

# THE INNER PRODUCT ON EXTERIOR POWERS OF A COMPLEX VECTOR SPACE

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ABSTRACT. We give a formula for the inner product of forms on a Hermitian vector space in terms of linear combinations of iterates of the adjoint of the Lefschetz operator. As an application, we reprove the Kobayashi–Lübke inequality for Hermite–Einstein bundles.

## INTRODUCTION

Let  $V$  be a complex vector space of dimension  $n$ , equipped with a Hermitian inner product  $h$  whose positive  $(1, 1)$ -form we denote by  $\omega = -\operatorname{Im} h$ . The inner product on  $V$  induces an inner product on the exterior algebra  $\bigwedge^* V^*$ . If we denote the Hodge star operator by  $*$ , then this inner product is also defined by

$$\langle u, v \rangle \omega^{[n]} = u \wedge * \bar{v},$$

where  $\omega^{[k]} := \omega^k/k!$  for  $k \geq 0$  and  $u, v$  are elements of the exterior algebra.

There are two cases where we can easily calculate this inner product without writing  $u$  and  $v$  in local coordinates and painfully calculating minors of the resulting matrices: If  $v$  is a primitive  $(p, q)$ -form, then

$$\langle u, v \rangle \omega^{[n]} = u \wedge \bar{v} \wedge \omega^{[n-p-q]};$$

and if  $u$  and  $v$  are  $(1, 1)$ -forms then we eventually find that

$$\langle u, v \rangle \omega^{[n]} = -u \wedge v \wedge \omega^{[n-2]} + \Lambda u \wedge \bar{\Lambda} v \wedge \omega^{[n]}$$

by decomposing the forms into primitive components. These formulas are no better for explicit calculations than the ones involving minors of matrices, but they come in handy when calculating things that let the inner product  $\omega$  vary and give amusing matrix identities when written in an orthonormal basis.

In this note we generalize the above formulas to arbitrary  $(p, q)$ -forms  $u$  and  $v$ . In Theorem 1.1 we show that there exist integers  $b_l$ , independent of the vector space  $V$ , such that

$$(-1)^{k(k+1)/2} \langle u, v \rangle \omega^{[n]} = \sum_{l=0}^n (-1)^l b_l \Lambda^{[l]} u \wedge \Lambda^{[l]} \bar{\mathbf{I}} v \wedge \omega^{[n-k+2l]}$$

for any  $k$ -forms  $u$  and  $v$ . The integers  $b_l$  have actually been studied in quite a different context [Car55, Rio64, OEI13]; they are the coefficients in the series expansion of the reciprocal of a Bessel function of the first kind.

The proof of this result is mostly by formal calculations with the Lefschetz operator and its adjoint on the exterior algebra of  $V$ , the only slight difficulty

is showing that the coefficients  $b_l$  are the indicated ones. We do this in Section 1. There I go into considerable detail in all calculations, which I hope the reader will forgive me, but I decided it was best to make all the steps explicit for the sake of error-checking. In Section 2 we apply our results to calculate the norm of a curvature form of a Hermitian vector bundle and reprove the Kobayashi–Lübke inequality for Hermite–Einstein vector bundles. The resulting proof is very likely the same as the usual differential-geometric one if we write all the calculations in local coordinates.

*Remark* — We state and prove all our results on a finite-dimensional vector space  $V$  equipped with a Hermitian inner product. However, since our calculations are formal the same proofs work verbatim on a vector space  $V$  equipped with a representation of  $\mathfrak{sl}(2)$ , such as the cohomology algebra of a compact Kähler manifold or a projective variety over a field  $k$ .

## 1. INNER PRODUCTS OF EXTERIOR FORMS

Let  $V$  be a complex vector space of dimension  $n$  and  $\omega$  a Hermitian inner product on  $V$ . The Hodge star operator of  $\omega$  is  $*$ , the Lefschetz operator is  $L$  and its adjoint is  $\Lambda$ . We write  $\mathbf{I} = \sum_{p,q} i^{p-q} \pi_{p,q}$ , where  $\pi_{p,q} : \Lambda^* V^* \rightarrow \Lambda^{p,q} V^*$  is the orthogonal projection.

Recall that a  $k$ -form  $u$  on  $V$  is *primitive* if  $\Lambda u = 0$ . This is equivalent to  $L^{n-k-1}u = 0$ . Any  $k$ -form  $u$  on  $V$  admits a *primitive decomposition*  $u = \sum L^{k-j}u_j$ , which is an orthogonal decomposition of  $u$  where each form  $u_j$  is a primitive  $(k-2j)$ -form.

If  $A$  is an element of an algebra then we define  $A^{[k]} = A^k/k!$  for  $k \geq 0$ . This entails that

$$A^{[j]} \cdot A^{[k]} = \binom{j+k}{j} A^{[j+k]}.$$

We will use this convention for the element  $\omega$  of the exterior algebra  $\Lambda^* V^*$  and the operators  $L$  and  $\Lambda$  on that algebra.

Consider the sequence of integers defined recursively by  $b_0 = 1$  and

$$(1.1) \quad \sum_{l=0}^p (-1)^l \binom{p}{l}^2 b_l = 0$$

for  $p \geq 1$ . This is sequence number A000275 in the *On-line encyclopedia of integer sequences* [OEI13]; see also [Car55] and [Rio64] for not unrelated information about the sequence. Its first few values are:

1 1 3 19 211 3651 90.921 3.081.513 136.407.699 7.642.177.651

Our objective in this section is to prove the following theorem.

**Theorem 1.1** *Let  $u$  be a  $k$ -form on an  $n$ -dimensional complex vector space  $V$ . Then*

$$(-1)^{k(k+1)/2} |u|^2 \omega^{[n]} = \sum_{l=0}^n (-1)^l b_l \Lambda^{[l]} u \wedge \Lambda^{[l]} \overline{\mathbf{I}u} \wedge \omega^{[n-k+2l]}.$$

We will actually prove this result for a  $(p, q)$ -form  $u$  with  $p \leq q$  and  $p + q = k$ . By conjugation the restriction  $p \leq q$  is irrelevant and it is just a matter of basic combinatorics and degree reasoning to see that if the result holds for  $(p, q)$ -forms with  $p + q = k$  then it also holds for  $k$ -forms, i.e., that the right-hand side respects the orthogonal decomposition of  $\wedge^k V^*$ . Proving this result demands a certain amount of preparation, all of which rests on the following formula.

**Proposition 1.2** ([Huy05, Proposition 1.67]) *Let  $u$  be a primitive  $(p, q)$ -form on  $V$ . Set  $k = p + q$ . Then*

$$*L^{[j]}u = (-1)^{k(k+1)/2} L^{[n-j-k]} \mathbf{I}u.$$

**Example 1.3** If  $u$  is a primitive  $(p, q)$ -form, then this formula gives

$$|L^{[j]}u|^2 \omega^{[n]} = \binom{n-k}{j} |u|^2 \omega^{[n]}$$

after some manipulations. Working out the details of this is a fine way to appreciate how error-prone these calculations become.

If  $u$  is a  $(p, q)$ -form with  $p \leq q$ , then we write  $u = \sum_{j=0}^p L^{[p-j]}u_j$  for its primitive decomposition, where each  $u_j$  is a primitive  $(j, j + q - p)$ -form. This decomposition is orthogonal, so

$$|u|^2 = \sum_{j=0}^p |L^{[p-j]}u_j|^2 = \sum_{j=0}^p \binom{n-2j-q+p}{p-j} |u_j|^2.$$

**Proposition 1.4** *Let  $u = \sum_{j=0}^p u_j \wedge \omega^{[p-j]}$  be the primitive decomposition of a  $(p, q)$ -form  $u$ , where  $p \leq q$  and each  $u_j$  is a primitive  $(j, j + q - p)$ -form. Then*

$$\Lambda^{[l]}u = \sum_{j=0}^{p-l} \binom{n-j-q+l}{l} L^{[p-j-l]}u_j,$$

and the decomposition of  $\Lambda^{[l]}u$  is primitive.

*Proof.* By linearity it is enough to prove this for a  $(p, q)$ -form  $u = L^{[p-j]}u_j$ , where  $u_j$  is a primitive  $(j, j + q - p)$ -form. Let  $v$  be a form of degree  $(n - p + l, n - q + l)$  and set  $k = 2j + q - p$ . Then

$$\begin{aligned} \langle v, \Lambda^{[l]}(L^{[p-j]}u_j) \rangle \omega^{[n]} &= \langle L^{[l]}v, L^{[p-j]}u_j \rangle L^{[n]} \\ &= L^{[l]}v \wedge i^{q-p} (-1)^{\binom{k+1}{2}} L^{[n-j-q]} \bar{u}_j \\ &= v \wedge i^{q-p} (-1)^{\binom{k+1}{2}} \binom{n-j-q+l}{l} L^{[n-j-q+l]} \bar{u}_j \\ &= v \wedge * \left( \binom{n-j-q+l}{l} L^{[p-j-l]} \bar{u}_j \right) \\ &= \langle v, \binom{n-j-q+l}{l} L^{[p-j-l]}u_j \rangle \omega^{[n]}. \end{aligned}$$

Since the equality holds for all  $v$ , the result is proved for forms of type  $L^{[p-j]}u_j$ , where  $u_j$  is primitive. Note that the decomposition of  $\Lambda^{[l]}(L^{[p-j]}u_j)$  is again primitive, so the same will hold for an arbitrary form  $u$ .  $\square$

**Proposition 1.5** *Let  $u = \sum_{j=0}^p u_j \wedge \omega^{p-j}$  be the primitive decomposition of a  $(p, q)$ -form  $u$ , where  $p \leq q$  and  $u_j$  is a primitive  $(j, j+q-p)$ -form. Then*

$$u \wedge \overline{\mathbf{I}u} \wedge \omega^{n-p-q} = \sum_{j=0}^p u_j \wedge \overline{\mathbf{I}u_j} \wedge \omega^{n-2j-q+p}.$$

*Proof.* We induct on  $p$ , the result being clear for  $p = 0$  or  $q = 0$ . If  $u$  is a  $(p+1, q+1)$ -form we have  $u = \omega \wedge v_p + u_{p+1}$ , where  $v_p$  is a  $(p, q)$ -form whose primitive decomposition is evident. Then

$$u \wedge \overline{\mathbf{I}u} = \omega^2 \wedge v_p \wedge \overline{\mathbf{I}v_p} + \omega \wedge v_p \wedge \overline{\mathbf{I}u_{p+1}} + \omega \wedge u_{p+1} \wedge \overline{\mathbf{I}v_p} + u_{p+1} \wedge \overline{\mathbf{I}u_{p+1}},$$

so

$$\begin{aligned} u \wedge \overline{\mathbf{I}u} \wedge \omega^{n-p-q-2} &= v_p \wedge \overline{\mathbf{I}v_p} \wedge \omega^{n-p-q} + v_p \wedge \overline{\mathbf{I}u_{p+1}} \wedge \omega^{n-p-q-1} \\ &\quad + u_{p+1} \wedge \overline{\mathbf{I}v_p} \wedge \omega^{n-p-q-1} + u_{p+1} \wedge \overline{\mathbf{I}u_{p+1}} \wedge \omega^{n-p-q-2}. \end{aligned}$$

The two middle terms are zero because  $u_{p+1}$  is primitive, so  $u_{p+1} \wedge \omega^{n-p-q-1} = 0$  and  $v_p \wedge \overline{\mathbf{I}v_p} \wedge \omega^{n-p-q}$  is of the announced form by induction.  $\square$

*Remark* — Here the reader may wonder what happens for a  $(p, q)$ -form with  $p+q > n$ . The Lefschetz theorems tell all: That  $L^k : \bigwedge^{n-k} V^* \rightarrow \bigwedge^{n+k} V^*$  is an isomorphism entails that there are no primitive  $(p, q)$ -forms with  $p+q > n$ , so no information is lost here by the wedge product of such forms.

**Proposition 1.6** *Let  $u = \sum_{j=0}^p u_j \wedge \omega^{[p-j]}$  be the primitive decomposition of a  $(p, q)$ -form  $u$ , where  $p \leq q$  and each  $u_j$  is a primitive  $(j, j+q-p)$ -form. Then*

$$\begin{aligned} \Lambda^{[l]} u \wedge \Lambda^{[l]} \overline{\mathbf{I}u} \wedge \omega^{[n-p-q+2l]} \\ = \sum_{j=0}^{p-l} (-1)^j (-1)^{(q-p)(q-p+1)/2} \binom{p-j}{l} \binom{n-j-q+l}{p-l-j} \binom{n-j-q+l}{l} |L^{[p-j]} u_j|^2 \wedge \omega^{[n]}. \end{aligned}$$

*Proof.* We first apply Proposition 1.5 to our  $(p, q)$ -form  $u = \sum_j L^{[p-j]} u_j = \sum_j (u_j / (p-j)!) \wedge \omega^{p-j}$ . That gives

$$\begin{aligned} u \wedge \overline{\mathbf{I}u} \wedge \omega^{[n-p-q]} &= \sum_{j=0}^p \frac{1}{(n-p-q)!} \left( \frac{u_j \wedge \overline{\mathbf{I}u_j}}{(p-j)!^2} \right) \wedge \omega^{n-2j-q+p} \\ &= \sum_{j=0}^p \frac{1}{(n-p-q)!} \frac{(n-2j-q+p)!}{(p-j)!^2} u_j \wedge \overline{\mathbf{I}u_j} \wedge \omega^{[n-2j-q+p]} \\ &= \sum_{j=0}^p \binom{n-j-q}{p-j} \binom{n-2j-q+p}{p-j} u_j \wedge \overline{\mathbf{I}u_j} \wedge \omega^{[n-2j-q+p]} \\ &= \sum_{j=0}^p (-1)^{\binom{2j+q-p+1}{2}} \binom{n-j-q}{p-j} \binom{n-2j-q+p}{p-j} |u_j|^2 \omega^{[n]} \\ &= \sum_{j=0}^p (-1)^j (-1)^{(q-p)(q-p+1)/2} \binom{n-j-q}{p-j} \binom{n-2j-q+p}{p-j} |u_j|^2 \omega^{[n]}. \end{aligned}$$

To get the general result, we apply this to the  $(p-l, q-l)$ -form  $\Lambda^{[l]}u$ . That gives (by letting  $p \mapsto p-l, q \mapsto q-l$ )

$$\begin{aligned} \Lambda^{[l]}u \wedge \Lambda^{[l]}\overline{\mathbf{I}}u \wedge \omega^{[n-p-q+l]} &= \sum_{j=0}^{p-l} (-1)^{\binom{2j+q-p+1}{2}} \binom{n-2j-q+p}{p-l-j} \binom{n-j-q+l}{p-l-j} \binom{n-j-q+l}{l}^2 |u_j|^2 \wedge \omega^{[n]} \\ &= \sum_{j=0}^{p-l} (-1)^{\binom{2j+q-p+1}{2}} \frac{\binom{n-2j-q+p}{p-l-j} \binom{n-j-q+l}{p-l-j} \binom{n-j-q+l}{l}^2}{\binom{n-2j-q+p}{p-j}} |L^{[p-j]}u_j|^2 \wedge \omega^{[n]}. \end{aligned}$$

Once we remark that

$$\frac{\binom{n-2j-q+p}{p-j-l} \binom{n-j-q+l}{l}}{\binom{n-2j-q+p}{p-j}} = \binom{p-j}{l}$$

the proof is finished.  $\square$

This last result is the key to proving what we want. It tells us how to write the square of the norm of a  $(p, q)$ -form  $u$  with  $p \leq q$  as a linear combination of traces of the form: Define two vector spaces

$$\begin{aligned} X &= \text{Span}(\Lambda^{[l]}u \wedge \Lambda^{[l]}\overline{\mathbf{I}}u \wedge \omega^{[n-p-q+2l]} \mid l = 0, \dots, p), \\ Y &= \text{Span}(|L^{[p-j]}u_j|^2 \omega^{[n]} \mid j = 0, \dots, p). \end{aligned}$$

(By a Zariski-open argument it is enough to prove our result on the open set of forms  $u$  where all the above symbols are nonzero.) Proposition 1.6 defines a linear morphism  $A : X \rightarrow Y$ ; a morphism that only depends on the dimension of  $V$  and the degree  $p$ , but is otherwise independent of the form  $u$ . Since  $|u|^2 \omega^{[n]} = \sum |L^{[p-j]}u_j|^2 \omega^{[n]}$ , the coefficients of the linear combination we seek are the coordinates of the vector  $A^{-1}(1, \dots, 1)$ . This observation shows that coefficients like the ones we seek exist, the task is now to show that they coincide with our integer sequence.

*Remark* — I'll say a little about how we originally found the main result of this paper in case the reader is curious. First, we guessed that some kind of linear combination like the one in Theorem 1.1 existed, but assumed that its coefficients at least depended on the dimension of the underlying vector space. Then we calculated our way to Proposition 1.6. Once there, we calculated  $A^{-1}(1, \dots, 1)$  for  $(1, 1), (2, 2), (3, 3), (4, 4)$  and  $(5, 5)$ -forms with the help of computer algebra software, from which we guessed that the coefficients were in fact independent of the vector space. Searching the OEIS then revealed the coefficients probably formed a known sequence, and from there it was not difficult to prove the main result.

*Proof of Theorem 1.1.* If  $u$  is a  $(p, q)$ -form with  $p \leq q$  then we set  $k = p + q$  and write

$$|u|^2 \omega^{[n]} = \sum_{l=0}^n (-1)^{l+k(k+1)/2} b_l(p, n) \Lambda^{[l]}u \wedge \Lambda^{[l]}\overline{\mathbf{I}}u \wedge \omega^{[n-p-q+2l]},$$

where  $b_l(p, n)$  is the coefficient whose existence is guaranteed by Proposition 1.6. We will prove that  $b_l(p, n) = b_l$  by induction on  $p$ .

We first remark that  $b_0(p, n) = b_0 = 1$  for all  $p, q, n$ , because the norm of a primitive  $(p, q)$ -form  $u$  is  $|u|^2 \omega^{[n]} = (-1)^{k(k+1)/2} u \wedge \overline{\mathbf{I}u} \wedge \omega^{[n-p-q]}$ .

For the induction step, we assume that  $b_l(p, n) = b_l$  for  $l = 0, \dots, p-1$  and want to prove that  $b_p(p, n) = b_p$ . For this, first recall that if  $u = \sum_j L^{[p-j]} u_j$  is the primitive decomposition of a  $(p, q)$ -form with  $p \leq q$ , then

$$|u|^2 \omega^{[n]} = \sum_j |L^{[p-j]} u_j|^2.$$

Let's record for immediate use that if  $p \leq q$  then  $(-1)^{(q-p)(q-p+1)/2} = (-1)^p (-1)^{(p+q)(p+q+1)/2}$ . Then we can also write the above as

$$\begin{aligned} & |u|^2 \omega^{[n]} \\ &= \sum_{l=0}^n (-1)^{k(k+1)/2+l} b_l(p, n) \Lambda^{[l]} u \wedge \Lambda^{[l]} \overline{\mathbf{I}u} \wedge \omega^{[n-2(p-l)]} \\ &= \sum_{l=0}^n (-1)^{k(k+1)/2+l} b_l(p, n) \sum_{j=0}^l (-1)^j (-1)^{(q-p)(q-p+1)/2} \\ &\quad \times \binom{p-j}{l} \binom{n-j-q+l}{n-p-q+2l} \binom{n-j-q+l}{l} |L^{[p-j]} u_j|^2 \wedge \omega^{[n]} \\ &= \sum_{l=0}^n (-1)^{l+p} b_l(p, n) \sum_{j=0}^l (-1)^j \binom{p-j}{l} \binom{n-j-q+l}{n-p-q+2l} \binom{n-j-q+l}{l} |L^{[p-j]} u_j|^2 \wedge \omega^{[n]}. \end{aligned}$$

By comparing the coefficients of  $|L^{[p]} u_0|^2$  in these two expressions we find

$$\begin{aligned} 1 &= \sum_{l=0}^p (-1)^{l+p} b_l(p, n) \binom{p}{l} \binom{n-q+l}{n-p-q+2l} \binom{n-q+l}{l} \\ &= \binom{n}{p} b_p(p, n) + \sum_{l=0}^{p-1} (-1)^{l+p} b_l \binom{p}{l} \binom{n-q+l}{p-l} \binom{n-q+l}{l} \end{aligned}$$

for all  $n \geq p+q$ . The binomial coefficient  $\binom{n}{k}$  is a polynomial of degree  $k$  in  $n$  whose leading term is  $1/k!$ . Comparing the top-degree coefficients of  $n$  in the above equation we find that

$$0 = \frac{1}{p!} b_p(p, n) + \sum_{l=0}^{p-1} (-1)^{l+p} b_l \binom{p}{l} \frac{1}{l!(p-l)!}.$$

Since this equation expresses  $b_p(p, n)$  in terms of things that do not depend on  $n$ , we conclude that  $b_p$  doesn't depend on  $n$  either. The defining recurrence relation (1.1) for the integers  $b_l$ , now shows that  $b_p(p, n) = b_p$ .

Finally, we remark that by conjugating the form  $u$  it is enough to prove our formula for forms  $u$  with  $p \leq q$ .  $\square$

**Corollary 1.7** *If  $u$  and  $v$  are complex  $(p, q)$ -forms on  $V$ , then*

$$(-1)^{k(k+1)/2} \langle u, v \rangle \omega^{[n]} = \sum_{l=0}^n (-1)^l b_l \Lambda^{[l]} u \wedge \Lambda^{[l]} \overline{\mathbf{I}v} \wedge \omega^{[n-p-q+2l]}.$$

*Proof.* Immediate from polarization.  $\square$

**Example 1.8** In addition to the well-known formula for the square of the norm of a  $(1, 1)$ -form, Theorem 1.1 gives these formulas for the norms of higher-degree forms, that have not appeared before to the best of my knowledge. We a few here, for real forms to simplify notation.

(i) For a real  $(2, 2)$ -form  $u$  on  $V$  we have

$$|u|^2 \omega^{[n]} = u^2 \wedge \omega^{[n-4]} - (\Lambda u)^2 \wedge \omega^{[n-2]} + 3(\Lambda^{[2]}u)^2 \wedge \omega^{[n]}.$$

(ii) For a real  $(3, 3)$ -form  $u$  our formula gives

$$|u|^2 \omega^{[n]} = -u^2 \wedge \omega^{[n-6]} + (\Lambda u)^2 \wedge \omega^{[n-4]} - 3(\Lambda^{[2]}u)^2 \wedge \omega^{[n-2]} + 19(\Lambda^{[3]}u)^2 \wedge \omega^{[n]}.$$

(iii) For a real  $(4, 4)$ -form  $u$  we get

$$\begin{aligned} |u|^2 \omega^{[n]} = & u^2 \wedge \omega^{[n-8]} - (\Lambda u)^2 \wedge \omega^{[n-6]} + 3(\Lambda^{[2]}u)^2 \wedge \omega^{[n-4]} \\ & - 19(\Lambda^{[3]}u)^2 \wedge \omega^{[n-2]} + 211(\Lambda^{[4]}u)^2 \wedge \omega^{[n]}. \end{aligned}$$

Our theorem allows us to express the scalar product of two forms as a wedge product of forms derived from the original ones. Doing things the other way around, or expressing a wedge product in terms of inner products is also possible:

**Corollary 1.9** *If  $u$  and  $v$  are  $(p, q)$ -forms on  $V$  and  $k = p + q$ , then*

$$(-1)^{k(k+1)/2} u \wedge \overline{\mathbf{I}v} \wedge \omega^{[n-k]} = \sum_{m=0}^n (-1)^m \langle \Lambda^{[m]}u, \Lambda^{[m]}\overline{\mathbf{I}v} \rangle \omega^{[n]}.$$

*Proof.* We remark that as usual it is enough to prove our statement for  $(p, q)$ -forms with  $p \leq q$ , so we assume this holds. Plugging  $\Lambda^{[m]}u$  and  $\Lambda^{[m]}\overline{\mathbf{I}v}$  into our formula gives

$$\begin{aligned} & (-1)^{k_m(k_m+1)/2} \langle \Lambda^{[m]}u, \Lambda^{[m]}\overline{\mathbf{I}v} \rangle \omega^{[n]} \\ &= \sum_{l=0}^n (-1)^l b_l \binom{l+m}{l}^2 (\Lambda^{[l+m]}u) \wedge (\Lambda^{[l+m]}\overline{\mathbf{I}v}) \wedge \omega^{[n-k+2(l+m)]}, \end{aligned}$$

where we write  $k_m = k - 2m$ . Remark that

$$(-1)^{k_m(k_m+1)/2} = (-1)^{k(k+1)/2} (-1)^m.$$

If we sum both sides of the above equation for the scalar product over  $m$  from 0 to  $n$  and then change the variable in the first sum from  $m$  to  $\nu = l + m$  we get

$$\begin{aligned} & (-1)^{k(k+1)/2} \sum_{m=0}^n (-1)^m \langle \Lambda^{[m]}u, \Lambda^{[m]}\overline{\mathbf{I}v} \rangle \omega^{[n]} \\ &= \sum_{\nu=0}^n \left( \sum_{l=0}^{\nu} (-1)^l b_l \binom{\nu}{l}^2 \right) (\Lambda^{[\nu]}u) \wedge (\Lambda^{[\nu]}\overline{\mathbf{I}v}) \wedge \omega^{[n-k+2\nu]} \\ &= u \wedge \overline{\mathbf{I}v} \wedge \omega^{[n-k]}, \end{aligned}$$

because  $\sum_{l=0}^{\nu} (-1)^l b_l \binom{\nu}{l}^2 = 0$  for all  $\nu \geq 1$  by definition.  $\square$

*Remark* — Our formula for the inner product is given by numbers related to Bessel functions. It is possible to write our results in compact form by letting holomorphic functions define sesquilinear operators on  $V$ . For this, let

$$f(z) = J_0(2\sqrt{z}) = \sum_{m \geq 0} (-1)^m \frac{1}{m!^2} z^m.$$

This is a Bessel function of the first kind and if we set  $z = xy$  and look at its reciprocal we find

$$\frac{1}{f(xy)} = \sum_{l \geq 0} (-1)^l b_l x^{[l]} y^{[l]},$$

see [Car55, Rio64]. This function defines a sesquilinear operator on  $\Lambda^* V^*$  if we declare that

$$x^{[a]} y^{[b]}(\Lambda, \Lambda) := (u, v) \mapsto \Lambda^{[a]} u \wedge \Lambda^{[b]} \bar{v}.$$

We can then write

$$\begin{aligned} (-1)^{k(k+1)/2} u \wedge \bar{\mathbf{I}v} \wedge \omega^{[n-p-q]} &= \pi_{n,n} \left( f(xy)(\Lambda, \Lambda)(u, \bar{\mathbf{I}v}) \wedge \exp(\omega) \right), \\ (-1)^{k(k+1)/2} \langle u, v \rangle \omega^{[n]} &= \pi_{n,n} \left( \frac{1}{f(xy)}(\Lambda, \Lambda)(u, \bar{\mathbf{I}v}) \wedge \exp(\omega) \right), \end{aligned}$$

where  $\pi_{n,n} : \Lambda^* V^* \rightarrow \Lambda^{n,n} V^*$  is the projection onto the subspace of  $(n, n)$ -forms and  $\exp(\omega) := \sum_{k \geq 0} \omega^{[k]}$ . In applications we would most likely consider differential forms on a manifold  $X$  and be interested in the global inner product  $\langle\langle u, v \rangle\rangle = \int_X \langle u, v \rangle \omega^{[n]}$  instead of the pointwise one. There integration kills all but the top-degree forms, so we can write

$$(-1)^{k(k+1)/2} \langle\langle u, v \rangle\rangle = \int_X \frac{1}{f(xy)}(\Lambda, \Lambda)(u, \bar{\mathbf{I}v}) \wedge \exp(\omega).$$

Consider now the set  $U$  of complexified Hermitian inner products on  $V$ , that is the set of  $(1, 1)$ -forms  $\alpha + i\omega$ , where  $\alpha$  and  $\omega$  are real and  $\omega$  is an inner product. We have a trivial holomorphic vector bundle  $F^k \rightarrow U$  whose fiber is the space of  $k$ -forms on  $V$ , and the inner products  $\omega$  define a Hermitian metric  $h$  on  $F^k$ . The words “mirror symmetry” may be waived around here in certain crowds (but then we should perhaps look at  $G^k = \bigoplus_{p+q=n-k} \Lambda^{p,q} V^*$  instead of  $F^k$ ).

It is tempting to use the differential equation that  $J_0$  satisfies to say something about the curvature tensor of  $h$ , but in practice this seems difficult at best and should in fact depend heavily on how the space of primitive  $k$ -forms varies with  $\omega$  inside  $\Lambda^k V^*$ . The preprint [Huy] may be useful here.

## 2. CURVATURE TENSORS AND THE KOBAYASHI–LÜBKE INEQUALITY

**Linear algebraic preliminaries** In this section, we will (morally speaking) be viewing the curvature form  $R$  as a form defined on the total space of its vector bundle  $E$  and equipping that space with a metric induced by the ones on the underlying space  $X$  and on  $E$ . This lets us use the Hodge



star and Lefschetz operators on that bigger space and apply the results from Section 1.

Let  $V$  and  $E$  be complex vector space of dimensions  $n$  and  $r$ , equipped with Hermitian metrics  $\omega$  and  $h$ . Let  $R$  be a curvature-type tensor, or an element of  $\Lambda^{1,1} V^* \otimes \text{End } E$  that is Hermitian. We view  $R$  as a  $(2, 2)$ -form on the space  $E \oplus V$ , equipped with the Hermitian metric  $\alpha = \omega + h$ , where we abuse notation and do not write  $\alpha = p_V^* \omega + p_E^* h$  as we should.

Let  $e$  be a form on  $E$  and  $v$  a form on  $V$ . By picking orthonormal coordinates we quickly verify that

$$*_\alpha(p_V^* v \wedge p_E^* e) = p_V^*(*_\omega v) \wedge p_E^*(*_h e).$$

Since the exterior algebra of  $V \oplus E$  is generated by elements of the type  $p_V^* v \wedge p_E^* e$  this lets us calculate with the Hodge star operator on that space.

Let  $\Lambda_\omega$  and  $\Lambda_h$  be the adjoints of the Lefschetz operators of  $\omega$  and  $h$ , pulled back to  $V \oplus E$ . These operators commute by general facts on trace operators in Coffman's [Cof] or by calculations in an orthonormal basis.

We also note that Newton's binomial formula gives

$$\alpha^{[l]} = \sum_{k=0}^l \omega^{[k]} \wedge h^{[l-k]}$$

and that many of those terms will be zero for  $l$  big for degree reasons. A similar formula expresses  $\Lambda_\alpha^{[l]}$  in terms of  $\Lambda_\omega^{[k]}$  and  $\Lambda_h^{[l-k]}$ .

Finally we set  $k!c_k := \Lambda_h^{[k]}(\Lambda^k R)$  for  $k = 0, \dots, r$ . The notation is so chosen because when  $R$  is the curvature tensor of an actual Hermitian metric on a vector bundle, the  $c_k$  will be the Chern forms defined by  $R$ . The  $k!$  factor deserves an explanation:

The inner product  $h$  is an isomorphism  $h : E \rightarrow \overline{E}^*$ . It induces inner products on both  $\text{End } E$  and  $\Lambda^{1,1} E^*$  and a morphism  $h \otimes \text{id}_{E^*} : \text{End } E \rightarrow \Lambda^{1,1} E^*$ . The trace of an endomorphism of  $E$  is just its scalar product again the identity morphism. Taking  $k$ -th exterior powers we get a canonical morphism  $h^k \otimes \text{id}_{\wedge^k E^*} : \text{End } \wedge^k E \rightarrow \wedge^{k,k} E^*$ . This morphism is however not an isometry, but  $h^{[k]}$  is; this can be seen by comparing the norms of  $\text{id}_{\wedge^k E}$  and its image  $h^{[k]}$ . We now want to find a morphism  $\wedge^{1,1} E \rightarrow \wedge^{k,k} E$  that makes the diagram

$$\begin{array}{ccc} \text{End } E & \xrightarrow{f \mapsto \wedge^k f} & \text{End } \wedge^k E \\ \downarrow h \otimes \text{id}_{E^*} & & \downarrow h^{[k]} \otimes \text{id}_{\wedge^k E^*} \\ \wedge^{1,1} E & \dashrightarrow & \wedge^{k,k} E \end{array}$$

commute. This morphism is clearly  $u \mapsto u^k/k!$ , whence the factor of  $k!$  above.

**The norm of a curvature tensor** Here we calculate the norm of a curvature tensor of a vector bundle. The identity we find is implicit in the literature on the Kobayashi–Lübke inequality (compare with [CO75], [Lüb82])

and [Siu87]); the inequality is actually a corollary of a simple application of Cauchy–Schwarz to the equation for the norm of the curvature tensor.

**Theorem 2.1** *Let  $E \rightarrow X$  be a holomorphic vector bundle of rank  $r$  over a complex manifold  $X$  of dimension  $n$ . Let  $\omega$  and  $h$  be Hermitian metrics on  $X$  and  $E$ , respectively. Let  $\frac{i}{2\pi}\Theta$  be the curvature form of  $(E, h)$  and let  $c_k$  be the Chern forms defined by the curvature form. Then*

$$\left| \frac{i}{2\pi}\Theta \right|^2 \omega^{[n]} = (2c_2 - c_1^2) \wedge \omega^{[n-2]} + \left| \text{tr}_\omega \frac{i}{2\pi}\Theta \right|^2 \omega^{[n]}$$

at every point of  $X$ . If  $(E, h)$  is Hermite–Einstein, then we also have

$$0 \leq (2rc_2 - (r-1)c_1^2) \wedge \omega^{[n-2]}$$

pointwise on  $X$  with equality if and only if  $\frac{i}{2\pi}\Theta = (\lambda/n)\text{id}_E \otimes \omega$ , where  $\lambda$  is the Hermite–Einstein constant of  $(E, h)$ .

*Proof.* The announced result is local on  $X$ , so we pick a point  $x \in X$  and write  $V = T_{X,x}$ , abuse notation to write  $E = E_x$  and write  $R$  for the image of  $\frac{i}{2\pi}\Theta$  under the isometry  $\wedge^{1,1}T_X \otimes \text{End } E \rightarrow \wedge^{2,2}(T_X \otimes E)^*$  defined by  $h$ . Then  $R$  is a  $(2, 2)$ –form on  $V \oplus E$ . We write  $\alpha = \omega + h$  for the induced inner product on  $V \oplus E$ , in slight abuse of notation.

The norm of  $R$  as a  $(2, 2)$ –form on  $V \oplus E$  is

$$|R|^2 \alpha^{[n+r]} = R^2 \wedge \alpha^{[n+r-4]} - (\Lambda_\alpha R)^2 \wedge \alpha^{[n+r-2]} + 3(\Lambda_\alpha^2 R)^2 \wedge \alpha^{[n+r]}.$$

We’ll indicate the general steps in the calculation of each of these factors but leave the details mostly to the reader. We have

$$\begin{aligned} R^2 \wedge \alpha^{[n+r-4]} &= R^2 \wedge \left( \sum_{k=0}^4 \omega^{[n-k]} \wedge h^{[r+k-4]} \right) \\ &= R^2 \wedge \omega^{[n-2]} \wedge h^{[r-2]} = 2c_2 \wedge \omega^{[n-2]} \wedge h^{[r]}, \end{aligned}$$

where the second equality holds for degree reasons. Similarly we get

$$\begin{aligned} (\Lambda_\alpha R)^2 \wedge \alpha^{[n+r-2]} &= (\Lambda_\omega R)^2 \wedge \omega^{[n]} \wedge h^{[r-2]} \\ &\quad + 2(\text{tr}_\omega c_1)^2 \omega^{[n]} \wedge h^{[r]} + c_1^2 \wedge \omega^{[n-2]} \wedge h^{[r]} \end{aligned}$$

because

$$\Lambda_\omega R \wedge \Lambda_h R \wedge \omega^{[n-1]} \wedge h^{[r-1]} = (\text{tr}_\omega c_1)^2 \omega^{[n]} \wedge h^{[r]}.$$

Finally,

$$(\Lambda_\alpha^2 R)^2 \wedge \alpha^{[n+r]} = (\Lambda_\omega \Lambda_h R)^2 \wedge \omega^{[n]} \wedge h^{[r]} = (\text{tr}_\omega c_1)^2 \wedge \omega^{[n]} \wedge h^{[r]},$$

again for degree reasons and commutativity of the adjoints of the Lefschetz operators. From this we reap

$$\begin{aligned} |R|^2 \omega^{[n]} \wedge h^{[r]} &= (2c_2 - c_1^2) \wedge \omega^{[n-2]} \wedge h^{[r]} \\ &\quad + (\text{tr}_\omega c_1)^2 \wedge \omega^{[n]} \wedge h^{[r]} - (\Lambda_\omega R)^2 \wedge \omega^{[n]} \wedge h^{[r-2]} \end{aligned}$$

which yields

$$|R|^2 \omega^{[n]} = (2c_2 - c_1^2) \wedge \omega^{[n-2]} + (\text{tr}_\omega c_1)^2 \omega^{[n]} - \Lambda_h^2 (\Lambda_\omega R)^2 \omega^{[n]}.$$

We now use the formula for the norm of a  $(1, 1)$ -form and see that

$$|\Lambda_\omega R|_h^2 = (\Lambda_h \Lambda_\omega R)^2 - \Lambda_h^{[2]}(\Lambda_\omega R)^2 = (\operatorname{tr}_\omega c_1)^2 - \Lambda_h^{[2]}(\Lambda_\omega R)^2,$$

thus obtaining our first announced result (in equivalent notation):

$$|R|^2 \omega^{[n]} = (2c_2 - c_1^2) \wedge \omega^{[n-2]} + |\Lambda_\omega R|_h^2 \omega^{[n]}.$$

Now assume that  $(E, h)$  is Hermite–Einstein. By definition, this means that  $\operatorname{tr}_\omega \frac{i}{2\pi} \Theta = \lambda \operatorname{id}_E$ . Under our isometries, this translates into  $\Lambda_\omega R = \lambda h$ . The factor  $\lambda$  satisfies  $r\lambda = \operatorname{tr}_\omega c_1$ , so we get

$$\begin{aligned} 0 \leq |R|^2 \omega^{[n]} &= (2c_2 - c_1^2) \wedge \omega^{[n-2]} + r|\lambda|^2 \omega^{[n]}. \\ &= (2c_2 - c_1^2) \wedge \omega^{[n-2]} + \frac{1}{r}(\operatorname{tr}_\omega c_1)^2 \omega^{[n]}. \end{aligned}$$

Multiplying by  $r$  and rearranging gives

$$0 \leq r|R|^2 \omega^{[n]} = (2rc_2 - (r-1)c_1^2) \wedge \omega^{[n-2]} + |c_1|^2 \omega^{[n]}.$$

Proposition 2.2 below, which is just the Cauchy–Schwarz inequality in disguise, says that

$$|c_1|^2 \leq r|R|^2,$$

with equality if and only if  $R = u \wedge \omega$ , where  $u$  is the pullback of a form on  $E$ . By the Hermite–Einstein condition we necessarily have  $u = (\lambda/n)h$  in that case. This proves the Kobayashi–Lübke inequality.  $\square$

**Proposition 2.2** *We have  $|\Lambda_\omega R|^2 \leq n|R|^2$  and  $|c_1|^2 = |\Lambda_h R|^2 \leq r|R|^2$ , with equalities if and only if  $R = u \wedge \omega$  or  $R = v \wedge h$ , where  $u$  and  $v$  are pullbacks of  $(1, 1)$ -forms from  $E$  and  $V$ , respectively.*

*Proof.* We just prove the first result since the proof of the second differs from that in notation only. The primitive decomposition of  $R$  as a  $(2, 2)$ -form on  $E \oplus V$  is

$$R = r_0 \omega \wedge h + r_1^\omega \wedge h + r_1^h \wedge \omega + r_2.$$

Here  $r_0$  is a scalar,  $r_1^h$  is a primitive form that's a pullback from  $E$ , similar for  $r_1^\omega$ . By orthogonality we see that  $\Lambda_\omega r_j = 0$  for these primitive forms. This gives  $\Lambda_\omega R = n(r_0 h + r_1^h)$ , so

$$\begin{aligned} |\Lambda_\omega R|^2 &= n^2(|r_0 h|^2 + |r_1^h|^2) = n(|r_0 \omega \wedge h|^2 + |r_1^h \wedge \omega|^2) \\ &\leq n(|r_0 \omega \wedge h|^2 + |r_1^h \wedge \omega + r_1^\omega \wedge h|^2 + |r_2|^2) = n|R|^2 \end{aligned}$$

with equality if and only if  $R = u \wedge \omega$  for a form  $u$  that's a pullback from  $E$ .  $\square$

**Example 2.3** (1) Let  $(X, \omega)$  be a Kähler–Einstein manifold, so that  $\operatorname{Ric} \omega = \lambda \omega$  for some  $\lambda \in \mathbb{R}$ . Then

$$|R|^2 \omega^{[n]} = c_2 \wedge \omega^{[n-2]} + (n\lambda)^2 \omega^{[n]},$$

where  $c_2$  is the second Chern form defined by  $\omega$ .

(2) Let  $(X, \omega)$  now be a Kähler manifold of constant sectional curvature  $\lambda$ ; like projective space with the Fubini–Study metric, a torus with its flat

metric or the unit ball with the Bergman metric. Then the curvature tensor of  $\omega$  is

$$R_{jklm} = \lambda(\omega_{jk}\omega_{lm} - \omega_{jl}\omega_{mk})$$

in local coordinates and we have  $\text{Ric}\omega = \lambda(n-1)\omega$ . Some calculations give

$$|R|^2 = 2n(n-1)\lambda^2,$$

so we see that

$$c_2 \wedge \omega^{[n-2]} = -\lambda^2(n-2)(n-1)n(n+1)\omega^{[n]}.$$

*Remark* — The original motivation for all of this was that I didn't understand where the differential-geometric proofs of the Kobayashi–Lübke inequality came from (see [CO75, Lüb82, Siu87]), since they all brutally calculate things in local coordinates. I also naively thought that if I found a more coordinate-invariant proof of the inequality it would be possible to use it to find inequalities involving higher Chern classes, because if calculating  $|\frac{i}{2\pi}\Theta|^2$  gives an inequality involving  $c_2$  then calculating  $|\wedge^k \frac{i}{2\pi}\Theta|^2$  should give an inequality involving  $c_{2k}$ . Unfortunately this does not seem to be possible, basically because we cannot calculate  $\Lambda(u \wedge v)$  or  $*(u \wedge v)$  in terms of  $u, v$  and  $\Lambda$  or  $*$ , which again is not possible because the wedge product of primitive forms is not primitive. I leave to the reader the pleasure of trying to estimate  $|\wedge^k \frac{i}{2\pi}\Theta|^2$  in terms of things we know and are interested in and seeing where things go wrong.

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