FULLY COMMUTATIVE ELEMENTS OF TYPE D AND HOMOGENEOUS REPRESENTATIONS OF KLR-ALGEBRAS

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ABSTRACT. In this paper, we decompose the set of fully commutative elements into natural subsets when the Coxeter group is of type D_n, and study combinatorics of these subsets, revealing hidden structures. (We do not consider type A_n first, since a similar decomposition for type A_n is trivial.) As an application, we classify and enumerate the homogeneous representations of the Khovanov–Lauda–Rouquier algebras of type D_n.

INTRODUCTION

An element w of a Coxeter group is said to be *fully commutative* if any reduced word for w can be obtained from any other by interchanges of adjacent commuting generators. These elements were first introduced by Fan [Fan95, Fan96] and Graham [Gra95] in their study of the generalized Temperley–Lieb algebras, where they showed that the generalized Temperley–Lieb algebras have a linear basis indexed by the fully commutative elements. Soon after, these elements were extensively studied by Stembridge in a series of papers [Ste96, Ste97, Ste98]. He gave a classification of the Coxeter groups having a finite number of fully commutative elements, which was previously done by Graham and Fan in the simply-laced case, and enumerated fully commutative elements for each of the finite types. The set of fully commutative elements also has connections to Kazhdan–Lusztig cells [Fan97, FG99, GL01].

In this paper, we study fully commutative elements of the Coxeter groups of type D_n. We decompose the set of fully commutative elements into natural subsets, and study combinatorial properties of these subsets. More precisely, the subsets obtained from the decomposition of the set of fully commutative elements are called *packets*. The main result (Theorem 2.12) shows that each set in the (n, k)-packet has size equal to $C(n, k)$, the (n, k)-entry of Catalan's triangle, and implies the following identity (Corollary 2.14):

$$(0.1) \qquad \sum_{k=0}^{n} C(n, k) \, |\mathcal{P}(n, k)| = \frac{n+3}{2} C_n - 1,$$

where C_n is the nth Catalan number and $|\mathcal{P}(n, k)|$ is the number of elements in the (n, k)-packet.

It is quite intriguing that the set of fully commutative elements of type D_n has such a structure. A similar decomposition of the set of fully commutative elements of type A_n would be trivial as there would be only one packet. Since the right-hand side of (0.1) is equal to the dimension

of the Temperley–Lieb algebra of type D_n, the identity (0.1) suggests that there might be a representation theoretic construction, in which $C(n, k)$ would correspond to the dimension of a representation and $|\mathcal{P}(n, k)|$ to its multiplicity.

As a related work, Lejczyk and Stroppel studied these fully commutative elements of type D_n and the Temperley–Lieb algebra action in their paper [LS13], where they also gave a diagrammatical description of the parabolic Kazhdan–Lusztig polynomials.

Actually, this research was motivated by the connection of the fully commutative elements to the homogeneous representations of the Khovanov–Lauda–Rouquier (KLR) algebras (also known as quiver Hecke algebras). Introduced by Khovanov and Lauda [KL09] and independently by Rouquier [Rou08], the KLR algebras have been the focus of many recent studies. In particular, these algebras categorify the lower (or upper) half of a quantum group. In the paper [KR11], Kleshchev and Ram significantly reduce the problem of describing the irreducible representations of the KLR algebras to the study of *cuspidal* representations for finite types. In the process of constructing the cuspidal representations, Kleshchev and Ram defined a class of representations known as *homogeneous representations* [KR10], those that are concentrated in a single degree. Homogeneous representations include most of the cuspidal representations for finite types with a suitable choice of ordering on words. In particular, for type D_n, a natural choice of ordering makes all the cuspidal representations homogeneous. Therefore it is important to completely understand homogeneous representations. The main point in our approach is that homogeneous representations can be constructed from the sets of reduced words of fully commutative elements in the corresponding Coxeter group as shown in [KR10].

As an application intended at the beginning of this research, our results classify and enumerate the homogeneous representations of KLR algebras according to the decomposition of the set of fully commutative elements. See Corollary 2.13.

The outline of this paper is as follows. In Section 1, we fix notations, briefly review the representations of KLR algebras, and explain the relationship between homogeneous representations and fully commutative elements of a Coxeter group. In the next section, we study the set of fully commutative elements of type D_n, construct explicit bijections among packets, prove the main theorem (Theorem 2.12) and obtain the identity (0.1).

1. Homogeneous Representations and Fully Commutative Elements

1.1. **Definitions.** To define a KLR algebra, we begin with a quiver Γ. In this paper, we will focus mainly on quivers of Dynkin type D_n, but for the definition, any finite quiver with no double bonds will suffice. Let I be the set indexing the vertices of Γ, and for indices $i \neq j$, we will say that i and j are neighbors if $i \to j$ or $i \leftarrow j$. Define $Q_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\, \alpha_i$ as the non-negative lattice with basis $\{\alpha_i | i \in I\}$. The set of all words in the alphabet I is denoted by $\langle I \rangle$, and for a fixed $\alpha = \sum_{i \in I} c_i \alpha_i \in Q_+$, let $\langle I \rangle_\alpha$ be the set of words $\boldsymbol{w}$ on the alphabet I such that each $i \in I$ occurs exactly c_i times in $\boldsymbol{w}$. We define the *height* of α to be $\sum_{i \in I} c_i$. We will write $\boldsymbol{w} = [w_1, w_2, \ldots, w_d]$, $w_j \in I$.

Now, fix an arbitrary ground field $\mathbb{F}$ and choose an element $\alpha \in Q_+$. Then the *Khovanov–Lauda–Rouquier algebra* R_α is the associative $\mathbb{F}$-algebra generated by:

- idempotents $\{e(\boldsymbol{w}) \mid \boldsymbol{w} \in \langle I \rangle_\alpha\}$,
- symmetric generators $\{\psi_1, \ldots, \psi_{d-1}\}$ where d is the height of the root α,
- polynomial generators $\{y_1, \ldots, y_d\}$,

subject to relations

$$(1.1) \qquad e(\boldsymbol{w})e(\boldsymbol{v}) = \delta_{\boldsymbol{w}\boldsymbol{v}}e(\boldsymbol{w}), \qquad \sum_{\boldsymbol{w} \in \langle I \rangle_\alpha} e(\boldsymbol{w}) = 1;$$

$$(1.2) \qquad y_k e(\boldsymbol{w}) = e(\boldsymbol{w})y_k;$$

$$(1.3) \qquad \psi_k e(\boldsymbol{w}) = e(s_k \boldsymbol{w})\psi_k;$$

$$(1.4) \qquad y_k y_\ell = y_\ell y_k;$$

$$(1.5) \qquad y_k \psi_\ell = \psi_\ell y_k \ (\text{for } k \neq \ell, \ell+1);$$

$$(1.6) \qquad (y_{k+1}\psi_k - \psi_k y_k)e(\boldsymbol{w}) = \begin{cases} e(\boldsymbol{w}) & \text{if } w_k = w_{k+1}, \\ 0 & \text{otherwise}; \end{cases}$$

$$(1.7) \qquad (\psi_k y_{k+1} - y_k \psi_k)e(\boldsymbol{w}) = \begin{cases} e(\boldsymbol{w}) & \text{if } w_k = w_{k+1}, \\ 0 & \text{otherwise}; \end{cases}$$

$$(1.8) \qquad \psi_k^2 e(\boldsymbol{w}) = \begin{cases} 0 & \text{if } w_k = w_{k+1}, \\ (y_k - y_{k+1})e(\boldsymbol{w}) & \text{if } w_k \to w_{k+1}, \\ (y_{k+1} - y_k)e(\boldsymbol{w}) & \text{if } w_k \leftarrow w_{k+1}, \\ e(\boldsymbol{w}) & \text{otherwise}; \end{cases}$$

$$(1.9) \qquad \psi_k \psi_\ell = \psi_\ell \psi_k \ (\text{for } |k - \ell| > 1);$$

$$(1.10) \qquad (\psi_{k+1}\psi_k\psi_{k+1} - \psi_k\psi_{k+1}\psi_k)e(\boldsymbol{w}) = \begin{cases} e(\boldsymbol{w}) & \text{if } w_{k+2} = w_k \to w_{k+1}, \\ -e(\boldsymbol{w}) & \text{if } w_{k+2} = w_k \leftarrow w_{k+1}, \\ 0 & \text{otherwise}. \end{cases}$$

Here $\delta_{\boldsymbol{w}\boldsymbol{v}}$ in (1.1) is the Kronecker delta and, in (1.3), s_k is the kth simple transposition in the symmetric group S_d, acting on the word $\boldsymbol{w}$ by swapping the letters in the kth and $(k+1)^{\text{st}}$ positions. If Γ is a Dynkin-type quiver, we will say that R_α is a KLR algebra of that type.

We impose a $\mathbb{Z}$-grading on R_α by

$$(1.11) \qquad \deg(e(\boldsymbol{w})) = 0, \quad \deg(y_i) = 2,$$

$$(1.12) \qquad \deg(\psi_i e(\boldsymbol{w})) = \begin{cases} -2 & \text{if } w_i = w_{i+1}, \\ 1 & \text{if } w_i, w_{i+1} \text{ are neighbors in } \Gamma, \\ 0 & \text{if } w_i, w_{i+1} \text{ are not neighbors in } \Gamma. \end{cases}$$

Set $R = \bigoplus_{\alpha \in Q_+} R_\alpha$, and let $\mathrm{Rep}(R)$ be the category of finite dimensional graded R-modules, and denote its Grothendieck group by $[\mathrm{Rep}(R)]$. Then $\mathrm{Rep}(R)$ categorifies one half of the quantum group. More precisely, let $\mathbf{f}$ and $'\mathbf{f}$ be the Lusztig's algebras defined in [Lus11, Section 1.2] attached to the Cartan datum encoded in the quiver Γ over the field $\mathbb{Q}(v)$. We put $q = v^{-1}$ and $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$, and let $'\mathbf{f}_\mathcal{A}$ and $\mathbf{f}_\mathcal{A}$ be the $\mathcal{A}$-forms of $'\mathbf{f}$ and $\mathbf{f}$, respectively. Consider the graded duals $'\mathbf{f}^*$ and $\mathbf{f}^*$, and their $\mathcal{A}$-forms

$$'\mathbf{f}_\mathcal{A}^* := \{x \in {}'\mathbf{f}^* : x({}'\mathbf{f}_\mathcal{A}) \subset \mathcal{A}\} \quad \text{and} \quad \mathbf{f}_\mathcal{A}^* := \{x \in \mathbf{f}^* : x(\mathbf{f}_\mathcal{A}) \subset \mathcal{A}\}.$$

Then we have:

Theorem 1.1. [KL09] *There is an $\mathcal{A}$-linear (bialgebra) isomorphism $\gamma^* : [\mathrm{Rep}(R)] \xrightarrow{\sim} \mathbf{f}_\mathcal{A}^*$.*

Since it is not directly related to our purpose, we omit defining the bialgebra structures. Details can be found in [KL09, KR11].

A word $\mathbf{i} \in \langle I \rangle_\alpha$ is naturally considered as an element of $'\mathbf{f}_\mathcal{A}^*$ to be dual to the corresponding monomial in $'\mathbf{f}_\mathcal{A}$. Let M be a finite dimensional graded R_α-module. Define the q-character of M by

$$\mathrm{ch}_q M := \sum_{\mathbf{i} \in \langle I \rangle_\alpha} (\dim_q M_\mathbf{i})\, \mathbf{i} \in {}'\mathbf{f}_\mathcal{A}^*,$$

where $M_\mathbf{i} = e(\mathbf{i})M$ and $\dim_q V := \sum_{n \in \mathbb{Z}} (\dim V_n)\, q^n \in \mathcal{A}$ for $V = \oplus_{n \in \mathbb{Z}} V_n$. A non-empty word $\mathbf{i}$ is called *Lyndon* if it is lexicographically smaller than all its proper right factors, or equivalently smaller that all its rotations. For $x \in {}'\mathbf{f}^*$ we denote by $\max(x)$ the largest word appearing in x. A word $\mathbf{i} \in \langle I \rangle$ is called *good* if there is $x \in \mathbf{f}^*$ such that $\mathbf{i} = \max(x)$. Given a module $L \in \mathrm{Rep}(R_\alpha)$, we say that $\mathbf{i} \in \langle I \rangle$ is the *highest weight* of L if $\mathbf{i} = \max(\mathrm{ch}_q L)$. An irreducible module $L \in R_\beta$ is called *cuspidal* if its highest weight is a good Lyndon word. It is known that the set of good Lyndon words is in bijection with the set of positive roots Δ_+ of the root system attached to Γ, when Γ is of finite type.

Using the techniques developed by Leclerc in [Lec04], Kleshchev and Ram, and then Melvin, Mondragon, and Hill showed:

Theorem 1.2 ([KR11]; [HMM12], 4.1.1)**.** *Assume that Γ is of finite Dynkin type. Then the good Lyndon words parameterize the cuspidal representations of the KLR algebra R_β, $\beta \in \Delta_+$. In*

turn, any irreducible graded R_α-module for $\alpha \in Q_+$ is given by an irreducible head of a standard representation induced from cuspidal representations up to isomorphism and degree shift.

The above theorem clearly explains the importance of cuspidal representations. In the next subsection, we will introduce another class of representations which contains most of the cuspidal representations with a suitable choice of ordering on words.

1.2. **Homogeneous representations.** We define a *homogeneous representation* of a KLR algebra to be an irreducible, graded representation fixed in a single degree (with respect to the $\mathbb{Z}$-grading described in (1.11) and (1.12)). Homogeneous representations form an important class of irreducible modules since most of the cuspidal representations are homogeneous with a suitable choice of ordering on $\langle I \rangle$ ([KR11, HMM12]). After introducing some terminology, we will describe these representations in a combinatorial way. We continue to assume that Γ is a simply-laced quiver.

Fix an $\alpha \in Q_+$ and let d be the height of α. For any word $\boldsymbol{w} \in \langle I \rangle_\alpha$, we say that the simple transposition $s_r \in S_d$ is an *admissible transposition for* $\boldsymbol{w}$ if the letters w_r and w_{r+1} are neither equal nor neighbors in the quiver Γ. Following Kleshchev and Ram [KR10], we define the *weight graph* G_α with vertices given by $\langle I \rangle_\alpha$. Two words $\boldsymbol{w}, \boldsymbol{v} \in \langle I \rangle_\alpha$ are connected by an edge if there is an admissible transposition s_r such that $s_r \boldsymbol{w} = \boldsymbol{v}$.

We say that a connected component C of the weight graph G_α is *homogeneous* if the following property holds for every $\boldsymbol{w} \in C$:

(1.13)   If $w_r = w_s$ for some $1 \leq r < s \leq d$, then there exist t, u

         with $r < t < u < s$ such that w_r is neighbors with both w_t and w_u.

A word satisfying condition (1.13) will be called a *homogeneous word*.

Example 1.3. Consider the A_3 quiver

$$\Gamma = \quad \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longleftarrow \underset{3}{\bullet}$$
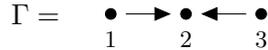
We then have $I = \{1, 2, 3\}$, and choose the element $\alpha = \alpha_1 + 2\alpha_2 + \alpha_3 \in Q_+$. Then the weight graph G_α is given by:

1322	2213	2132	1223	2123	3212
3122	2231	2312	1232	2321	3221
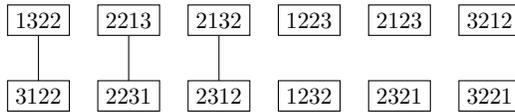
One can see that the only homogeneous component is

2132
2312
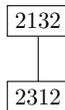
In this case, the two instances of the letter 2 have the neighbors 1 and 3 occurring between them. Note that, for some $\alpha \in Q_+$ we may have that every component of the weight graph is homogeneous (e.g. $\alpha = \alpha_1 + \alpha_2 + \alpha_3$), while for others we may see that no components are homogeneous (e.g. $\alpha = 2\alpha_1 + \alpha_2$).

A main theorem of [KR10] shows that the homogeneous components of G_α exactly parameterize the homogeneous representations of the KLR algebra R_α:

Theorem 1.4 ([KR10], Theorem 3.4). *Let C be a homogeneous component of the weight graph G_α. Define an $\mathbb{F}$-vector space $S(C)$ with basis $\{v_{\boldsymbol{w}} \mid \boldsymbol{w} \in C\}$ labeled by the vertices in C. Then we have an R_α-action on $S(C)$ given by*

$$
\begin{aligned}
e(\boldsymbol{w}')v_{\boldsymbol{w}} &= \delta_{\boldsymbol{w},\boldsymbol{w}'}v_{\boldsymbol{w}} \quad (\boldsymbol{w}' \in \langle I \rangle_\alpha, \boldsymbol{w} \in C), \\
y_r v_{\boldsymbol{w}} &= 0 \quad (1 \leq r \leq d, \boldsymbol{w} \in C), \\
\psi_r v_{\boldsymbol{w}} &= \begin{cases} v_{s_r \boldsymbol{w}} & \text{if } s_r \boldsymbol{w} \in C \\ 0 & \text{otherwise} \end{cases} \quad (1 \leq r \leq d-1, \boldsymbol{w} \in C),
\end{aligned}
$$

which gives $S(C)$ the structure of a homogeneous, irreducible R_α-module. Further $S(C) \not\cong S(C')$ if $C \neq C'$, and this construction gives all of the irreducible homogeneous modules, up to isomorphism.

As a result, the task of identifying homogeneous modules of a KLR algebra is reduced to identifying homogeneous components in a weight graph. This is simplified further by the following lemma:

Lemma 1.5 ([KR10], Lemma 3.3). *A connected component C of the weight graph G_α is homogeneous if and only if an element $\boldsymbol{w} \in C$ satisfies the condition (1.13).*

Recall that we call a word satisfying condition (1.13) a homogeneous word. The homogeneous words have other combinatorial characterizations, which we explore in the next subsection.

1.3. **Fully commutative elements of Coxeter groups.** Since the homogeneity of $\boldsymbol{w} \in \langle I \rangle$ does not depend on the orientation of a quiver, it is enough to consider Dynkin diagrams and the corresponding Coxeter groups. Given a simply laced Dynkin diagram, the corresponding Coxeter group will be denoted by W and the generators by s_i, $i \in I$. A reduced expression $s_{i_1} \cdots s_{i_r}$ will be identified with the word $[i_1, \ldots, i_r]$ in $\langle I \rangle$. For example, in the type A_4, the reduced expressions

$$s_3 s_1 s_2 s_3 s_4 = s_1 s_3 s_2 s_3 s_4 = s_1 s_2 s_3 s_2 s_4 = s_1 s_2 s_3 s_4 s_2$$

are identified with the words

$$[3,1,2,3,4], \ [1,3,2,3,4], \ [1,2,3,2,4], \ [1,2,3,4,2], \quad \text{respectively.}$$

The identity element will be identified with the empty word $[\,]$.

Assume that W is a simply-laced Coxeter group. An element $w \in W$ is said to be *fully commutative* if any reduced word for w can be obtained from any other by interchanges of adjacent commuting generators, or equivalently if no reduced word for w has $[i, i', i]$ as a subword where i and i' are neighbors in the Dynkin diagram. We make several observations, which are important for our study of homogeneous representations, and list them in the following lemma. These observations were first made by Kleshchev and Ram.

Lemma 1.6. [KR10]

(1) *A homogeneous component of the weight graph G_α contains as its vertices exactly the set of reduced expressions for a fully commutative element in W.*

(2) *The set of homogeneous components is in bijection with the set of fully commutative elements in W.*

(3) *Any KLR algebra of type A_n $(n \geq 1)$, D_n $(n \geq 4)$, or E_n $(n = 6, 7, 8)$ has finitely many irreducible homogeneous representations.*

Proof. Part (1) follows from the definitions; in particular, the condition (1.13) implies that no word in a homogeneous component has $[i, i', i]$ as a subword where i and i' are neighbors. Parts (2) and (3) are consequences of (1). $\square$

Stembridge [Ste96] classified all of the Coxeter groups that have finitely many fully commutative elements. The list includes the infinite families of types E_n, F_n and H_n. His results completed the work of Fan [Fan96], who had done this for the simply-laced types. In the same paper [Fan96], Fan showed that the fully commutative elements parameterized natural bases for corresponding quotients of Hecke algebras. In type A_n, these give rise to the Temperley–Lieb algebras (see [Jon87]). Fan and Stembridge also enumerated the set of fully commutative elements. In particular, they showed the following.

Proposition 1.7 ([Fan96, Ste98])**.** *Let C_n be the nth Catalan number, i.e. $C_n = \frac{1}{n+1}\binom{2n}{n}$. Then the number of fully commutative elements in the Coxeter group of type A_n is C_{n+1}, and that of type D_n is $\frac{n+3}{2}C_n - 1$.*

We immediately obtain a consequence on homogeneous representations from Lemma 1.6.

Corollary 1.8. *A KLR algebra $R = \bigoplus_{\alpha \in Q_+} R_\alpha$ of type A_n has C_{n+1} irreducible homogeneous representations, while a KLR algebra of type D_n has $\frac{n+3}{2}C_n - 1$ irreducible homogeneous representations.*

In [KR10], Kleshchev and Ram parameterized homogeneous representations using skew shapes. In this paper, we will decompose the set of fully commutative elements to give a finer enumeration of homogeneous representations in type D_n. More precisely, our main theorem (Theorem 2.12) proves that these homogeneous representations can be organized naturally into *packets* (defined in Section 2.2), and counted by Catalan's triangle. Note that these results contribute not only

to the combinatorics of the representation theory of KLR algebras, but also to the study of fully commutative elements of Coxeter groups.

2. Packets in Type D_n

In this section, we will focus on fully commutative elements and representations of KLR-algebras of type D_n. That is, we shall assume that Γ is a quiver whose underlying graph is of the form:
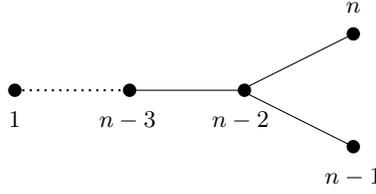
We begin with canonical reduced words of type D_n.

2.1. **Canonical reduced words.** For $1 \leq i \leq n-1$, we define the words s_{ij} by:

$$s_{ij} = \begin{cases} [i, i-1, \ldots, j] & \text{if } i \geq j, \\ [\,] & \text{if } i < j. \end{cases}$$

When $i = n$, we define

$$s_{nj} = \begin{cases} [n, n-2, \ldots, j] & \text{if } j \leq n-2, \\ [n] & \text{if } j = n, n-1, \\ [\,] & \text{if } j > n. \end{cases}$$

We will often write $s_{ii} = s_i$. The following lemma provides a canonical form we need.

Lemma 2.1 ([BS01], Lemma 5.2). *Any element of the Coxeter group of type D_n can be uniquely written in the reduced form*

$$s_{1i_1} s_{2i_2} \cdots s_{n-1i_{n-1}} s_{nj_1} s_{n-1j_2} s_{nj_3} s_{n-1j_4} \cdots s_{n-1+[\ell]_2 j_\ell}$$

where $1 \leq i_k \leq k+1$ for $1 \leq k \leq n-1$, and $1 \leq j_1 < j_2 < \cdots < j_\ell \leq n-1$ for $\ell \geq 0$, and $[\ell]_2 = 1$ when ℓ is odd, $[\ell]_2 = 0$ when ℓ is even.

The left factor $s_{1i_1} s_{2i_2} \cdots s_{n-1i_{n-1}}$ will be called the *prefix*, and similarly the right factor $s_{nj_1} s_{n-1j_2} s_{nj_3} s_{n-1j_4} \cdots s_{n-1+[\ell]_2 j_\ell}$ will be called the *suffix* of the reduced word. For example, in the case of D_5, the word $s_{21} s_3 s_4 s_{52} s_{43} s_5 = [2, 1, 3, 4, 5, 3, 2, 4, 3, 5]$ has prefix $s_{21} s_3 s_4 = [2, 1, 3, 4]$ and suffix $s_{52} s_{43} s_5 = [5, 3, 2, 4, 3, 5]$. Given a reduced word $\boldsymbol{w}$ in canonical form, we will denote by $\boldsymbol{w}_0$ the prefix of $\boldsymbol{w}$ and by $\boldsymbol{w}'$ the suffix, and write $\boldsymbol{w} = \boldsymbol{w}_0 \boldsymbol{w}'$. Generally, a word of the form $s_{nj_1} s_{n-1j_2} s_{nj_3} s_{n-1j_4} \cdots s_{n-1+[\ell]_2 j_\ell}$ with $1 \leq j_1 < j_2 < \cdots < j_\ell \leq n-1$ for $\ell \geq 0$ will be called a *suffix*.

Remark 2.2. Notice that choosing a suffix is equivalent to choosing a (possibly empty) subset of $\{1, 2, \ldots, n-1\}$. There are 2^{n-1} ways to do this. Since there are $n!$ prefixes, we have $n! \cdot 2^{n-1}$

elements in the canonical reduced form. We recall that there are the same number of elements in the type-D_n Coxeter group.

Lemma 2.3. *Every suffix is a homogeneous word, and so represents a fully commutative element.*

Proof. A suffix $\boldsymbol{w}'$ cannot have $[1, \ldots, 1]$ as a subword. If a suffix $\boldsymbol{w}'$ has a subword $[k, \ldots, k]$ for $2 \le k \le n - 2$, then $\boldsymbol{w}'$ must have $[k, k - 1, \ldots, k + 1, k]$ as a subword from the conditions on a suffix, and $\boldsymbol{w}'$ is a homogeneous word from the definition (1.13). If $\boldsymbol{w}'$ has $[n - 1, \ldots, n - 1]$ as a subword, then $\boldsymbol{w}'$ must have $[n - 1, n - 2, \ldots, n, n - 2, \ldots, n - 1]$. Similarly, if $\boldsymbol{w}'$ has $[n, \ldots, n]$, then $\boldsymbol{w}'$ must have $[n, n - 2, \ldots, n - 1, n - 2, \ldots, n]$. In both cases, $\boldsymbol{w}'$ is a homogeneous word. $\square$

Let $\mathcal{W}_n$ be the set of canonical reduced words of type D_n given in Lemma 2.1. Each homogeneous word in $\mathcal{W}_n$ uniquely represents a fully commutative element of the Coxeter group of type D_n and also a homogeneous component of a weight graph by Lemma 1.6. The homogeneous words in $\mathcal{W}_n$ will be grouped based on their suffixes:

Definition 2.4. A *collection* $\mathbf{c}_{\boldsymbol{w}'}^n \subset \mathcal{W}_n$ labeled by a suffix $\boldsymbol{w}'$ is defined to be the set of homogeneous words in $\mathcal{W}_n$ whose suffix is $\boldsymbol{w}'$. A collection will be identified with the set of corresponding fully commutative elements in the Coxeter group of type D_n.

Some of the collections have the same number of elements as we will see in the following lemma and proposition.

Lemma 2.5. *For a fixed k, $0 \le k \le n - 3$, any collection labeled by a suffix of the form*

$$(2.1) \qquad s_{n\,k+1} s_{n-1\,j_2} s_{n\,j_3} s_{n-1\,j_4} \cdots s_{n-1+[\ell]_2\,j_\ell} \quad (\ell \ge 2)$$

has the same set of prefixes. In particular, these collections have the same number of elements.

Proof. Let $\boldsymbol{w}'$ be a suffix of the form (2.1). Then $\boldsymbol{w}'$ has the suffix $\boldsymbol{w}_1 := s_{n\,k+1} s_{n-1}$ as a subword. Since removing letters from the end of a word will not affect its homogeneity, it is clear that any prefix appearing in the collection $\mathbf{c}_{\boldsymbol{w}'}^n$ also appears in $\mathbf{c}_{\boldsymbol{w}_1}^n$. We need to show, then, the opposite inclusion.

Suppose now that $\boldsymbol{w}_0$ is a prefix of a homogeneous word appearing in the collection labeled by $\boldsymbol{w}_1$. Since the prefix and suffix of a homogeneous word are individually homogeneous words, we only assume that there is some letter r which appears in both $\boldsymbol{w}_0$ and $\boldsymbol{w}'$. From the condition

$$1 \le j_1 < j_2 < \cdots < j_\ell \le n - 1$$

on the suffix $\boldsymbol{w}'$, we see that the letter r also appears in $\boldsymbol{w}_1$. The homogeneity of $\boldsymbol{w}_0 \boldsymbol{w}_1$ requires that two neighbors of r appear between the instance of r in $\boldsymbol{w}_0$ and the instance of r in $\boldsymbol{w}_1$, and these two neighbors of r also satisfy the homogeneity condition for $\boldsymbol{w}_0 \boldsymbol{w}'$ since $\boldsymbol{w}_1$ is a left factor of $\boldsymbol{w}'$. This proves that $\boldsymbol{w}_0$ is a prefix of $\boldsymbol{w}'$ for any suffix $\boldsymbol{w}'$ of the form (2.1). $\square$

Proposition 2.6. *For $1 \leq k \leq n-3$, the collection labeled by the suffix s_{nk} has the same number of elements as any of the collections labeled by the suffix of the form*

$$s_{n\,k+1} s_{n-1 j_2} s_{nj_3} s_{n-1 j_4} \cdots s_{n-1+[\ell]_2 \, j_\ell} \quad (\ell \geq 2).$$

Proof. Let $\boldsymbol{w}_1 = s_{n\,k+1} s_{n-1}$ and $\boldsymbol{w}_2 = s_{nk}$. By Lemma 2.5, it is enough to establish a bijection between the collections $\mathbf{c}_{\boldsymbol{w}_1}^n$ and $\mathbf{c}_{\boldsymbol{w}_2}^n$. We define a map $\sigma : \mathbf{c}_{\boldsymbol{w}_2}^n \to \mathbf{c}_{\boldsymbol{w}_1}^n$ as follows. Suppose that $\boldsymbol{w}_0$ is the prefix of the word $\boldsymbol{w} = \boldsymbol{w}_0 \boldsymbol{w}_2 = \boldsymbol{w}_0[n, n-2, \ldots, k] \in \mathbf{c}_{\boldsymbol{w}_2}^n$, and let r be the last letter of $\boldsymbol{w}_0$. Then by the condition of homogeneity, we must have $r < k$ or $r = n-1$. If $r < k$, we define $\sigma(\boldsymbol{w}) = \boldsymbol{w}_0 \boldsymbol{w}_1$, i.e. the map σ will simply replace the suffix $\boldsymbol{w}_2 = [n, n-2, \ldots, k+1, k]$ with the suffix $\boldsymbol{w}_1 = [n, n-2, \ldots, k+1, n-1]$. To see that this image is actually in $\mathbf{c}_{\boldsymbol{w}_1}^n$, we need to check that changing the last letter of the suffix from k to $n-1$ does not violate homogeneity. In turn, we need only to consider the case when $\sigma(\boldsymbol{w})$ has $[n-1, n-2, \ldots, i_{n-1}, n, n-2, \ldots, k+1, n-1]$ as a right factor. Clearly the neighbor $n-2$ appears twice between the two occurrences of $n-1$ and $\sigma(\boldsymbol{w}) \in \mathbf{c}_{\boldsymbol{w}_1}^n$ in this case.

If $r = n-1$, we take $m \geq k$ to be the smallest letter such that the string $[m, m+1, \ldots, n-1]$ is a right factor of $\boldsymbol{w}_0$. Then we have $\boldsymbol{w} = s_{1i_1} \cdots s_{m-1 i_{m-1}} s_m s_{m+1} \cdots s_{n-1} \boldsymbol{w}_2$, and we define

$$\sigma(\boldsymbol{w}) = s_{1i_1} \cdots s_{m-1 i_{m-1}} s_{mk} \boldsymbol{w}_1.$$

In other words, the map σ replaces the factor $s_m s_{m+1} \cdots s_{n-1} = [m, m+1, \ldots, n-1]$ with the factor $s_{mk} = [m, m-1, \ldots, k]$ in addition to changing the suffix from $\boldsymbol{w}_2$ to $\boldsymbol{w}_1$.

We show now that the image $\sigma(\boldsymbol{w}) = s_{1i_1} \cdots s_{m-1 i_{m-1}} s_{mk} \boldsymbol{w}_1$ is still a homogeneous word. It is not hard to check that the right factor $s_{mk} \boldsymbol{w}_1$ is homogeneous, but suppose that some letter t appears in both the segment s_{mk} and in some segment $s_{j i_j}$, $j \leq m-1$. Then we have $k \leq t \leq j \leq m-1 \leq n-2$. Since $t < m$, the letter t does not appear in the ascending string $[m, \ldots, n-1]$ in the word $\boldsymbol{w}$, but since $k \leq t \leq n-2$, it does appear in the suffix, $[n, \ldots, t+1, t, \ldots, k]$, along with one neighbor $t+1$. Since $\boldsymbol{w}$ is homogeneous, there must have been another neighbor in the prefix $\boldsymbol{w}_0$ which is not touched by the action of σ. Thus $\sigma(\boldsymbol{w})$ is also a homogeneous word, and we have shown that $\sigma(\mathbf{c}_{\boldsymbol{w}_2}^n) \subset \mathbf{c}_{\boldsymbol{w}_1}^n$.

To summarize, we have

$$\sigma(\boldsymbol{w}) = \begin{cases} s_{1i_1} \cdots s_{m-1 i_{m-1}} s_{mk} \boldsymbol{w}_1 & \text{if } \boldsymbol{w}_0 \text{ ends with } [m, m+1, \ldots, n-1], \\ \boldsymbol{w}_0 \boldsymbol{w}_1 & \text{otherwise,} \end{cases}$$

where $m \geq k$. Note that $\boldsymbol{w}_0$ can not end in $n-1$ in the second case. For example, when $n = 5$ and $k = 2$, we have

$\boldsymbol{w} \in \mathbf{c}_{[5,3,2]}^5$	$\sigma(\boldsymbol{w}) \in \mathbf{c}_{[5,3,4]}^5$
$[3, 2, 1, 5, 3, 2]$	$[3, 2, 1, 5, 3, 4]$
$[4, 3, 2, 1, 5, 3, 2]$	$[4, 3, 2, 1, 5, 3, 4]$
$[1, 2, 3, 4, 5, 3, 2]$	$[1, 2, 5, 3, 4]$
$[2, 1, 4, 5, 3, 2]$	$[2, 1, 4, 3, 2, 5, 3, 4]$

Next, we define a map that goes in the other direction, $\tau : \mathbf{c}^n_{\boldsymbol{w}_1} \to \mathbf{c}^n_{\boldsymbol{w}_2}$. Suppose that $\boldsymbol{w} = \boldsymbol{w}_0 \boldsymbol{w}_1 = \boldsymbol{w}_0[n, n-2, \ldots, k+1, n-1] \in \mathbf{c}^n_{\boldsymbol{w}_1}$, and let r be the last letter of $\boldsymbol{w}_0$. Then by the condition of homogeneity, we must have $1 \leq r \leq k$.

If $r < k$, then we define $\tau(\boldsymbol{w}_0 \boldsymbol{w}_1) = \boldsymbol{w}_0 \boldsymbol{w}_2$, i.e. τ simply replaces the suffix $[n, \ldots, k+1, n-1]$ with the suffix $[n, \ldots, k+1, k]$. To see that this results in a homogeneous word, we need to check only that the letter k at the end of the suffix does not violate the homogeneity condition. Assume that another k appears in $\boldsymbol{w}_0$, and consider the last non-empty segment s_{mr} of $\boldsymbol{w}_0$. Then we have $r < k \leq m$. Since there are two neighbors ($k+1$ and $k-1$) between the two occurrences of k, homogeneity is preserved.

If $r = k$, then the final non-empty segment of the prefix is s_{mk} for some m with $k \leq m \leq n-1$. We define

$$\tau(\boldsymbol{w}) = \tau(s_{1i_1} \cdots s_{m-1i_{m-1}} s_{mk} \boldsymbol{w}_1) = s_{1i_1} \cdots s_{m-1i_{m-1}} s_m s_{m+1} \cdots s_{n-1} \boldsymbol{w}_2.$$

That is, τ replaces s_{mk} with the ascending string $[m, m+1, \cdots, n-1]$ and the suffix $\boldsymbol{w}_1$ with $\boldsymbol{w}_2$. It remains to see that $\tau(\boldsymbol{w})$ is in fact a homogeneous word. Notice that the left factor $s_{1i_1} \cdots s_{m-1i_{m-1}}$ and the ascending string $[m, \ldots, n-1]$ have no letters in common, so there is nothing here to check. Also the right factor $[m, \ldots, n-1, n, n-2, \ldots, k]$ is easily checked to be homogeneous. Suppose that some letter r occurs in the left factor $s_{1i_1} \cdots s_{m-1i_{m-1}}$, and also in the suffix $[n, n-2, \cdots, k]$. Since r does not appear in the ascending string $[m, \ldots, n-1]$ but does appear in the suffix $[n, n-2, \ldots, k]$, it follows that $k \leq r < m$, so the letter r appears in the segment s_{mk} in $\boldsymbol{w}_0$. Since the word $\boldsymbol{w}_0$ is homogeneous, it must be the case that two neighbors of r appear between these two instances of r in $\boldsymbol{w}_0$. One of them may be in the segment s_{mk} which is replaced by the map τ, but at least one of them must be in the left factor which remains fixed under τ. Clearly another neighbor occurs in the suffix, so the condition for homogeneity is satisfied.

To summarize, we have, for $\boldsymbol{w} = \boldsymbol{w}_0 \boldsymbol{w}_1 \in \mathbf{c}^n_{\boldsymbol{w}_1}$

$$\tau(\boldsymbol{w}) = \begin{cases} s_{1i_1} \cdots s_{m-1i_{m-1}} s_m s_{m+1} \cdots s_{n-1} \boldsymbol{w}_2 & \text{if } \boldsymbol{w}_0 \text{ ends with } s_{mk}, \\ \boldsymbol{w}_0 \boldsymbol{w}_2 & \text{otherwise,} \end{cases}$$

where $k \leq m \leq n-1$. Note that $\boldsymbol{w}_0$ can not end in k in the second case. For example, when $n = 5$ and $k = 2$, we have

$\boldsymbol{w} \in \mathbf{c}^5_{[5,3,4]}$	$\tau(\boldsymbol{w}) \in \mathbf{c}^5_{[5,3,2]}$
$[2, 1, 5, 3, 4]$	$[2, 1, 5, 3, 2]$
$[2, 5, 3, 4]$	$[2, 3, 4, 5, 3, 2]$
$[1, 4, 3, 2, 5, 3, 4]$	$[1, 4, 5, 3, 2]$

Now one can check that τ is both a left and a right inverse of σ, so the bijection is established.

$\square$

2.2. **Packets.** The results in the previous subsection show that some collections have the same cardinality. It is natural, then, to group them together, which lead us to the following definition.

Definition 2.7. For $0 \leq k \leq n$, we define the (n, k)-*packet* of collections:
- The $(n, 0)$-packet is the set of collections labeled by suffixes of the form
$$s_{n1} s_{n-1 j_2} s_{n j_3} s_{n-1 j_4} \cdots s_{n-1+[\ell]_2 j_\ell} \quad (\ell \geq 2).$$
- The (n, k)-packet, $1 \leq k \leq n - 3$, is the set of collections labeled by s_{nk} or suffixes of the form $s_{n\,k+1} s_{n-1 j_2} s_{n j_3} s_{n-1 j_4} \cdots s_{n-1+[\ell]_2 j_\ell} \quad (\ell \geq 2)$.
- The $(n, n-2)$-packet contains only the collection labeled by $s_{n\,n-2} = [n, n-2]$.
- The $(n, n-1)$-packet contains only the collection labeled by $s_n = [n]$.
- The (n, n)-packet contains only the collection labeled by the empty suffix [ ].

We will denote the (n, k)-packet by $\mathcal{P}(n, k)$. As an example, Table 2.1 shows all of the packets in the case of D_4.

We record an important property of a packet:

Corollary 2.8. *The collections in a packet have the same number of elements.*

Proof. The assertion follows from Lemma 2.5 and Proposition 2.6. $\qquad\square$

We count the number of collections in a packet and obtain:

Proposition 2.9. *The size of the packet $\mathcal{P}(n, k)$ is*
$$|\mathcal{P}(n, k)| = \begin{cases} 2^{n-2} - 1 & \text{if } k = 0, \\ 2^{n-k-2} & \text{if } 1 \leq k \leq n - 2, \\ 1 & \text{if } k = n - 1, n. \end{cases}$$

It may be convenient to visualize these values in an array as in Table 2.2, where the row is given by n (starting at 0), and the column is given by k (also beginning at 0).

Proof. If $k = 0$, a collection in the $(n, 0)$-packet is determined by a sequence $1 < j_2 < \cdots < j_\ell \leq n - 1$, $\ell \geq 2$. The number of such sequences is the same as the number of non-empty subsets of $\{2, 3, \ldots, n - 1\}$, which is $2^{n-2} - 1$. For $1 \leq k \leq n - 3$, a collection in the (n, k)-packet is labeled by s_{nk} or is determined by a sequence $k + 1 < j_2 < \cdots < j_\ell \leq n - 1$, $\ell \geq 2$. The number of such sequences is the same as the number of non-empty subsets of $\{k + 2, k + 3, \ldots, n - 1\}$, which is $2^{n-k-2} - 1$. Hence the total number of collections in the (n, k)-packet is 2^{n-k-2}. The remaining cases $k = n - 2, n - 1, n$ are obvious. $\qquad\square$

Remark 2.10. Recall that there are 2^{n-1} suffixes in total, and as a check, we see that
$$\sum_{k=0}^{n} |\mathcal{P}(n, k)| = 2^{n-1}.$$

$\mathcal{P}(4,0)$

$\mathbf{c}^4_{\underline{[4,2,1,3]}}$	$\mathbf{c}^4_{\underline{[4,2,1,3,2]}}$	$\mathbf{c}^4_{\underline{[4,2,1,3,2,4]}}$
$[4,2,1,3]$	$[4,2,1,3,2]$	$[4,2,1,3,2,4]$

$\mathcal{P}(4,1)$

$\mathbf{c}^4_{\underline{[4,2,1]}}$		$\mathbf{c}^4_{\underline{[4,2,3]}}$	
$[4,2,1]$	$[2,3,4,2,1]$	$[3,2,1,4,2,3]$	$[1,4,2,3]$
$[3,4,2,1]$	$[1,2,3,4,2,1]$	$[2,1,4,2,3]$	$[4,2,3]$

$\mathcal{P}(4,2)$

$\mathbf{c}^4_{\underline{[4,2]}}$		
$[4,2]$	$[1,4,2]$	$[3,4,2]$
$[3,2,1,4,2]$	$[2,1,3,4,2]$	$[2,3,4,2]$
$[2,1,4,2]$	$[1,3,4,2]$	$[1,2,3,4,2]$

$\mathcal{P}(4,3)$

$\mathbf{c}^4_{\underline{[4]}}$			
$[1,2,3,4]$	$[1,4]$	$[2,3,4]$	$[3,4]$
$[1,2,4]$	$[2,1,3,2,4]$	$[2,4]$	$[4]$
$[1,3,2,4]$	$[2,1,3,4]$	$[3,2,1,4]$	
$[1,3,4]$	$[2,1,4]$	$[3,2,4]$	

$\mathcal{P}(4,4)$

$\mathbf{c}^4_{\underline{[\ ]}}$			
$[1,2,3]$	$[1]$	$[2,3]$	$[3]$
$[1,2]$	$[2,1,3,2]$	$[2]$	$[\ ]$
$[1,3,2]$	$[2,1,3]$	$[3,2,1]$	
$[1,3]$	$[2,1]$	$[3,2]$	

TABLE 2.1. The packets of D_4

```
1
1   1
1   1   1
1   1   1   1
3   2   1   1   1
7   4   2   1   1   1
15  8   4   2   1   1   1
31  16  8   4   2   1   1   1
⋮   ⋮   ⋮   ⋮   ⋮   ⋮   ⋮   ⋮   ⋱
```

TABLE 2.2. Triangle of Packet Sizes

2.3. **Catalan's Triangle.** In this subsection, we will compute the size of a collection in a given packet, allowing us to classify and enumerate all homogeneous representations. We begin by presenting a seemingly unrelated sequence.

The array shown in Table 2.3 is known as *Catalan's Triangle* [OEI]. The entry in the nth row and k^{th} column is denoted by $C(n, k)$, for $0 \leq k \leq n$. It can be built recursively: set the first entry $C(0, 0) = 1$, and then each subsequent entry is the sum of the entry above it and the entry to the left. All entries outside of the range $0 \leq k \leq n$ are considered to be 0.

```
1
1   1
1   2   2
1   3   5    5
1   4   9    14   14
1   5   14   28   42    42
1   6   20   48   90    132   132
1   7   27   75   165   297   429   429
⋮   ⋮   ⋮   ⋮    ⋮     ⋮     ⋮     ⋮     ⋱
```

TABLE 2.3. Catalan's Triangle

More precisely, for $n \geq 0$ and $0 \leq k \leq n$, we define

(2.2)
$$C(n, k) = \begin{cases} 1 & \text{if } n = 0; \\ C(n, k-1) + C(n-1, k) & \text{if } 0 < k < n; \\ C(n-1, 0) & \text{if } k = 0; \\ C(n, n-1) & \text{if } k = n. \end{cases}$$

The closed form for entries in this triangle is well known [OEI]. For $n \geq 0$ and $0 \leq k \leq n$, we have

$$(2.3) \qquad C(n,k) = \frac{(n+k)!(n-k+1)}{k!(n+1)!}$$

One can also see that

$$(2.4) \qquad C_n = C(n, n-1) = C(n,n),$$

where C_n is the nth Catalan number.

Lemma 2.11. *For $n \geq 4$, each collection in the packets $\mathcal{P}(n, n-1)$ and $\mathcal{P}(n,n)$ contains C_n homogeneous words in $\mathcal{W}_n$.*

Proof. Recall that each of these packets contains only one collection: $\mathcal{P}(n, n-1) = \{\mathbf{c}^n_{[n]}\}$ and $\mathcal{P}(n,n) = \{\mathbf{c}^n_{[\,]}\}$. Since neither of the suffixes labeling these packets contains any of the letters $1, 2, \ldots, n-1$, there are no restrictions on the homogeneous prefixes that can be included. Therefore, any homogeneous word of type A_{n-1} can be a prefix. Since a fully commutative element is represented by a unique homogeneous word in canonical reduced form, there are exactly C_n of homogeneous words of type A_{n-1} by Proposition 1.7. $\qquad \square$

The previous lemma shows that the diagonal and subdiagonal of Catalan's triangle do in fact count the sizes of collections in the corresponding packets. We now give the main theorem of this section, which says that entries in the rest of Catalan's triangle also count the sizes of collections, naturally extending Lemma 2.11. This theorem allows us to classify the homogeneous representations.

Theorem 2.12. *Assume that $n \geq 4$ and $0 \leq k \leq n$. Then any collection in the packet $\mathcal{P}(n,k)$ contains exactly $C(n,k)$ elements.*

Proof. Note that Lemma 2.11 proves the case when $k = n-1$ or n. We next consider the packet $\mathcal{P}(n,0)$, which consists of the collections labeled by the prefixes $s_{n1}s_{n-1j_2}s_{nj_3}s_{n-1j_4} \cdots s_{n-1+[\ell]_2 j_\ell}$ ($\ell \geq 2$). By Lemma 2.5, it is enough to consider the collection $\mathbf{c}$ labeled by $s_{n1}s_{n-1}$. We claim that no prefix is possible except the empty word $[\,]$. Indeed, if a word $\boldsymbol{w}$ in $\mathbf{c}$ contains a non-empty prefix $\boldsymbol{w}_0$ ending with the letter r for $1 \leq r \leq n-1$, then the word $\boldsymbol{w}$ contains the factor $[r, n, n-2, \ldots, 1, n-1]$ and we immediately reach a contradiction to homogeneity. Thus the collection $\mathbf{c}$, and hence every collection in $\mathcal{P}(n,0)$, contains only one element, the suffix itself. Since $C(n,0) = 1$, we are done with this case.

We see in Table 2.1 that any collection in the packet $\mathcal{P}(4,k)$ contains exactly $C(4,k)$ elements. Thus the assertion of the theorem is true for $n = 4$. Recall the recursive definition of $C(n,k)$: for $1 \leq k \leq n-1$,

$$(2.5) \qquad C(n,k) = C(n, k-1) + C(n-1, k).$$

We will prove that the collection sizes of the packets satisfy the relation given in equation (2.5). Since we already checked the base cases ($k = 0$, $n = 4$), the theorem will be established by

induction. Further, by Corollary 2.8, it will be enough to check that the relation holds for a single collection in each of the packets.

Now assume that $n > 4$ and $1 \le k \le n - 2$. We define

$$\boldsymbol{w}_1 = s_{nk}s_{n-1} = [n, n-2, \ldots, k, n-1], \quad \boldsymbol{w}_2 = s_{nk} = [n, n-2, \ldots, k],$$

$$\boldsymbol{w}_3 = \begin{cases} [n-1, n-3, \ldots, k] & \text{if } k < n - 2, \\ [n-1] & \text{if } k = n - 2. \end{cases}$$

Then $\mathbf{c}_{\boldsymbol{w}_1}^n \in \mathcal{P}(n, k-1)$, $\mathbf{c}_{\boldsymbol{w}_2}^n \in \mathcal{P}(n, k)$ and $\mathbf{c}_{\boldsymbol{w}_3}^{n-1} \in \mathcal{P}(n-1, k)$. We will give an explicit bijection from $\mathbf{c}_{\boldsymbol{w}_1}^n \cup \mathbf{c}_{\boldsymbol{w}_3}^{n-1}$ to $\mathbf{c}_{\boldsymbol{w}_2}^n$. Since $\mathbf{c}_{\boldsymbol{w}_1}^n \cap \mathbf{c}_{\boldsymbol{w}_3}^{n-1} = \emptyset$ as sets of formal words, we will have

(2.6) $$|\mathbf{c}_{\boldsymbol{w}_1}^n| + |\mathbf{c}_{\boldsymbol{w}_3}^{n-1}| = |\mathbf{c}_{\boldsymbol{w}_2}^n|,$$

and the proof will be completed by induction since the collection sizes satisfy the same recursive relation as (2.5).

Define the map $\varphi_1 : \mathbf{c}_{\boldsymbol{w}_1}^n \to \mathbf{c}_{\boldsymbol{w}_2}^n$ by

$$\varphi_1(\boldsymbol{w}_0\boldsymbol{w}_1) = \varphi(\boldsymbol{w}_0[n, n-2, \ldots, k, n-1]) = \boldsymbol{w}_0\boldsymbol{w}_2 = \boldsymbol{w}_0[n, n-2, \ldots, k].$$

Clearly, removing the last letter of the suffix will not affect the homogeneity of a word, and hence φ_1 maps $\mathbf{c}_{\boldsymbol{w}_1}^n$ into $\mathbf{c}_{\boldsymbol{w}_2}^n$. Similarly, we define the map $\varphi_2 : \mathbf{c}_{\boldsymbol{w}_3}^{n-1} \to \mathbf{c}_{\boldsymbol{w}_2}^n$ to be

$$\varphi_2(\boldsymbol{w}_0\boldsymbol{w}_3) = \boldsymbol{w}_0 s_{n-1}\boldsymbol{w}_2 = \boldsymbol{w}_0 s_{n-1}[n, n-2, n-3, \ldots, k].$$

To see that the image is homogeneous, we first consider the case when $\boldsymbol{w}_0$ contains the letter $n - 2$. In such a case, there are two neighbors $n - 1$ and n between two occurrences of $n - 2$. Now consider the other case when $\boldsymbol{w}_0$ contains $i < n - 2$. In this case, there is a neighbor to the right of i in $\boldsymbol{w}_0$ by homogeneity of $\boldsymbol{w}_0\boldsymbol{w}_3$ and also an $i + 1$ in the suffix to the left of i. Hence, φ_2 maps $\mathbf{c}_{\boldsymbol{w}_3}^{n-1}$ into $\mathbf{c}_{\boldsymbol{w}_2}^n$. Note that the images of φ_1 and φ_2 are disjoint: the words in the image of φ_2 all have prefixes that end with the letter $n - 1$, while none of the words in the image of φ_1 do since $\boldsymbol{w}_0$ cannot end with $n - 1$ due to the homogeneity condition on $\boldsymbol{w}_0\boldsymbol{w}_1$.

Finally, we define the map $\varphi : \mathbf{c}_{\boldsymbol{w}_1}^n \cup \mathbf{c}_{\boldsymbol{w}_3}^{n-1} \to \mathbf{c}_{\boldsymbol{w}_2}^n$ to be the combination of ϕ_1 and ϕ_2, i.e. the restriction of ϕ to $\mathbf{c}_{\boldsymbol{w}_1}^n$ is defined to be φ_1 and the restriction of ϕ to $\mathbf{c}_{\boldsymbol{w}_3}^{n-1}$ is defined to be φ_2. Figure 2.1 shows an example of the maps φ_1 and φ_2 in the case of $n = 5$ and $k = 2$.

Now we define the map $\rho : \mathbf{c}_{\boldsymbol{w}_2}^n \to \mathbf{c}_{\boldsymbol{w}_1}^n \cup \mathbf{c}_{\boldsymbol{w}_3}^{n-1}$ to be given by the rule:

$$\rho(\boldsymbol{w}_0\boldsymbol{w}_2) = \rho(\boldsymbol{w}_0[n, n-2, \ldots, k]) = \begin{cases} \boldsymbol{w}_0[n-3, \ldots, k] \in \mathbf{c}_{\boldsymbol{w}_3}^{n-1} & \text{if } \boldsymbol{w}_0 \text{ ends with } n - 1, \\ \boldsymbol{w}_0\boldsymbol{w}_1 \in \mathbf{c}_{\boldsymbol{w}_1}^n & \text{otherwise.} \end{cases}$$

In the case where $\boldsymbol{w}_0$ ends with $n - 1$, we first check homogeneity with the letter $n - 3$ in $\boldsymbol{w}_0$. In passing to D_{n-1}, the letters $n - 1$ and $n - 3$ become neighbors and homogeneity follows because there is another neighbor of $n - 3$ further to the right in $\boldsymbol{w}_0$. Now we check homogeneity when $\boldsymbol{w}_0$ contains $i < n - 3$. Note that there is a neighbor to the right of i in $\boldsymbol{w}_0$ by homogeneity of $\boldsymbol{w}_0\boldsymbol{w}_2$ and also an $i + 1$ in the suffix to the left of i. In the case where $\boldsymbol{w}_0$ does not end with $n - 1$,
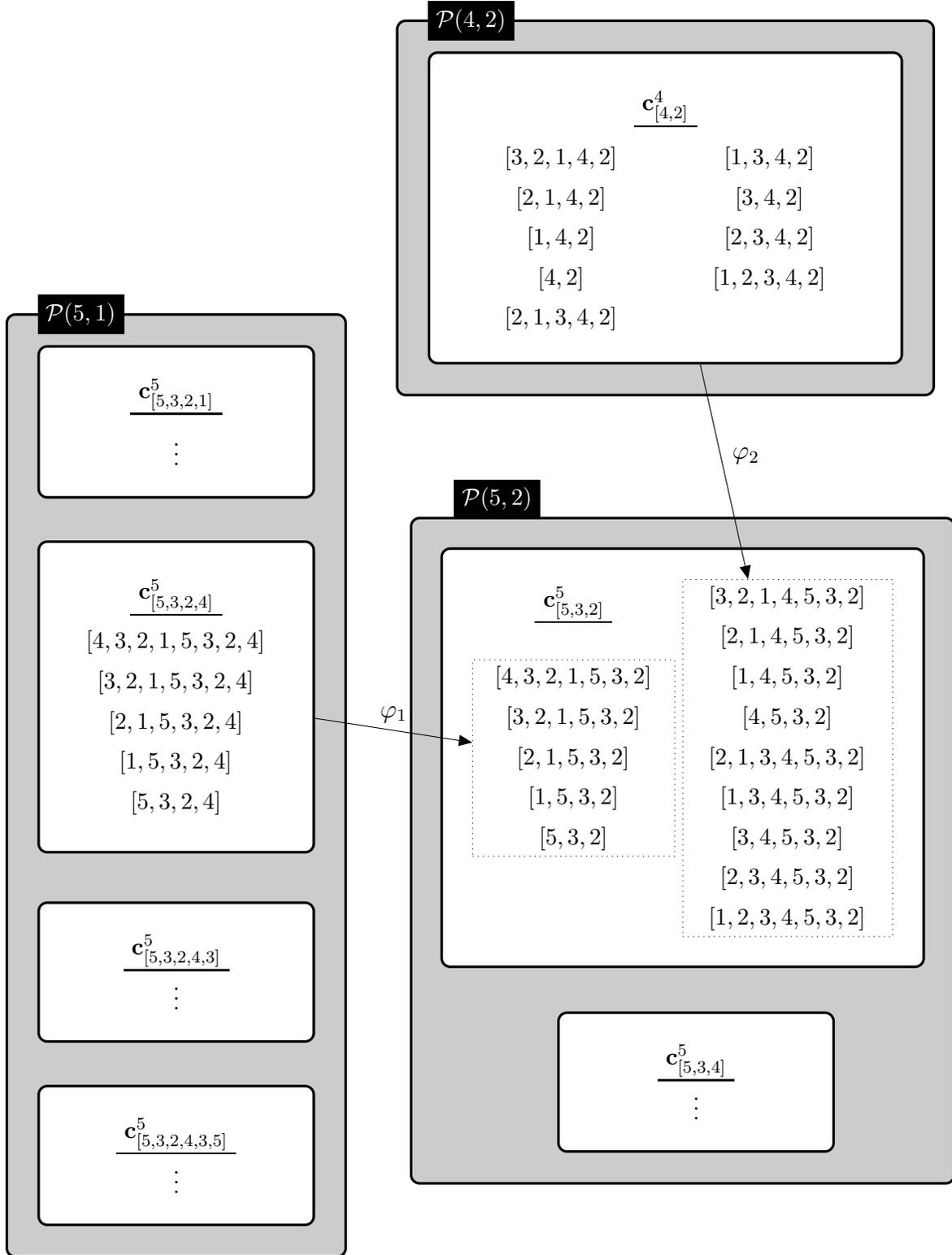
FIGURE 2.1. The maps φ_1 and φ_2 into the packet $\mathcal{P}(5,2)$.

it is still possible that $\boldsymbol{w}_0$ contains $n - 1$, but since it would be part of a descending sequence at the end of the prefix, there will always be two instances of the neighbor $n - 2$ between the two occurrences of $n - 1$. Thus the map ρ is well defined.

Now one can check that ρ is the two-sided inverse of ϕ. In particular, if we restrict ρ to the words whose prefixes end with $n - 1$, then we obtain the inverse for φ_2, while if we restrict to the prefixes not ending in $n - 1$, we have the inverse for φ_1. This establishes (2.6) and completes the proof. $\square$

We obtain immediate consequences for homogeneous representations of KLR algebras.

Corollary 2.13. *Assume that $R = \bigoplus_{\alpha \in Q_+} R_\alpha$ is the type-D_n KLR algebra.*

(1) *The set of irreducible homogeneous representations of R is decomposed into packets and collections according to the decomposition of the set of fully commutative elements (or of the set of homogeneous words in $\mathcal{W}_n$).*

(2) *Each entry in Catalan's triangle counts the number of homogeneous representations of R in the same collection.*

Corollary 2.14. *For $n \geq 4$, we obtain the identity:*

$$(2.7) \qquad \sum_{k=0}^{n} C(n,k)\,|\mathcal{P}(n,k)| = \frac{n+3}{2} C_n - 1.$$

Proof. The identity follows from Proposition 1.7 and Theorem 2.12. $\square$

Remark 2.15. As pointed out in Introduction, the identity (2.7) suggests that there might be a representation theoretic construction, in which $C(n,k)$ would correspond to the dimension of a representation and $|\mathcal{P}(n,k)|$ to its multiplicity. Also recall that the right-hand side of (2.7) is equal to the dimension of the Temperley–Lieb algebra of type D_n.

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