On generalized Ramanujan primes

Christian Axler

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Abstract

In this paper we establish several results concerning the generalized Ramanujan primes. For $n \in \mathbb{N}$ and $k \in \mathbb{R}_{>1}$ we give estimates for the nth k-Ramanujan prime which lead both to generalizations and to improvements of the results presently in the literature. Moreover, we obtain results about the distribution of k-Ramanujan primes. In addition, we find explicit formulae for certain nth k-Ramanujan primes. As an application, we prove that a conjecture of Mitra, Paul and Sarkar [9] concerning the number of primes in certain intervals holds for every sufficiently large positive integer.

1 Introduction

Ramanujan primes, named after the Indian mathematician Srinivasa Ramanujan, were introduced by Sondow [17] in 2005 and have their origin in Bertrand's postulate.

Bertrand's Postulate. *For each $n \in \mathbb{N}$ there is a prime number p with $n < p \leq 2n$.*

Bertrand's postulate was proved, for instance, by Tchebychev [19] and by Erdös [6]. In 1919, Ramanujan [13] proved an extension of Bertrand's postulate by showing that

$$\pi(x) - \pi\left(\frac{x}{2}\right) \geq 1 \text{ (respectively } 2, 3, 4, 5, \ldots)$$

for every

$$x \geq 2 \text{ (respectively } 11, 17, 29, 41, \ldots).$$

Motivated by the fact $\pi(x) - \pi(x/2) \to \infty$ as $x \to \infty$ by the Prime Number Theorem (PNT), Sondow [17] defined the number $R_n \in \mathbb{N}$ for each $n \in \mathbb{N}$ as the smallest positive integer such that the inequality $\pi(x) - \pi(x/2) \geq n$ holds for every $x \geq R_n$. He called the number R_n the *nth Ramanujan prime*, because $R_n \in \mathbb{P}$ for every $n \in \mathbb{N}$, where $\mathbb{P}$ denotes the set of prime numbers.

This can be generalized as follows. Let $k \in (1, \infty)$. Again, the PNT implies that $\pi(x) - \pi(x/k) \to \infty$ as $x \to \infty$ and Shevelev [16] introduced the nth k-Ramanujan prime as follows.

Definition. Let $k > 1$ be real. For every $n \in \mathbb{N}$, let

$$R_n^{(k)} = \min\{m \in \mathbb{N} \mid \pi(x) - \pi(x/k) \geq n \text{ for every } x \geq m\}.$$

This number is prime and it is called the *nth k-Ramanujan prime*. Since $R_n^{(2)} = R_n$ for every $n \in \mathbb{N}$, the numbers $R_n^{(k)}$ are also called *generalized Ramanujan primes.*

In 2009, Sondow [17] showed that

$$R_n \sim p_{2n} \qquad (n \to \infty), \tag{1}$$

where p_n denotes the nth prime number. Further, he proved that

$$R_n > p_{2n} \tag{2}$$

for every $n \geq 2$. In 2011, Amersi, Beckwith, Miller, Ronan and Sondow [1] generalized the asymptotic formula (1) to k-Ramanujan primes by showing that

$$R_n^{(k)} \sim p_{\lceil kn/(k-1) \rceil} \qquad (n \to \infty). \tag{3}$$

In view of (3), one may ask whether the inequality (2) can also be generalized to k-Ramanujan primes. We prove that this is indeed the case. In fact, we derive further inequalities concerning the nth k-Ramanujan prime, by constructing explicit constants $n_0, n_1, n_2, n_3 \in \mathbb{N}$ depending on a series of parameters including k (see (10), (24), Theorem 4.4, Theorem 5.3, respectively), such that the following theorems hold.

Theorem A. *Let $t \in \mathbb{Z}$ with $t > -\lceil k/(k-1) \rceil$. Then for every $n \geq n_0$,*

$$R_n^{(k)} > p_{\lceil kn/(k-1) \rceil + t}. \tag{4}$$

Another problem which arises is to find a minimal bound $m = m(k,t)$ such that the inequality (4) holds for all $n \geq m$. For the case $t = 0$, we introduce the following

Definition. For $k > 1$ let

$$N(k) = \min\{m \in \mathbb{N} \mid R_n^{(k)} > p_{\lceil kn/(k-1) \rceil} \text{ for every } n \geq m\}.$$

In Section 3.2 we prove the following theorem giving an explicit formula for $N(k)$.

Theorem B. *If $k \geq 745.8$, then*

$$N(k) = \pi(3k) - 1.$$

Theorem A is supplemented by the following upper bound for nth k-Ramanujan prime.

Theorem C. *Let $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$ and $\varepsilon_1 + \varepsilon_2 \neq 0$. Then for every $n \geq n_1$,*

$$R_n^{(k)} \leq (1 + \varepsilon_1) p_{\lceil (1+\varepsilon_2) kn/(k-1) \rceil}.$$

By [1], there exists a positive constant $\beta_1 = \beta_1(k)$ such that for every sufficiently large n,

$$|R_n^{(k)} - p_{\lfloor kn/(k-1) \rfloor}| < \beta_1 n \log\log n. \tag{5}$$

In Theorem A, we actually obtain a lower bound for $R_n^{(k)} - p_{\lceil kn/(k-1) \rceil}$ improving the lower bound given in (5). The next theorem yields an improvement of the upper bound.

Theorem D. *There exists a positive constant γ, depending on a series of parameters including k, such that for every $n \geq n_2$,*

$$R_n^{(k)} - p_{\lceil kn/(k-1) \rceil} < \gamma n.$$

Let $\pi_k(x)$ be the number of k-Ramanujan primes less than or equal to x. Using PNT, Amersi, Beckwith, Miller, Ronan and Sondow [1] proved that there exists a positive constant $\beta_2 = \beta_2(k)$ such that for every sufficiently large n,

$$\left| \frac{k-1}{k} - \frac{\pi_k(n)}{\pi(n)} \right| \leq \frac{\beta_2 \log\log n}{\log n}. \tag{6}$$

In Section 4 we prove the following two theorems which lead to an improvement of the lower and upper bound in (6).

Theorem E. *If $x \geq R_{N(k)}^{(k)}$ with $N(k)$ defined above, then*

$$\frac{\pi_k(x)}{\pi(x)} < \frac{k-1}{k}.$$

Theorem F. *There exists a positive constant c, depending on a series of parameters including k, such that for every $x \geq n_3$,*

$$\frac{k-1}{k} - \frac{\pi_k(x)}{\pi(x)} < \frac{c}{\log x}.$$

In 2009, Mitra, Paul and Sarkar [9] stated a conjecture concerning the number of primes in certain intervals, namely that

$$\pi(mn) - \pi(n) \geq m - 1$$

for all $m, n \in \mathbb{N}$ with $n \geq \lceil 1.1 \log(2.5m) \rceil$. In Section 5 we confirm the conjecture for large m.

Theorem G. *If m is sufficiently large and $n \geq \lceil 1.1 \log(2.5m) \rceil$, then $\pi(mn) - \pi(n) \geq m - 1$.*

2 Some simple properties of k-Ramanujan primes

We begin with

Proposition 2.1. *The following three properties hold for* $R_n^{(k)}$*.*

(i) *Let* $k_1, k_2 \in \mathbb{R}$ *with* $k_2 > k_1 > 1$*. Then* $R_n^{(k_1)} \geq R_n^{(k_2)}$*.*

(ii) $R_n^{(k)} \geq p_n$ *for every* $n \in \mathbb{N}$ *and every* $k > 1$*.*

(iii) *For each* k*, the sequence* $(R_n^{(k)})_n$ *is strictly increasing.*

Proof. The assertions follow directly from the definition of $R_n^{(k)}$. □

Proposition 2.2. *Let* $k > 1$ *and let* $n \in \mathbb{N}$ *so that* $R_n^{(k)} = p_n$*. Then* $R_m^{(k)} = p_m$ *for every* $m \leq n$*.*

Proof. The assertion follows from the fact that $R_n^{(k)} \in \mathbb{P}$ and Proposition 2.1. □

For the next property we need the following

Lemma 2.3. *Let* $m, n \in \mathbb{N}$*. Then*

$$\pi(m) + \pi(n) \leq \pi(mn).$$

Proof. By [7], we have $\pi(m) + \pi(n) < \pi(m+n)$ for every $m, n \in \mathbb{N}$ with $m, n \geq 2$ and $\max\{m, n\} \geq 6$. Now it is easy to check that the desired inequality holds in the remaining cases. □

Proposition 2.4. *If* $k \geq 2$*, then*

$$R_{\pi(k)}^{(k)} = p_{\pi(k)}.$$

Proof. Let $t = \lfloor k \rfloor$ and $x \geq k$. Let $m \in \mathbb{N}$ be such that $mk \leq x < (m+1)k$. Using Lemma 2.3, we get

$$\pi(x) - \pi\left(\frac{x}{k}\right) \geq \pi(mt) - \pi(m) \geq \pi(k),$$

i.e. $R_{\pi(k)}^{(k)} \leq k < p_{\pi(k)+1}$. Using Proposition 2.1(ii), we obtain the required equality. □

Corollary 2.5. *If* $k \geq 2$*, then* $R_n^{(k)} = p_n$ *for every* $n = 1, \ldots, \pi(k)$*.*

Proof. The claim follows from Proposition 2.2 and Proposition 2.4. □

For $1 < k < 2$ we can give more information on $R_n^{(k)}$.

Proposition 2.6. *We have:*

(i) *If* $1 < k < 5/3$*, then* $R_n^{(k)} > p_n$ *for every* $n \in \mathbb{N}$*.*

(ii) *If* $5/3 \leq k < 2$*, then* $R_n^{(k)} = p_n$ *if and only if* $n = 1$*.*

Proof. (i) If $1 < k < 3/2$, we set $x = 2k$ and obtain $\pi(x) - \pi(x/k) = 0$, i.e. $R_1^{(k)} > 2k > p_1$. It remains to use Proposition 2.1(iii). If $3/2 \leq k < 5/3$, we set $x = 3k$ and proceed as before.

(ii) Let $n = 1$ and $5/3 \leq k < 2$. By Proposition 2.1(ii) we get $p_1 \leq R_1^{(5/3)} = p_1$, i.e. $R_1^{(k)} = p_1$. Let $n \geq 2$. Then $R_2^{(k)} \geq R_2^{(2)} > p_2$ and we use Proposition 2.1(iii) as in the previous case. □

The following property will be useful in Section 4.

Proposition 2.7. *For every* n *and* k*,*

$$\pi(R_n^{(k)}) - \pi\left(\frac{R_n^{(k)}}{k}\right) = n.$$

Proof. This easily follows from the definition of $R_n^{(k)}$. □

Finally, we formulate an interesting property of the k-Ramanujan primes.

Proposition 2.8. *If $p \in \mathbb{P} \setminus \{2\}$, then for every $n \in \mathbb{N}$*

$$R_n^{(k)} \neq kp - 1.$$

Proof. It suffices to consider the case $kp \in \mathbb{N}$. Assume $R_n^{(k)} = kp - 1$ for some $n \in \mathbb{N}$. Since $kp - 1 > 2$, we obtain $kp \notin \mathbb{P}$. Let $r \in \mathbb{R}$ with $0 \leq r < 1$. Using Proposition 2.7, we get

$$\pi(kp + r) - \pi\left(\frac{kp + r}{k}\right) = \pi(R_n^{(k)}) - \pi\left(\frac{R_n^{(k)}}{k}\right) - 1 = n - 1,$$

which contradicts the definition of $R_n^{(k)}$. □

3 Estimates for the nth k-Ramanujan prime

From here on, we use the following notation. Let $m_1 \in \mathbb{N}$ and $s, a_1, \ldots, a_{m_1} \in \mathbb{R}$ with $s \geq 0$. We define

$$A(x) = \sum_{j=1}^{m_1} \frac{a_j}{\log^j x}$$

and $Y_s = Y_s(a_1, \ldots, a_{m_1})$ so that

$$\pi(x) > \frac{x}{\log x - 1 - A(x)} + s \tag{7}$$

for every $x \geq Y_s$. Further, for $m_2 \in \mathbb{N}$ and $b_1, \ldots, b_{m_2} \in \mathbb{R}_{\geq 0}$ we define

$$B(x) = \sum_{j=1}^{m_2} \frac{b_j}{\log^j x}$$

and $X_0 = X_0(b_1, \ldots, b_{m_2})$ so that

$$\pi(x) < \frac{x}{\log x - 1 - B(x)} \tag{8}$$

for every $x \geq X_0$. In addition, let $X_1 = X_1(k, a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2})$ be such that

$$\log k - B(kx) + A(x) \geq 0 \tag{9}$$

for every $x \geq X_1$.

Remark. It is clear that $B(x) > A(x)$ for every $x \geq \max\{X_0, X_1\}$. Hence $b_1 > a_1$.

3.1 A lower bound for the nth k-Ramanujan prime

The theorem below implies that the inequality (2) can be generalized, in view of (3), to k-Ramanujan primes.

Theorem 3.1. *Let $t \in \mathbb{Z}$ with $t > -\lceil k/(k-1) \rceil$ and let $r = (t+1)(k-1)/k$. Then*

$$R_n^{(k)} > p_{\lceil kn/(k-1) \rceil + t}$$

for every $n \in \mathbb{N}$ with

$$n \geq n_0 = \frac{k-1}{k}(\pi(X_2) - t + 1), \tag{10}$$

where $X_2 = X_2(k, t, m_1, m_2, a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2}) = \max\{X_0, kX_1, kY_r\}$.

Proof. Let $x \geq X_2/k$. Then the inequality (9) is equivalent to

$$\frac{x}{\log x - 1 - A(x)} \geq \frac{x}{\log(kx) - 1 - B(kx)}$$

and we get

$$\pi(x) > \frac{x}{\log x - 1 - A(x)} + r \geq \frac{x}{\log kx - 1 - B(kx)} + r > \frac{\pi(kx)}{k} + r. \tag{11}$$

By setting $x = p_{\lceil kn/(k-1) \rceil + t}/k$ in (11), we obtain

$$\pi\left(\frac{1}{k}\, p_{\lceil kn/(k-1) \rceil + t}\right) > \frac{1}{k}\left(\left\lceil \frac{kn}{k-1} \right\rceil + t\right) + r.$$

Hence,

$$\pi(p_{\lceil kn/(k-1) \rceil + t}) - \pi\left(\frac{1}{k}\, p_{\lceil kn/(k-1) \rceil + t}\right) < n,$$

and we apply the definition of $R_n^{(k)}$. □

Corollary 3.2. *We have*

$$\liminf_{n \to \infty} (R_n^{(k)} - p_{\lceil kn/(k-1) \rceil}) = \infty.$$

Proof. From Theorem 3.1, it follows that for every $t \in \mathbb{N}$ there is an $N_0 \in \mathbb{N}$ such that for every $n \geq N_0$,

$$R_n^{(k)} - p_{\lceil kn/(k-1) \rceil} \geq p_{\lceil kn/(k-1) \rceil + t} - p_{\lceil kn/(k-1) \rceil} \geq 2t.$$

This proves our corollary. □

Remark. In 2013, Sondow [18] raised the question whether the sequence $(R_n - p_{2n})_n$ is unbounded. Corollary 3.2 implies that this is indeed the case.

Corollary 3.3. *If* $n \geq \max\{2, (k-1)\pi(X_2)/k\}$, *then*

$$R_n^{(k)} - p_{\lceil kn/(k-1) \rceil} \geq 6.$$

Proof. We set $t = 1$ in Theorem 3.1. Then for every $n \geq (k-1)\pi(X_2)/k$ we obtain

$$R_n^{(k)} \geq p_{\lceil kn/(k-1) \rceil + 2} \geq p_{\lceil kn/(k-1) \rceil + 1} + 2 \geq p_{\lceil kn/(k-1) \rceil} + 4.$$

Since there is no prime triple of the form $(p, p+2, p+4)$ for $p > 3$, we are done. □

To find an explicit value for X_2 in the case $t = 0$, we need the following

Lemma 3.4. *If* $x \geq 470077$, *then*

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x}} + 1.$$

Proof. We consider the function $g(x) = 2.65x - \log^4 x$. Then $g(x) > 0$ for every $x \geq e^7$ and we get

$$\frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{2.65}{\log^2 x}} > \frac{x}{\log x - 1 - \frac{1}{\log x}} + 1$$

for every $x \geq e^7$. By Corollary 3.11 of [2], the inequality $\pi(x) > x/(\log x - 1 - 1/\log x) + 1$ holds for every $x \geq 38168363$. We set $h(x) = x/(\log x - 1 - 1/\log x)$. Then $h'(x) \geq 0$ for every $x \geq 12.8$ and we check with a computer that $\pi(p_i) \geq h(p_{i+1}) + 1$ for every $\pi(470077) \leq i \leq \pi(38168363)$. □

Proposition 3.5. *Let* $X_3 = X_3(k) = \max\{470077k, kr(k)\}$, *where*

$$r(k) = \frac{1}{k}\exp\left(\sqrt{\max\left\{\frac{3.83}{\log k} - 1, 0\right\}}\right).$$

Then

$$R_n^{(k)} > p_{\lceil kn/(k-1)\rceil}$$

for every

$$n \geq \frac{k-1}{k}(\pi(X_3)+1).$$

Proof. We choose $t = 0$ in Theorem 3.1. Then $r = (k-1)/k$. We set $A(x) = 1/\log x$ and $Y_r = 470077$. By Lemma 3.4, we get that the inequality (7) holds for every $x \geq Y_r$. By choosing $b_1 = 1$, $b_2 = 3.83$ and $X_0 = 9.25$, we can use the third inequality in Corollary 3.9 of [2]. Let $x \geq r(k)$. Then it is easy to show that the inequality (9) holds. Now our proposition follows from Theorem 3.1. □

Corollary 3.6. *If* $n \geq 4$, *then*

$$R_n - p_{2n} \geq 6.$$

Proof. We set $t = 1$ and $k = 2$ in Theorem 3.1. Then $r = 1$. From Corollary 3.3 and from the proof of Proposition 3.5, if follows that $R_n - p_{2n} \geq 6$ for all $n \geq \pi(X_3(2))/2 = \pi(\max\{940154, 2r(2)\})/2 = 37098$. We check with a computer that the inequality $R_n - p_{2n} \geq 6$ also holds for every $4 \leq x \leq 37097$. □

Remark. Since $R_2 - p_4 = 4$ and $R_3 - p_6 = 4$, Corollary 3.6 gives a positive answer to the question raised by Sondow [18], whether $\min\{R_n - p_{2n} \mid n \geq 2\} = 4$.

3.2 An explicit formula for $N(k)$

In the introduction we defined $N(k)$ to be the smallest positive integer so that

$$R_n^{(k)} > p_{\lceil kn/(k-1)\rceil}$$

for every $n \geq N(k)$. By Proposition 3.5, we get

$$N(k) \leq \left\lceil \frac{k-1}{k}(\pi(\max\{470077k, kr(k)\})+1)\right\rceil$$

for every $k > 1$. We can significantly improve this inequality in the following case.

Theorem 3.7. *If* $k \geq 745.8$, *then*

$$N(k) \leq \pi(3k) - 1.$$

Proof. We have $R^{(k)}_{\pi(3k)-1} > 3k$. Since $3k \geq p_{\pi(3k)}$, we obtain by Proposition 2.2 that

$$R_n^{(k)} > p_{n+1} \tag{12}$$

for every $n \geq \pi(3k) - 1$. We set $A(x) = -7.1/\log x$, $s = 1$ and $Y_1 = 3$. Then, as in the proof of Lemma 3.4, we obtain that the inequality (7) holds for every $x \geq Y_1$. By setting $B(x) = 1.17/\log x$ and $X_0 = 5.43$ and using Corollary 3.9 of [2], we see that the inequality (8) holds. Let

$$\widetilde{r}(k) = \exp\left(\sqrt{7.1 + \frac{1}{4}\left(\log k - \frac{8.27}{\log k}\right)^2} - \frac{1}{2}\left(\log k - \frac{8.27}{\log k}\right)\right).$$

It is easy to see that $x \geq \widetilde{r}(k)$ implies the inequality (9). By Theorem 3.1, we obtain

$$N(k) \leq \left\lceil \frac{k-1}{k}(\pi(X_4)+1)\right\rceil, \tag{13}$$

where $X_4 = X_4(k) = \max\{5.43, 3k, k\widetilde{r}(k)\}$. Since $\widetilde{r}(k)$ is decreasing, from $\widetilde{r}(745.8) \leq 2.999966$ we get that $\widetilde{r}(k) \leq 3$. Hence, $X_4 = 3k$ for every $k \geq 745.8$. Since $\pi(3k) + 1 \leq k$ for every $k \geq 745.8$, we obtain $N(k) \leq \pi(3k) + 1$ by (13). Finally, we apply (12). □

Next, we find a lower bound for $N(k)$.

Proposition 3.8. *For every $k > 1$,*

$$N(k) > \pi(k).$$

Proof. First, let $k \geq 2$. Using Proposition 2.4, we get

$$R^{(k)}_{\pi(k)} < p_{\lceil k\pi(k)/(k-1)\rceil}. \tag{14}$$

Hence, $N(k) > \pi(k)$ for every $k \geq 2$. The asserted inequality clearly holds for every $1 < k < 2$. □

In order to prove a sharper lower bound for $N(k)$, see Theorem 3.11, we need the following lemma.

Lemma 3.9. *Let $r, s \in \mathbb{R}$ with $r > s > 0$. If $t \geq s/r \cdot R^{(r/s)}_{\pi(r)}$, then*

$$\pi(r) + \pi(t) \leq \pi\left(\frac{rt}{s}\right).$$

Proof. Since $rt/s \geq R^{(r/s)}_{\pi(r)}$, the claim follows from the definition of $R^{(k)}_n$. □

Proposition 3.10. *If $m, n \in \mathbb{N}$ with $m, n \geq 5$ and $\max\{m, n\} \geq 18$, then*

$$\pi(m) + \pi(n) \leq \pi\left(\frac{mn}{3}\right).$$

Proof. Without loss of generality, let $m \geq n$. First, we consider the case $m \geq n \geq 20$. By [20], we have $\pi(x) < 8x/(5\log x)$ for every $x > 1$. Using an estimate from [15] for $\pi(x)$, we get

$$\pi\left(\frac{mn}{3}\right) - \pi(m) \geq \frac{mn}{3\log(mn/3)} - \frac{8m}{5\log m} \geq \frac{8m}{5\log m} > \pi(n).$$

So the proposition is proved, when $m \geq n \geq 20$. Now let $m \geq 18$ and $\min\{m, 20\} \geq n \geq 5$. We have:

r	20	19	18
$\lceil 3/r \cdot R^{(r/3)}_{\pi(r)}\rceil$	5	5	4

We apply Lemma 3.9 with $s = 3$, $r = m$ and $t = n$. □

Theorem 3.11. *For every $k > 1$,*

$$N(k) \geq \pi(3k) - 1.$$

Proof. For every $1 < k < 5/3$ the claim is obviously true. For every $5/3 \leq k < 7/3$, we have

$$R^{(k)}_{\pi(3k)-2} \leq R^{(5/3)}_1 < p_{\lceil k(\pi(3k)-2)/(k-1)\rceil};$$

i.e., $N(k) > \pi(3k) - 2$. Similarly, for every $p_i/3 \leq k < p_{i+1}/3$, where $i = 4, \ldots, 8$, we check that

$$R^{(k)}_{\pi(3k)-2} \leq p_{\lceil k(\pi(3k)-2)/(k-1)\rceil}.$$

Hence our theorem is proved for every $1 < k < 19/3$. Now, let $k \geq 19/3$. For $p_{\pi(3k)-1} \leq x < 3k$ and for $3k \leq x < 5k$ it is easy to see that $\pi(x) - \pi(x/k) \geq \pi(3k) - 2$. So let $x \geq 5k$ and let $m \in \mathbb{N}$ be such that $m \geq 5$ and $mk \leq x < (m+1)k$. Since $3k \geq 19$, we use Proposition 3.10 to get the inequality

$$\pi(x) - \pi\left(\frac{x}{k}\right) \geq \pi\left(\frac{m(3k)}{3}\right) - \pi(m) \geq \pi(3k).$$

Hence, altogether we have

$$R^{(k)}_{\pi(3k)-2} \leq p_{\pi(3k)-1} \leq p_{\lceil k(\pi(3k)-2)/(k-1)\rceil} \tag{15}$$

and therefore $N(k) > \pi(3k) - 2$. □

Remark. The proof of Theorem 3.11 yields $R^{(k)}_{\pi(3k)} \leq p_{\pi(5k)}$ for every $k \geq 19/3$.

From Theorem 3.7 and Theorem 3.11, we obtain the following explicit formula for $N(k)$.

Corollary 3.12. *If $k \geq 745.8$, then*

$$N(k) = \pi(3k) - 1.$$

3.3 An explicit formula for $N_0(k)$

By replacing ">" with "≥" in the definition of $N(k)$, we get the following

Definition. For $k > 1$, let

$$N_0(k) = \min\{m \in \mathbb{N} \mid R_n^{(k)} \geq p_{\lceil kn/(k-1)\rceil} \text{ for every } n \geq m\}.$$

Since $N_0(k) > \pi(k)$ for every $1 < k < 2$, it follows from (14) that

$$N_0(k) > \pi(k)$$

is fulfilled for every $k > 1$. In the following case we obtain a sharper lower bound for $N_0(k)$.

Theorem 3.13. *If $k \geq 11/3$, then*

$$N_0(k) \geq \pi(2k).$$

Proof. First, we show that

$$\pi(x) - \pi\left(\frac{x}{k}\right) \geq \pi(2k) - 1 \tag{16}$$

for every $x \geq p_{\pi(2k)-1}$. For $p_{\pi(2k)-1} \leq x < 2k$ and for $2k \leq x < 3k$, the inequality (16) is obviously true. Let $3k \leq x < 5k$. Since $\pi(3t) - \pi(2t) \geq 1$ for every $t \geq 11/3 = 1/3 \cdot R_1^{(3/2)}$, it follows $\pi(x) - \pi(x/k) \geq \pi(2k) - 1$. So let $x \geq 5k$ and let $l \in \mathbb{N}$ be such that $l \geq 5$ and $lk \leq x < (l+1)k$. Similarly to the proof of Proposition 3.10, we get that

$$\pi(m) + \pi(n) \leq \pi\left(\frac{mn}{2}\right) \tag{17}$$

for every $m, n \geq 4$ with $\max\{m, n\} \geq 6$. Since $2k > 7$, using (17) we obtain the inequality

$$\pi(x) - \pi\left(\frac{x}{k}\right) \geq \pi\left(\frac{2lk}{2}\right) - \pi(l) \geq \pi(2k). \tag{18}$$

Hence, we proved that the inequality (16) holds for every $x \geq p_{\pi(2k)-1}$. So,

$$R_{\pi(2k)-1}^{(k)} \leq p_{\pi(2k)-1} < p_{\lceil k(\pi(2k)-1)/(k-1)\rceil}, \tag{19}$$

which gives the required inequality. □

Using (19), we get an improvement of Corollary 2.5.

Corollary 3.14. *If $k \geq 11/3$, then $R_n^{(k)} = p_n$ for every $1 \leq n \leq \pi(2k) - 1$.*

Proof. Follows from Proposition (2.1)(ii), the left inequality in (19) and Proposition 2.2. □

To prove an upper bound for $N_0(k)$, the following proposition will be useful.

Proposition 3.15. *If $k \geq 29/3$, then*

$$R_{\pi(2k)}^{(k)} = p_{\pi(2k)+1}.$$

Proof. Since $\pi(2k) - \pi(2k/k) < \pi(2k)$ and $2k < p_{\pi(2k)+1}$, we have $R_{\pi(2k)}^{(k)} \geq p_{\pi(2k)+1}$ for every $k > 1$. To prove $R_{\pi(2k)}^{(k)} \leq p_{\pi(2k)+1}$, it suffices to show that

$$\pi(x) - \pi\left(\frac{x}{k}\right) \geq \pi(2k) \tag{20}$$

for every $x \geq p_{\pi(2k)+1}$. It is clear that (20) is true for every $p_{\pi(2k)+1} \leq x < 3k$. Let $3k \leq x < 5k$. We have $\pi(3t) - \pi(2t) \geq 2$ for every $t \geq 29/3 = 1/3 \cdot R_2^{(3/2)}$ and thus (20) holds. By (18) we already have that the inequality (20) also holds for every $x \geq 5k$. □

We can show more than in Corollary 3.14 for the following case.

Corollary 3.16. *If $k \geq 29/3$, then:*

(i) $R_n^{(k)} = p_n$ *if and only if* $1 \leq n \leq \pi(2k) - 1$.

(ii) $R_n^{(k)} = p_{n+1}$ *if and only if* $\pi(2k) \leq n \leq \pi(3k) - 2$.

Proof. (i) From Proposition 3.15, we get $R_n^{(k)} > p_n$ for every $n \geq \pi(2k)$ and then use Corollary 3.14.

(ii) Let $\pi(2k) \leq n \leq \pi(3k) - 2$. By (i) and (15), we obtain $R_n^{(k)} = p_{n+1}$. Now let $R_n^{(k)} = p_{n+1}$. Since

$$\pi(p_{\pi(3k)}) - \pi\left(\frac{p_{\pi(3k)}}{k}\right) \leq \pi(3k) - \pi(3) < \pi(3k) - 1,$$

we obtain

$$R_{\pi(3k)-1}^{(k)} > p_{\pi(3k)}. \tag{21}$$

Hence, $n \leq \pi(3k) - 2$. By (i), we get $n \geq \pi(2k)$. □

Remark. Similarly to the proof of (21), we obtain in general that for every real $r \geq 2/k$,

$$R_{\pi(rk)-\pi(r)+1}^{(k)} > p_{\pi(rk)}.$$

Remark. Corollary 3.16 implies that for every $k \geq 29/3$ the prime numbers $p_{\pi(2k)}$ and $p_{\pi(3k)-1}$ are not k-Ramanujan primes.

Corollary 3.17. *For each $m \in \mathbb{N}$ there exists $k = k(m) \geq 29/3$ such that $p_{\pi(2k)+1}, \ldots, p_{\pi(2k)+m}$ are all k-Ramanujan primes.*

Proof. By PNT, we obtain $\pi(3k) - 2 - \pi(2k) \to \infty$ as $k \to \infty$. Then use Corollary 3.16(ii). □

The next lemma provides an upper bound for $N_0(k)$.

Lemma 3.18. *Let $X_5 = X_5(k) = \max\{X_0, kX_1, kY_0\}$. Then the inequality*

$$R_n^{(k)} \geq p_{\lceil kn/(k-1) \rceil}$$

holds for every

$$n \geq \frac{k-1}{k}(\pi(X_5) + 2).$$

Proof. We just set $t = -1$ in Theorem 3.1. □

The inequality in Theorem 3.13 becomes an equality in the following case.

Theorem 3.19. *If $k \geq 143.7$, then*

$$N_0(k) = \pi(2k).$$

Proof. We set $A(x) = -3.3/\log x$, $s = 0$ and $Y_0 = 2$. Similarly to the proof of Corollary 3.11 from [2], we get that (7) is fulfilled for every $x \geq Y_0$. By setting $B(x) = 1.17/\log x$ and $X_0 = 5.43$ and using Corollary 3.9 from [2], we see that the inequality (8) is fulfilled for every $x \geq X_0$. Let

$$z(k) = \exp\left(\sqrt{3.3 + \frac{1}{4}\left(\log k - \frac{4.47}{\log k}\right)^2} - \frac{1}{2}\left(\log k - \frac{4.47}{\log k}\right)\right).$$

It is easy to show that $x \geq z(k)$ is equivalent to the inequality (9). By setting $X_6 = X_6(k) = \max\{2k, 5.43, kz(k)\}$ and using Lemma 3.18, we get that

$$N_0(k) \leq \left\lceil \frac{k-1}{k}(\pi(X_6) + 2) \right\rceil. \tag{22}$$

In the proof of Theorem 3.7, we showed that $\widetilde{r}(k) \leq 3$ for every $k \geq 745.8$. Analogously, we get $z(k) \leq 2$ for every $k \geq 143.7$. Hence we obtain $X_6 = 2k$ and therefore $N_0(k) \leq \pi(2k) + 2$ for every $k \geq 143.7$. Proposition 3.15 and Theorem 3.13 finish the proof. □

3.4 An upper bound for the nth k-Ramanujan prime

After finding a lower bound for the nth k-Ramanujan prime, we find an upper bound by using the following two propositions, where $\Upsilon_k(x)$ is defined by

$$\Upsilon_k(x) = \Upsilon_{k,a_1,\ldots,a_{m_1},b_1,\ldots,b_{m_2}}(x) = \frac{x}{\log x - 1 - A(x)} \left(1 - \frac{1}{k} - \frac{1}{k} \frac{\log k - A(x) + B(x/k)}{\log(x/k) - 1 - B(x/k)} \right).$$

Proposition 3.20. *If $x \geq \max\{Y_0, kX_0\}$, then*

$$\pi(x) - \pi\left(\frac{x}{k}\right) > \Upsilon_k(x).$$

Proof. We have

$$\pi(x) - \pi\left(\frac{x}{k}\right) > \frac{x}{\log x - 1 - A(x)} - \frac{x/k}{\log(x/k) - 1 - B(x/k)}$$

and see that the term on the right hand side is equal to $\Upsilon_k(x)$. □

Proposition 3.21. *For every sufficiently large x, the derivative $\Upsilon_k'(x) > 0$.*

Proof. We set

$$F(x) = F_{a_1,\ldots,a_{m_1}}(x) = \frac{x}{\log x - 1 - A(x)}$$

and

$$G(x) = G_{k,a_1,\ldots,a_{m_1},b_1,\ldots,b_{m_2}}(x) = 1 - \frac{1}{k} - \frac{1}{k} \frac{\log k - A(x) + B(x/k)}{\log(x/k) - 1 - B(x/k)}.$$

It is clear that there exists an $X_7 = X_7(k, m_1, m_2, a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2})$ such that $G(x) > 0$ for every $x \geq X_7$. Since $k > 1$, we obtain $B(x/k) > B(x) > A(x)$ as well as $\log(x/k) \leq \log x$ and $\log(x/k) - 1 - B(x/k) > 0$ for every $x \geq \max\{Y_0, kX_0\}$. It follows that

$$G'(x) > \frac{1}{k(\log(x/k) - 1 - B(x/k))} \left(\sum_{j=1}^{m_2} \frac{j \cdot b_j}{\log^{j+1} x} - \sum_{i=1}^{m_1} \frac{i \cdot a_i}{\log^{i+1} x} \right) \tag{23}$$

for every $x \geq \max\{Y_0, kX_0\}$. Since $b_1 > a_1$, there is an $X_8 = X_8(m_1, m_2, a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2})$ such that

$$\sum_{j=1}^{m_2} \frac{j \cdot b_j}{\log^{j+1} x} - \sum_{i=1}^{m_1} \frac{i \cdot a_i}{\log^{i+1} x} \geq 0$$

for every $x \geq X_8$. Hence, $G'(x) > 0$ for every $x \geq \max\{Y_0, kX_0, X_8\}$. We have $F(x) > 0$ for every $x \geq Y_0$. Further, there exists an $X_9 = X_9(m_1, a_1, \ldots, a_{m_1})$ such that

$$\log x - 2 - A(x) - \sum_{i=1}^{m_1} \frac{i \cdot a_i}{\log^{i+1} x} > 0$$

for every $x \geq X_9$. Therefore,

$$F'(x) = \frac{1}{(\log x - 1 - A(x))^2} \left(\log x - 2 - A(x) - \sum_{i=1}^{m_1} \frac{i \cdot a_i}{\log^{i+1} x} \right) > 0$$

for every $x \geq \max\{Y_0, X_9\}$. So, for every $x \geq \max\{Y_0, kX_0, X_7, X_8, X_9\}$, we get $\Upsilon_k'(x) = F'(x)G(x) + F(x)G'(x) > 0$. □

Now, let $m_1 = m_2 = 1$. By Proposition 3.21, there exists an $X_{10} = X_{10}(k, a_1, b_1)$ such that $\Upsilon_k'(x) > 0$ for every $x \geq X_{10}$. Let $X_{11} = X_{11}(b_1) \in \mathbb{N}$ be such that

$$p_n \geq n(\log p_n - 1 - b_1/\log p_n)$$

for every $n \geq X_{11}$. Clearly, $X_{11} \leq \pi(X_0)+1$. Let $\varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$ be such that $\varepsilon_1+\varepsilon_2 \neq 0$. We define

$$\varepsilon = \begin{cases} \varepsilon_1 & \text{if } \varepsilon_1 \neq 0, \\ \varepsilon_2 & \text{otherwise} \end{cases}$$

and

$$\lambda = \frac{\varepsilon}{2} + \varepsilon_2 \cdot \operatorname{sign}(\varepsilon_1)\left(1+\frac{\varepsilon}{2}\right).$$

Let $S = S(k, a_1, b_1, X_0, \varepsilon_1, \varepsilon_2)$ be defined by

$$S = \exp\left(\sqrt{b_1 + \frac{2(1+\varepsilon)}{(k-1)\varepsilon}\left(b_1 - a_1 + \frac{a_1 \log k}{\log kX_0}\right) + \left(\frac{1}{2} + \frac{(1+\varepsilon)\log k}{(k-1)\varepsilon}\right)^2} + \frac{1}{2} + \frac{(1+\varepsilon)\log k}{(k-1)\varepsilon}\right),$$

and let $T = T(a_1, b_1, \varepsilon_1, \varepsilon_2)$ be defined by

$$T = \exp\left(\sqrt{b_1 + \frac{b_1 - a_1}{\lambda} + \frac{a_1 \log(1+\varepsilon_1)}{\lambda} + \left(\frac{1}{2} + \frac{\log(1+\varepsilon_1)}{2\lambda}\right)^2} + \frac{1}{2} + \frac{\log(1+\varepsilon_1)}{2\lambda}\right).$$

By defining $X_{12} = X_{12}(k, a_1, b_1, Y_0, X_0, \varepsilon_1, \varepsilon_2, X_{10})$ by

$$X_{12} = \max\left\{\frac{Y_0}{1+\varepsilon_1}, \frac{kX_0}{1+\varepsilon_1}, \frac{k\,S(k, a_1, b_1, \varepsilon_1, \varepsilon_2)}{1+\varepsilon_1}, T(a_1, b_1, \varepsilon_1, \varepsilon_2), \frac{X_{10}}{1+\varepsilon_1}\right\},$$

we get, in view of (3), the following result.

Theorem 3.22. *The inequality*

$$R_n^{(k)} \leq (1+\varepsilon_1) p_{\lceil (1+\varepsilon_2)kn/(k-1)\rceil}$$

holds for every

$$n \geq n_1 = \frac{k-1}{k(1+\varepsilon_2)} \max\{\pi(X_{12})+1, X_{11}\}. \tag{24}$$

Proof. For convenience, we write $t = t(n, k, \varepsilon_2) = \lceil (1+\varepsilon_2)nk/(k-1)\rceil$. Using Proposition 3.20, we obtain

$$\pi(x) - \pi\left(\frac{x}{k}\right) > \Upsilon((1+\varepsilon_1)p_t)$$

for every $x \geq (1+\varepsilon_1)p_t$. So to prove the claim, it is enough to show that

$$\Upsilon((1+\varepsilon_1)p_t) \geq n. \tag{25}$$

For this, we first show that

$$1 - \frac{1}{k} - \frac{1}{k}\,\frac{\log k - a_1/\log((1+\varepsilon_1)p_t) + b_1/\log((1+\varepsilon_1)p_t/k)}{\log((1+\varepsilon_1)p_t/k) - 1 - b_1/\log((1+\varepsilon_1)p_t/k)} > \frac{k-1}{k}\left(1 - \frac{\varepsilon}{2(1+\varepsilon)}\right). \tag{26}$$

We have $(1+\varepsilon_1)p_t/k \geq S(k, a_1, b_1, X_0, \varepsilon_1, \varepsilon_2)$ and therefore

$$\frac{\varepsilon(k-1)}{2k(1+\varepsilon)}\left(\log((1+\varepsilon_1)p_t/k) - 1 - \frac{b_1}{\log((1+\varepsilon_1)p_t/k)}\right) > \frac{\log k}{k} + \frac{b_1}{k\log((1+\varepsilon_1)p_t/k)} - \frac{a_1}{k\log(1+\varepsilon_1)p_t}.$$

From this inequality, we obtain (26). So for the proof of (25), it suffices to show that the inequality

$$\frac{k-1}{k}\left(1 - \frac{\varepsilon}{2(1+\varepsilon)}\right) \cdot \frac{(1+\varepsilon_1)p_t}{\log(1+\varepsilon_1)p_t - 1 - a_1/\log((1+\varepsilon_1)p_t)} \geq n \tag{27}$$

is fulfilled. Since $p_t \geq T(a_1, b_1, \varepsilon_1, \varepsilon_2)$, we get

$$\frac{k-1}{k}\left(1 - \frac{\varepsilon}{2(1+\varepsilon)}\right) \cdot \frac{(1+\varepsilon_1)\,t\,(\log p_t - 1 - b_1/\log(p_t))}{\log((1+\varepsilon_1)p_t) - 1 - a_1/\log((1+\varepsilon_1)p_t)} \geq n. \tag{28}$$

Since $t \geq X_{11}$, we have $p_t > t(\log p_t - 1 - b_1/\log(p_t))$. Using (28), we get (27) and therefore (25). □

Now, let $m_1 = m_2 = 1, a_1 = 1$ and $b_1 = 1.17$. In the next lemma, we determine an explicit $X_{13} = X_{13}(k)$ such that $\Upsilon'_k(x) > 0$ for every $x \geq X_{13}$.

Lemma 3.23. *Let $X_{13} = X_{13}(k) = \max\{kX_{14}, e^{2.547}, 5.43k\}$, where*

$$X_{14} = X_{14}(k) = \exp\left(\sqrt{1.17 + \frac{1.17}{k-1} + \left(\frac{1}{2} + \frac{\log k}{2(k-1)}\right)^2} + \frac{1}{2} + \frac{\log k}{2(k-1)}\right).$$

Then, $\Upsilon'_k(x) > 0$ for every $x \geq X_{13}$.

Proof. We set $F(x) = x/(\log x - 1 - 1/\log x)$ and

$$G(x) = \frac{k-1}{k} - \frac{1}{k}\,\frac{\log k - 1/\log x + 1.17/\log(x/k)}{\log(x/k) - 1 - 1.17/\log(x/k)}.$$

We have $F'(x) \geq 0$ for every $x \geq e^{2.547}$. Since $F(e^{2.547}) > 0$, we get that $F(x) > 0$ for every $x \geq e^{2.547}$. For every $x \geq 5.43k$ we have $\log(x/k) - 1 - 1.17/\log(x/k) > 0$ and, using (23) we obtain $G'(x) > 0$. Further, it is easy to see that $x \geq kX_{14}$ implies $G(x) > 0$ and we get $\Upsilon'_k(x) = F'(x)G(x) + F(x)G'(x) > 0$ for every $x \geq X_{13}$. □

For Ramanujan primes, we obtain the following result.

Proposition 3.24. *If $t > 48/19$, then for every $n \in \mathbb{N}$*

$$R_n \leq p_{\lceil tn \rceil}.$$

Proof. By [2], we choose $Y_0 = 468049$ and $X_0 = 5.43$, and by Lemma 3.23 we choose $X_{10} = X_{13}$ in Theorem 3.22. We have $p_n > n(\log p_n - 1 - 1.17/\log p_n)$ for every $n \geq 4$. Since this inequality is also true for $1 \leq n \leq 3$, we choose $X_{11} = 1$ in Theorem 3.22. Then we get

$$R_n^{(k)} \leq (1 + \varepsilon_1) p_{\lceil (1+\varepsilon_2)kn/(k-1) \rceil}$$

for every

$$n \geq \frac{k-1}{k(1+\varepsilon_2)}(\pi(X_{15}) + 1),$$

where $X_{15} = X_{15}(k, \varepsilon_1, \varepsilon_2) = X_{12}(k, 1, 1.17, 468049, e^{2.547}, \varepsilon_1, \varepsilon_2, X_{13})$. Let $s = 48/19$ and $t > s$. We set $k = 2$, $\varepsilon_1 = 0$ and $\varepsilon_2 = 5/19$ in Theorem 3.22, and we get $R_n \leq p_{\lceil sn \rceil}$ for every $n \geq 19536$. By using a computer, we check that $R_n \leq p_{\lceil sn \rceil}$ for every $20 \leq n \leq 19535$ and for every $1 \leq n \leq 18$. For $n = 19$, we have $p_{\lceil 19s \rceil} < R_{19} = p_{49} \leq p_{\lceil 19t \rceil}$. □

Remark. By Proposition 3.24, we obtain that $R_n \leq p_{\lceil 2.53n \rceil}$ for every $n \in \mathbb{N}$, which improves the current best upper bound $R_n \leq 41p_{3n}/47$ found by Nicholson, Noe and Sondow [12] for every $n \geq 11$.

4 On the difference $R_n^{(k)} - p_{\lceil nk/(k-1) \rceil}$

Another question that arises in view of (3), is the size of

$$R_n^{(k)} - p_{\lceil nk/(k-1) \rceil}. \tag{29}$$

In Proposition 3.5, we yield a lower bound for (29), which improves the lower bound in (5). The goal in this section is to improve the upper bound in (5). In order to do this, we set $A(x) = 0$ and $B(x) = b_1/\log x$. By [3], we choose $Y_0 = 5393$. Let ε_1, ε_2, δ_1 and δ_2 all be positive and let

$$\eta(k) = k\left(\sqrt{b_1\left(1 + \frac{1}{\delta_1}\right) + \left(\frac{1}{2} + \frac{\log k}{2\delta_1}\right)^2} + \frac{1}{2} + \frac{\log k}{2\delta_1}\right).$$

In addition, we set $X_{16} = X_{16}(k, b_1, \delta_1, \delta_2) = \max\{7477, kX_0, \eta(k), ke^{b_1/\delta_2}\}$. As in the proof of Lemma 3.23, we get that $\Upsilon_k'(x) \geq 0$ for every $x \geq X_{17} = X_{17}(k) = \max\{5393, kX_0, kX_{14}\}$. Further, let

$$X_{18} = X_{18}(k, b_1, \varepsilon_2) = X_{12}(k, 0, b_1, 5393, X_0, 0, \varepsilon_2, X_{17})$$

as well as

$$X_{19} = X_{19}(k, b_1, X_0, \varepsilon_2, \delta_1, \delta_2) = \max\left\{\pi(X_{16}) + 1, \frac{k-1}{k(1+\varepsilon_2)}(\pi(X_{18}) + 1), \frac{(k-1)X_{11}}{k(1+\varepsilon_2)}\right\}. \tag{30}$$

4.1 On the difference $kn \log R_n^{(k)}/(k-1) - R_n^{(k)}$

We consider the difference

$$\frac{kn}{k-1} \log R_n^{(k)} - R_n^{(k)}.$$

The results below for this difference will be useful to find an upper bound of the difference (29).

Proposition 4.1. *If $n \geq X_{19}$, then*

$$\frac{kn}{k-1} \log R_n^{(k)} - R_n^{(k)} > \left(1 - \frac{(1+\varepsilon_2)(1+\delta_1)(\log k + \delta_2)}{k-1}\right) \frac{kn}{k-1}.$$

Proof. Using an estimate for $\pi(x)$ proved by Dusart [3], we get

$$\pi(x) - \pi\left(\frac{x}{k}\right) > \frac{(k-1)x}{k(\log x - 1)} - \frac{x}{k(\log x - 1)} \cdot \frac{\log k + b_1/\log(x/k)}{\log x/k - 1 - b_1/\log(x/k)} \tag{31}$$

for every $x \geq \max\{5393, kX_0\}$. Since $\log x - 1 \leq (1+\delta_1)(\log x/k - 1 - b_1/\log(x/k))$ for every $x \geq \eta(k)$, we can use (31) and the inequality $b_1/\log(x/k) \leq \delta_2$, which is fulfilled for every $x \geq ke^{b_1/\delta_2}$, to see that

$$\pi(x) - \pi\left(\frac{x}{k}\right) > \frac{(k-1)x}{k(\log x - 1)} - \frac{(1+\delta_1)(\log k + \delta_2)}{k} \cdot \frac{x}{(\log x - 1)^2} \tag{32}$$

for every $x \geq X_{16}$. We choose $x = R_n^{(k)}$ in (32) and, using Proposition 2.7, we get the inequality

$$\frac{kn}{k-1} \log R_n^{(k)} - R_n^{(k)} > \frac{kn}{k-1} - \frac{(1+\delta_1)(\log k + \delta_2)}{k-1} \cdot \frac{R_n^{(k)}}{\log R_n^{(k)} - 1}. \tag{33}$$

From Lemma 3.4, it follows that

$$\pi(x) > \frac{x}{\log x - 1} + 1 \tag{34}$$

for every $x \geq 470077$. We check with a computer that (34) also holds for every $7477 \leq x \leq 470077$. Using (33) and (34), we get

$$\frac{kn}{k-1} \log R_n^{(k)} - R_n^{(k)} > \frac{kn}{k-1} - \frac{(1+\delta_1)(\log k + \delta_2)}{k-1}(\pi(R_n^{(k)}) - 1). \tag{35}$$

By Theorem 3.22, we obtain $\pi(R_n^{(k)}) \leq (1+\varepsilon_2)nk/(k-1) + 1$ and then use (35). □

Corollary 4.2. *Let ε_2, δ_1 and δ_2 all be positive so that*

$$(1+\varepsilon_2)(1+\delta_1)(\log k + \delta_2) < k - 1.$$

If $n \geq X_{19}$, where X_{19} is defined by (30), then

$$\frac{kn}{k-1} \log R_n^{(k)} > R_n^{(k)}.$$

Proof. Follows directly from Proposition 4.1. □

Remark. Nicholson [11] proved that $R_n < 2n \log R_n$ is fulfilled for every $n \geq 33$. Corollary 4.2 generalizes the result of Nicholson to k-Ramanujan primes.

Remark. Amersi, Beckwith, Miller, Ronan and Sondow [1] showed that there exists a positive constant $c = c(k)$ such that

$$\left| \frac{kn}{k-1} \log R_n^{(k)} - R_n^{(k)} \right| \leq \frac{cR_n^{(k)}}{\log R_n^{(k)}} \tag{36}$$

for every sufficiently large n. With Corollary 4.2, we obtain an improvement of the lower bound in (36). We end this section by finding an upper bound for $kn \log(R_n^{(k)})/(k-1) - R_n^{(k)}$.

Proposition 4.3. *Let* $\varepsilon > 0$. *If*

$$n \geq \frac{k-1}{k}(\pi(X_{23}) + 1),$$

where

$$X_{23} = X_{23}(k, b_1, \varepsilon) = \max\{X_0, 5393k, e^{b_1/\varepsilon}, e^{b_1/\log k}, X_2(k, 0, 1, 1, 0, b_1)\},$$

then

$$\frac{kn}{k-1} \log R_n^{(k)} - R_n^{(k)} < \left(1 - \frac{\log k - \varepsilon k}{k-1}\right) \frac{kn}{k-1}.$$

Proof. Using an estimate for $\pi(x)$ given by Dusart [3], we obtain

$$\pi(x) - \pi\left(\frac{x}{k}\right) < \frac{(k-1)x}{k(\log x - 1 - b_1/\log x)} - \frac{x}{k(\log x - 1 - b_1/\log x)} \cdot \frac{\log k - b_1/\log x}{\log x/k - 1}$$

for every $x \geq \max\{5393k, X_0\}$. Since $\log x - 1 - b_1/\log x \geq \log(x/k) - 1$ for every $x \geq e^{b_1/\log k}$, we get

$$\pi(x) - \pi\left(\frac{x}{k}\right) < \frac{(k-1)x}{k(\log x - 1 - b_1/\log x)} - \frac{x(\log k - b_1/\log x)}{k(\log x - 1 - b_1/\log x)^2} \tag{37}$$

for every $x \geq \max\{5393k, X_0, e^{b_1/\log k}\}$. We set $x = R_n^{(k)}$ in (37) and, using Proposition 2.7, get

$$R_n^{(k)} - \frac{kn}{k-1} \log R_n^{(k)} > \frac{\log k - b_1/\log R_n^{(k)}}{k-1} \cdot \frac{R_n^{(k)}}{\log R_n^{(k)} - 1 - b_1/\log R_n^{(k)}} - \frac{kn}{k-1} - \frac{kn}{k-1} \cdot \frac{b_1}{\log R_n^{(k)}}. \tag{38}$$

Since $R_n^{(k)} \geq X_0$ and $\pi(R_n^{(k)}) \geq \pi(p_{\lceil nk/(k-1)\rceil}) \geq nk/(k-1)$, it follows from (38) that

$$R_n^{(k)} - \frac{kn}{k-1} \log R_n^{(k)} > \frac{\log k - b_1/\log R_n^{(k)}}{k-1} \cdot \frac{kn}{k-1} - \frac{kn}{k-1} - \frac{kn}{k-1} \cdot \frac{b_1}{\log R_n^{(k)}}.$$

It remains to notice that $b_1/\log R_n^{(k)} < \varepsilon$. □

4.2 An upper bound for $R_n^{(k)} - p_{\lceil nk/(k-1)\rceil}$

Now, we find an upper bound for (29) which improves the upper bound in (5). We define

$$X_{20} = X_{20}(k, b_1, \varepsilon_1) = X_{12}(k, 0, b_1, 5393, X_0, \varepsilon_1, 0, X_{17}).$$

Let $\varepsilon_3 > 0$ and let $X_{21} = X_{21}(\varepsilon_3)$ be such that $\log\log x < \varepsilon_3 \log x$ for every $x \geq X_{21}$. By setting

$$\gamma = \gamma(k, \varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2) = \left(\frac{(1+\varepsilon_2)(1+\delta_1)(\log k + \delta_2)}{k-1} + \log((1+\varepsilon_1)(1+\varepsilon_3)) \right) \frac{k}{k-1} \tag{39}$$

and

$$X_{22} = X_{22}(k, b_1, X_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2) = \max\left\{ \frac{k-1}{k} X_{11}, X_{19}, \frac{k-1}{k}(\pi(X_{20}) + 1), \frac{k-1}{k} X_{21} \right\},$$

we obtain the following result.

Theorem 4.4. *If $n \geq n_2 = X_{22}$, then*

$$R_n^{(k)} - p_{\lceil nk/(k-1) \rceil} < \gamma n.$$

Proof. By Theorem 3.22, we get the inequality $R_n^{(k)} \leq (1 + \varepsilon_1) p_{\lceil nk/(k-1) \rceil}$. By [15], we have

$$p_n \leq n(\log n + \log\log n) \tag{40}$$

for every $n \geq 20$. Hence,

$$\frac{kn}{k-1} \log R_n^{(k)} \leq \frac{kn}{k-1} \log(1+\varepsilon_1) + \frac{kn}{k-1} \log \left\lceil \frac{kn}{k-1} \right\rceil + \frac{kn}{k-1} \log \left(\log \left\lceil \frac{kn}{k-1} \right\rceil + \log\log \left\lceil \frac{kn}{k-1} \right\rceil \right).$$

Since $nk/(k-1) \geq X_{21}$, we get

$$\frac{kn}{k-1} \log R_n^{(k)} \leq \frac{kn}{k-1} \log((1+\varepsilon_1)(1+\varepsilon_3)) + \frac{kn}{k-1} \log \left\lceil \frac{kn}{k-1} \right\rceil + \frac{kn}{k-1} \log\log \left\lceil \frac{kn}{k-1} \right\rceil. \tag{41}$$

Using an estimate for p_n proved by Dusart [4], we get

$$\frac{kn}{k-1} \log R_n^{(k)} - p_{\lceil nk/(k-1) \rceil} \leq \frac{kn}{k-1} \left(\log((1+\varepsilon_1)(1+\varepsilon_3)) + 1 \right).$$

Now use Proposition 4.1. □

Corollary 4.5. *The sequence $((R_n^{(k)} - p_{\lceil nk/(k-1) \rceil})/n)_n$ is bounded.*

Proof. Follows from Proposition 3.5 and Theorem 4.4. □

Remark. In particular, Corollary 4.5 gives a positive answer to the question raised by Sondow [18] in 2013, whether the sequence $((R_n - p_{2n})/n)_n$ is bounded.

5 On the number of k-Ramanujan primes $\leq x$

Let $\pi_k(x)$ be the number of k-Ramanujan primes less than or equal to x. Amersi, Beckwith, Miller, Ronan and Sondow [1] proved that

$$\frac{\pi_k(x)}{\pi(x)} \sim \frac{k-1}{k} \qquad (x \to \infty)$$

by showing that there exists a positive constant $\beta_2 = \beta_2(k)$ such that for every sufficiently large n,

$$|\rho_k(n)| \leq \frac{\beta_2 \log\log n}{\log n}, \tag{42}$$

where

$$\rho_k(x) = \frac{k-1}{k} - \frac{\pi_k(x)}{\pi(x)}.$$

Now we improve the lower bound in (42).

Proposition 5.1. *If $x \geq R_{N(k)}^{(k)}$, then*

$$\rho_k(x) > 0.$$

Proof. Let $n \geq N(k)$ be such that $R_n^{(k)} \leq x < R_{n+1}^{(k)}$. Hence, $R_n^{(k)} > p_{\lceil nk/(k-1) \rceil}$, and we get $\pi(x) > nk/(k-1)$. Since $\pi_k(x) = n$, our proposition is proved. □

With the same method as in [1], we improve the upper bound of (42) by using the following lemma.

Lemma 5.2. *If $s \geq 0$ and*

$$c_0(s) = \max\{4, 4s, (\pi(2+s)-1)\log 2\},$$

then for every $n \in \mathbb{N}$,

$$\pi(p_n + sn) - \pi(p_n) \leq \frac{c_0 n}{\log p_n}.$$

Proof. The claim obviously holds for $s = 0$. So let $s > 0$. If $n = 1$, we get

$$\pi(p_n + sn) - \pi(p_n) = \pi(2+s) - 1.$$

Let $n \geq 2$. If $n < 3/s$, we obtain

$$\pi(p_n + sn) - \pi(p_n) \leq 1.$$

So let $n \geq \max\{2, 3/s\}$. Montgomery and Vaughan [10] proved that

$$\pi(M+N) - \pi(M) \leq \frac{2N}{\log N}$$

for every $M, N \in \mathbb{N}$ with $N \geq 2$. By setting $M = p_n$ and $N = \lfloor sn \rfloor$, we get

$$\pi(p_n + sn) - \pi(p_n) \leq \frac{2sn}{\log sn}.$$

Using the inequality $p_n \leq n^2$, which holds for every $n \geq 2$, we obtain

$$\pi(p_n + sn) - \pi(p_n) < \frac{2\max\{1,s\}n}{\log n} \leq \frac{4\max\{1,s\}n}{\log p_n}.$$

So the lemma is proved. □

Now we prove the following result, which leads to an improvement of the upper bound in (42). Let $\varepsilon_4 > 0$ and $X_{24} = X_{24}(\varepsilon_3, \varepsilon_4)$ be such that $\log(1+\varepsilon_3) + \log(x+1) + \log(x + \log(x+1)) \leq \varepsilon_4 x$ for every $x \geq X_{24}$. In addition, we define

$$c_1 = c_1(k, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \delta_1, \delta_2) = 1 + \varepsilon_4 + c_0(\gamma),$$

where γ is defined by (39), and take $c_2 \in \mathbb{R}$ with $c_2 > c_1$. Further, define

$$X_{25} = X_{25}(k, b_1, X_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \delta_1, \delta_2) = \max\left\{X_{22}, X_{24}, \log\frac{k}{k-1}\right\}$$

and

$$X_{26} = X_{26}(k, b_1, X_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \delta_1, \delta_2, c_2) = \max\left\{X_{21}, \left\lceil \frac{k}{k-1} X_{25} \right\rceil + 1, \pi\left(2^{c_2/(c_2-c_1)}\right)\right\}.$$

Theorem 5.3. *If $x \geq n_3 = p_{X_{26}}$, then*

$$\rho_k(x) \leq \frac{c_2}{\log x}.$$

Proof. First, we prove the claim for $x = p_n$ with $n \geq X_{26}$. By Theorem 4.4, we have

$$R_n^{(k)} < p_{\lceil nk/(k-1)\rceil} + \gamma n. \tag{43}$$

Let $m \in \mathbb{N}$ be such that $\lceil mk/(k-1)\rceil = n$. Then $m \geq X_{22}$. Hence by (43), we have $R_m^{(k)} < p_n + \gamma m$. Since $m \leq n$ and $\gamma > 0$, we get $R_m^{(k)} < p_n + \gamma n$. It follows $\pi_k(p_n) \geq m - (\pi_k(p_n + \gamma n) - \pi_k(p_n))$. Since every k-Ramanujan prime is prime, we obtain

$$\pi_k(p_n) \geq m - (\pi(p_n + \gamma n) - \pi(p_n))$$

and using Lemma 5.2, we get the inequality

$$\frac{\pi_k(p_n)}{\pi(p_n)} \geq \frac{m}{n} - \frac{c_0(\gamma)}{\log p_n}.$$

Since

$$\frac{m}{n} \geq \frac{k-1}{k} - \frac{1}{m},$$

we get

$$\rho_k(p_n) \leq \frac{1}{m} + \frac{c_0(\gamma)}{\log p_n}. \tag{44}$$

Using (40) as well as $X_{21} \leq n \leq (m+1)k/(k-1)$, we obtain

$$\log p_n \leq \log \frac{k}{k-1} + \log(1+\varepsilon_3) + \log(m+1) + \log\left(\log \frac{k}{k-1} + \log(m+1)\right).$$

Since $m \geq \log(k/(k-1))$, it follows that

$$\log p_n \leq m + \log(1+\varepsilon_3) + \log(m+1) + \log(m + \log(m+1)).$$

Using $m \geq X_{24}$, we get $\log p_n \leq (1+\varepsilon_4)m$ and using (44), we get the inequality

$$\rho_k(p_n) \leq \frac{c_1}{\log p_n}. \tag{45}$$

So the theorem is proved in the case $x = p_n$.

Now let $x \in \mathbb{R}$ with $x \geq p_{X_{26}}$ and let $n \geq X_{26}$ be such that $p_n \leq x < p_{n+1}$. Using (45) we obtain

$$\rho_k(x) = \rho_k(p_n) \leq \frac{c_1}{\log p_n}.$$

From Bertrand's postulate, it follows that $2p_n \geq p_{n+1}$ for every $n \in \mathbb{N}$. Since $n \geq \pi(2^{c_2/(c_2-c_1)})$, we get $\rho_k(x) \leq c_2/\log p_{n+1} \leq c_2/\log x$ and the theorem is proved in general. □

6 On a conjecture of Mitra, Paul and Sarkar

In 2009, Mitra, Paul and Sarkar [9] made the following

Conjecture 6.1. *If $m, n \in \mathbb{N}$ with $n \geq \lceil 1.1 \log 2.5m \rceil$, then*

$$\pi(mn) - \pi(n) \geq m - 1.$$

We prove this conjecture for every sufficiently large m by using the following proposition.

Proposition 6.2. *Let $\varepsilon > 0$. Then*

$$\pi(mx) - \pi(x) \geq m - 1$$

for every sufficiently large m and every $x \geq (1+\varepsilon)p_m/m$.

Proof. We set $A(x) = 0$, $B(x) = b_1/\log x$ and choose $Y_0 = 5393$ by [3]. By Theorem 3.22, we get

$$R_n^{(k)} \leq (1+\varepsilon)p_{\lceil kn/(k-1)\rceil}$$

for every $n \in \mathbb{N}$ with $n \geq (k-1)\max\{\pi(X_{27})+1, X_{11}\}/k$, where

$$X_{27} = X_{27}(k, b_1, \varepsilon) = \max\left\{\frac{5393}{1+\varepsilon}, \frac{kX_0}{1+\varepsilon}, \frac{kS(k,0,b_1,\varepsilon,0)}{1+\varepsilon}, T(0,b_1,\varepsilon,0), \frac{X_{13}}{1+\varepsilon}\right\}.$$

Now let $m \in \mathbb{N}$ be sufficiently large, so that

$$m \geq \pi(X_{27}(m, b_1, \varepsilon)) + 1.$$

Then we get the inequality $R_{m-1}^{(m)} \leq (1+\varepsilon)p_m$. □

Corollary 6.3. *The conjecture of Mitra, Paul and Sarkar holds for every sufficiently large m.*

Proof. Let $0 < \varepsilon < 0.1$. By [15], we have $p_m \leq m(\log m + \log\log m - 0.5)$ for every $m \geq 20$. So, we get

$$(1+\varepsilon)p_m/m \leq (1+\varepsilon)(\log m + \log\log m - 0.5) \leq \lceil 1.1 \log 2.5m \rceil$$

for every sufficiently large m. It remains to apply Proposition 6.2. □

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References

[1] Amersi, N., Beckwith, O., Miller, S.J., Ronan, R., Sondow, J., *Generalized Ramanujan Primes*, arXiv:1108.0475v4 (2011).

[2] Axler, C., *New bounds for the prime counting function* $\pi(x)$, in progress.

[3] Dusart, P., *Autour de la fonction qui compte le nombre de nombres premiers*, Dissertation, Université de Limoges, 1998.

[4] —, *The kth prime is greater than* $k(\ln k + \ln\ln k - 1)$ *for* $k \geq 2$, Math. Comp. **68** (1999), no. 225, 411–415.

[5] —, *Estimates of some functions over primes without R.H.*, arXiv:1002.0442v1 (2010).

[6] Erdös, P., *Beweis eines Satzes von Tschebyschef*, Acta Litt. Sci. Szeged **5** (1932), 194–198.

[7] Ishikawa, H., *Über die Verteilung der Primzahlen*, Sci. Rep. Tokyo Bunrika Daigaku 2 (1934), 27–40.

[8] Laishram, S., *On a conjecture on Ramanujan primes*, Int. J. Number Theory **6** (2010), 1869–1873.

[9] Mitra, A., Paul, G., Sarkar, U., *Some conjectures on the number of primes in certain intervals*, arXiv:0906.0104v1 (2009).

[10] Montgomery, H.L., Vaughan, R.C., *The large sieve*, Mathematika **20** (1973), 119–134.

[11] Nicholson, J.W., *Sequence* A214934, The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A214934.

[12] Nicholson, J.W., Noe, T.D., Sondow, J., *Ramanujan primes: Bounds, Runs, Twins, and Gaps*, J. Integer Seq. **14** (2011), Article 11.6.2.

[13] Ramanujan, S., *A proof of Bertrand's postulate*, J. Indian Math. Soc. **11** (1919), 181–182.

[14] Rosser, J.B., *The n-th prime is greater than* $n \log n$, Proc. London Math. Soc., ser. 2, **45** (1939), 21–44.

[15] Rosser, J.B., Schoenfeld, L., *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6**:1 (1962), 64–94.

[16] Shevelev, V., *Ramanujan and Labos primes, their generalizations, and classifications of primes*, J. Integer Seq. **15** (2012), no. 5, Article 12.1.1, 15 pp.

[17] Sondow, J., *Ramanujan primes and Bertrand's Postulate*, Amer. Math. Monthly **116** (2009), 630–635.

[18] —, *Sequence* A233739, The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A233739.

[19] Tchebychev, P., *Mémoire sur les nombres premiers*, Mémoires des savants étrangers de l'Acad. Sci. St.Pétersbourg **7** (1850), 17–33. [Also in: Journal de mathématiques pures et appliques **17** (1852), 366–390.]

[20] Trost, E., *Primzahlen*, Birkhäuser, Basel/Stuttgart, 1953.

Mathematisches Institut, Heinrich-Heine-Universität Düsseldorf, 40225 Düsseldorf, Germany

E-mail address: axler@math.uni-duesseldorf.de