this definition in [18], page 950, for  $A = \mathbb{Z}_p$ , and this definition was given in [10].

- A function  $\phi : X \times X \rightarrow A$  for an abelian group  $A$  is called a *quandle 2-cocycle* [6] if it satisfies

$$\phi(x, y) - \phi(x, z) + \phi(x * y, z) - \phi(x * z, y * z) = 0$$

and  $\phi(x, x) = 0$  for any  $x, y, z \in X$ . For a quandle 2-cocycle  $\phi$ ,  $X \times A$  becomes a quandle by  $(x, a) * (y, b) = (x * y, a + \phi(x, y))$  for  $x, y \in X$ ,  $a, b \in A$ , and it is called an *abelian extension* of  $X$  by  $A$ , see [5].

Let  $X$  be a quandle. The *right translation*  $\mathcal{R}_a : X \rightarrow X$ , by  $a \in X$ , is defined by  $\mathcal{R}_a(x) = x * a$  for  $x \in X$ . Similarly the *left translation*  $\mathcal{L}_a$  is defined by  $\mathcal{L}_a(x) = a * x$ . Then  $\mathcal{R}_a$  is a permutation of  $X$  by Axiom (2). The subgroup of  $\text{Sym}(X)$  generated by the permutations  $\mathcal{R}_a$ ,  $a \in X$ , is called the *inner automorphism group* of  $X$ , and is denoted by  $\text{Inn}(X)$ . A quandle is *connected* if  $\text{Inn}(X)$  acts transitively on  $X$ . The operation  $\bar{*}$  on  $X$  defined by  $a \bar{*} b = \mathcal{R}_b^{-1}(a)$  is a quandle operation, and  $(X, \bar{*})$  is called the *dual* quandle of  $(X, *)$ . We also denote the dual of  $X$  by  $X^*$ . If  $X^*$  is isomorphic to  $X$ , then  $X$  is called *self-dual*.

We write  $K = K'$  to denote that there is an orientation preserving homeomorphism of  $\mathbb{S}^3$  that takes  $K$  to  $K'$  preserving the orientations of  $K$  and  $K'$ . A *coloring* of an oriented knot diagram by

a quandle  $X$  is a map  $\mathcal{C} : \mathcal{A} \rightarrow X$  from the set of arcs  $\mathcal{A}$  of the diagram to  $X$  such that the image of the map satisfies the relation depicted in Figure 1 at each crossing. More details can be found in [7, 13], for example. A coloring that assigns the same element of  $X$  to all the arcs is called trivial, otherwise non-trivial. The number of colorings of a knot diagram by a finite quandle is known to be independent of the choice of a diagram, and hence is a knot invariant. In Figure 2, a coloring of a figure-eight knot is indicated. In the figure, the elements  $x, y, z$  of a quandle are assigned at the top arcs. The coloring conditions are applied to color other arcs as depicted. Since the bottom arcs are connected to the top arcs, we need the conditions

$$\begin{aligned} x &= z \bar{*} (x * y), \\ y &= (x * y) \bar{*} (y * (z \bar{*} (x * y))), \\ z &= y * (z \bar{*} (x * y)). \end{aligned}$$

to obtain a coloring. Each crossing corresponds to a standard generator  $\sigma_i$  of the 3-strand braid group or its inverse, and the closure of  $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$  represents the figure-eight knot. The dotted lines indicate the closure, see [44] for more details on braids. We give the orientation downward for braids, see the figure. *For the rest of this paper all knots will be prime and oriented.*

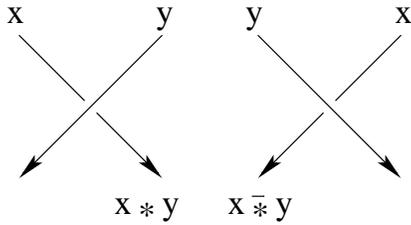


Figure 1: The coloring rule at crossings

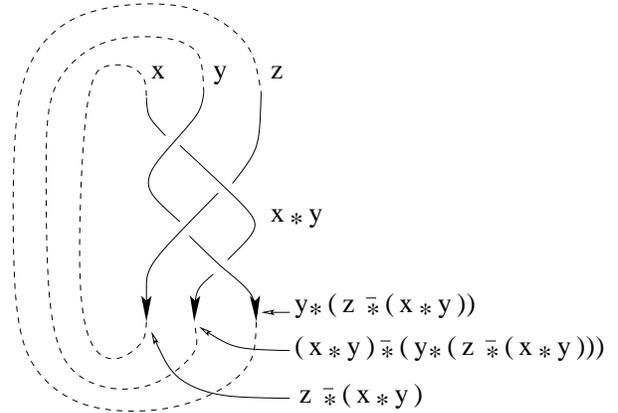


Figure 2: Figure-eight knot as the closure of the braid  $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$

The fundamental quandle is defined in a manner similar to the fundamental group [25, 32]. A *presentation* of a quandle is defined in a manner similar to groups as well, and a presentation of the fundamental quandle is obtained from a knot diagram (see, for example, [16]), by assigning generators to arcs of a knot diagram, and relations corresponding to crossings. The set of colorings of a knot diagram  $K$  by a quandle  $X$ , then, is in one-to-one correspondence with the set of quandle homomorphisms from the fundamental quandle of  $K$  to  $X$ .

If  $Q$  is a quandle and  $K$  is a knot we denote by  $\text{Col}_Q^K(K)$  the number of non-trivial colorings of  $K$  by  $Q$ . The number of colorings including trivial colorings is  $\text{Col}_Q(K) = \text{Col}_Q^K(K) + |Q|$ , where  $|Q|$  is the order of  $Q$ . It is well known that for each quandle  $Q$ ,  $\text{Col}_Q(K)$  is an invariant of knots ([43], for example). We say that the quandle  $Q$  *distinguishes* knots  $K$  and  $K'$  if  $\text{Col}_Q(K) \neq \text{Col}_Q(K')$ .

Let  $m : \mathbb{S}^3 \rightarrow \mathbb{S}^3$  be an orientation reversing homeomorphism. For a knot  $K$ ,  $m(K)$  is the mirror image of  $K$ . Let  $r(K)$  denote the same knot  $K$  with its orientation reversed. We regard  $m$

and  $r$  as maps on equivalence classes of knots. We consider the group  $\mathcal{G} = \{1, r, m, rm\}$  acting on the set of all oriented knots. For each knot  $K$  let  $\mathcal{G}(K) = \{K, r(K), m(K), rm(K)\}$  be the orbit of  $K$  under the action of  $\mathcal{G}$ .

By a symmetry we mean that a knot (type)  $K$  remains unchanged under one of  $r, m, rm$ . As in the definition of *symmetry type* in [8] we say that a knot  $K$  is

- *reversible* if the only symmetry it has is  $K = r(K)$ ,
- *negative amphicheiral* if the only symmetry it has is  $K = rm(K)$ ,
- *positive amphicheiral* if the only symmetry it has is  $K = m(K)$ ,
- *chiral* if it has none of these symmetries,
- *fully amphicheiral* if it has all three symmetries, that is,  $K = r(K) = m(K) = rm(K)$ .

The symmetry type of each knot on at most 12 crossings is given at [8]. Thus each of the 2977 knots  $K$  given there represents as many as four knots  $K, m(K), r(K)$  and  $rm(K)$ .

It is known [30, 32] that the fundamental quandles of  $K$  and  $K'$  are isomorphic if and only if  $K = K'$  or  $K = rm(K')$ .

### 3 Distinguishing the knots $K, m(K), r(K), rm(K)$

For the problem of distinguishing knots by quandle coloring it suffices to consider only connected quandles by the following lemma (see Ohtsuki [41], Section 5.2).

**Lemma 3.1** *The image of a quandle homomorphism of the fundamental quandle of any knot in another quandle is connected. In particular, given two knots  $K_1$  and  $K_2$ , a quandle  $Q$  of smallest order satisfying  $\text{Col}_Q(K_1) \neq \text{Col}_Q(K_2)$  is connected.*

**Lemma 3.2** *The following equations hold for all quandles  $Q$  and all knots  $K$ :*

- (1)  $\text{Col}_Q(K) = \text{Col}_Q(rm(K))$ ,
- (2)  $\text{Col}_Q(r(K)) = \text{Col}_Q(m(K))$ ,
- (3)  $\text{Col}_Q(m(K)) = \text{Col}_{Q^*}(K)$ ,

where  $Q^*$  in (3) denotes the dual quandle of  $Q$ .

*Proof:* For (1) and (2) see Kamada [26]. Item (3) follows easily from the definitions. ■

**Corollary 3.3** *For all quandles  $Q$  and for knots  $K$  of any symmetry type except chiral or negative amphicheiral,*

$$\text{Col}_Q(rm(K)) = \text{Col}_Q(m(K)) = \text{Col}_Q(r(K)) = \text{Col}_Q(K).$$

**Conjecture 3.4** *If  $\mathcal{K}$  is any finite set of knots closed under the action of  $\mathcal{G}$ , there exists a finite sequence of finite quandles  $\mathcal{S} = (Q_1, Q_2, \dots, Q_n)$  such that the invariant*

$$C(K) := (\text{Col}_{Q_1}(K), \text{Col}_{Q_2}(K), \dots, \text{Col}_{Q_n}(K))$$

*satisfies for all  $K, K' \in \mathcal{K}$ :*

$$C(K') = C(K) \text{ if and only if } K' = K \text{ or } K' = mr(K).$$

Our computations verify the truth of the following weaker conjecture where  $\mathcal{K}$  is the set of all knots with at most 12 crossings. We also verify the truth of Conjecture 3.4 for smaller sets of knots (see Section 4).

**Conjecture 3.5** *If  $\mathcal{K}$  is any finite set of knots closed under the action of  $\mathcal{G}$ , there exists a finite sequence of finite quandles  $\mathcal{S} = (Q_1, Q_2, \dots, Q_n)$  such that the invariant*

$$C(K) := (\text{Col}_{Q_1}(K), \text{Col}_{Q_2}(K), \dots, \text{Col}_{Q_n}(K))$$

*satisfies for all  $K, K' \in \mathcal{K}$ :*

$$C(K') = C(K) \text{ implies } \mathcal{G}(K') = \mathcal{G}(K)$$

It is clear that if Conjecture 3.4 is true then the following conjecture is true.

**Conjecture 3.6** *There exists a sequence of finite quandles  $\mathcal{S} = (Q_1, Q_2, \dots, Q_n, \dots)$  such that the invariant*

$$C(K) := (\text{Col}_{Q_1}(K), \text{Col}_{Q_2}(K), \dots, \text{Col}_{Q_n}(K), \dots)$$

*satisfies for all knots  $K, K'$ ,*

$$C(K') = C(K) \text{ if and only if } K' = K \text{ or } K' = mr(K).$$

This conjecture is similar to a conjecture made by Fenn and Rourke [16], page 385, for racks.

Let  $\mathcal{K}$  be a finite set of knots closed under the action of the group  $\mathcal{G}$ . Let  $\mathcal{R} = \{k_1, k_2, \dots, k_n\}$  be a set of representatives of the orbits of  $\mathcal{G}$  acting on  $\mathcal{K}$ . Let  $\mathcal{S} = (Q_1, Q_2, \dots, Q_n)$  be a list of finite quandles and for any knot  $K$  let

$$C(K) := (\text{Col}_{Q_1}(K), \text{Col}_{Q_2}(K), \dots, \text{Col}_{Q_n}(K)).$$

To simplify the following proof, for knots  $K$  and  $K'$  we write

$$K \sim K' \text{ if and only if } C(K) = C(K').$$

Thus  $K \approx K'$  if and only if there exist  $Q_i \in \mathcal{S}$  such that  $\text{Col}_{Q_i}(K) \neq \text{Col}_{Q_i}(K')$ . The analysis of our computations is based on the following.

**Proposition 3.7** *Staying with the notation above, suppose for all pairs of knots  $k, k' \in \mathcal{R}$  the list  $\mathcal{S}$  satisfies*

- (A) *if  $k \neq k'$  then  $k \approx k'$ ,*
- (B) *if  $k \neq k'$  then  $k \approx m(k')$ , and*
- (C) *if  $k \neq k'$  then  $m(k) \approx m(k')$ .*

*Then if  $K, K' \in \mathcal{K}$  and  $K \sim K'$  we have  $\mathcal{G}(K) = \mathcal{G}(K')$ . If in addition  $\mathcal{S}$  satisfies*

- (D)  *$m(k) \approx k$  when  $k$  is chiral or negative amphicheiral,*

*then if  $K, K' \in \mathcal{K}$  and  $K \sim K'$  we have  $K' = K$  or  $K' = rm(K)$ .*

*Proof:* Assume first that (A), (B), (C) hold and that  $K, K' \in \mathcal{K}$  and  $K \sim K'$ . We must prove that  $\mathcal{G}(K) = \mathcal{G}(K')$ . Let  $k, k'$  be elements of  $\mathcal{R}$  such that

$$K \in \mathcal{G}(k) \text{ and } K' \in \mathcal{G}(k').$$

Note that  $\mathcal{G}(k) = \{k, rm(k)\} \cup \{r(k), m(k)\}$  and  $\mathcal{G}(k') = \{k', rm(k')\} \cup \{r(k'), m(k')\}$ . By Lemma 3.2 (1) and (2) we have  $k \sim rm(k)$  and  $r(k) \sim m(k)$ , and similarly for  $k$  replaced by  $k'$ . We consider four cases and show that each case leads to  $\mathcal{G}(k) = \mathcal{G}(k')$ .

Case(i)  $K \in \{k, rm(k)\}$  and  $K' \in \{k', rm(k')\}$ . Then we have  $k \sim K \sim K' \sim k'$ . This implies by (A) that  $k = k'$ , and hence  $\mathcal{G}(k) = \mathcal{G}(k')$ .

Case(ii)  $K \in \{k, rm(k)\}$  and  $K' \in \{m(k'), r(k')\}$ . Then  $k \sim K \sim K' \sim m(k')$  which implies by (B) that  $k = k'$ , hence  $\mathcal{G}(k) = \mathcal{G}(k')$ .

Case(iii)  $K \in \{m(k), r(k)\}$  and  $K' \in \{k', rm(k')\}$ . This is similar to Case(ii).

Case(iv)  $K \in \{m(k), r(k)\}$  and  $K' \in \{m(k'), r(k')\}$ . Then  $m(k) \sim K \sim K' \sim m(k')$ , and by (C) we have  $k = k'$ , hence  $\mathcal{G}(k) = \mathcal{G}(k')$ .

This proves the first part of the proposition. Now assume that (A), (B), (C), (D) hold and  $K \sim K'$ . Let  $k$  and  $k'$  be as above. By what we just proved we have  $\mathcal{G}(k) = \mathcal{G}(k')$  and so we have  $K, K' \in \mathcal{G}(k)$ . If  $K = K'$  there is nothing to prove so we can assume that one of the following 4 cases holds and that  $K \neq K'$ .

Case(1)  $k$  is reversible or positive amphicheiral. Hence  $\mathcal{G}(k) = \{k, rm(k)\}$ . In this case we may take  $K = k$  and  $K' = rm(k)$  so  $K' = rm(K)$ .

Case(2)  $k$  is negative amphicheiral. Hence  $\mathcal{G}(k) = \{k, m(k)\}$ . In this case by (D) we have that  $k \approx m(k)$  so  $K \neq K'$ ,  $K \sim K'$  and  $K, K' \in \mathcal{G}(k)$  is impossible.

Case(3)  $k$  is chiral. Hence  $\mathcal{G}(k) = \{k, r(k), m(k), rm(k)\}$ . Of the four elements of  $\mathcal{G}(k)$  there are 6 possible pairs that might be  $\{K, K'\}$ . Note that the pair  $\{k, m(k)\}$  is ruled out by (D). Since we always have  $k \sim rm(k)$  and  $r(k) \sim m(k)$ , the pairs  $\{k, r(k)\}$ ,  $\{rm(k), m(k)\}$ ,  $\{rm(k), r(k)\}$ , are also ruled out. This leaves the only possibilities to be  $\{k, rm(k)\}$  and  $\{r(k), rm(k)\}$ . If  $r(k) \sim rm(k)$  then  $m(k) \sim r(k) \sim rm(k) \sim k$  which is a contradiction to (D), so we are left with  $\{k, rm(k)\}$  as the only possibility for  $\{K, K'\}$ , as desired.

Case(4)  $k$  is fully amphicheiral. Hence  $\mathcal{G}(k) = \{k\}$ . This is impossible since there is only a single knot in  $\mathcal{G}(k)$ .

This completes the proof. ■

## 4 Computational support for Conjectures 3.4 and 3.5

Let  $\mathcal{Q}_{26} = (Q_1, \dots, Q_{26})$  be the list of 26 quandles in Table B.1 in Appendix B. Matrices for these 26 quandles may be found at [48], where they are denoted by  $Q[i], i = 1, \dots, 26$ . The list  $\mathcal{Q}_{26}$  is closed under taking dual quandles. This is to simplify computing the number of colorings of the mirror image of a knot. Included in the file is the list *dual* satisfying  $dual[i] = j$  if and only if  $Q_i^*$  is isomorphic to  $Q_j$ .

If  $i \neq 15$  each quandle  $Q_i$  is faithful and hence isomorphic to a conjugation quandle on some conjugacy class in its inner automorphism group  $\text{Inn}(Q_i)$ . The quandle  $Q_{15}$  is not faithful and is not a conjugation quandle [Vendramin, personal communication]. We note that some non-faithful quandles are conjugation quandles, so this is not trivial. However,  $Q_{15}$  is a generalized Alexander quandle on GAP's  $\text{SmallGroup}(32, 50) = (C2 \times Q8) : C2$ . The quandles in  $\mathcal{Q}_{26}$  of order less than 36 are RIG quandles and the remaining 11 were found by searching for conjugation quandles on conjugacy classes of finite groups using GAP. In retrospect all of these 26 quandles could have been found by searching for conjugation quandles and generalized Alexander quandles.

We denote the knots in KnotInfo up to 12 crossings by

$$\mathcal{R}_{12} = \{K[i] : i = 1, \dots, 2977\}.$$

In our notation, the knot  $K[i]$  represents the knot with *name rank* =  $i + 1$  in [8]. In particular,  $K[1]$  is the trefoil, which has name rank 2 in [8].

The  $26 \times 2977$  matrix  $M$  found at [48] has the property that

$$M_{i,j} = \text{Col}_{Q_i}^N(K[j]).$$

See Appendix D for comments on the computation of  $\text{Col}_Q^N(K)$ .

Since  $\mathcal{R}_{12}$  is a set of representatives of the set  $\mathcal{K}_{12}$  of (isotopy classes of prime, oriented) knots with at most 12 crossings, including all symmetry types, to verify that Conjecture 3.4 holds when  $\mathcal{K} = \mathcal{K}_{12}$  and the list of quandles  $\mathcal{S} = \mathcal{Q}_{26}$ , it suffices, by Proposition 3.7, to verify that the matrix  $M$  and the list *dual* has the following properties for  $i, j \in \{1, 2, \dots, 2977\}$  and  $k \in \{1, 2, \dots, 26\}$ .

- (1) For all  $i \neq j$  there exists  $k$  such that  $M_{k,i} \neq M_{k,j}$ .
- (2) For all  $i \neq j$  there exists  $k$  such that  $M_{k,i} \neq M_{s,j}$ , where  $s = \text{dual}[k]$ .
- (3) For all  $i \neq j$  there exists  $k$  such that  $M_{s,i} \neq M_{s,j}$ , where  $s = \text{dual}[k]$ .

These conditions are easy to verify directly from the matrix  $M$  and the list *dual*. This shows that the set  $\mathcal{Q}_{26}$  distinguishes the knots in  $\mathcal{K}_{12}$  up to mirror images and orientation.

**Remark 4.1** Note that the unknot is not included among the knots  $K[i]$ ,  $i = 1, \dots, 2977$ . Since the unknot has only trivial colorings, to show that we can distinguish it from all of the 2977 knots it suffices to note that for every knot  $K[j]$  there is a quandle  $Q_i$  such that  $\text{Col}_{Q_i}^N(K[j]) \neq 0$ , that is, no column of the matrix  $M$  contains only 0s. This is confirmed from the matrix  $M$ .

As pointed out above the only knots we can hope to distinguish from their mirror images (equivalently from their reversals) are chiral knots and negative amphicheiral knots. Among 2977 knots in KnotInfo [8] up to 12 crossings, 1319 knots are chiral and 47 are negative amphicheiral. Denote this set of 1366 knots by  $\mathcal{K}^c$ . We were not able to find quandles to distinguish all of these 1366 knots from their mirror images. Thus we cannot verify Conjecture 3.5 when  $\mathcal{K} = \mathcal{K}_{12}$ . However, we were able to find quandles that distinguish 1058 of the knots in  $\mathcal{K}^c$  from their mirror images. The quandles that distinguish these 1058 knots are the quandles  $T_1, T_2, \dots, T_{23}$  in Table B.2. Matrices for the quandles can be found in [48]. A list of 1058 pairs  $(i, n)$  where  $K[n]$  is one of 1058 knots in  $\mathcal{K}^c$  and  $T_i$  is a quandle that distinguished knot  $K[n]$  from  $m(K[n])$  can be found in [48].

With the exception of quandles  $T_2$  and  $T_3$  the quandles  $T_i$  are conjugation quandles on conjugation classes in the group  $\text{Inn}(T_i)$ .  $T_2$  and  $T_3$  are generalized Alexander quandles, both on the group  $PSL(2, 8)$ . We note that  $Q_{17} = T_{22}$  and  $Q_{24} = T_{23}$ . Otherwise the quandles in Table B.2 are different from those in Table B.1

In Table B.1 and Table B.2 the structure descriptions are given by GAP's *StructureDescription* function. In particular,  $A \times B$  is the direct product,  $N : H$  is a semidirect product and  $A.B$  is a non-split extension. The group ID, *SmallGroup*( $n, i$ ) is the  $i$ th group of order  $n$  in the GAP Small Groups library for  $n$  at most 2000. If a group has order  $n > 2000$  we write *SmallGroup*( $n, 0$ ). Nevertheless for all of the inner automorphism groups we have computed, GAP does give a Structure Description.

## 5 Similarity of quandles over knot colorings

For two quandles  $Q_1$  and  $Q_2$ , and a set  $\mathcal{K}$  of knots, we write  $Q_1 \approx_{\mathcal{K}} Q_2$  if  $\text{Col}_{Q_1}(K) = \text{Col}_{Q_2}(K)$  for all  $K \in \mathcal{K}$ .

This equivalence relation was considered in [10]. We omit the subscript  $\mathcal{K}$  when it is the set of all knots. The following is immediate from the definitions and a property of the number of colorings.

**Lemma 5.1** *If  $Q_1 \approx Q'_1$  and  $Q_2 \approx Q'_2$  then  $(Q_1 \times Q_2) \approx (Q'_1 \times Q'_2)$ .*

Let  $\mathcal{K}$  be the set of all 2977 knots in the table in KnotInfo [8] up to 12 crossings. We observe the following for  $\approx_{\mathcal{K}}$  and for the 431 RIG quandles.

- There are 151 classes of  $\approx_{\mathcal{K}}$  consisting of more than one quandle.
- Among these classes, 145 classes consist of two quandles.
- Among 145 classes of pairs, all but 35 classes consist of a pair of dual Alexander quandles.
- Among 35 classes of pairs, one class is a pair of self-dual Alexander quandles, some are non-Alexander dual quandles, and some are non-Alexander self-dual quandles.

**Conjecture 5.2** For every connected Alexander quandle  $Q$ ,  $Q \approx Q^*$ .

We prove the above conjecture for simple Alexander quandles of prime power order. (Following [1] we say that a quandle  $Q$  is *simple* if it is not trivial and whenever  $f : Q \rightarrow Q'$  is an epimorphism then  $f$  is an isomorphism or  $|Q'| = 1$ . This implies that  $|Q| > 2$  since the quandles of orders 1 and 2 are trivial.) The following proposition gives alternative characterizations of these quandles.

**Proposition 5.3** *The following statements are equivalent for a quandle  $Q$  and a prime  $p$ .*

- (1)  $Q$  is a simple quandle with  $p^k$  elements.
- (2)  $Q$  is isomorphic to an Alexander quandle on  $\mathbb{Z}_p[t, t^{-1}]/(h(t))$  where  $h(t) \in \mathbb{Z}_p[t]$  is irreducible, of degree  $k$  and different from  $t$  and  $t - 1$  and the quandle operation is defined by  $x * y = \bar{t}x + (1 - \bar{t})y$  where  $\bar{t}$  is the coset  $t + (h(t))$ .
- (3)  $Q$  is isomorphic to an Alexander quandle on the vector space  $(\mathbb{F}_p)^k$  where  $T$  is the companion matrix of a monic irreducible polynomial  $h(t) \in \mathbb{Z}_p[t]$  of degree  $k$ , different from  $t$  and  $t - 1$  and the quandle operation is given by  $x * y = Tx + (1 - T)y$ .

(4)  $Q$  is isomorphic to a quandle defined on the finite field  $\mathbb{F}_q$  where  $q = p^k$ ,  $a \in \mathbb{F}_q$ ,  $a \neq 1$ ,  $a$  generates  $\mathbb{F}_q$ , and the quandle operation is given by  $x * y = ax + (1 - a)y$  for  $x, y \in \mathbb{F}_q$ .

*Proof.* The equivalence of (1), (3) and (4) is proved in [1] Section 3. The equivalence of (2) and (4) is proved in [2].  $\square$

**Proposition 5.4** *If  $Q$  is a simple quandle of prime power order, then  $Q \approx Q^*$ .*

*Proof.* By Proposition 5.3, the quandle  $Q$  is equal to  $(\mathbb{F}_q, *)$  where  $\mathbb{F}_q = \mathbb{Z}_p[t, t^{-1}]/(h(t))$ ,  $q = p^k$ ,  $h(t) \in \mathbb{Z}_p[t]$  is irreducible of degree  $k$ . The quandle operation is defined by  $x * y = \bar{t}x + (1 - \bar{t})y$  where  $\bar{t}$  is the coset  $t + (h(t))$ . Let  $S \in M_n(\mathbb{F}_q)$  denote the reduction modulo  $(p, h(t))$  of a Seifert matrix of  $K$ , and let  $N = \bar{t}S - S^T$ , which is denoted by  $N(1, p, h(t))$  in [2]. Although  $p$  is restricted to odd primes in [2], we claim that their results hold for  $p = 2$  also. The authors of [2] agree (personal communication).

Let  $\varphi : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$  be the linear map that  $N$  represents. Then in [2] it was shown that the set of colorings  $\text{SCol}_Q(K)$  is in one-to-one correspondence with the module  $\mathbb{F}_q \oplus \ker(\varphi)$ .

The dual quandle  $Q^*$  is equal to  $(\mathbb{F}_q, \bar{*})$  where  $a \bar{*} b = \bar{t}^{-1}a + (1 - \bar{t}^{-1})b$ . Let  $N^* = \bar{t}^{-1}S - S^T$ , and  $\varphi^* : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$  be the corresponding linear map. Then the number of colorings by  $Q^*$  is the cardinality of  $\mathbb{F}_q \oplus \ker(\varphi^*)$ . It remains to show that the dimensions of  $\ker(\varphi)$  and  $\ker(\varphi^*)$  are the same. One computes

$$(-\bar{t}^{-1})N^T = -\bar{t}^{-1}(\bar{t}S^T - S) = -S^T + \bar{t}^{-1}S = N^*.$$

Since  $\text{rank}(N^T) = \text{rank}(N)$ , we have  $\text{rank}(N) = \text{rank}(N^*)$  and hence the dimension of the kernel of  $\varphi^*$  is equal to the dimension of the kernel of  $\varphi$ , so the number of colorings is the same for  $Q$  and  $Q^*$ .  $\square$

This proof is similar to the proof of the fact that the Alexander polynomial is symmetric,  $\Delta(t^{-1}) = \Delta(t)$  modulo a power of  $t$ .

From Proposition 5.4 and Lemma 5.1 we obtain the following corollary.

**Corollary 5.5** *If a quandle  $Q$  is isomorphic to a product of simple quandles of prime power order then  $Q \approx Q^*$ .*

There are 273 connected Alexander quandles of order at most 35. Only 45 of these quandles are not a product of simple quandles of prime power order.

**Remark 5.6** All connected quandles of prime order clearly satisfy the conditions of Proposition 5.3. Among RIG quandles, other than quandles of prime orders, the following quandles fall into this family.

$$\begin{aligned} & C[4, 1], C[8, 2], C[8, 3], C[9, 3], C[9, 7], C[9, 8], C[16, 3], C[16, 8], C[16, 9], C[25, 3], \\ & C[25, 11], C[25, 12], C[25, 13], C[25, 19], C[25, 20], C[25, 31], C[25, 32], C[25, 33], C[25, 34], \\ & C[27, 31], C[27, 32], C[27, 33], C[27, 34], C[27, 62], C[27, 63], C[27, 64], C[27, 65], C[32, 10], \\ & C[32, 11], C[32, 12], C[32, 13], C[32, 14], C[32, 15]. \end{aligned}$$

We now explain to the extent possible the 6 non-trivial classes of  $\approx_{\mathcal{K}}$  that have more than two elements. Among these 6 classes, the first three classes listed below form equivalence classes under  $\approx$  by Lemma 5.1 and Proposition 5.4. In all cases unless otherwise stated we have no explanation for an equivalence class.

- $\{C[25, 6], C[25, 7], C[25, 8]\}$ : This situation is explained as follows:  $C[25, 6] = C[5, 2] \times C[5, 2]$ ,  $C[25, 7] = C[5, 3] \times C[5, 3]$ ,  $C[25, 8] = C[5, 2] \times C[5, 3]$ , where  $C[5, 2] = \mathbb{Z}_5[t]/(t+2)$  is dual to  $C[5, 3] = \mathbb{Z}_5[t]/(t+3)$ .

- $\{C[35, 8], C[35, 9], C[35, 10], C[35, 11]\}$ : This situation is explained as follows:  $C[35, 8] = C[5, 2] \times C[7, 4]$ ,  $C[35, 9] = C[5, 2] \times C[7, 5]$ ,  $C[35, 10] = C[5, 3] \times C[7, 4]$ ,  $C[35, 11] = C[5, 3] \times C[7, 5]$ , where  $C[7, 4] = \mathbb{Z}_7[t]/(t+2)$  is dual to  $C[7, 5] = \mathbb{Z}_7[t]/(t+4)$ .

- $\{C[35, 12], C[35, 13], C[35, 14], C[35, 15]\}$ : This situation is explained as follows:  $C[35, 12] = C[5, 3] \times C[7, 3]$ ,  $C[35, 13] = C[5, 3] \times C[7, 2]$ ,  $C[35, 14] = C[5, 2] \times C[7, 3]$ ,  $C[35, 15] = C[5, 2] \times C[7, 2]$ , where  $C[7, 2] = \mathbb{Z}_7[t]/(t+3)$  is dual to  $C[7, 3] = \mathbb{Z}_7[t]/(t+5)$ .

- $\{C[30, 7], C[30, 8], C[30, 9], C[30, 10]\}$ : This situation is explained as follows:  $C[30, 7] = C[5, 2] \times C[6, 1]$ ,  $C[30, 8] = C[5, 3] \times C[6, 1]$ ,  $C[30, 9] = C[5, 2] \times C[6, 2]$ ,  $C[30, 10] = C[5, 3] \times C[6, 2]$ , where  $C[5, 2] = \mathbb{Z}_5[t]/(t+2)$  is dual to  $C[5, 3] = \mathbb{Z}_5[t]/(t+3)$ . As mentioned before it is conjectured [10] that  $C[6, 1] \approx C[6, 2]$ . So we conjecture that this forms a class for  $\approx$ .

- $\{C[24, 5], C[24, 6], C[24, 16], C[24, 17]\}$ : These quandles are abelian extensions of  $C[12, 8]$ ,  $C[12, 9]$ ,  $C[12, 8]$ ,  $C[12, 8]$ , respectively, and none is Alexander. The quandles  $C[12, 8]$ ,  $C[12, 9]$  have  $C[6, 1]$ ,  $C[6, 2]$  as subquandles, respectively, and have  $C[6, 1]$  as an epimorphic image. These facts alone, however, do not seem to imply that they constitute a class for  $\approx$ . We conjecture that this forms a class for  $\approx$ .

- $\{C[24, 29], C[24, 30], C[24, 31]\}$ : These three quandles are self-dual Galkin quandles. This class was reported in [10].

The remaining classes consist of pairs, that are not dual Alexander quandles, and we categorize them as follows. We conjecture that they form equivalence classes under  $\approx$ .

There is one class with a pair of Alexander self-dual quandles:  $C[27, 17]$  and  $C[27, 22]$ , both of which are Alexander quandles on the abelian group  $\mathbb{Z}_3 \times \mathbb{Z}_9$ .

Each of the following classes is a pair of non-Alexander quandles that are duals of each other:

$$\begin{aligned} & \{C[27, 35], C[27, 36]\}, \{C[27, 37], C[27, 38]\}, \{C[27, 41], C[27, 42]\}, \\ & \{C[27, 43], C[27, 46]\}, \{C[27, 44], C[27, 45]\}, \{C[27, 56], C[27, 57]\}, \\ & \{C[27, 58], C[27, 59]\}, \{C[28, 11], C[28, 12]\}, \{C[30, 17], C[30, 18]\}, \\ & \{C[32, 7], C[32, 8]\}. \end{aligned}$$

The remaining classes are pairs of self-dual non-Alexander quandles. In [10], the relation  $\approx$  was studied for a family called Galkin quandles. Galkin quandles are known to be self-dual.

The following are two element  $\approx_{\mathcal{K}}$ -equivalence classes of (non-Alexander) Galkin quandles that

were reported in [10]:

$$\begin{aligned} & \{C[6, 1], C[6, 2]\}, \{C[12, 5], C[12, 6]\}, \{C[12, 8], C[12, 9]\}, \\ & \{C[18, 1], C[18, 4]\}, \{C[18, 5], C[18, 8]\}, \{C[24, 27], C[24, 28]\}, \\ & \{C[24, 38], C[24, 39]\}, \{C[30, 12], C[30, 14]\}, \{C[30, 13], C[30, 15]\}. \end{aligned}$$

The following classes are pairs of self-dual non-Alexander, non-Galkin quandles:

$$\begin{aligned} & \{C[18, 2], C[18, 3]\}, \{C[18, 6], C[18, 10]\}, \{C[18, 7], C[18, 9]\}, \{C[20, 5], C[20, 6]\}, \\ & \{C[20, 9], C[20, 10]\}, \{C[24, 3], C[24, 4]\}, \{C[24, 15], C[24, 18]\}, \{C[24, 22], C[24, 23]\}, \\ & \{C[24, 34], C[24, 36]\}, \{C[24, 35], C[24, 37]\}, \{C[24, 41], C[24, 42]\}, \\ & \{C[30, 2], C[30, 6]\}, \{C[30, 23], C[30, 24]\}, \{C[32, 5], C[32, 6]\}. \end{aligned}$$

## 6 Quandle homomorphisms and knot colorings

We write  $\text{SCol}_Q(K)$  to denote the set of all colorings of the knot  $K$  by the quandle  $Q$ .

Let  $f : Q_1 \rightarrow Q_0$  be a quandle homomorphism, and  $K$  be a knot. Note that for any coloring  $y \in \text{SCol}_{Q_1}(K)$  regarded as a quandle homomorphism  $Q(K) \rightarrow Q_1$ , the composition  $x = f \circ y : Q(K) \rightarrow Q_0$  is a coloring in  $\text{SCol}_{Q_0}(K)$ . In this case we say that  $x$  lifts to  $y$ . This correspondence defines a map  $f_{\#} : \text{SCol}_{Q_1}(K) \rightarrow \text{SCol}_{Q_0}(K)$ .

For  $x \in \text{SCol}_{Q_0}(K)$ , denote by  $f_{\#}^{-1}(x) \subset \text{SCol}_{Q_1}(K)$  the set of colorings  $y \in \text{SCol}_{Q_1}(K)$  such that  $f_{\#}(y) = x$ . (This set may be empty.) A pair of colorings  $x \in \text{SCol}_{Q_0}(K)$  and  $y \in \text{SCol}_{Q_1}(K)$  such that  $x = f \circ y$  for a knot diagram determines another such a pair after each Reidemeister move, and hence for each  $x \in \text{SCol}_{Q_0}(K)$ , the cardinality  $|f_{\#}^{-1}(x)|$  does not depend on the choice of diagrams. Thus we obtain the following knot invariant.

**Definition 6.1** For a given quandle homomorphism  $f : Q_1 \rightarrow Q_0$ , we denote by  $\text{Col}_f(K)$  the multiset  $\{|f_{\#}^{-1}(x)| \mid x \in \text{SCol}_{Q_0}(K)\}$ , that does not depend on the choice of a diagram of a knot.

We use the notation  $[a, k]$  for  $k$  copies of the element  $a$ . For example,  $\{[0, 3], [2, 4]\}$  represents  $\{0, 0, 0, 2, 2, 2, 2\}$ . We observe the following.

- If  $\text{Col}_f(K) = \{[h_1, k_1], \dots, [h_n, k_n]\}$ , then  $\text{Col}_{Q_0}(K) = k_1 + \dots + k_n$  and  $\text{Col}_{Q_1}(K) = h_1 k_1 + \dots + h_n k_n$ .
- Suppose  $Q_1$  is connected. If  $\text{Col}_{Q_0}^N(K) = 0$  (i.e.  $K$  is only trivially colored by  $Q_0$ ) and  $f$  is an epimorphism, then  $\text{Col}_f(K) = \{[h, k]\}$ , where  $k = |Q_0|$  and  $h = \text{Col}_{Q_1}(K)/|Q_0|$ .
- Suppose  $Q_1$  is connected. If  $\text{Col}_{Q_1}^N(K) = 0$ , then  $\text{Col}_f(K) = \{[0, \ell], [h, k]\}$ , where  $\ell = \text{Col}_{Q_0}^N(K)$ ,  $k = |Q_0|$ , and  $h = |Q_1|/|Q_0|$ .
- If  $f : Q_1 \rightarrow Q_0$  has image  $Q'_0$ , let  $f' : Q_1 \rightarrow Q'_0$  be the same map with the codomain  $Q'_0$ , then  $\text{Col}_f(K) = \text{Col}_{f'}(K) \cup \{[0, h]\}$  where  $h$  is the number of colorings by  $Q_0$  such that some colors are not from  $Q'_0$ .

- If  $f : Q_1 = Q_0 \times Q'_0 \rightarrow Q_0$  is the projection from a product quandle, then  $\text{Col}_f(K) = \{[h, k]\}$ , where  $k = \text{Col}_{Q_0}(K)$  and  $h = \text{Col}_{Q'_0}(K)$ .

**Example 6.2** We computed the invariant  $\text{Col}_f(K)$  for  $i = 1, 2$  and for 2977 knots up to 12 crossings, where  $f$  is the unique (up to isomorphisms of  $C[6, i]$  and  $C[3, 1]$ ) epimorphism from  $C[6, i]$  to  $C[3, 1]$ . The values of the invariants for  $i = 1, 2$  are the same for each  $K$  up to 12 crossings.

For 2977 knots, the types of multisets are

$$\begin{aligned} & \{[2, 3]\}, \{[2, 3], [4, 6]\}, \{[2, 3], [4, 24]\}, \{[2, 3], [8, 6]\}, \{[2, 3], [8, 24]\}, \\ & \{[2, 3], [4, 6], [8, 18]\}, \{[2, 3], [4, 12], [8, 12]\}, \{[2, 3], [4, 18], [8, 6]\}, \\ & \{[2, 3], [8, 18], [16, 6]\}, \{[2, 3], [4, 36], [8, 24], [16, 18]\}. \end{aligned}$$

**Example 6.3** There is an epimorphism  $f : C[8, 1] \rightarrow C[4, 1]$  such that there are 12 non-trivial colorings of the trefoil  $K = 3_1$  by  $C[4, 1]$ , none of which lifts to a coloring by  $C[8, 1]$ . Hence for this  $f : C[8, 1] \rightarrow C[4, 1]$  and for  $K = 3_1$ , the invariant  $\text{Col}_{f:Q_1 \rightarrow Q_0}(K)$  contains 12 copies of 0s. There are 4 and 8 trivial colorings of  $K$  by  $C[4, 1]$  and  $C[8, 1]$ , respectively, and every trivial coloring by  $C[4, 1]$  has two lifts. Therefore we have  $\text{Col}_f(K) = \{[0, 12], [2, 4]\}$ .

We note that  $C[8, 1]$  is not Alexander but is an abelian extension of  $C[4, 1] = \mathbb{Z}_2[t]/(t^2 + t + 1)$ . Hence this computation is equivalent to quandle cocycle invariant. All possible patterns over all of the first 1000 knots are

$$\{[2, 4]\}, \{[2, 16]\}, \{[2, 64]\}, \{[0, 12], [2, 4]\}, \{[0, 24], [2, 40]\}, \{[0, 48], [2, 16]\}.$$

**Example 6.4** For  $f : C[9, 1] = R_9 \rightarrow C[3, 1] = R_3$ , the patterns over all of the first 1000 knots are

$$\{[3, 3]\}, \{[9, 9]\}, \{[27, 27]\}, \{[0, 6], [9, 3]\}, \{[0, 18], [27, 9]\}, \{[0, 24], [27, 3]\}.$$

**Example 6.5** There is an epimorphism  $f : C[18, 5] \rightarrow C[6, 2]$ . The quandle  $C[18, 5]$  is not Alexander, not a Kei, not Latin, self-dual, and faithful. It is not an abelian extension of  $C[6, 2]$ . It has 3 subquandles,  $C[3, 1]$ , a trivial quandle of 2 elements, a 6-element non-connected quandle, and does not have  $C[6, 1]$  nor  $C[6, 2]$  as subquandles. The RIG quandles onto which it has epimorphisms are  $C[3, 1]$ ,  $C[6, 2]$ , and  $C[9, 6]$ .

For this  $f : C[18, 5] \rightarrow C[6, 2]$  based on the first 100 knots, the patterns of the invariant are

$$\{[3, 6]\}, \{[9, 30]\}, \{[9, 54]\}, \{[27, 102]\}, \{[0, 96], [27, 6]\}, \{[0, 120], [27, 6]\}.$$

**Remark 6.6** For all examples of epimorphisms  $f : Q_1 \rightarrow Q_0$  we computed, the invariant  $\text{Col}_f(K)$  is determined by  $(\text{Col}_{Q_1}(K), \text{Col}_{Q_0}(K))$ . Specifically, for these epimorphisms and all pairs of knots  $K$  and  $K'$  with at most 12 crossings, if  $(\text{Col}_{Q_1}(K), \text{Col}_{Q_0}(K)) = (\text{Col}_{Q_1}(K'), \text{Col}_{Q_0}(K'))$  then  $\text{Col}_f(K) = \text{Col}_f(K')$ . This holds for epimorphisms  $C[6, 1] \rightarrow C[3, 1]$ ,  $C[8, 1] \rightarrow C[4, 1]$ , and  $C[9, 1] \rightarrow C[3, 1]$ . It also holds for  $C[18, 5] \rightarrow C[6, 2]$ , for many knots (not necessarily all 2977 knots) we computed.

Furthermore, for  $C[6, 1] \rightarrow C[3, 1]$ , the invariant is determined by  $\text{Col}_{C[6,1]}(K)$  alone for many knots, while for  $C[8, 1] \rightarrow C[4, 1]$  and  $C[9, 1] \rightarrow C[3, 1]$ , both  $\text{Col}_{Q_1}(K)$  and  $\text{Col}_{Q_0}(K)$  are needed to determine the invariant.

**Remark 6.7** From the considerations in the preceding remark, it is of interest whether the number of colorings of a quandle determines that of another. There are many such pairs among RIG quandles over all 2977 knots. We note that many of the pairs do not have epimorphisms between them, and more studies on this phenomenon may be desirable.

**Remark 6.8** The quandle 2-cocycle invariant was defined in [6] as follows. Let  $\phi : X \times X \rightarrow A$  be a quandle 2-cocycle, for a finite quandle  $X$  and an abelian group  $A$ . For a knot diagram  $K$  and a coloring  $x \in \text{SCol}_X(K)$ , the weight at a crossing  $\tau$  as depicted in Figure 1 is defined by  $B_\phi(\tau, x) = \epsilon\phi(x, y)$ , where  $\epsilon = \pm 1$  is the sign of  $\tau$ , defined as  $+1$  if the under-arc points right in the figure, otherwise  $-1$ . The 2-cocycle invariant  $\Phi_\phi(K)$  is defined as the multiset  $\{\sum_\tau B_\phi(\tau, x) \mid x \in \text{SCol}_X(K)\}$ .

The cocycle invariant and the invariant  $\text{Col}_f(K)$ , which we defined for an epimorphism  $f : Q_1 \rightarrow Q_0$  for quandles  $Q_i$ ,  $i = 0, 1$ , can be naturally combined as follows. Let  $A$  be an abelian group, and  $\phi_i$ ,  $i = 0, 1$ , be 2-cocycles of  $Q_i$ . Then define  $\Phi_{f;\phi_1,\phi_0}(K)$  to be a multiset

$$\left\{ \left( \sum_\tau B_{\phi_0}(\tau, x), \left\{ \sum_\tau B_{\phi_1}(\tau, y) \mid y \in f_\#^{-1}(x) \right\} \right) \mid x \in \text{SCol}_{Q_0}(K) \right\}.$$

## 7 Applications to other knot invariants

In this section we give applications of the number of quandle colorings to a few invariants for the 2977 knots in [8]. In particular, we determine the tunnel number for some of the knots with 11 and 12 crossings in [8]. For definitions of knot invariants, we refer to [8, 44].

Recall that  $\text{Col}_X(K)$  denotes the number of colorings of a knot  $K$  by a finite quandle  $X$ . Let  $\text{Lq}_X(K)$  denote  $\lceil \log_{|X|}(\text{Col}_X(K)) \rceil$ , where  $|X|$  denotes the order of  $X$ .

We computed  $\text{Lq}_Q(K)$  for the 2977 knots over a set  $\mathcal{QL}$  of 439 quandles consisting of all 431 RIG quandles and the 8 quandles  $Q_i$ ,  $i = 16, 18, 19, 20, 22, 23, 25$  from the 26 quandles  $Q_i$  in Table B.1. The additional 8 quandles are not Alexander. Let  $\text{MLq}(K)$  be the maximum of  $\text{Lq}_Q(K)$  over all of the 439 quandles in  $\mathcal{QL}$ . Among 2977 knots with 12 crossings or less, there are 1473 knots  $K$  with  $\text{MLq}(K) = 2$ , 1441 knots with  $\text{MLq} = 3$ , and 63 (15 11-crossing and 48 12-crossing) knots with  $\text{MLq} = 4$ . Let  $\text{MLq}^F(K)$  denote the maximum of  $\text{Lq}_X(K)$  over all Alexander RIG quandles of the form  $\Lambda_p/(h(t))$  where  $h(t)$  is irreducible in  $\mathbb{Z}_p[t]$  for prime  $p$  and different from  $t$  and  $t-1$ . Note that  $\text{MLq}$  and  $\text{MLq}^F$  are defined over all quandles in the particular set  $\mathcal{QL}$ . The computational results on these are listed in [49].

### Bridge index

Let  $\text{Bg}(K)$  denote the bridge index of a knot  $K$ . In [43], a lower bound of the bridge index is given in terms of the number of quandle colorings: Let  $X$  be a finite quandle. For an  $n$ -bridge presentation of a knot, an assignment of colors at the  $n$  maxima determines the colors of the remaining arcs of the diagram, if it extends to the entire diagram and gives a well-defined coloring. Hence we have  $|\text{Col}_X(K)| \leq |X|^n$ , where  $|X|$  denotes the order of  $X$ . Thus we obtain  $\text{Lq}_X(K) \leq \text{Bg}(K)$ , and therefore,  $\text{MLq}(K) \leq \text{Bg}(K)$ .

It is proved in [3] that a Montesinos knot  $K$  with  $r$  rational tangle summands has  $\text{Bg}(K) = r$ . In [35], Montesinos knots were used for determining the bridge indices of knots with 11 crossings. In

particular, the set of 4-bridge 11 crossing knots identified in [35] agreed with the list of 11 crossing knots with  $\text{MLq} = 4$ . These are:

$$\begin{aligned} &11a\_43, 11a\_44, 11a\_47, 11a\_57, 11a\_231, 11a\_263, \\ &11n\_71, 11n\_72, 11n\_73, 11n\_74, 11n\_75, 11n\_76, 11n\_77, 11n\_78, 11n\_81. \end{aligned}$$

At the time of writing, KnotInfo [8] did not have information on the bridge index for 12 crossing knots. The set of 12 crossing Montesinos knots with 4 rational tangle summands were computed by Slavik Jablan using LinKnot [24], and the list was provided to us by Chad Musick. There are 48 knots in their table. Our computation showed that there are 48 knots with 12 crossings with  $\text{MLq} = 4$ , and the list can be found in [49]. These two lists coincide.

### Nakanishi index

We use the notations  $\Lambda = \mathbb{Z}[t, t^{-1}]$  and  $\Lambda_p = \mathbb{Z}_p[t, t^{-1}]$ . The Nakanishi index  $\text{NI}(K)$  of a knot  $K$  is the minimum size of square presentation matrices of the Alexander module  $(H_1(\tilde{Y}))$  as a  $\Lambda$ -module, where  $\tilde{Y}$  is the infinite cyclic covering of the complement  $Y$  of  $K$  [36]. The Nakanishi index for 11 and 12 crossing knots was blank in [8] when this project was started (June, 2012). We examine how much of the Nakanishi index can be determined for these knots using quandle colorings.

**Lemma 7.1** *For any knot  $K$  and Alexander quandle  $Q$  of the form  $\Lambda_p/(h(t))$  where  $h(t)$  is irreducible in  $\mathbb{Z}_p[t]$  for prime  $p$  and different from  $t$  and  $t - 1$ , it holds that*

$$\text{Lq}_Q(K) \leq \text{NI}(K) + 1 \leq \text{Bg}(K).$$

*Proof.* As in the proof of Proposition 5.4,  $\Lambda_p/(h(t))$  is a finite field  $\mathbb{F}_q$  of order  $q$  denoted by  $\mathbb{F}(p, h(t))$  in [2], where  $q = p^k$  for a prime  $p$  and  $k = \deg(h(t))$ .

It is shown in [2] that if  $A$  is a presentation matrix of the Alexander module, then the same matrix reduced modulo  $p$ ,  $A^{(p)}$ , is a presentation matrix of  $H_1(\tilde{Y}; \mathbb{Z}_p)$  as a  $\Lambda_p$ -module. Let  $\overline{A^{(p)}}$  be the matrix  $A^{(p)}$  with entries reduced modulo  $h(t)$ .

Let  $\psi : \Lambda_p^n \rightarrow \Lambda_p^n$  denote the map corresponding to  $A^{(p)}$ , and  $\overline{\psi} : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$  denote the map on the vector space  $\mathbb{F}_q^n$  corresponding to  $\overline{A^{(p)}}$ . Then in [2] it was shown that the set of colorings  $\text{SCol}_Q(K)$  is in one-to-one correspondence with the vector space  $\mathbb{F}_q \oplus \ker(\overline{\psi})$ .

If  $\text{NI}(K) = n$ , then there is an  $n \times n$  presentation matrix  $A$  for the Alexander module. Then  $\ker(\overline{\psi})$  has dimension at most  $n$ . Hence

$$\text{Col}_Q(K) = |\mathbb{F}_q \oplus \ker(\overline{\psi})| = q^{1+\dim \ker(\overline{\psi})} \leq q^{1+n},$$

so that we obtain  $\text{Lq}_Q(K) \leq 1 + n$ . It is well known that  $\text{NI}(K) + 1 \leq \text{Bg}(K)$ .  $\square$

**Corollary 7.2** *For any knot  $K$ ,  $\text{MLq}^F(K) \leq \text{NI}(K) + 1 \leq \text{Bg}(K)$ .*

For 10 crossing knots, in KnotInfo [8] as of October 2013, the following knots are listed as having  $\text{NI} = 2$ :  $10\_k$  for

$$k = 61, 63, 65, 69, 74, 75, 98, 99, 101, 103, 122, 123, 140, 142, 144, 155, 157, 160.$$

The list of 10 crossing knots such that Corollary 7.2 determines that  $\text{NI} = 2$  consists of  $10_{\ell}$  for

$$\ell = 61, 63, 65, 74, 75, 98, 99, 103, 115, 122, 123, 140, 142, 144, 155, 157, 163.$$

Thus Corollary 7.2 determines knots with  $\text{NI} = 2$  for all but  $k = 69, 101,$  and  $160$ . We point out that Corollary 7.2 implies  $\text{NI} = 2$  for  $k = 115$  and  $163$ , these values of  $\text{NI}$  are posted in [8] incorrectly as having  $\text{NI} = 1$ . This error was corrected in [28] (the knot  $10_{163}$  is denoted as  $10_{164}$  in [28]).

The 11 and 12 crossing knots with  $\text{MLq}^F(K) = 3$ , and therefore,  $\text{NI}(K) \geq 2$  by Corollary 7.2, are listed in [49]. Among these, knots with  $\text{Bg}(K) = 3$  are determined to have  $\text{NI}(K) = 2$  by Corollary 7.2. In particular, 11 crossing knots in this list (46 knots) have bridge index 3 by [35], and therefore, have  $\text{NI} = 2$ .

### Tunnel number

Let  $\tau(K)$  denote the tunnel number of a knot  $K$ . At the writing of this article (October 2013), KnotInfo [8] did not contain the tunnel number for knots with 11 or more crossings.

Let  $Q$  be a quandle of the form  $\Lambda_p/(h(t))$  for a prime  $p$  and an irreducible polynomial  $h(t)$  as before. It was shown in [23] (see also [2]) that  $\text{Lq}_Q(K) \leq \tau(K) + 1$  for any knot  $K$ . It is well known that  $\tau(K) + 1 \leq \text{Bg}(K)$  holds for any knot  $K$ . Hence we obtain the following lemma.

**Lemma 7.3** *If  $\text{Lq}_Q(K) = \text{Bg}(K)$  for some quandle  $Q$  of the form  $\Lambda_p/(h(t))$  where  $h(t)$  is irreducible in  $\mathbb{Z}_p[t]$  for prime  $p$  and different from  $t$  and  $t - 1$ , then  $\tau(K) = \text{Bg}(K) - 1$ .*

As we mentioned earlier the bridge index was determined for all 11 crossing knots in [35]. For RIG Alexander quandles  $Q$  of the form  $\Lambda_p/(h(t))$ , the following 11 crossing knots satisfy the condition  $\text{Lq}_Q(K) = \text{Bg}(K) = 3$ , and therefore they have tunnel number 2:  $11a_{\underline{k}}$  and  $11n_{\underline{\ell}}$  where

$$\begin{aligned} k &= 87, 97, 107, 123, 132, 133, 135, 143, 155, 157, 165, 173, 181, 196, 239, 249, \\ &277, 291, 293, 297, 314, 317, 321, 322, 329, 332, 340, 347, 352, 354, 366. \\ \ell &= 49, 83, 90, 91, 126, 133, 148, 157, 162, 164, 165, 167, 175, 184, 185. \end{aligned}$$

These are 46 knots among 552 total of 11 crossing knots.

### Unknotting number

It is known [36] that the unknotting number  $u(K)$  of a knot  $K$  is bounded below by  $\text{NI}(K)$ :  $\text{NI}(K) \leq u(K)$ . We use this fact to examine the unknotting number of knots in the KnotInfo table.

We consider the knots with  $\text{MLq}^F(K) = 3$ . The knots in the table with  $\text{MLq}^F(K) = 3$  are posted at [49], together with the first simple prime power Alexander quandle  $X$  that gives  $\text{Lq}_X(K) = 3$ . A file containing knots with  $\text{MLq}^F(K) = 3$  and their unknotting numbers is also found at [48]. In the list, the notation  $[2, 3]$  means  $u(K) = 2$  or  $3$ , which we also denote by  $u(K) = [2, 3]$  below for shorthand. By Corollary 7.2 and the inequality  $\text{NI}(K) \leq u(K)$ , for these knots with  $\text{MLq}^F(K) = 3$  we obtain  $2 = \text{MLq}^F(K) - 1 \leq \text{NI}(K) \leq u(K)$ . Hence we obtain the following information on the Nakanishi index and the unknotting number for these knots with  $\text{MLq}^F(K) = 3$  and with crossings 11 or 12:

(U1) If  $u(K) = 2$ , then it is determined that  $\text{NI}(K) = 2$ . The knots that satisfy this condition are listed below.

- $11a_-$  : 87, 97, 107, 132, 133, 135, 143, 155, 157, 165, 173, 181, 196, 239, 249, 277, 293, 297,  
 314, 317, 321, 322, 332, 347, 352.
- $11n_-$  : 90, 164, 175, 185.
- $12a_-$  : 216, 253, 408, 444, 466, 679, 701, 987, 1183, 1206.
- $12n_-$  : 273, 332, 403, 436, 508, 510, 526, 549, 565, 570, 592, 604, 617, 643, 666, 839, 887.
- (U2) If  $u(K) = [1, 2]$ , then it is determined that  $\text{NI}(K) = 2$  and  $u(K) = 2$ . The knots that satisfy this condition are listed below.
- $11a_-$ : None.
- $11n_-$ : 49, 83, 91, 157, 162, 165, 167.
- $12a_-$ : There are 63 knots in this category. See Appendix C for the list.
- $12n_-$ : There are 74 knots in this category. See Appendix C for the list.
- (U3) If  $u(K) = 3$ , then we obtain the information that  $\text{NI}(K) = [2, 3]$ . The knots that satisfy this condition are listed below.
- $11a_-$ : 123.
- $11n_-$ : 126, 133, 148, 183.
- $12a_-$ : 295, 311, 327, 386, 433, 561, 563, 569, 576, 615, 664, 683, 725, 780, 907, 921, 1194.
- $12n_-$ : 147, 276, 387, 402, 494, 496, 581, 626, 654, 660.
- (U4) If  $u(K) = [2, 3]$ , then  $\text{NI}(K) = [2, 3]$  and no new information is obtained for  $u(K)$ . The knots that satisfy this condition are listed below.
- $11a_-$ : 329.
- $11n_-$ : None.
- $12a_-$ : 297, 970, 1286.
- $12n_-$ : 294, 509, 881.
- (U5) If  $u(K) = [1, 2, 3]$ , then  $\text{NI}(K) = [2, 3]$  and the possibility for the unknotting number is narrowed to  $u(K) = [2, 3]$ . The knots that satisfy this condition are listed below.
- $11a_-$ : None.
- $11n_-$ : None.
- $12a_-$ : 244, 291, 376, 381, 481, 493, 494, 634, 886, 1124, 1142, 1202, 1205, 1269, 1288.
- $12n_-$ : 270, 601, 602, 630, 701, 844, 873.
- (U6) If  $u(K) = [2, 3, 4]$  ;  $[3, 4]$  ; 4, respectively, then  $\text{NI}(K) = [2, 3, 4]$  and no new information is obtained for  $u(K)$ . The knots that satisfy these conditions are listed below.
- $11a_-$  : 354, 366 ; 291, 340 ; None, respectively.
- $11n_-$  : None ; None ; None.
- $12a_-$  : 1097, 1164 ; 973 ; 574, 647.
- $12n_-$  : None ; 600, 764, 806 ; 386, 518.

We observed no other cases in the list. In particular, in our list of knots with  $\text{MLq}^F = 3$ , there is no knot  $K$  with  $u(K)$  containing 5, nor  $u(K) = [1, 2, 3, 4]$ . All the other positive integer intervals containing less than 5 are listed above.

**Remark 7.4** Among 12 crossing knots  $K$  with  $\text{MLq}(K) = 4$  (recall that there are 48 of them), the following 7 knots have  $\text{MLq}^F(K) = 4$ , and for each of the 7 knots, the only RIG quandle  $Q$  of type  $F(p, h(t))$  such that  $Lq_Q(K) = 4$  is  $Q = C[3, 1]$ :

12a\_: 554, 750.

12n\_: 553, 554, 555, 556, and 642.

These knots have the Nakanishi index at least 3 from Corollary 7.2. The bridge indices of these knots were determined to be 4. Hence by Corollary 7.2, we obtain that  $\text{NI}(K) = 3$  for these knots.

The unknotting number for all of these knots except 12n\_642 is listed in [8] as  $[1, 2, 3]$ , and it is posted as  $[2, 3, 4]$  for 12n\_642. Since  $\text{NI} = 3$ , the unknotting number of these knots except 12n\_642 is determined to be 3, and it is  $[3, 4]$  for 12n\_642.

*Acknowledgment:* We were greatly helped in this project by the construction of all connected quandles of order at most 35 by Leandro Vendramin [46]. We also acknowledge help from Michael Kinyon in showing us how to use the program Mace4 [33] to compute the number of coloring of a knot by a quandle. We also thank Larry Dunning for showing us how to minimize the number of quandles needed by use of binary integer linear programming. Thanks also to James McCarron for his help with the Maple Package Magma and especially with the use of the Maple procedure Magma:-AreIsomorphic for determining when two quandles are isomorphic. We would also like to thank David Stanovsky for useful information about connected quandles, and Slavik Jablan and Chad Musick for valuable information on the bridge index.

MS was partially supported by NSF DMS-0900671 and NIH 1R01GM109459-01.

# Appendix

## A Enumeration of quandles and properties of RIG quandles

Sequences enumerating quandles of order  $n$  with various properties may be found in the Online Encyclopedia of Integer Sequences [42]. We list the OEIS identification numbers of some of these sequences below. We also list below some properties and characterizations of RIG quandles that may be found in files at [48].

- <http://oeis.org/A193024>. The number of isomorphism classes of Alexander quandles of order  $n$ .
- <http://oeis.org/A193067>. The number of isomorphism classes of connected Alexander quandles of order  $n$ .
- <http://oeis.org/A225744>. The number of isomorphism classes of connected Generalized Alexander quandles of order  $n$ .
- <http://oeis.org/A196111> The number of isomorphism classes of simple quandles of order  $n$ . A quandle is *simple* if it has more than one element, and if it has no homomorphic images other than itself or the singleton quandle. However, in [1], it is further assumed that simple quandles are not trivial. This also rules out the quandles of order 2.
- <http://oeis.org/A177886> The number of isomorphism classes of latin quandles of order  $n$ . A *Latin quandle* is a quandle such that for each  $a \in X$ , the left translation  $\mathcal{L}_a$  is a bijection. That is, the multiplication table of the quandle is a Latin square.
- <https://oeis.org/A226172> The number of isomorphism classes of faithful connected quandles of order  $n$ . A quandle is *faithful* if the mapping  $a \mapsto \mathcal{R}_a$  from  $X$  to  $\text{Inn}(X)$  is an injection.
- <https://oeis.org/A226173> The number of isomorphism classes of connected keis (involutory quandles) of order  $n$ . A quandle  $X$  is *involutory*, or a *kei*, if the right translations are involutions:  $\mathcal{R}_a^2 = \text{id}$  for all  $a \in X$ .
- <https://oeis.org/A226174> The number of isomorphism classes of self-dual connected quandles of order  $n$ .

We also obtained the following information about RIG quandles. This is available at [48]:

- Subquandles of RIG quandles that are RIG quandles and trivial quandles. We also identified the orders of subquandles that are neither connected nor trivial.
- The connected RIG Alexander quandles are presented as  $\mathbb{Z}[t]/J$  where  $J = (f(t))$  or  $J = (f(t), g(t))$  except for one Alexander RIG quandle, namely  $C[27, 17]$ , which is not a cyclic  $\mathbb{Z}[t, t^{-1}]$ -module.
- We identified RIG quandles that are products of other RIG quandles.

- We identified dual quandles of RIG quandles. In particular, if the dual is itself, then it is self-dual.
- A function  $\phi : X \times X \rightarrow A$  for an abelian group  $A$  is called a *quandle 2-cocycle* if it satisfies

$$\phi(x, y) - \phi(x, z) + \phi(x * y, z) - \phi(x * z, y * z) = 0$$

and  $\phi(x, x) = 0$  for any  $x, y, z \in X$ . For a quandle 2-cocycle  $\phi$ ,  $X \times A$  becomes a quandle by  $(x, a) * (y, b) = (x * y, a + \phi(x, y))$  for  $x, y \in X$ ,  $a, b \in A$ , and it is called an *abelian extension* of  $X$  by  $A$ . See [5], for example, for more details of extensions. A list of RIG quandles that are abelian extensions of RIG quandles is found in [48].

- For each RIG quandle  $C[n, i]$  we identified a set  $\text{Gen}[n, i]$  which is a smallest set of generators for  $C[n, i]$ . We also produced the set of all  $[n, i]$  such that any two elements of  $C[n, i]$  will generate  $C[n, i]$ . Note that if  $[n, i]$  is in this set then there are no non-trivial subquandles of  $C[n, i]$ .

**Remark A.1** It is known that connected quandles of prime order  $p$  and those of order  $p^2$  are Alexander quandles. By examining characterizations and properties of RIG quandles, listed above, one might wonder if these results might generalize. For example, RIG quandles of order  $2^n$  for some  $n$ , and those with order  $pq$  for primes  $p, q > 3$  are Alexander. However, inspections of higher order conjugation quandles reveal that there are quandles with order 64 and 55 that are not Alexander. The former is a size 64 conjugacy class of GAP's `SmallGroup(1344, 11699)`. Two of the latter are a conjugacy class of `PSL(2, 11)` and a conjugacy class of `SmallGroup(1210, 7)`.

**Remark A.2** As in [20] and [46], for computational purposes we represent quandles by matrices. The entries of the matrix  $A$  of a quandle of order  $n$  are integers  $1, \dots, n$  and the quandle operation is given by  $i * j = A_{i,j}$ . We note that the quandles in the RIG package are left distributive whereas for knot colorings we prefer right distributive quandles. Hence we have reversed the order of the product in the RIG quandle by taking the transposes of the matrices representing the RIG quandles. In the RIG package the  $i$ -th quandle of order  $n$  is denoted by `SmallQuandle(n, i)`, for which we use the notation  $C[n, i]$ .

## B Tables of quandles used to distinguish knots and their mirrors

**Table B.1** List of 26 quandles that distinguish 2977 knots up to orientation and mirror image. The quandle matrices can be found at [48].

<i>Quandle</i>	<i>order</i>	<i>GAP ID for Inn(Q)</i>	<i>StructureDescription(Inn(Q))</i>
$Q_1$	12	<i>SmallGroup</i> (60, 5)	$A_5$
$Q_2$	13	<i>SmallGroup</i> (52, 3)	$C_{13} : C_4$
$Q_3$	13	<i>SmallGroup</i> (52, 3)	$C_{13} : C_4$
$Q_4$	13	<i>SmallGroup</i> (156, 7)	$(C_{13} : C_4) : C_3$
$Q_5$	13	<i>SmallGroup</i> (156, 7)	$(C_{13} : C_4) : C_3$
$Q_6$	15	<i>SmallGroup</i> (60, 5)	$A_5$
$Q_7$	17	<i>SmallGroup</i> (136, 12)	$C_{17} : C_8$
$Q_8$	17	<i>SmallGroup</i> (136, 12)	$C_{17} : C_8$
$Q_9$	20	<i>SmallGroup</i> (120, 34)	$S_5$
$Q_{10}$	24	<i>SmallGroup</i> (168, 42)	$PSL(3, 2)$
$Q_{11}$	25	<i>SmallGroup</i> (75, 2)	$(C_5 \times C_5) : C_3$
$Q_{12}$	27	<i>SmallGroup</i> (702, 47)	$((C_3 \times C_3 \times C_3) : C_{13}) : C_2$
$Q_{13}$	27	<i>SmallGroup</i> (702, 47)	$((C_3 \times C_3 \times C_3) : C_{13}) : C_2$
$Q_{14}$	30	<i>SmallGroup</i> (120, 34)	$S_5$
$Q_{15}$	32	<i>SmallGroup</i> (160, 199)	$((C_2 \times Q_8) : C_2) : C_5$
$Q_{16}$	40	<i>SmallGroup</i> (320, 1635)	$((C_2 \times C_2 \times C_2 \times C_2) : C_5) : C_4$
$Q_{17}$	40	<i>SmallGroup</i> (320, 1635)	$((C_2 \times C_2 \times C_2 \times C_2) : C_5) : C_4$
$Q_{18}$	42	<i>SmallGroup</i> (168, 42)	$PSL(3, 2)$
$Q_{19}$	48	<i>SmallGroup</i> (288, 1025)	$(A_4 \times A_4) : C_2$
$Q_{20}$	60	<i>SmallGroup</i> (660, 13)	$PSL(2, 11)$
$Q_{21}$	72	<i>SmallGroup</i> (576, 8652)	$(A_4 \times A_4) : C_4$
$Q_{22}$	72	<i>SmallGroup</i> (504, 156)	$PSL(2, 8)$
$Q_{23}$	84	<i>SmallGroup</i> (1512, 779)	$PSL(2, 8) : C_3$
$Q_{24}$	84	<i>SmallGroup</i> (1512, 779)	$PSL(2, 8) : C_3$
$Q_{25}$	90	<i>SmallGroup</i> (720, 765)	$A_6.C_2$
$Q_{26}$	182	<i>SmallGroup</i> (1092, 25)	$PSL(2, 13)$

**Table B.2** List of 23 quandles that distinguish 1058 knots from their mirror images. The quandle matrices can be found at [48].

<i>Quandle</i>	<i>order</i>	<i>GAP ID for Inn(Q)</i>	<i>StructureDescription(Inn(Q))</i>
$T_1$	351	<i>SmallGroup</i> (2106, 0)	$((C_3 \times C_3 \times C_3) : C_{13}) : C_3 : C_2$
$T_2$	504	<i>SmallGroup</i> (3024, 0)	$C_2 \times (PSL(2, 8) : C_3)$
$T_3$	504	<i>SmallGroup</i> (4536, 0)	$PSL(2, 8) : C_9$
$T_4$	18	<i>SmallGroup</i> (216, 90)	$((C_2 \times C_2) : C_9) : C_3 : C_2$
$T_5$	27	<i>SmallGroup</i> (216, 86)	$((C_3 \times C_3) : C_3) : C_8$
$T_6$	27	<i>SmallGroup</i> (486, 41)	$((C_3 \cdot ((C_3 \times C_3) : C_3) = (C_3 \times C_3) \cdot (C_3 \times C_3)) : C_3) : C_2$
$T_7$	28	<i>SmallGroup</i> (168, 43)	$((C_2 \times C_2 \times C_2) : C_7) : C_3$
$T_8$	720	<i>SmallGroup</i> (7920, 0)	$M11$
$T_9$	112	<i>SmallGroup</i> (1344, 816)	$((C_2 \times C_2 \times C_2) \cdot (C_2 \times C_2 \times C_2)) : C_7 : C_3$
$T_{10}$	112	<i>SmallGroup</i> (1344, 816)	$((C_2 \times C_2 \times C_2) \cdot (C_2 \times C_2 \times C_2)) : C_7 : C_3$
$T_{11}$	117	<i>SmallGroup</i> (1053, 51)	$((C_3 \times C_3 \times C_3) : C_{13}) : C_3$
$T_{12}$	162	<i>SmallGroup</i> (1296, 2890)	$(C_3 \cdot (((C_3 \times C_3) : Q_8) : C_3) = (((C_3 \times C_3) : C_3) : Q_8) \cdot C_3) : C_2$
$T_{13}$	162	<i>SmallGroup</i> (1296, 2891)	$((((C_3 \times C_3) : C_3) : Q_8) : C_3) : C_2$
$T_{14}$	192	<i>SmallGroup</i> (1344, 814)	$(C_2 \times C_2 \times C_2) \cdot PSL(3, 2)$
$T_{15}$	125	<i>SmallGroup</i> (1500, 36)	$((C_5 \times C_5) : C_5) : C_4 : C_3$
$T_{16}$	135	<i>SmallGroup</i> (1620, 421)	$((C_3 \times C_3 \times C_3 \times C_3) : C_5) : C_4$
$T_{17}$	135	<i>SmallGroup</i> (1620, 421)	$((C_3 \times C_3 \times C_3 \times C_3) : C_5) : C_4$
$T_{18}$	168	<i>SmallGroup</i> (1512, 779)	$PSL(2, 8) : C_3$
$T_{19}$	64	<i>SmallGroup</i> (448, 179)	$((C_2 \times C_2 \times C_2) \cdot (C_2 \times C_2 \times C_2)) : C_7$
$T_{20}$	64	<i>SmallGroup</i> (768, 1083508)	$((((C_8 \times C_4) : C_2) : C_2) : C_2) : C_3$
$T_{21}$	64	<i>SmallGroup</i> (768, 1083509)	$((((C_8 \times C_4) : C_2) : C_2) : C_2) : C_3$
$T_{22}$	40	<i>SmallGroup</i> (320, 1635)	$((C_2 \times C_2 \times C_2 \times C_2) : C_5) : C_4$
$T_{23}$	84	<i>SmallGroup</i> (1512, 779)	$PSL(2, 8) : C_3$

## C Knots with unknotting number 2

In this section we list knots with 12 crossings such that their unknotting number is listed as 1 or 2 in KnotInfo [8], and our computation that  $MLq^F = 3$  determines that their unknotting number is in fact 2.

The list of alternating knots is  $12a_k$  for  $k =$   
100, 177, 215, 218, 245, 248, 249, 265, 270, 279, 298, 312, 332, 347, 348, 396, 413, 427, 429, 435,  
448, 465, 475, 503, 594, 703, 712, 742, 769, 787, 806, 808, 810, 868, 873, 895, 904, 905, 906, 941,  
949, 960, 975, 990, 1019, 1022, 1026, 1053, 1079, 1092, 1093, 1102, 1105, 1123, 1152, 1167, 1181,  
1225, 1229, 1251, 1260, 1280, 1283.

The list of non-alternating knots is  $12n_\ell$  for  $\ell =$   
144, 145, 257, 268, 269, 274, 297, 333, 334, 355, 356, 357, 379, 380, 388, 389, 393, 394, 397, 414,  
420, 440, 442, 460, 462, 480, 495, 498, 505, 533, 546, 567, 571, 582, 583, 598, 605, 611, 622, 636,  
637, 651, 652, 669, 706, 714, 717, 737, 742, 745, 746, 752, 756, 757, 760, 779, 781, 798, 813, 817,  
837, 838, 840, 843, 846, 847, 869, 874, 876, 877, 878, 879, 883.

## D Remarks on the computation of $\text{Col}_Q(K)$

We used several techniques to compute  $\text{Col}_Q(K)$  for quandles  $Q$  and knots  $K$ . We take for each knot the braid representation given at KnotInfo [8]. We take the braids to be oriented from top to bottom. This induces an orientation of the knots. The two parameters that are most important for computing the number of colorings are the braid index,  $b(K)$  (the number of strands) and the order  $|Q|$  of the quandle. A straightforward computation requires  $|Q|^{b(K)}$  steps. Using the fact that our quandles are connected, i.e., the group  $\text{Inn}(Q)$  acts transitively on  $Q$  we may assume that one of the arcs has a fixed color. This allowed us to reduce the number of steps to  $|Q|^{b(K)-1}$ , which reduces the time by a factor of  $1/|Q|$ . In the case of quandles of large order this makes a noticeable difference in program running times (hours vs months).

We initially used Mace4 [33] to find the number of colorings. Later on, we wrote a program in C to verify that the results we obtained using Mace4 were correct. The C program uses the same braid representations as those used for the Mace4 computations.

Let  $K$  be a knot with a braid index  $b(K)$ . Let  $Q$  be a quandle of order  $n$ . Let  $a_i, i = 1, \dots, b$ , represent the element of a given quandle assigned to the top of the  $i$ -th strand of a braid, where  $b = b(K)$  is the braid index. To count the number of knot colorings, we find the colorings at the bottom strands of the braid, and check assignments such that the top and bottom colors are the same for each strand, compare Figure 2. The following pseudo code demonstrates that our algorithm runs in  $O(n^{b-1})$  time,  $n = |Q|$ .

```
/* fix strand a1 to be 0. This allows us to get away *
 * with coloring b-1 strands instead of b strands. */
a1=0;
for(a2=0; a2 < n; a2++) {
  for(a3 ...) {
    for(a4 ...) {
      for(a5 ...) {
        ... up to the braid index 'b'
        /* constant time quandle coloring of knot strands */
      }
    }
  }
}
```

We implemented a multithreaded version of the program using POSIX threads to further improve performance. Combined with fixing a color of one strand, we found that multithreading reduces the computation time considerably on multicore processors. The unexpectedly large speedup of multithreading over trivial parallelization may be due to operating system characteristics and prevention of CPU cache swapping to the slower RAM when the operating system is switching tasks.

The computations were performed on a GNU/Linux computer equipped with 24 Gigabytes of RAM and an Intel Core i7 processor with 4 cores. The kernel was compiled with a task switching latency of 100hz.

quandle	order	serial seconds	multithread seconds	speedup factor
$Q_1$	12	1	4	$0.25 \times$ (slower)
$Q_2$	13	2	4	$0.5 \times$ (slower)
$Q_3$	13	2	4	$0.5 \times$ (slower)
$Q_4$	13	2	4	$0.5 \times$ (slower)
$Q_5$	13	2	4	$0.5 \times$ (slower)
$Q_6$	15	3	4	$0.75 \times$ (slower)
$Q_7$	17	6	4	$1.5 \times$ (faster)
$Q_8$	17	6	4	$1.5 \times$ (faster)
$Q_9$	20	12	5	$2.4 \times$ (faster)
$Q_{10}$	24	31	5	$6.2 \times$ (faster)
$Q_{11}$	25	38	5	$7.6 \times$ (faster)
$Q_{12}$	27	60	5	$12 \times$ (faster)
$Q_{13}$	27	60	5	$12 \times$ (faster)
$Q_{14}$	30	105	7	$15 \times$ (faster)
$Q_{15}$	32	125	7	$17.86 \times$ (faster)
$Q_{16}$	40	455	9	$50.56 \times$ (faster)
$Q_{17}$	40	455	9	$50.56 \times$ (faster)
$Q_{18}$	42	542	10	$54.2 \times$ (faster)
$Q_{19}$	48	1215	14	$86.79 \times$ (faster)
$Q_{20}$	60	(hours)	28	?
$Q_{21}$	72	(days)	56	?
$Q_{22}$	72	(days)	56	?
$Q_{23}$	84	(days)	112	?
$Q_{24}$	84	(days)	112	?
$Q_{25}$	90	(weeks)	155	?
$Q_{26}$	182	(months)	5602	(a lot faster)

Table 1: Computational time comparison

In Table 1, the first column indicates 26 quandles in Table B.1, the second column gives the order of the quandle, the third column is the computation time for all 2977 knots with at most 12 crossings using a single thread, the fourth column is time using multiple threads, and the last column is the ratio of column 3 and column 4. A log plot of timings of three different coloring programs is given in Figure 3.

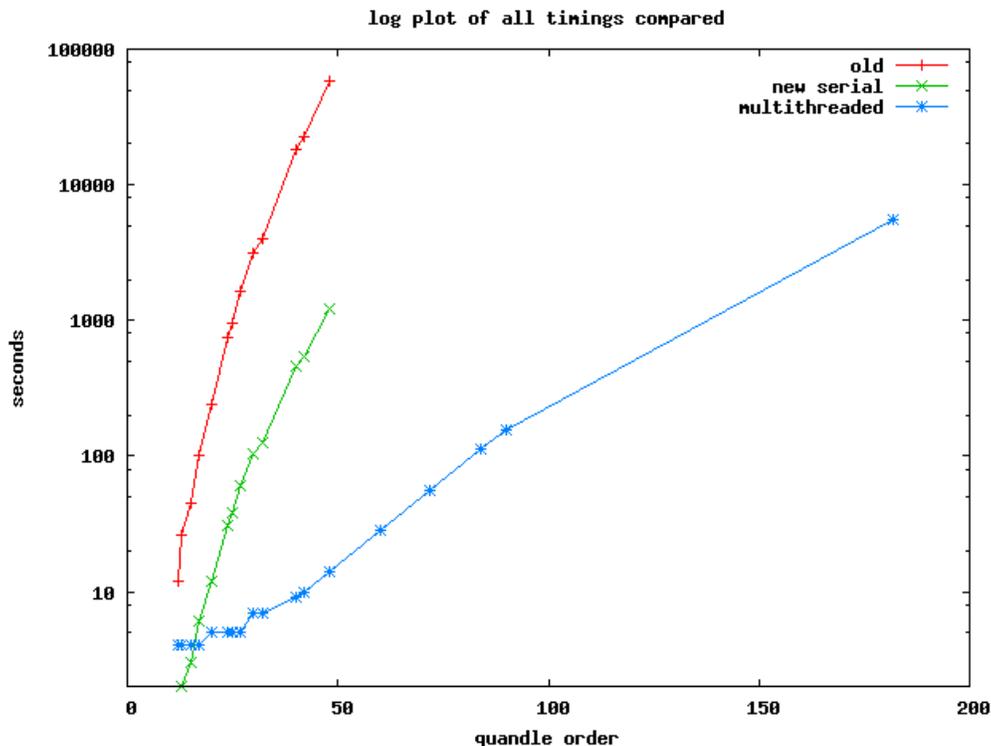


Figure 3: Log plot of timings

Multithreading was used to perform computations in parallel for the outermost loop of our knot coloring program. We set the number of threads to be the order of the quandle. None of the child threads communicate with each other. The parent thread waits for all of the child threads to return the number of colorings from its search. The parent sums each of the results from the child threads. The number of colorings of the knot is the sum of the counts from the child threads. The multithreaded version of the program is required to compute the colorings by the quandle of order 182.

```

/* fix strand a1 to be 0. This allows us to get away *
 * with coloring b-1 strands instead of b strands. */
a1=0;

/* when multithreading, we compute b-2 strands since a2 is fixed */
a2=thread_id;
/* for each thread */
for(a3=0; a3 < n; a3++) {
  for(a4 ...) {
    for(a5 ...) {
      ... up to the braid index 'b'
      /* constant time quandle coloring of knot strands */
    }
  }
}

```

$\left. \begin{array}{l} \} \\ \} \\ \} \end{array} \right\}$
---

The output of our program is the number  $\text{Col}_Q(K)_0$  of colorings with the color of the first strand fixed. The result is then converted to the number of non-trivial colorings  $\text{Col}_Q^N(K)$  using the formula  $\text{Col}_Q^N(K) = |Q|(\text{Col}_Q(K)_0 - 1)$ .

All of the programs we used are provided on-line at [48] with source and data sets available. We made numerous checks, but would be pleased if our computations can be independently confirmed.

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