IDENTITIES FOR GENERALIZED EULER POLYNOMIALS

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ABSTRACT. For $N \in \mathbb{N}$, let T_N be the Chebyshev polynomial of the first kind. Expressions for the sequence of numbers $p_\ell^{(N)}$, defined as the coefficients in the expansion of $1/T_N(1/z)$, are provided. These coefficients give formulas for the classical Euler polynomials in terms of the so-called generalized Euler polynomials. The proofs are based on a probabilistic interpretation of the generalized Euler polynomials recently given by Klebanov et al. Asymptotics of $p_\ell^{(N)}$ are also provided.

1. Introduction

The Euler numbers E_n, defined by the generating function

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!} \tag{1.1}$$

and the Euler polynomials $E_n(x)$ that generalize them

$$\sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} = \frac{2e^{xz}}{e^z+1} \tag{1.2}$$

([2, 9.630,9.651]) are examples of basic special functions. It follows directly from the definition that $E_n = 0$ for n odd. Morever, the relation $E_n = 2^n E_n\left(\frac{1}{2}\right)$ follows by setting $x = \frac{1}{2}$ in (1.2), replacing z by $2z$ and comparing with (1.1).

Moreover, the identity

$$\frac{2e^{xz}}{e^z+1} = \frac{2e^{(x-1/2)z}}{e^{z/2}+e^{-z/2}} \tag{1.3}$$

produces

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x-\tfrac{1}{2}\right)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} E_k\left(\tfrac{1}{2}\right)\left(x-\tfrac{1}{2}\right)^{n-k}, \tag{1.4}$$

that gives $E_n(x)$ in terms of the Euler numbers (see [2, 9.650]).

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The *generalized Euler polynomials* $E_n^{(p)}(z)$, defined by the generating function

$$\sum_{n=0}^{\infty} E_n^{(p)}(x)\frac{z^n}{n!} = \left(\frac{2}{1+e^z}\right)^p e^{xz}, \qquad \text{for } p \in \mathbb{N} \tag{1.5}$$

are polynomials extending $E_n(x)$, the case $p=1$. These appear in Section 24.16 of [5]. The definition leads directly to the expression

$$E_n^{(p)}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k E_{n-k}^{(p)}(0), \tag{1.6}$$

where the *generalized Euler numbers* $E_n^{(p)}(0)$ are defined recursively by

$$E_n^{(p)}(0) = \sum_{k=0}^{n} \binom{n}{k} E_k^{(p-1)}(0) E_{n-k}(0), \tag{1.7}$$

for $p > 1$ and initial condition $E_n^{(1)}(0) = E_n(0)$.

2. A probabilistic representation of Euler polynomials and their generalizations

This section discusses probabilistic representations of the Euler polynomials and their generalizations. The results involve the expectation operator $\mathbb{E}$ defined by

$$\mathbb{E}g(L) = \int g(x) f_L(x)\, dx, \tag{2.1}$$

with $f_L(x)$ the probability density of the random variable L and for any function g such that the integral exists.

Proposition 2.1. *Let L be a random variable with hyperbolic secant density*

$$f_L(x) = \text{sech } \pi x, \qquad \text{for } x \in \mathbb{R}. \tag{2.2}$$

Then the Euler polynomial is given by

$$E_n(x) = \mathbb{E}\left(x + \imath L - \tfrac{1}{2}\right)^n. \tag{2.3}$$

Proof. The right hand-side of (2.3) is

$$\begin{aligned}
\mathbb{E}\left(x + \imath L - \tfrac{1}{2}\right)^n &= \int_{-\infty}^{\infty} \left(x - \tfrac{1}{2} + \imath t\right)^n \text{sech } \pi t\, dt \\
&= \sum_{j=0}^{n} \binom{n}{j} \left(x - \tfrac{1}{2}\right)^{n-j} \imath^j \int_{-\infty}^{\infty} t^j \text{sech } \pi\, dt
\end{aligned}$$

The identity

$$\int_{-\infty}^{\infty} t^k \text{sech } \pi t\, dt = \frac{|E_k|}{2^k} \tag{2.4}$$

holds for k odd, since both sides vanish and for k even, it appears as entry 3.523.4 in [2]. A proof of this entry may be found in [1]. Then, using $|E_{2n}| = (-1)^n E_{2n}$ (entry 9.633 in [2])

$$\mathbb{E}(x + \imath L - \tfrac{1}{2})^n = \sum_{j=0}^{n} \binom{n}{j} (x - \tfrac{1}{2})^{n-j} \frac{E_j}{2^j} = E_n(x). \tag{2.5}$$

□

There is a natural extension to the case of $E_n^{(p)}(x)$. The proof is similar to the previous case, so it is omitted.

Theorem 2.2. *Let $p \in \mathbb{N}$ and L_j, $1 \leq j \leq p$ a collection of independent identically distributed random variables with hyperbolic secant distribution. Then*

$$E_n^{(p)}(x) = \mathbb{E}\left[x + \sum_{j=1}^{p} \left(\imath L_j - \tfrac{1}{2} \right) \right]^n . \tag{2.6}$$

In a recent paper, L. B. Klebanov et al. [3] considered random sums of independent random variables of the form

$$\frac{1}{N} \sum_{j=1}^{\mu_N} L_j \tag{2.7}$$

where the random number of summands μ_N is independent of the $Lj's$ and is described below.

Definition 2.3. Let $N \in \mathbb{N}$ and $T_N(z)$ be the Chebyshev polynomial of the first kind. The random variable μ_N taking values in $\mathbb{N}$, is defined by its generating function

$$\mathbb{E}z^{\mu_N} = \frac{1}{T_N(1/z)}. \tag{2.8}$$

Information about the Chebyshev polynomials appears in [2] and [5].

Example 2.4. Take $N = 2$. Then $T_2(z) = 2z^2 - 1$ gives

$$\mathbb{E}z^{\mu_2} = \frac{1}{T_2(1/z)} = \frac{z^2}{2 - z^2} = \sum_{\ell=1}^{\infty} \frac{z^{2\ell}}{2^\ell}. \tag{2.9}$$

Therefore μ_2 takes the value 2ℓ, with $\ell \in \mathbb{N}$, with probability

$$\Pr(\mu_2 = 2\ell) = 2^{-\ell}. \tag{2.10}$$

In [3], Klebanov et al. prove the following result.

Theorem 2.5 (Klebanov et al.). *Assume $\{L_j\}$ is a sequence of independent identically distributed random variables with hyperbolic secant distribution. Then, for all $N \geq 2$ and μ_N defined in (2.8), the random variable*

$$L := \frac{1}{N}\sum_{j=1}^{\mu_N} L_j \tag{2.11}$$

has the same hyperbolic secant distribution.

3. The Euler polynomials in terms of the generalized ones

The identifies (1.6) and (1.7) can be used to express the generalized Euler polynomial $E_n^{(p)}(x)$ in terms of the standard Euler polynomials $E_n(x)$. However, to the best of our knowledge, there is no formula that allows to express $E_n(x)$ in terms of $E_n^{(p)}(x)$. This section presents such a formula.

Definition 3.1. Let $N \in \mathbb{N}$. The sequence $\{p_\ell^{(N)} : \ell = 0, 1, \cdots\}$ is defined as the coefficients in the expansion

$$\frac{1}{T_N(1/z)} = \sum_{\ell=0}^{\infty} p_\ell^{(N)} z^\ell. \tag{3.1}$$

Definition 2.3 shows that

$$p_\ell^{(N)} = \Pr(\mu_N = \ell), \quad \text{for } \ell \in \mathbb{N}. \tag{3.2}$$

The numbers $p_\ell^{(N)}$ will be referred as the *probability numbers.*

Example 3.2. For $N = 2$, Example 2.4 gives

$$p_\ell^{(2)} = \begin{cases} 0 & \text{if } \ell \text{ is odd} \\ 2^{-\ell/2} & \text{if } \ell \text{ is even}, \ell \neq 0. \end{cases} \tag{3.3}$$

The coefficients $p_\ell^{(N)}$ are now used to produce expansions of $E_n(x)$, one for each $N \in \mathbb{N}$, in terms of the generalized Euler polynomials.

Theorem 3.3. *The Euler polynomials satisfy, for all $N \in \mathbb{N}$,*

$$E_n(x) = \frac{1}{N^n}\mathbb{E}\left[E_n^{(\mu_N)}\left(\tfrac{1}{2}\mu_N + N(x - \tfrac{1}{2})\right)\right] \tag{3.4}$$

Proof. From (2.3) and (2.11)

$$E_n\left(\tfrac{1}{2}\right) = \mathbb{E}(\imath L)^n = \frac{1}{N^n}\mathbb{E}\left[\imath\sum_{j=1}^{\mu_N} L_j\right]^n, \tag{3.5}$$

with Theorem 2.2, this yields

$$\mathbb{E}\left[E_n^{(\mu_N)}\left(\frac{\mu_N}{2}\right)\right] = \mathbb{E}\left[\imath\sum_{j=1}^{\mu_N} L_j\right]^n = N^n E_n\left(\tfrac{1}{2}\right). \tag{3.6}$$

Using identity (1.4), it follows that

$$
\begin{aligned}
E_n(x) &= \sum_{k=0}^{n} \binom{n}{k} E_k\left(\tfrac{1}{2}\right)\left(x-\tfrac{1}{2}\right)^{n-k} \\
&= \mathbb{E}\left[\sum_{k=0}^{n} \binom{n}{k} N^{-k} E_k^{(\mu_N)}\left(\tfrac{1}{2}\mu_N\right)\left(x-\tfrac{1}{2}\right)^{n-k}\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{n} \binom{n}{k} N^{-k} (\imath L_1 + \cdots + \imath L_{\mu_N})^k \left(x-\tfrac{1}{2}\right)^{n-k}\right] \\
&= \mathbb{E}\left[\frac{1}{N^n}\sum_{k=0}^{n} \binom{n}{k} (\imath L_1 + \cdots + \imath L_{\mu_N})^k \left(N(x-\tfrac{1}{2})\right)^{n-k}\right] \\
&= \mathbb{E}\left[\frac{1}{N^n}\left(\imath L_1 + \cdots + \imath L_{\mu_N} + N(x-\tfrac{1}{2})\right)^{n}\right] \\
&= \mathbb{E}\left[\frac{1}{N^n}\left(\imath L_1 + \cdots + \imath L_{\mu_N} + z - \tfrac{1}{2}\mu_N\right)^{n}\right] \\
&= \frac{1}{N^n}\mathbb{E}\left[E_n^{(\mu_N)}(z)\right],
\end{aligned}
$$

where $z = \frac{1}{2}\mu_N + N\left(x - \frac{1}{2}\right)$. This completes the proof. □

The next result is established using the fact that the expectation operator $\mathbb{E}$ satisfies

$$\mathbb{E}[h(\mu_N)] = \sum_{k=0}^{\infty} p_k^{(N)} h(k), \tag{3.7}$$

for any function h such that the right-hand side exists.

Corollary 3.4. *The Euler polynomials satisfy*

$$E_n(x) = \frac{1}{N^n}\sum_{k=N}^{\infty} p_k^{(N)} E_n^{(k)}\left(\tfrac{1}{2}k + N\left(x-\tfrac{1}{2}\right)\right). \tag{3.8}$$

Note 3.5. Corollary 3.4 gives an infinite family of expressions for $E_n(x)$ in terms of the generalized Euler polynomials $E_n^{(k)}(x)$, one for each value of $N \geq 2$.

Example 3.6. The expansion (3.8) with $N = 2$ gives

$$E_n(x) = \frac{1}{2^n}\sum_{\ell=1}^{\infty} \frac{1}{2^\ell} E_n^{(2\ell)}(\ell + 2x - 1). \tag{3.9}$$

For instance, when $n = 1$,

$$E_1(x) = \frac{1}{2}\sum_{\ell=1}^{\infty} \frac{1}{2^\ell} E_1^{(2\ell)}(\ell + 2x - 1) \tag{3.10}$$

and the value $E_1^{(\ell)}(x) = x - \frac{\ell}{2}$ gives

$$E_1(x) = \frac{1}{2}\sum_{\ell=1}^{\infty} \frac{1}{2^{\ell}}(\ell + 2x - 1 - \ell) = x - \tfrac{1}{2} \tag{3.11}$$

as expected.

4. The probability numbers

For fixed $N \in \mathbb{N}$, the random variable μ_N has been defined by its moment generating function

$$\mathbb{E}z^{\mu_N} = \frac{1}{T_N(1/z)} = \sum_{\ell=0}^{\infty} p_\ell^{(N)} z^\ell. \tag{4.1}$$

This section presents properties of the probability numbers $p_\ell^{(N)}$ that appear in Corollary 3.4.

For small N, the coefficients $p_\ell^{(N)}$ can be computed directly by expanding the rational function $1/T_N(1/z)$ in partial fractions. Example 2.4 gave the case $N = 2$. The cases $N = 3$ and $N = 4$ are presented below.

Example 4.1. For $N = 3$, the Chebyshev polynomial is

$$T_3(z) = 4z^3 - 3z = 4z(z-\alpha)(z+\alpha), \tag{4.2}$$

with $\alpha = \sqrt{3}/2$. This yields

$$\frac{1}{T_3(1/z)} = \frac{z^3}{4(1-\alpha z)(1+\alpha z)} = \sum_{k=0}^{\infty} \frac{3^k}{2^{2k+2}} z^{2k+3}. \tag{4.3}$$

It follows that $p_\ell^{(3)} = 0$ unless $\ell = 2k+3$ and

$$p_{2k+3}^{(3)} = \frac{3^k}{2^{2k+2}}. \tag{4.4}$$

Corollary 3.4 now gives

$$E_n(x) = \frac{1}{3^n}\sum_{k=0}^{\infty} \frac{3^k}{2^{2k+2}} E_n^{(2k+3)}(3x+k), \tag{4.5}$$

a companion to (3.9).

Example 4.2. The probability numbers for $N = 4$ are computed from the expression

$$\frac{1}{T_4(1/z)} = \frac{z^4}{z^4 - 8z^2 + 8}. \tag{4.6}$$

The factorization

$$z^4 - 8z^2 + 8 = (z^2 - \beta)(z^2 - \gamma) \tag{4.7}$$

with $\beta = 2(2+\sqrt{2})$ and $\gamma = 2(2-\sqrt{2})$ and the partial fraction decomposition

$$\frac{z^4}{z^4 - 8z^2 + 8} = \frac{\beta}{\beta - \gamma}\frac{1}{1-\beta/z^2} - \frac{\gamma}{\beta-\gamma}\frac{1}{1-\gamma/z^2} \tag{4.8}$$

show that $p_\ell^{(4)} = 0$ for ℓ odd or $\ell = 2$ and

$$p_{2\ell}^{(4)} = \frac{\sqrt{2}}{2^{2\ell+1}}\left[(2+\sqrt{2})^{\ell-1} - (2-\sqrt{2})^{\ell-1}\right] \tag{4.9}$$

for $\ell \geq 2$. Corollary 3.4 now gives

$$E_n(x) = \sqrt{2}\sum_{\ell=2}^{\infty}\frac{\left[(2+\sqrt{2})^{\ell-1} - (2-\sqrt{2})^{\ell-1}\right]}{2^{2\ell+1}}E_n^{(2\ell)}(4x+\ell-2). \tag{4.10}$$

Some elementary properties of the probability numbers are presented next.

Proposition 4.3. *The probability numbers $p_\ell^{(N)}$ vanish if $\ell < N$.*

Proof. The Chebyshev polynomial $T_N(z)$ has the form $2^{N-1}z^N +$ lower order terms. Then the expansion of $1/T_N(1/z)$ has a zero of order N at $z = 0$. This proves the statement. □

Proposition 4.4. *The probability numbers $p_\ell^{(N)}$ vanish if $\ell \not\equiv N \bmod 2$.*

Proof. The polynomial $T_N(z)$ has the same parity as N. The same holds for the rational function $1/T_N(1/z)$. □

An expression for the probability numbers is given next.

Theorem 4.5. *Let $N \in \mathbb{N}$ be fixed and define*

$$\theta_k^{(N)} = \frac{(2k-1)\pi}{2N}. \tag{4.11}$$

Then

$$p_\ell^{(N)} = \frac{1}{N}\sum_{k=1}^{N}(-1)^{k+1}\sin\theta_k^{(N)}\cos^{\ell-1}\theta_k^{(N)}. \tag{4.12}$$

Proof. The Chebyshev polynomial is defined by $T_N(\cos\theta) = \cos(N\theta)$, so its roots are $z_k^{(N)} = \cos\theta_k^{(N)}$, with $\theta_k^{(N)}$ as above. The leading coefficient of $T_N(z)$ is 2^{N-1}, thus

$$\frac{1}{T_N(z)} = \frac{2^{1-N}}{\prod_{k=1}^N (z-z_k)}. \tag{4.13}$$

In the remainder of the proof, the superscript N has been dropped from $z_k^{(N)}$ and $\theta_k^{(N)}$, for clarity. Define

$$Q(z) = \prod_{k=1}^{N}(z-z_k). \tag{4.14}$$

The roots z_k of Q are distinct, therefore

$$\frac{1}{Q(z)} = \sum_{k=1}^{N} \frac{1}{Q'(z_k)} \frac{1}{z - z_k}. \tag{4.15}$$

The identity $T_N'(z) = NU_{N-1}(z)$ gives

$$Q'(z_k) = N2^{1-N} U_{N-1}(z_k) \tag{4.16}$$

where $U_j(z)$ is the Chebyshev polynomial of the second kind defined by

$$U_N(\cos\theta) = \frac{\sin(N+1)\theta}{\sin\theta}. \tag{4.17}$$

Then

$$U_{N-1}(z_k) = U_{N-1}(\cos\theta_k) = \frac{\sin N\theta_k}{\sin\theta_k}. \tag{4.18}$$

and the value $\sin N\theta_k = (-1)^{k+1}$ yields

$$Q'(z_k) = \frac{(-1)^{k+1}}{\sin\theta_k} N 2^{1-N}. \tag{4.19}$$

Therefore (4.15) now gives

$$\frac{1}{Q(z)} = \frac{2^{N-1}}{N} \sum_{k=1}^{N} \frac{(-1)^{k+1}\sin\theta_k}{z - \cos\theta_k}. \tag{4.20}$$

It follows that

$$\begin{aligned}
\frac{1}{T_N(1/z)} = \frac{2^{1-N}}{Q(1/z)} &= \frac{1}{N}\sum_{k=1}^{N} (-1)^{k+1} \frac{z\sin\theta_k}{1 - z\cos\theta_k} \\
&= \frac{1}{N}\sum_{k=1}^{N} (-1)^{k+1} \sin\theta_k \sum_{\ell=0}^{\infty} z^{\ell+1}\cos^{\ell}\theta_k \\
&= \frac{1}{N}\sum_{\ell=0}^{\infty} z^{\ell+1} \sum_{k=1}^{N} (-1)^{k+1} \sin\theta_k \cos^{\ell}\theta_k.
\end{aligned}$$

The proof is complete. □

The next result provides another explicit formula for the probability numbers. The coefficients $A(n,k)$ appear in OEIS entry A008315, as entries of the Catalan triangle.

Theorem 4.6. *Let $A(n,k) = \binom{n}{k} - \binom{n}{k-1}$. Then, if $N \equiv \ell \bmod 2$,*

$$p_{\ell}^{(N)} = \frac{1}{2^{\ell}} \sum_{t=\left\lfloor \frac{1}{2}\left(\frac{2-\ell}{N}-1\right)\right\rfloor}^{\left\lfloor \frac{1}{2}\left(\frac{\ell}{N}-1\right)\right\rfloor} (-1)^t A(\ell-1, \tfrac{1}{2}(\ell-(2t+1)N)),$$

indent when ℓ is not an odd multiple of N and

$$p_\ell^{(N)} = \frac{1}{2^\ell}\left[\sum_{s=1}^{\lfloor \ell/N-1\rfloor}(-1)^{k-s}A(\ell-1,sN)\right] + \frac{(-1)^k}{2^{\ell-1}},\ \text{with } k=\tfrac{1}{2}\left(\ell/N-1\right)$$

otherwise.

The proof begins with a preliminary result.

Lemma 4.7. *Let $N\in\mathbb{N}$ and $\theta_k = \frac{\pi}{2}\frac{(2k-1)}{N}$. Then*

$$f_N(z) = \sum_{k=1}^{N}(-1)^{k+1}e^{\imath\theta_k z} \tag{4.21}$$

is given by

$$f_N(z) = \frac{1-(-1)^N e^{\pi\imath z}}{2\cos\left(\frac{\pi z}{2N}\right)} \qquad \text{if } z \neq (2t+1)N \text{ with } t\in\mathbb{Z} \tag{4.22}$$

and

$$f_N(z) = (-1)^t N\imath \qquad \text{if } z=(2t+1)N \text{ for some } t\in\mathbb{Z}. \tag{4.23}$$

In particular

$$f_N(k) = \begin{cases} (-1)^{(k/N-1)/2}N\imath & \text{if } \frac{k}{N} \text{ is an odd integer} \\ \frac{1-(-1)^{N+k}}{2\cos\left(\frac{\pi k}{2N}\right)} & \text{otherwise.}\end{cases} \tag{4.24}$$

Proof. The function f_N is the sum of a geometric progression. The formula (4.23) comes from (4.22) by passing to the limit. □

The proof of Theorem 4.6 is given now.

Proof. The expression for $p_\ell^{(N)}$ given in Theorem 4.5 yields

$$\begin{aligned} p_\ell^{(N)} &= \frac{1}{N}\sum_{k=1}^{N}(-1)^{k+1}\frac{(e^{\imath\theta_k}-e^{-\imath\theta_k})}{2i}\left(\frac{e^{\imath\theta_k}+e^{-\imath\theta_k}}{2}\right)^{\ell-1} \\ &= \frac{1}{2^\ell N i}\sum_{k=1}^{N}(-1)^{k+1}\sum_{r=0}^{\ell-1}\binom{\ell-1}{r}\left[e^{\imath(\ell-2r)\theta_k}-e^{\imath(\ell-2r-2)\theta_k}\right] \\ &= \frac{1}{2^\ell N\imath}\sum_{r=0}^{\ell-1}\binom{\ell-1}{r}\left[f_N(l-2r)-f_N(l-2r-2)\right] \\ &= \frac{1}{2^\ell N\imath}\left[\sum_{r=1}^{\ell-1}A(\ell-1,r)f_N(\ell-2r)+f_N(\ell)-f_N(-\ell)\right]. \end{aligned}$$

Now $f_N(\ell)=f_N(-\ell)=0$ if ℓ/N is not an odd integer. On the other hand, if $\ell=(2t+1)N$, with $t\in\mathbb{Z}$, then

$$f_N(\ell) = (-1)^t N\imath \text{ and } f_N(-\ell) = -(-1)^t N\imath. \tag{4.25}$$

Thus

$$f_N(\ell) - f_N(-\ell) = \begin{cases} 2N\imath(-1)^{(\ell/N-1)/2} & \text{if } \ell \text{ is an odd multiple of } N \\ 0 & \text{otherwise.} \end{cases}$$

The simplification of the previous expression for $p_\ell^{(N)}$ is divided in two cases, according to whether ℓ is an odd multiple of N or not.

Case 1. Assume ℓ is not an odd multiple of N. Then

$$p_\ell^{(N)} = \frac{1}{2^\ell N\imath} \sum_{r=0}^{\ell-1} A(\ell-1,r) f_N(\ell-2r). \tag{4.26}$$

Morever,

$$f_N(\ell-2r) = \begin{cases} (-1)^t N\imath & \text{if } \frac{l-2r}{N} = 2t+1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.27}$$

Therefore

$$p_\ell^{(N)} = \frac{1}{2^\ell} \sum_{\substack{t=\frac{1}{2}\left(\frac{2-\ell}{N}-1\right) \\ \ell-2r=(2t+1)N}}^{\frac{1}{2}\left(\frac{\ell}{N}-1\right)} (-1)^t A(\ell-1,r). \tag{4.28}$$

Observe that $\ell-(2t+1)N$ is always an even integer, thus the index r may be eliminated from the previous expression to obtain

$$p_\ell^{(N)} = \frac{1}{2^\ell} \sum_{t=\left\lfloor \frac{1}{2}\left(\frac{2-\ell}{N}-1\right)\right\rfloor}^{\left\lfloor \frac{1}{2}\left(\frac{\ell}{N}-1\right)\right\rfloor} (-1)^t A(\ell-1, \tfrac{1}{2}(\ell-(2t+1)N)). \tag{4.29}$$

Case 2. Assume ℓ is an odd multiple of N, say $\ell = (2k+1)N$. Then

$$\begin{aligned} p_\ell^{(N)} &= \frac{1}{2^\ell N i}\left[\sum_{r=0}^{\ell-1} A(\ell-1,r) f_N(\ell-2r) + 2Ni(-1)^k\right] \\ &= \frac{1}{2^\ell N i}\left[\sum_{r=0}^{\ell-1} A(\ell-1,r) f_N(\ell-2r)\right] + \frac{(-1)^k}{2^{\ell-1}}. \end{aligned}$$

The term $f_N(\ell-2r)$ vanishes unless $\ell-2r$ is an odd multiple of N. Given that $\ell=(2k+1)N$, the term is non-zero provided $2r$ is an even multiple of N; say $r = sN$ for $s \in \mathbb{N}$. The range of s is $1 \leq s \leq \frac{\ell-1}{N} = 2k+1-\frac{1}{N}$. This implies $1 \leq s \leq 2k = \ell/N - 1$, and it follows that

$$p_\ell^{(N)} = \frac{1}{2^\ell}\left[\sum_{s=1}^{\ell/N-1} (-1)^{k-s} A(\ell-1, sN)\right] + \frac{(-1)^k}{2^{\ell-1}}, \text{ with } k = \tfrac{1}{2}\left(\ell/N-1\right).$$

The proof is complete. □

Note 4.8. The expression in Theorem 4.6 shows that $p_\ell^{(N)}$ is a rational number with a denominator a power of 2 of exponent at most ℓ. Arithmetic properties of these coefficients will be described in a future publication [4]. Moreover, the probability numbers $p_\ell^{(N)}$ appear in the description of a random walk on N sites. Details will appear in [4].

5. An asymptotic expansion

The final result deals with the asymptotic behavior of the probability numbers $p_\ell^{(N)}$.

Theorem 5.1. *Let $\varphi_N(z) = \mathbb{E}\left[z^{\mu_N}\right]$. Then, for fixed z in the unit disk $|z| < 1$,*

$$\varphi_N(z) \sim \left(\frac{z}{1+\sqrt{1-z^2}} \right)^N, \text{ as } N \to \infty. \tag{5.1}$$

Proof. The generating function satisfies

$$\varphi_N(z) = 1/T_N(1/z) = \frac{z^N}{2^{N-1}} \prod_{k=1}^{N} \left(1 - z\, \cos \theta_k^{(N)} \right)^{-1} \tag{5.2}$$

with $\theta_k^{(N)} = (2k-1)\pi/2N$ as before. Then

$$\log \varphi_N(z) = \log 2 + N \log \frac{z}{2} - \sum_{k=1}^{N} \log \left(1 - z \cos \theta_k^{(N)} \right). \tag{5.3}$$

The last sum is approximated by a Riemann integral

$$\frac{1}{N} \sum_{k=1}^{N} \log \left(1 - z \cos \theta_k^{(N)} \right) \sim \frac{1}{\pi} \int_0^\pi \log(1 - z\cos\theta)\, d\theta = \log \left(\frac{1+\sqrt{1-z^2}}{2} \right).$$

The last evaluation is elementary. It appears as entry 4.224.9 in [2]. It follows that

$$\log \varphi_N(z) \sim \log 2 + N \log \left(\frac{z}{2} \right) - N \log \left(\frac{1+\sqrt{1-z^2}}{2} \right) \tag{5.4}$$

indent and this is equivalent to the result. □

The function

$$A(z) = \frac{2}{1+\sqrt{1-4z}} = \sum_{n=0}^{\infty} C_n z^n \tag{5.5}$$

is the generating function for the Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n}. \tag{5.6}$$

The final result follows directly from the expansion of Binet's formula for Chebyshev polynomial

$$T_N(z) = \frac{(z-\sqrt{z^2-1})^N + (z+\sqrt{z^2-1})^N}{2}. \tag{5.7}$$

Some standard notation is recalled. Given two sequences $\mathbf{a} = \{a_n\}$, $\mathbf{b} = \{b_n\}$, their convolution $\mathbf{c} = \mathbf{a} * \mathbf{b}$ is the sequence $\mathbf{c} = \{c_n\}$, with

$$c_n = \sum_{j=0}^{n} a_j b_{n-j}. \tag{5.8}$$

The *convolution power* $\mathbf{c}^{(*N)}$ is the convolution of $\mathbf{c}$ with itself, N times.

Theorem 5.2. *For $N \in \mathbb{N}$ fixed, the first N nonzero terms of the sequence $q_\ell^{(N)} = 2^{\ell-1} p_\ell^{(N)}$ agree with the first N terms of the N-th convolution power $C_n^{(*N)}$ of the Catalan sequence:*

$$q_N^{(N)} = C_0^{(*N)},\ q_{N+2}^{(*N)} = C_1^{(*N)},\ \cdots,\ q_{N+2k}^{(N)} = C_k^{(*N)},\ \cdots,\ q_{3N-2}^{(N)} = C_{N-1}^{(*N)}.$$

In terms of generating functions, this is equivalent to

$$\left(\sum_{n=0}^{\infty} C_n z^{2n+1} \right)^N - \sum_{\ell=0}^{\infty} q_\ell^{(N)} z^\ell \sim 2^N z^{3N}. \tag{5.9}$$

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