CONGRUENCES FOR THE FISHBURN NUMBERS

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Abstract. The Fishburn numbers, $\xi(n)$, are defined by a formal power series expansion

$$\sum_{n=0}^{\infty} \xi(n)q^n = 1 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} (1-(1-q)^j).$$

For half of the primes p, there is a non–empty set of numbers $T(p)$ lying in $[0, p-1]$ such that if $j \in T(p)$, then for all $n \geq 0$,

$$\xi(pn+j) \equiv 0 \pmod{p}.$$

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1. Introduction

The Fishburn numbers $\xi(n)$ are defined by the formal power series

$$\sum_{n=0}^{\infty} \xi(n)q^n = \sum_{n=0}^{\infty} (1-q;1-q)_n \tag{1}$$

where

$$(A;q)_n = (1-A)(1-Aq)\dots(1-Aq^{n-1}). \tag{2}$$

The Fishburn numbers have arisen in a wide variety of combinatorial settings. One can gain some sense of the extent of their applications in [9, Sequence A022493]. Namely, these numbers arise in such combinatorial settings as linearized chord diagrams, Stoimenow diagrams, nonisomorphic interval orders, unlabeled $(2+2)$-free posets, and ascent sequences. They were first defined in the work of Fishburn (cf. [6, 7, 8]), and have recently found a connection with mock modular forms [4].

It turns out that the Fishburn numbers satisfy congruences reminiscent of those for the partition function $p(n)$ [2, Chapter 1]. Surprisingly, in contrast to $p(n)$, we shall see in Section 4 that there are congruences of the form $\xi(pn+b) \equiv 0 \pmod{p}$ for half of all the primes p. For example, for all $n \geq 0$,

$$\xi(5n+3) \equiv \xi(5n+4) \equiv 0 \pmod{5}, \tag{3}$$

$$\xi(7n+6) \equiv 0 \pmod{7}, \tag{4}$$

$$\xi(11+8) \equiv \xi(11n+9) \equiv \xi(11n+10) \equiv 0 \pmod{11}, \tag{5}$$

$$\xi(17n+16) \equiv 0 \pmod{17}, \quad \text{and} \tag{6}$$

$$\xi(19n+17) \equiv \xi(19n+18) \equiv 0 \pmod{19}. \tag{7}$$

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These results all follow from a general result stated as Theorem 3.1 in Section 3. The next section is devoted to background lemmas. Theorem 3.1 is then proved in Section 3. In Section 4 we discuss an infinite family of primes p for which these congruences hold. We conclude with some open problems.

2. Background Lemmas

The sequence of pentagonal numbers is given by

$$\{n(3n-1)/2\}_{n=-\infty}^{\infty} = \{0, 1, 2, 5, 7, 12, 15, 22, \dots\}. \tag{8}$$

Throughout this work the symbol λ will be used to designate a pentagonal number.

In our first lemma, $f(q)$ will denote an arbitrary polynomial in $\mathbb{Z}[q]$, and p will be a fixed prime. Then we separate the terms in $f(q)$ according to the residue of the exponent modulo p. Thus,

$$f(q) = \sum_{i=0}^{p-1} q^i \phi_i(q^p). \tag{9}$$

We also suppose that for every p^{th} root of unity ζ (including $\zeta = 1$),

$$f(\zeta) = \sum_{\lambda} c_\lambda \zeta^\lambda$$

where the λ's sum over some set of pentagonal numbers that includes 0. The c's are thus defined to be 0 outside this prescribed set of pentagonal numbers, and the c's are independent of the choice of ζ.

Lemma 2.1. *Under the above conditions, $\phi_j(1) = 0$ if j is not a pentagonal number.*

Proof. The assertion is not immediate because the p^{th} roots of unity are not linearly independent. In particular, if ζ is a primitive p^{th} root of unity, then

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{p-1} = 0.$$

However, we know that the ring of integers in $\mathbb{Q}(\zeta)$ has $1, \zeta, \zeta^2, \dots, \zeta^{p-2}$ as a basis [1, page 187]. Hence,

$$\phi_0(1)(-\zeta - \zeta^2 - \cdots - \zeta^{p-1}) + \sum_{j=1}^{p-1} \zeta^j \phi_j(1) = c_0(-\zeta - \zeta^2 - \cdots - \zeta^{p-1}) + \sum_{\lambda \neq 0} c_\lambda \zeta^\lambda.$$

Therefore, if $1 \leq j \leq p-1$,

$$\phi_j(1) - \phi_0(1) = \begin{cases} c_\lambda - c_0 & \text{if } j \text{ is one of the designated pentagonal numbers,} \\ -c_0 & \text{otherwise} \end{cases}$$

is a linear system of $p-1$ equations in p variables $\phi_j(1)$, $0 \leq j \leq p-1$. However, the $\zeta = 1$ case adds one further equation

$$\phi_0(1) + \phi_1(1) + \cdots + \phi_{p-1}(1) = \sum_{\lambda} c_\lambda.$$

We now have a linear system of p equations in p variables, and the determinant of the system is p. Hence, there is a unique solution which is the obvious solution

$$\phi_j(1) = \begin{cases} c_\lambda & \text{if } j \text{ is one of the designated pentagonal numbers,} \\ 0 & \text{otherwise.} \end{cases}$$

∎

In the next three lemmas, we require some variations on Leibniz's rule for taking the n^{th} derivative of a product. Each is probably in the literature, but is included here for completeness.

Lemma 2.2.

$$\left(q\frac{d}{dq}\right)^n (A(q)B(q)) = \sum_{j=1}^{n} q^j c_{n,j} \left(\frac{d}{dq}\right)^j (A(q)B(q)),$$

where the $c_{n,j}$ are the Stirling numbers of the second kind given by $c_{n,0} = c_{n,n+1} = 0$, $c_{1,1} = 1$, and $c_{n+1,j} = jc_{n,j} + c_{n,j-1}$ for $1 \leq j \leq n+1$.

Proof. The result is a tautology when $n = 1$. To pass from n to $n+1$, we note

$$\begin{aligned}
\left(q\frac{d}{dq}\right)^{n+1} (A(q)B(q)) &= q\frac{d}{dq}\left(\left(q\frac{d}{dq}\right)^n (A(q)B(q))\right) \\
&= q\frac{d}{dq}\sum_{j=1}^{n} q^j c_{n,j} \left(\frac{d}{dq}\right)^j (A(q)B(q)) \\
&= q\frac{d}{dq}\sum_{j=1}^{n} q^j c_{n,j} \left(\frac{d}{dq}\right)^j (A(q)B(q)) \\
&= \sum_{j=1}^{n} jq^j c_{n,j} \left(\frac{d}{dq}\right)^j (A(q)B(q)) \\
&\qquad + \sum_{j=1}^{n} q^{j+1} c_{n,j} \left(\frac{d}{dq}\right)^{j+1} (A(q)B(q)) \\
&= \sum_{j=1}^{n+1} q^j (jc_{n,j} + c_{n,j-1}) \left(\frac{d}{dq}\right)^j (A(q)B(q)) \\
&= \sum_{j=1}^{n+1} q^j c_{n+1,j} \left(\frac{d}{dq}\right)^j (A(q)B(q)).
\end{aligned}$$

∎

Lemma 2.3.

$$\left(\frac{d}{dt}\right)^n f(qe^t)\bigg|_{t=0} = \left(q\frac{d}{dq}\right)^n f(q).$$

Proof. By Lemma 2.2 with $A(q) = f(q)$ and $B(q) = 1$, we see that

$$\left(q\frac{d}{dq}\right)^n f(q) = \sum_{j=1}^{n} q^j c_{n,i} f^{(j)}(q). \tag{10}$$

On the other hand, we claim

$$\left(\frac{d}{dt}\right)^n f(qe^t) = \sum_{j=1}^{n} q^j e^{jt} c_{n,j} f^{(j)}(qe^t). \tag{11}$$

When $n = 1$, this is just the chain rule applied to $f(qe^t)$. To pass from n to $n+1$, we note

$$\begin{aligned}
\left(\frac{d}{dt}\right)^{n+1} f(qe^t) &= \frac{d}{dt}\left(\frac{d}{dt}\right)^{n} f(qe^t) \\
&= \frac{d}{dt}\sum_{j=1}^{n} q^j e^{jt} c_{n,j} f^{(j)}(qe^t) \\
&= \sum_{j=1}^{n} j q^j e^{jt} c_{n,j} f^{(j)}(qe^t) \\
&\qquad + \sum_{j=1}^{n} q^{j+1} e^{(j+1)t} c_{n,j} f^{(j+1)}(qe^t) \\
&= \sum_{j=1}^{n+1} (j c_{n,j} + c_{n,j-1}) q^j e^{jt} f^{(j)}(qe^t) \\
&= \sum_{j=1}^{n+1} c_{n+1,j} q^j e^{jt} f^{(j)}(qe^t).
\end{aligned}$$

Comparing (11) with $t = 0$ to (10), we see that our lemma is established. ∎

We now turn to the generating function for the Fishburn numbers as given by Zagier [10, page 946]. Namely,

$$F(1-q) = \sum_{n=0}^{\infty} \xi(n) q^n = \sum_{n=0}^{\infty} (1-q;1-q)_n. \tag{12}$$

To facilitate the study, we concentrate on

$$F(q) = \sum_{n=0}^{\infty} (q;q)_n \tag{13}$$

and

$$F(q,N) = \sum_{n=0}^{N} (q;q)_n = \sum_{i=0}^{p-1} q^i A_p(N,i,q^p), \tag{14}$$

where $A_p(N,i,q^p)$ is a polynomial in q^p. We note that if ζ is a p^{th} root of unity

$$F(\zeta) = F(\zeta, m) = F(\zeta, p-1) \tag{15}$$

for all $m \geq p$. Furthermore,

$$\left(q\frac{d}{dq}\right)^r F(q)\Big|_{q=\zeta} = \left(q\frac{d}{dq}\right)^r F(q,m)\Big|_{q=\zeta} = \left(q\frac{d}{dq}\right)^r F(q,(r+1)p-1)\Big|_{q=\zeta} \tag{16}$$

for all $m \geq (r+1)p$ because $(1-q^p)^{r+1}$ divides $(q;q)_j$ for all $j \geq (r+1)p$.

Similarly, for all $m \geq (r+1)p$,

$$F^{(r)}(q)\Big|_{q=\zeta} = F^{(r)}(q,m)\Big|_{q=\zeta} = F^{(r)}(q,(r+1)p-1)\Big|_{q=\zeta}. \tag{17}$$

In the next lemma, we require a Stirling–like array of numbers $C_{N,i,j}(p)$ given by $C_{N,i,0}(p) = i^N$ ($C_{0,0,0}(p) = 1$), $C_{N,i,N+1}(p) = 0$, and for $1 \leq j \leq N$,

$$C_{N+1,i,j}(p) = (i + jp)C_{N,i,j}(p) + pC_{N,i,j-1}(p). \tag{18}$$

Lemma 2.4.

$$\left(q\frac{d}{dq}\right)^N F(q,n) = \sum_{j=0}^{N}\sum_{i=0}^{p-1} C_{N,i,j}(p)q^{i+jp}A_p^{(j)}(n,i,q^p).$$

Proof. In light of the fact that $C_{0,i,0}(p) = 1$ for all i, the $N = 0$ assertion is

$$F(q,n) = \sum_{i=0}^{p-1} q^i A_p(n,i,q^p),$$

which is just the definition of the A's given in (14). To pass from N to $N + 1$, we note

$$\begin{aligned}
\left(q\frac{d}{dq}\right)^{N+1} F(q,n) &= q\frac{d}{dq}\sum_{j=0}^{N}\sum_{i=0}^{p-1} C_{N,i,j}(p)q^{i+jp}A_p^{(j)}(n,i,q^p) \\
&= \sum_{j=0}^{N}\sum_{i=0}^{p-1} C_{N,i,j}(p)(i+jp)q^{i+jp}A_p^{(j)}(n,i,q^p) \\
&\qquad + \sum_{j=0}^{N}\sum_{i=0}^{p-1} C_{N,i,j}(p)q^{i+jp}pq^pA_p^{(j+1)}(n,i,q^p) \\
&= \sum_{j=0}^{N+1}\sum_{i=0}^{p-1}((i+jp)C_{N,i,j}(p) + pC_{N,i,j-1}(p))q^{i+jp}A_p^{(j)}(n,i,q^p) \\
&= \sum_{j=0}^{N+1}\sum_{i=0}^{p-1} C_{N+1,i,j}(p)q^{i+jp}A_p^{(j)}(n,i,q^p).
\end{aligned}$$

∎

We now define, for any positive integer p, two special sets of integers:

$$S(p) = \{j \,|\, 0 \leq j \leq p-1 \text{ such that } n(3n-1)/2 \equiv j \pmod{p} \text{ for some } n\} \tag{19}$$

and

$$T(p) = \{k \,|\, 0 \leq k \leq p-1 \text{ such that } k \text{ is larger than every element of } S(p)\}. \tag{20}$$

For example, for $p = 11$, we have

$$S(11) = \{0, 1, 2, 4, 5, 7\} \quad \text{and} \quad T(11) = \{8, 9, 10\}.$$

Lemma 2.5. *If $i \notin S(p)$, then*

$$A_p(pn-1,i,q) = (1-q)^n \alpha_p(n,i,q)$$

where the $\alpha_p(n,i,q)$ are polynomials in $\mathbb{Z}[q]$.

Proof. This result is equivalent to the assertion that for $0 \le j < n$,

$$A_p^{(j)}(pn-1, i, 1) = 0,$$

and by (17) we need only prove for $j \ge 0$,

$$A_p^{(j)}((j+1)p-1, i, 1) = 0 \tag{21}$$

because $n \ge (j+1)$.

We proceed to prove (21) by induction on j. When $j = 0$, we only need show that if $i \notin S(p)$,

$$A_p(p-1, i, 1) = 0.$$

Following [10, Section 5], we define (where ζ is now an N^{th} root of unity)

$$\begin{aligned} F(\zeta e^t) &= \sum_{n=0}^{\infty} \frac{b_n(\zeta) t^n}{n!} \\ &= e^{t/24} \sum_{n=0}^{\infty} \frac{c_n(\zeta) t^n}{24^n n!} \\ &= \sum_{M=0}^{\infty} \frac{t^M}{24^M M!} \sum_{n=0}^{M} \binom{M}{n} c_n(\zeta), \end{aligned} \tag{22}$$

where we have replaced Zagier's ξ with ζ to avoid confusion with $\xi(n)$. In [10, Section 5], we see that

$$c_n(\zeta) = \frac{(-1)^n N^{2n+1}}{2n+2} \sum_{m=1}^{N/2} \chi(m) \zeta^{(m^2-1)/24} B_{2n+2}\left(\frac{m}{N}\right), \tag{23}$$

where the B's are Bernoulli polynomials and $\chi(m) = \left(\frac{12}{m}\right)$. Note that the only non–zero terms in the sum in (23) have

$$\zeta^{((6m\pm1)^2-1)/24} \chi(6m \pm 1) = (-1)^m \zeta^{m(3m\pm1)/2}, \tag{24}$$

i.e., $c_n(\zeta)$ is a linear combination of powers of ζ where each exponent is a pentagonal number. Hence, by (22) we see that $b_n(\zeta)$ is a linear combination of powers of ζ where each exponent is a pentagonal number.

Hence, if ζ is now a p^{th} root of unity,

$$\begin{aligned} F(\zeta) &= F(\zeta, p-1) \\ &= b_0(\zeta) \\ &= \sum_{\lambda} c_\lambda \zeta^\lambda, \end{aligned}$$

where the sum over λ is restricted to a subset of the pentagonal numbers. On the other hand,

$$\begin{aligned} F(\zeta) &= F(\zeta, p-1) \\ &= \sum_{i=0}^{p-1} \zeta^i A_p(p-1, i, 1). \end{aligned}$$

Hence, by Lemma 2.1, for $i \notin S(p)$,

$$A_p(p-1, i, 1) = 0$$

which is (21) when $j=0$. Now let us assume that

$$A_p^{(j)}(p(j+1)-1,i,1)=0 \tag{25}$$

for $0\le j<\nu<n$. By Lemma 2.4,

$$\left(q\frac{d}{dq}\right)^{\nu} F(q,p(\nu+1)-1)=\sum_{j=0}^{\nu}\sum_{i=0}^{p-1} C_{\nu,i,j}(p)\zeta^i A_p^{(j)}(p(\nu+1)-1,i,1). \tag{26}$$

But for $j<\nu$,

$$A_p^{(j)}(p(\nu+1),i,1)=A_p^{(j)}(p(j+1)-1,i,1)=0.$$

Hence the only terms in the sum in (26) where ζ is raised to a non–pentagonal power, i, arise from the terms with $j=\nu$, namely

$$C_{\nu,i,\nu}(p)\zeta^i A_p^{(\nu)}(p(\nu+1)-1,i,1), \tag{27}$$

and we note that $C_{\nu,i,\nu}(p)\neq 0$.

Applying Lemma 2.3 to the left side of (26), we see that by (22)

$$\begin{aligned} b_\nu(\zeta) &= \left(q\frac{d}{dq}\right)^{\nu} F(q)\bigg|_{q=\zeta} \\ &= \left(q\frac{d}{dq}\right)^{\nu} F(q,(\nu+1)p-1) \\ &= \sum_{j=0}^{\nu}\sum_{i=0}^{p-1} C_{\nu,i,j}(p)\zeta^i A_p^{(j)}(p(\nu+1)-1,i,1). \end{aligned} \tag{28}$$

Recall that $b_\nu(\zeta)$ is a linear combination of powers of ζ where the exponents are pentagonal numbers. Hence the expression given in (27) must be zero by Lemma 2.1. Therefore,

$$A_p^{(\nu)}(p(\nu+1)-1,i,1)=0,$$

and this proves (21) and thus proves Lemma 2.5. ∎

3. The Main Theorem

We recall from (12) that

$$\begin{aligned} \sum_{n=0}^{\infty}\xi(n)q^n &= \sum_{j=0}^{\infty}(1-q;1-q)_j \\ &= 1+\sum_{j=1}^{\infty}\prod_{i=1}^{j}\sum_{h=1}^{i}(-1)^{h-1}q^h\binom{i}{h} \\ &= 1+\sum_{j=1}^{\infty}(q^j+O(q^{j+1})). \end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty}\xi(n)q^n = F(1-q,N)+O(q^{N+1}). \tag{29}$$

We are now in a position to state and prove the main theorem of this paper.

Theorem 3.1. *If p is a prime and $i \in T(p)$ (as defined in (20)), then for all $n \geq 0$,*

$$\xi(pn+i) \equiv 0 \pmod{p}.$$

Remark 3.2. *Congruences (3)–(7) are the cases $p = 5, 7, 11, 17$ and 19 of Theorem 3.1.*

Proof. We begin with a simple observation derived from Lucas's theorem for the congruence class of binomial coefficients modulo p [5, page 271]. Namely if π is any integer congruent to a pentagonal number modulo p, and $i \in T(p)$, then

$$\binom{\pi}{i} \equiv 0 \pmod{p}, \tag{30}$$

because the final digit in the p–ary expansion of π is smaller than i because i is in $T(p)$.

Now by Lemma 2.5, we may write

$$\begin{aligned} F(q, pn-1) &= \sum_{i=0}^{p-1} q^i A_p(pn-1, i, q^p) \\ &= \sum_{\substack{i=0 \\ i \in S(p)}}^{p-1} q^i A_p(pn-1, i, q^p) + \sum_{\substack{i=0 \\ i \notin S(p)}}^{p-1} q^i (1-q^p)^n \alpha_p(n, i, q^p). \end{aligned}$$

So

$$\begin{aligned} F(1-q, pn-1) &= \sum_{\substack{i=0 \\ i \in S(p)}}^{p-1} (1-q)^i A_p(pn-1, i, (1-q)^p) \\ &\quad + \sum_{\substack{i=0 \\ i \notin S(p)}}^{p-1} (1-q)^i (1-(1-q)^p)^n \alpha_p(n, i, (1-q)^p) \\ &:= \Sigma_1 + \Sigma_2. \end{aligned}$$

Now modulo p,

$$\begin{aligned} \Sigma_2 &\equiv \sum_{\substack{i=0 \\ i \notin S(p)}}^{p-1} (1-q)^i q^{pn} \alpha_p(n, i, 1) \\ &= O(q^{pn}). \end{aligned}$$

Therefore, modulo p,

$$F(1-q, pn-1) \equiv \sum_{\substack{i=0 \\ i \in S(p)}}^{p-1} (1-q)^i A_p(pn-1, i, 1-q^p) \pmod{p}.$$

Let us look at the terms in this sum where q is raised to a power that is congruent to an element of $T(p)$. Such a term must arise from the expansion of some $(1-q)^i$ where $i \in S(p)$ because $A_p(pn-1, 1, 1-q^p)$ is a polynomial in q^p.

By (30) all such terms have a coefficient congruent to 0 modulo p. Therefore, every term q^j in $F(1-q, pn-1)$ where j is congruent to an element of $T(p)$ must have a coefficient congruent to 0 modulo p.

To conclude the proof, we let $n \to \infty$. ∎

4. An Infinite Set of Primes With Congruences

At this stage, one might ask whether one can identify an infinite set of primes p for which congruences such as those described in Theorem 3.1 are found. The answer to this question can be answered affirmatively.

Theorem 4.1. *Let $R = \{5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22\}$. (The elements of R are those numbers r, $0 < r < 23$, such that $\left(\frac{r}{23}\right) = -1$.) Let p be a prime of the form $p = 23k + r$ for some nonnegative integer k and some $r \in R$. Then $T(p)$ is not empty, i.e., at least one congruence such as those described in Theorem 3.1 must hold modulo p.*

Remark 4.2. *From the Prime Number Theorem for primes in arithmetic progression, we see that, asymptotically, $T(p)$ is not empty for half of the primes and $T(p)$ equals the empty set for half of the primes.*

Proof. Assume p is a prime for which $T(p)$ **is** empty. That means there is a pentagonal number which is congruent to -1 modulo p. Then $n(3n-1)/2 \equiv -1 \pmod{p}$ for some integer n. By completing the square we then obtain $(6n-1)^2 \equiv -23 \pmod{p}$. Thus, by contrapositive, if we know that -23 is a quadratic nonresidue modulo p, then we know that such a pentagonal number does not exist (which means $T(p)$ is not empty).

Thus, if $\left(\frac{-23}{p}\right) = -1$, then $T(p)$ is not empty. But thanks to properties of the Legendre symbol, we know

$$\begin{aligned} \left(\frac{-23}{p}\right) &= \left(\frac{-1}{p}\right)\left(\frac{23}{p}\right) \\ &= (-1)^{\frac{p-1}{2}}(-1)^{\frac{23-1}{2}\frac{p-1}{2}}\left(\frac{p}{23}\right) \quad \text{by quadratic reciprocity} \\ &= (-1)^{\frac{12(p-1)}{2}}\left(\frac{r}{23}\right) \quad \text{since } p = 23k + r \\ &= \left(\frac{r}{23}\right) \end{aligned}$$

and we want this value to be -1. The theorem then follows by the nature of the construction of R. ∎

Thus, we clearly have infinitely many primes p for which the Fishburn numbers will exhibit at least one congruence modulo p.

5. Conclusion

There are many natural open questions that could be answered at this point.

- First, we believe that Theorem 3.1 lists all the congruences of the form $\xi(pn + b) \equiv 0 \pmod{p}$, but we have not proved this at this time.

- Numerical evidence seems to indicate that Theorem 3.1 can be strengthened. Namely, for certain values of $j > 1$ and certain primes p, it appears that
$$\xi(p^j n + b) \equiv 0 \pmod{p^j}$$
for certain values b and all n.

- Numerical evidence suggests that Lemma 2.5 could be strengthened as follows: If $i \notin S(p)$, then
$$A_p(pn-1,i,q) = (q;q)_n \beta_p(n,i,q)$$
for some polynomial $\beta_p(n,i,q)$. That is to say, in Lemma 5, it was proved that $(1-q)^n$ divides $A_p(pn-1,i,q)$; it appears that the factor $(1-q)^n$ can be strengthened to $(q;q)_n$.

- With an eye towards the recent work of Andrews and Jelínek [3], consider the power series given by
$$\sum_{n=0}^{\infty} a(n)q^n \quad := \quad \sum_{n=0}^{\infty} \left(\frac{1}{1-q}, \frac{1}{1-q}\right)_n$$
which begins
$$1 - q + q^2 - 2q^3 + 5q^4 - 16q^5 + 61q^6 - 271q^7 + 1372q^8 - 7795q^9 + \dots$$
We conjecture that, for all $n \geq 0$, $a(5n+4) \equiv 0 \pmod 5$.

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