LATIN POLYTOPES

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ABSTRACT. Latin squares are well studied combinatorial objects. In this paper we generalize the concept and propose designs (*Latin triangles*, *Latin tetrahedra*, etc.) that feature similar properties.
We start with a classic definition of Latin squares followed by one based on concepts of modern design theory. A Latin square appears then as a combinatorial design whose points are geometric. Its rows and columns are now symmetric lines that intersect in specific ways, while its "labelled lines" intersect the former also in a particular manner.
The generalization that follows proceeds by 1. broadening the inherent symmetry of the Latin square 2. considering more general configurations of points and 3. admitting symmetric and labelled lines that intersect more freely. The resulting concept is the *Latin board*. Finally, we particularize this object to define *Latin polytopes*, *Latin polygons* and *Latin polyhedra*.

Keywords: Latin square, augmentation, board, design symmetrization problem, DSP, woof board, weft board, source, kaleidoscopic source, warp class, woven board, Latin board, Latin geometric design, LGD, Latin polytope, biregular polygon, Latin polygon, Latin polyhedron, Latin triangle, generalized Latin square, Latin hexagon, Latin tetrahedron, Latin cube, Latin octahedron, Latin icosahedron, Latin dodecahedron, Latin isochora.

1. COMBINATORIAL DESIGN

Combinatorial design theory is the part of combinatorial mathematics that deals with the existence, construction and properties of systems of finite sets whose arrangements satisfy generalized concepts of balance and symmetry. For an excellent illustration of the key role of these two properties in designs see [cd6].

With roots in XIXth century's recreational mathematics, design theory grew in the XXth with applications to experimental design. Later, its deep and rich connections with finite geometry, number theory, finite fields, and group theory were recognized, anticipating its importance in the theory of codes for error detection and correction in data telecommunication. After a period in which its pure mathematical aspects took over practical applications, the field has found new uses in cryptography, optical communications, storage system design, algorithm design and wireless communications (see [sVII] for a short history of design theory). Geometry is very present in design-theoretic terminology [s2]:

Definition 1.1 A *simple combinatorial design* is a pair (P, L) where P is a set of *points* and L is a set of subsets of P called *lines*.

It is worth to point out that the terms "points" and "lines" here are immaterial, as the key elements are evidently sets. David Hilbert is reported to have said in a similar context: "one must always be able to say *table, chair, beer mug* instead of *point, line, plane*". In fact no reference whatsoever to geometric objects is made or needed in many combinatorial designs.

In a general design, L is not a set but a multiset in which subsets may appear more than once. In what follows "design" will always mean simple design. When a point p belongs to a line l we say that *p lies on l* and that *l goes through p*. In design theory lines are also called *blocks* or *treatments*. In all designs considered here every point will lie at least on one line. If we call points *vertices* and lines *edges,* designs so defined are also *hypergraphs* [b1]. Of particular interest here are hypergraph with geometric vertices, also known as *geometric hypergraphs* [h105].

Definition 1.2 Let l be a line in a design. We define $| l |$, the number of points in the line, to be the *size* of the line.

Definition 1.3 A design is *k-uniform* if all its lines have size k.

Definition 1.4 By exchanging the roles of points and lines of a design we obtain its *dual design*.

Definition 1.5 A simple design with an explicit incidence relationship between points and lines is called a *finite geometry* or a *simple incidence structure*.

The most studied finite geometries are projective (in which, among other conditions, any two lines have a point in common) and affine (in which a necessary condition is: if a point p is not on a line l_1 then there is a unique line l_2 through p having no points in common with l_1). Additional concepts of design theory will be defined in the rest of the paper.

1.1 The Fano plane

Some well known designs are linked to geometry. Consider the picture on Figure 1.1.1 (left). Let P be the set of points marked in red and L the set of sets of 3 points on a line with the same color (center). The resulting design

$$FP = (P, L) \tag{1.1.1}$$

is called *the Fano plane* [cd3].

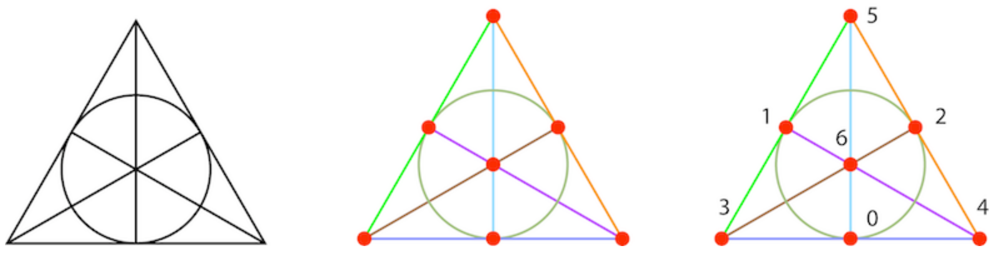


FIGURE 1.1.1

This design is indeed balanced: it has 7 design points and 7 design lines, with 3 points on each line (the design is thus 3-uniform) and 3 lines through each point; every 2 points define a line and every 2 lines meet on a point. As a finite geometry the Fano plane has another interesting property: it is the smallest finite projective plane.

In Figure 1.1.1 (right) we have labeled the points with numbers. The Fano plane has now $P = \{0, 1, 2, 3, 4, 5, 6\}$ and $L = \{\{0, 1, 2\}, \{0, 3, 4\}, \{0, 5, 6\}, \{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\}$.

Definition 1.1.2 Two designs $B = (P, L)$ and $B' = (P', L')$ are *isomorphic* iff there is a bijection from P to P' that preserves the lines. Such bijection is called an *isomorphism*.

Definition 1.1.3 A design *is self-dual* if it is isomorphic to its dual design.

The Fano plane has symmetry beyond the geometric one: it is not difficult to prove that it is self-dual. An isomorphism can be established also within the design itself

Definition 1.1.4 An *automorphism* of a design $B = (P, L)$ is a bijection from P to P that preserves the lines in L.

An automorphism is then a line-preserving permutation of the set of design points.

1.1.1 **Design symmetry from geometric symmetry**. Let's assume now that the triangle in 1.1.1 (center) is equilateral, and that the lines through its vertices are straight and go through the orthocenter. Then both the set of points and the set of lines are invariant under a 0º, 120º or 240º rotation around the orthocenter, and also under a reflection around each axis that goes through each vertex and the orthocenter. These 6 transformations form the so-called *dihedral group* D_3, with composition of transformations as the group operation. Just by looking at the picture we see that all elements in D_3 induce permutations in P that are automorphisms of FP. For example a 120º counter-clockwise rotation is equivalent to the permutation in P

$$0 \to 2,\ 1 \to 0,\ 2 \to 1,\ 3 \to 4,\ 4 \to 5,\ 5 \to 3,\ 6 \to 1$$

which is an automorphism of FP. Consider now this other permutation

$$0 \to 0,\ 1 \to 5,\ 2 \to 6,\ 3 \to 4,\ 4 \to 3,\ 5 \to 2,\ 6 \to 1$$

It is easy to verify that it leaves invariant the lines of FP and so that it is an automorphism. But no element of D_3 or composition thereof can reproduce it, as they all leave point 6 invariant and the permutation takes point 6 to point 1. So not all automorphisms of FP come from transformations in D_3.

1.2 Latin squares

Another combinatorial and geometric object is the Latin square (see Figure 1.2.1), one of whose classic definitions is

Definition 1.2.1 A *Latin square* of order n with entries from an n-set X is an n x n array in which every row and every column contains all elements of X.

1	2	3
2	3	1
3	1	2

FIGURE 1.2.1

Latin squares are so named because Leonhard Euler used latin letters inside the cells [e141]. It is easy to obtain Latin squares of any given order: write any permutation of the symbols in the top row, then shift it by one cell to the left in successive rows (see Figure 1.2.1). An alternative description of Latin squares follows.

1.2.1 **Designs and Latin squares**. Figure 1.2.2 (left) shows a square tiled with 9 congruent squares we call *faces*. Let P be the set of face centers and L the set of sets of 3 face centers on a line with the same color (center). We have then a 3-uniform design

$$B_1 = (P, L) \tag{1.2.1}$$

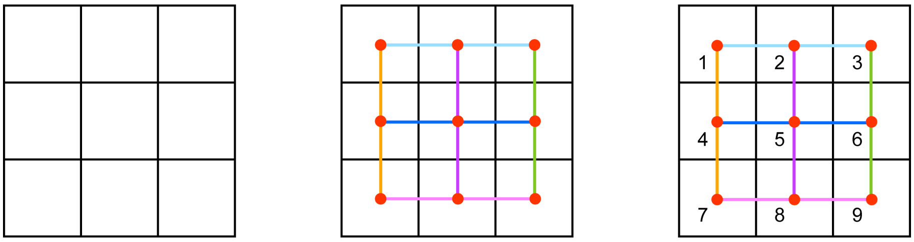


FIGURE 1.2.2

If we label the points with numbers (right), then $P = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $L = \{\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}, \{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\}$. B_1 has a few automorphisms. Using cycle notation we have for example that

$$(12)(45)(78) \tag{1.2.2}$$

is one that exchanges in parallel the points in the first and second column. Some basic definitions of design theory follow.

Definition 1.2.2 A *parallel class* or *resolution class* in a design $B = (P, L)$ is a set of lines that partitions P.

Definition 1.2.3 A parallel class is *k-uniform* iff all its lines have size k.

A k-uniform parallel class is then a subset of the set of lines with $|P| / k$ lines. A design can thus have a k-uniform parallel class only if $|P| \equiv 0 \ mod\ k$. Again in B_1, we see that each 3-set of lines

$$\begin{aligned} H &= \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\} \\ V &= \{\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}\} \end{aligned} \tag{1.2.3}$$

partitions the set of points and has lines of size 3: each is thus a 3-uniform parallel class.

Definition 1.2.4 In a design, two parallel classes Q, R are said to be *orthogonal* iff for any two lines l_1, l_2, with $l_1 \in Q$ and $l_2 \in R$, $|l_1 \cap l_2| = 1$.

It is immediate to see that every line in H intersect at a single point every line in V, thus H and V are orthogonal.

Definition 1.2.6 A *resolution* of a design is a set of parallel classes that partition the set of lines. A design with a resolution is said to be *resolvable*.

Definition 1.2.7 A resolution is *k-uniform* iff each of its parallel classes is *k*-uniform.

B_1 above is then resolvable: it has a 3-uniform resolution with 2 orthogonal parallel classes. In Euclidean space, orthogonality between geometric lines requires intersection but also equality of angles. "Parallelism" and "orthogonality" are then special cases of a concept more appropriate to design theory

Definition 1.2.8 Given a design $B = (P, L)$ we call $\{|l \cap l'| : l, l' \in L, l \neq l'\}$ its *set of intersection numbers* or *SIN*.

It is trivial to form one or more parallel classes among a general set of points, but it is not so if we demand a particular SIN in the resulting design. In the running example, the tiled square has provided an interesting intersection pattern from the outset: parallelism (within each parallel class) and orthogonality (among classes).

1.2.2 **Design symmetry from geometric symmetry**. Here again there is a connection between geometry and design: the set of lines in B_1 is invariant under counterclockwise rotations of 0º, 90º, 180º and 270º and under 4 reflections (along the two diagonals and the vertical and horizontal symmetry axis). These transformations with composition thereof form the dihedral group D_4. Each of its elements induce a permutation in the set of points that is an automorphism of the design. For example a 90º counter-clockwise rotation is equivalent to the automorphism

$$1 \to 2, 2 \to 4, 3 \to 1, 4 \to 8, 5 \to 5, 6 \to 2, 7 \to 9, 8 \to 6, 9 \to 3$$

And again, there are automorphisms with no correspondence in D_4. For example those that exchange the points in two lines of the same parallel class, i.e: a permutation of rows or one of columns.

Besides making possible an interesting resolution and intersection pattern, we see that the tiled square provides also some automorphisms of the design. All this suggests that non-trivial designs could be derived from geometric objects.

1.2.3 Nets from Latin squares. Let's make now a new line with the points that are labelled with the same number in Figure 1.2.1. Figure 1.2.3 shows design B_2 with the set S of 3 new lines, each indicated by a dashed line of a particular color. We see that S constitute yet another 3-uniform parallel class orthogonal to both H and V.

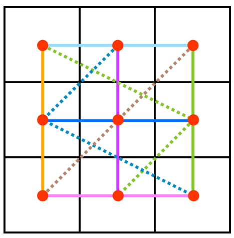


FIGURE 1.2.3

The new class with labels is

(1.2.4) $$S = \{\{1, 6, 8\}, \{2, 4, 9\}, \{3, 5, 7\}\}$$

The resulting 3-uniform design is known as a 3-*net* [cd639]

$$B_2 = (P, L \cup S) \tag{1.2.5}$$

Definition 1.2.9 A design $B' = (P', L')$ is a *subdesign* of another design $B = (P, L)$, $B' \subseteq B$, if $P' \subseteq P$ and $L' \subseteq L$. The subdesign is proper if $P' \subset P$ and improper if $P' = P$.

So we have $B_1 \subseteq B_2$. B_2 has a larger 3-uniform resolution and are more balanced: 9 points, 9 lines, with each point lying on 3 lines and each line going through 3 points.

1.2.4 **Equivalence classes of Latin squares**. As per Definition 1.2.1, if we permute the symbols, the rows or the columns of a Latin square we obtain another Latin square. Both squares are said to be *isotopic*. *Isotopism* is an equivalence relation that partitions the set of Latin squares of a particular order into *isotopy classes*.

A Latin square of order n with symbols $\{1,..., n\}$ may be represented with an n^2-set of 3-tuples (i, j, k), one for each cell, where i is the index of the row, j that of the column and k the symbol in the cell. This set, called the *orthogonal array representation* of the Latin square [dk190] makes apparent that any pair of components in the tuple may become row and column indices and the remaining one the cell content and obtain yet another Latin square. As there are 6 possible ways to do this, from one Latin square we can obtain a maximum of five others called the *conjugates* of the original one (there may be less than five as different choices of indices may result in the same Latin square).

We can combine both transformations: two Latin squares are said to be *paratopic* if one of them is isotopic to a conjugate of the other. This is again an equivalence relation that partitions the squares into *paratopy classes*, each one containing up to 6 isotopy classes. For the number of paratopy classes as a function of the order see [cd136] and [n].

For what will come next, we keep in mind that certain elements –or subsets thereof– in this larger set of paratopic transformations have the same effect as elements of D_4 have. For example a reflection of a Latin square along the vertical axis produces the same Latin square than a sequence of permutations of the columns.

1.3 Augmentation

Definition 1.3.1 We *augment* a design by specifying, finding and adding *extra lines* to it.

Example 1.3.1 Instead of taking the Latin square in Figure 1.2.1 as a reference to augment design B_1 (1.2.1) we could have specified an additional uniform parallel class orthogonal to the existing ones, then tried to find it.

In general, the specification for extra lines may include a mix of requirements like line size, resolution and SIN, with parallelism and orthogonality as special cases. To find an instance of the specified class for B_1 is relatively easy, but augmenting a design (i.e.: proving the *existence* of a complying one) is in general difficult and often requires computers.

So more so is *counting* or *enumerating* all complying designs, especially if the number of design points and lines is large and the specification a demanding one. It may also happen that the required design simply does not exist.

Definition 1.3.2 An augmentation is *viable* if there exists at least one design that complies with the specification.

Techniques like *dancing links* [k] or backtrack algorithms [cd757] may be used to augment a design. Backtrack algorithms explore systematically the tree of possibilities with a depth-first search. They assign points to the extra lines one at a time and progress forward, going back –backtracking– to a branching point when an incompatibility occurs. Different *pruning techniques* may help to speed up the process. It makes sense then an alternative definition of viability with respect to a set of computer resources thresholds (time, memory, complexity, etc.): the augmentation is viable if the search for extra lines does not exceed any resource in the set.

2. BOARDS

Both the Fano plane (1.1.1) and design B_1 (1.2.1) show a connection between geometry and design. For one, the design points were actual geometric points. We define

Definition 2.1 A *board* is a combinatorial design whose points are geometric.

We come back then here to the original meaning of "point". Then, in the same way that a design relates to a hypergraph, a board relates to a geometric hypergraph.

According to the definition, both the Fano plane and B_1 are boards. There are different ways to specify points and lines in a board as the next example shows

Example 2.1 Figure 1.1.1 (center) shows the Fano plane with extra graphical elements that help specify a line as being made up of points lying on either a straight drawn line or on the circle. We could as well say that a design line is the set of points on a drawn line of a particular color. The Fano plane is shown again in Figure 2.1. Here the points have been numbered so that each design line is made up of points whose numbers share a digit.

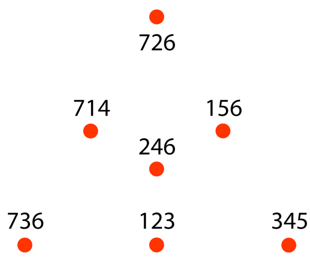


FIGURE 2.1

2.1 Symmetric boards

We define now some relevant objects before turning our attention to the link between geometric and design symmetry.

Definition 2.1.1 The (*full*) *symmetry group* T of a geometric object embedded in an space S is the group of all isometries under which the object is invariant, with composition of isometries as the group operation.

T is then a subgroup of all isometries of S. T was D_3 in the Fano plane (1.1.1) and D_4 in B_1 (1.2.1).

Definition 2.1.2 The *symmetric group* on a finite set P, Sym(P), is the group whose elements are all bijective functions from P to P with function composition as the group operation.

Sym(P) is then the group of all possible permutations of the elements of P, so its order is |P|! A subgroup of Sym(P) is called a *permutation group*.

Definition 2.1.3 A group obtained under composition of automorphisms is an *automorphism group*. The group of all automorphisms of a design B is the *full automorphism group* of B, denoted by Aut(B).

Being any isomorphism a permutation of P, it belongs too to Sym(P), so we have

$$\mathrm{Aut}(B) \subseteq \mathrm{Sym}(P). \tag{2.1.1}$$

Definition 2.1.4 If T is a group and P is a finite set then a *group action on P* is a group homomorphism –called the *action homomorphism*– from T to Sym(P). The action assigns a permutation of P to each element of the group such that

- the identity element of T is the identity permutation of P
- a product $t_1 \cdot t_2$ of two elements of T is the composition of the permutations assigned to t_1 and t_2

A group T may then act on P in different ways: as many as action homomorphisms can be established between T and Sym(P). But if T is not just a group, but the symmetry group of a finite set of geometric points P, from among all possible actions of T on P there is a natural one: that which assigns each transformation to a permutation of the points that encodes their position before and after the transformation.

In a board the set of design points may be acted upon by a symmetry group, but also the set of design lines. We now define

Definition 2.1.5 A *symmetric board* is a board whose set of lines is acted upon by a symmetry group. The lines in the board are called *symmetric lines*.

In a symmetric board the elements of the group take lines to lines. The lines in both the Fano plane (1.1.1) and board B_1 (1.2.1) are acted upon by D_3 and D_4 respectively, so they are symmetric boards. Unless explicitly stated, in what follows we will consider only non-trivial symmetry groups.

Lemma 2.1.1 *Let $B = (P, L)$ be a symmetric board and T the symmetry group acting on L. Then the permutations of P linked to the action form a group with composition of permutations as the group operation.*

Proof. Let h be the action homomorphism between T and Sym(P). If T acts on L then, being each line a subset of points, T also acts on P. As a group homomorphism preserves the subgroups, and as T is an improper subgroup of itself, then $h(T)$ is a subgroup of Sym(P), hence a group. □

Lemma 2.1.2 *Let $B = (P, L)$ be a symmetric board and T the symmetry group acting on L. Let $A_T(B)$ be the group of permutations of P linked to the action. Then $A_T(B) \subseteq Aut(B)$.*

Proof. As T acts on L then $A_T(B)$, being its image under the action homomorphism, preserves the lines, and so its elements are isomorphisms of B, hence $A_T(B) \subseteq Aut(B)$. □

If we want then interesting designs with large Aut(B), a symmetric board whose lines are acted upon by a symmetry group T provides at least $\mathrm{A_T}(B)$ upfront. If we combine this result with (2.1.1) above we have

$$A_T(B) \subseteq Aut(B) \subseteq Sym(P) \tag{2.1.2}$$

Example 2.1.1 Let $B = (P, L)$ be a board with with $P = \{v_i, i = 1,..., 3\}$ being the set of vertices of an equilateral triangle and $L = \{\{v_i\}, i = 1,..., 3\}$ (i.e.: each line has just a single point). As D_3 acts on L, B is a symmetric board. We have in this case $A_{D3}(B) = \mathrm{Aut}(B) = \mathrm{Sym}(P)$, all with order 6.

Example 2.1.2 Let $B = (P, L)$ be a board with with $P = \{v_i, i = 1,..., 4\}$ being the set of vertices of the square and $L = \{\{v_i\}, i = 1,..., 4\}$ (i.e.: each line has just a single point). As D_4 acts on L, B is a symmetric board. We have in this case $A_T(B) = \mathrm{Aut}(B)$ and $\mathrm{Aut}(B) \subset \mathrm{Sym}(P)$. $\mathrm{Sym}(P)$ has order $4! = 24$, while $A_{D4}(B)$ and $\mathrm{Aut}(B)$ have both order 8.

Example 2.1.3 The Fano plane (1.1.1) is a symmetric board with D_3 acting on its lines. We have now $A_{D3}(B) \subset \mathrm{Aut}(B) \subset \mathrm{Sym}(P)$, with respective orders 8, 168 [m] and $7! = 5040$.

Example 2.1.4 If instead of the tiled square we had used the fundamental polygon of a square-tiled torus (see Figure 2.2) as a source for board B_1 (1.2.1) then $A_T(B_1)$ would have been larger, as the symmetry group of the torus has D_4 as a subgroup but also the circular permutations of rows and columns coming respectively from 120° turns of the torus around its vertical and internal axis.

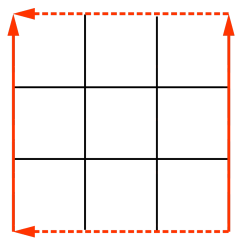


FIGURE 2.2

Definition 2.1.6 (design symmetrization problem, DSP). Let $B = (P, L)$ be a generic design and S_{GD} the set of all symmetric boards $B' = (P', L')$ isomorphic to B. The *design symmetrization problem* (*DSP*) asks for the subset of S_{GD} containing the boards whose lines are acted upon by the highest order symmetry groups.

DSP may be approached by establishing a bijection between the points in P and a set of geometric points in a space of k dimensions, with $1 \leqq k \leqq |P|-1$, then arranging the geometric points in all relevant ways, and finally looking for the largest symmetry group that acts on the lines.

This exploration is to be carried out for all values of k in the range, so the problem does not look trivial. But if $\mathrm{Aut}(P)$ is known, Lemma 2.1.2 provides an alternative: find first all subgroups in $\mathrm{Aut}(P)$, as one of them may be $A_T(B)$, then see which one may correspond to actions of symmetry groups, and finally select the ones among them with the highest order.

For practical reasons, the exploration may use symmetry groups together with homothetic transformations, so as to rather work with representatives of the equivalence class they induce in the set of boards.

2.2 Weft and woof boards

Definition 2.2.1 The action of group T on set P is *transitive* if P is non-empty and if for any p_1, p_2 in P there exists a t in T such that $t{\cdot}p_1 = p_2$.

So any element in P can be reached from any other if the action is transitive. We note that neither D_4 on the set of lines of B_1 nor D_3 on the set of lines of the Fano plane act transitively. If the symmetric board is resolvable (Definition 1.2.5) then symmetry and balance can be present at another level in the design

Definition 2.2.2 Let B be a resolvable symmetric board with symmetry group T and a singleton SIN. If T acts transitively on the parallel classes then B is a *weft board*.

In a weft board the elements of the group not only take classes to classes: there is also always a sequence of groups elements that take any class in the board to any other. We note that trivial boards with a single parallel class are avoided by the very definition of SIN (Definition 1.2.8), and also that a board may be a weft one even if the group does not act transitively *on its lines*. This condition for the SIN is inspired by the weft board inside the Latin square, whose SIN is $\{1\}$, a singleton; the condition aims at rendering the resulting boards more balanced.

Example 2.2.1 D_4 acts on the set of parallel classes of B_1 (it takes the set of rows to the set of columns and vice versa) and its SIN is $\{1\}$, so B_1 is a weft board. This example shows that the geometric nature of boards may produce upfront non-trivial resolutions.

Definition 2.2.3 A *woof board* is a symmetric board that is not a *weft board*.

In a woof board either there are no parallel classes, or being the classes there the group does not act upon them, or even when the action is there it is not transitive, or the condition for the SIN is not fulfilled. The Fano plane is an example of a woof board.

2.3 Symmetry groups and sources of points

A way to find symmetric boards featuring a symmetry group T is to start with a geometric object having T symmetry. We then define points and lines in such a way that T acts just on the lines (for a woof board) or the classes (weft board). The resulting board B is assured to have at least $A_T(B)$ as an automorphism group as per Lemma 2.1.2.

Definition 2.3.1 A *source of points* is a symmetric geometric object.

A source of points (or *source* for short) may be given analytically when it is very complicated or has many dimensions.

Examples 2.3.1 The right picture in Figure 1.1.1 (right) is a source, as it has D_3 symmetry. Figure 1.2.2 (left) is also a source as it has D_4 symmetry. These two sources are usually drawn together with the boards based on them to render the design clearer. We will also do this with other boards and their sources in the rest of the paper.

Example 2.3.2 A source with rich structure gives more options to define points on. Consider the $\mathbf{E}^2$ objects in Figure 2.3.1. Each one is a regular polygon tiled with regular polygons. They are sources because they are symmetric. The also feature structure and scalability: they may have any number of tile's sides on each perimetric side and still keep the symmetry, which means that entire families of symmetric boards can be defined in a very compact way.

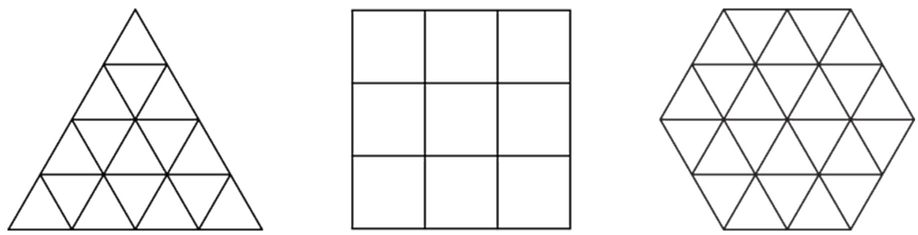


FIGURE 2.3.1

Natural candidates for T are the symmetry groups of finite regular polytopes, which are themselves natural sources of points as we will see later.

2.3.1 **Kaleidoscopic sources**. Sources can also be built from scratch in any space, or have more structure added to them while preserving the symmetry. We use for this group actions (Definition 2.1.4) and ideas from [c75]

Definition 2.3.2 Let T be a group acting on a set of points P. The *orbit* of a point p is the set of points in P to which p is moved by the elements of T.

Definition 2.3.3 Let T be a group acting on an space S. A *fundamental region* is a subset of S which contains exactly one point from the orbit of every point in S.

Definition 2.3.4 Let T be a symmetry group acting on an space S. Let E be a set of points inside a fundamental region. Then $K = \{\text{orbit of } p, p \in E\}$ is the *kaleidoscopic set of E by T.*

A kaleidoscopic set so constructed is evidently a source. The concepts defined have clear counterparts in a kaleidoscope: the points in E are the colored beads inside the tube and the fundamental region the chamber in which they are confined; the set of images of a bead is the orbit of a point; S is the plane in which the beads move; the kaleidoscopic set is the image seen for a certain rotation of the tube and T's generators are the inside mirrors.

Example 2.3.3 Let S be $\mathbf{E}^2$ and T be D_3. Figure 2.3.2 shows the 6 fundamental regions here. E is in the center and K on the right.

FIGURE 2.3.2

Example 2.3.4 Let S be the plane and T the group D_4. The 8 fundamental regions are shown in Figure 2.3.3 (left), E is in the center and K on the right.

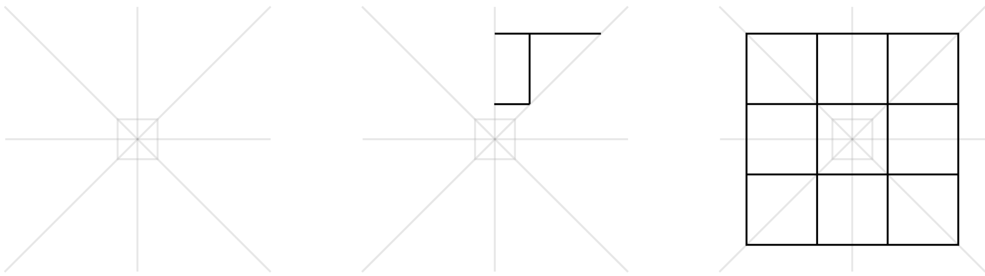


FIGURE 2.3.3

2.4 Design points and lines

Once a source is available, we obtain a symmetric board after defining points and lines in such a way that the symmetry group of the source acts just on the lines (woof board) or on the parallel classes (weft board) if the board is resolvable.

Example 2.4.1 In the source in Figure 2.4.1 (left), let's turn into design points its 15 vertices and make a design line out of each pair of points on the same geometric line (30 design lines in all). The board including the source is shown on the right. As D_3 acts on the lines we have a symmetric board. Every design point lies on 4 design lines and each design line goes through 2 design points: a design with some balance indeed, but not as much as the Fano plane.

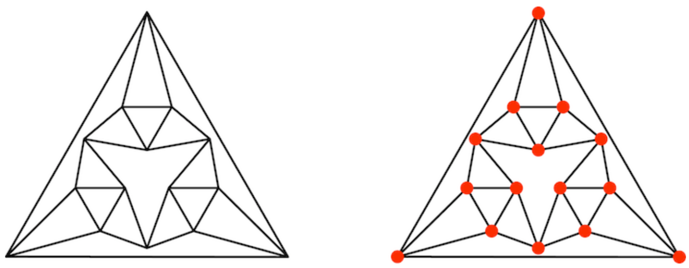


FIGURE 2.4.1

Example 2.4.2 Figure 2.4.2 (left) shows a 9x9 tiled square with D_4 symmetry, hence a source. Let's make a design point out of each face center (center), and a design line with the 9 face centers lying

- on each row (9 design lines that we group in set H)
- on each column (9 design lines that we group in set V)
- on each 3x3 subsquare (9 design lines that we group in set Q)

The board is shown on the right. As D_4 acts on the lines the board is symmetric. All three sets have lines of size 9 and partition the set of points, so each is a 9-uniform parallel class. This board is thus resolvable. H and V are orthogonal, whereas each Q line intersects in 3 points 3 H lines and 3 V lines, having an empty intersection with the rest. Although resolvable, the design is not a weft board as D_4 does not act transitively on the parallel classes: it can't take class Q to either H or V. It is then a woof board.

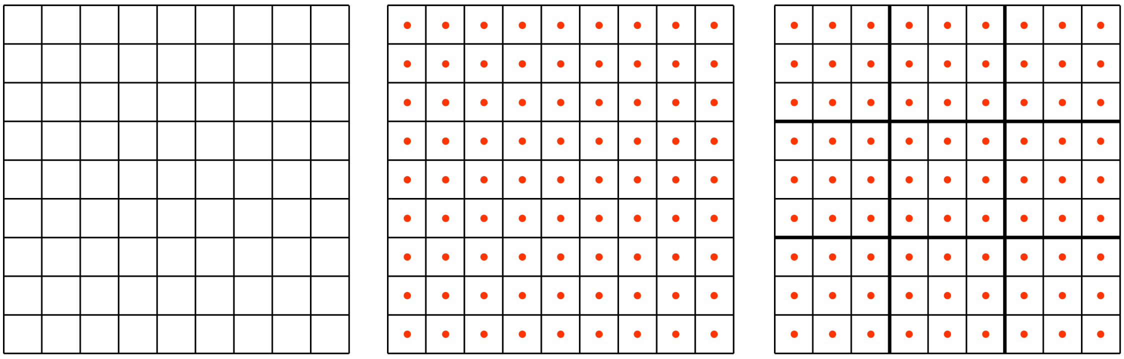


FIGURE 2.4.2

Example 2.4.3 Figure 2.4.3 (left) has dihedral symmetry D_{12}, so it can be a source. Let's take as design points the intersections between geometric lines (36 intersections in all, right) and let's define a 6-uniform line from each 6-set of points lying on the same geometric line (12 design lines in all). The board is shown on the right. The lines are acted upon by D_{12} and hence it is a symmetric board, but not a weft one one as the design is not resolvable (there are no parallel classes here). The board is then a woof board.

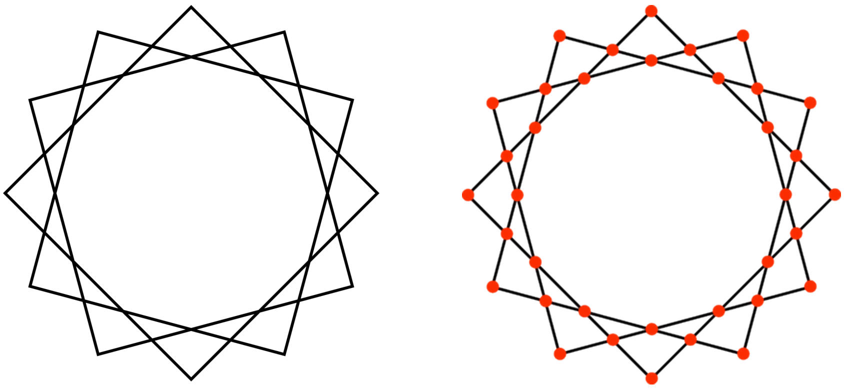


FIGURE 2.4.3

2.5 Three steps to a symmetric board

We summarize the steps to obtain a symmetric board:

1. Choose a symmetry group T
2. Choose or construct a source of points with symmetry T
3. Define design points and lines so that a woof or weft board is obtained

3. WOVEN BOARDS

We consider now nets [cd639] that result after a particular augmentation (Definition 1.3.1) of symmetric boards

Definition 3.1 Let B be a symmetric board. A *k-warp class through B* is a parallel class each of whose lines intersects at k points any line in B. The lines in the class are called *warp lines*.

Definition 3.2 A 1-warp class is an *orthogonal warp class*.

Example 3.1 We saw that board B_1 (1.2.1) is a symmetric board. Class S (Section 1.2.3) is a parallel class that intersects all lines in B_1 in a single point, hence S is an orthogonal warp class through B_1.

Example 3.2 A symmetric board may have more than one warp class. In Figure 1.2.2 (right), let's form the 3 lines of geometric points $\{\{3,4,8\},\{2,6,7\},\{1,5,9\}\}$. The result is another orthogonal warp class through B_1 (as a parallel class it happens to be orthogonal too to warp class S).

In a Latin square each warp line intersect any symmetric line at one point; on the other side the single element in the SIN of its symmetric board is 1. The parameter k in the definition renders the two values independent so that more general woven boards are possible.

Theorem 3.1 *If W is a k-warp class through symmetric board B then every symmetric line has size $k \cdot |W|$.*

Proof: Being W a parallel class (i.e.: a partition of the set of points), every point of every symmetric line l must lie exactly on a single warp line. As every warp intersects l in k points, l has size $k \cdot |W|$. □

So only uniform symmetric boards may feature a warp class. The next theorem gives a necessary condition for the warp class to be uniform too

Theorem 3.2 *If W is a k-warp class through weft board $B = (P, L)$ then W is uniform.*

Proof: Being weft, B is resolvable. Let PC be one of its parallel classes, with $|PC| = m$. Every warp line intersects in k points every line in PC, but as lines in PC are disjoint l must have as many points as there are lines in PC times k, hence W is $(k \cdot m)$-uniform. □

Definition 3.3 A board $B = (P, L \cup L')$ is a *woven board* if $B' = (P, L)$ is a symmetric board and L' is a k-warp class through B'.

The symmetric board is then contained (Definition 1.2.9) in the woven board. Like in a loom, weft/woof lines are interwoven with warp lines to produce woven objects.

Example 3.3 Design B_1 (1.2.1) is a symmetric board contained in board B_2 (Figure 1.2.3). Class S (Section 1.2.3) is an orthogonal warp class through B_1 made up of lines in B_2 but not in B_1, hence B_2 is a woven board.

Theorem 3.3 *No woven board contains the Fano plane.*

Proof: Each line in the Fano plane has 3 points, so the only possible value for k in a warp class through the Fano plane is 1 and every warp line must be orthogonal to every symmetric line. Let's consider an hypothetical warp line that goes through the Fano plane's bottom-center point (see Figure 3.1)

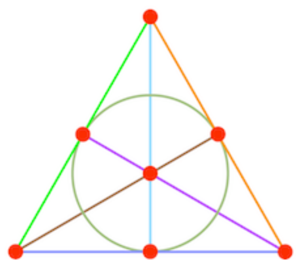


FIGURE 3.1

This warp line intersects at that point the dark blue, light blue and dark green lines. It should intersect too at one point the purple line as it must be orthogonal to it. But it can't do it at the purple line's rightmost point because it would intersect the dark blue line twice. For the same reason –this time because of the light blue line– it can't go through the purple's line center point. And again, due to the dark green line, it can't go through its leftmost point. So there are no warp classes through the Fano plane and hence no woven board contains it. □

4. LATIN BOARDS

Definition 4.1 Let B be a woven board with n warp lines. A *Latin board* with entries from an n-set X is the result of labeling (the points of) each warp line in B with a different element of X.

So from the *same* woven board we can generate *different* Latin boards using different labelings. A Latin board is also called a *Latin combinatorial design* (*LCD*).

Example 4.1 If we label the 3 warp lines in woven board B_2 (Figure 1.2.3) with symbols from a 3-set we obtain a Latin square, so Latin squares are Latin boards. As the warp class here is orthogonal, every symmetric line in board B_1 (1.2.1) intersects every warp line once. So "write the symbols of a 3-set on each symmetric line of board B_1 is equivalent to "find an orthogonal warp class through board B_1" and to "label board B_1 with a 3-set"

Similarly, given a generic l-uniform symmetric board B the specification "find an orthogonal warp class through B" is equivalent to "on each line of B write all symbols of an l-set" and to "label board B with an l-set".

Example 4.2 Write all numbers from 1 to 9 in each line (i.e.: rows, columns and 3x3 subsquares) of the board in Example 2.4.2. Figure 4.2 (left) shows a compliant board taken from [rt165]. These Latin boards are called *sudoku boards* [rtIX]

9	3	7	6	4	5	8	2	1
8	5	2	9	1	3	4	7	6
6	1	4	2	8	7	3	5	9
7	6	3	8	2	9	1	4	5
2	4	9	5	3	1	6	8	7
1	8	5	4	7	6	9	3	2
4	9	6	3	5	2	7	1	8
3	2	1	7	9	8	5	6	4
5	7	8	1	6	4	2	9	3

				4			2	
	5		9					
	1							
			8			1		5
2				3				
						9		
4	9				2			
3							6	
			1					

FIGURE 4.1

Example 4.3 Write all symbols in set {H, E, L, I, O, S} on each line of the board in Example 2.4.3. Using a backtrack algorithm a compliant board was found (Figure 4.1, left). These Latin boards are called *Helios boards* [mh].

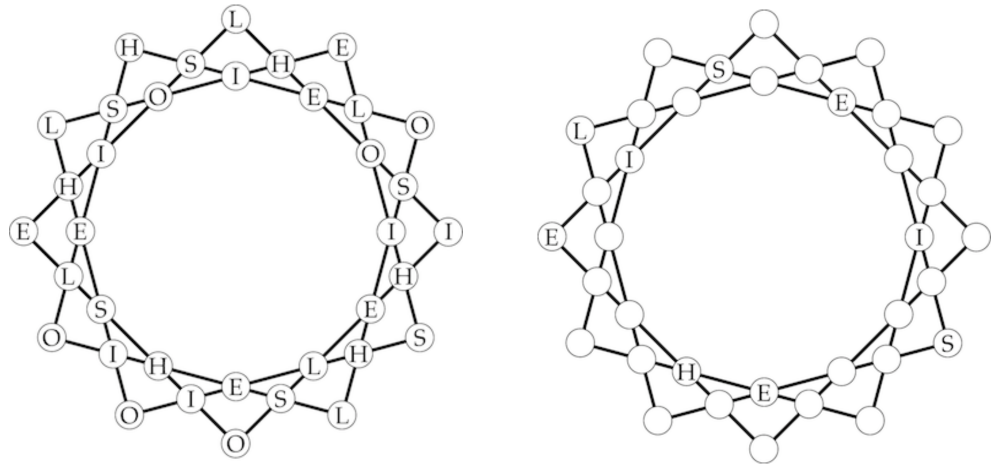


FIGURE 4.2

4.1 Equivalence classes of Latin boards

We saw in Section 1.2.4 that a transformation from the isotopic or paratopic set applied to a Latin square results in another Latin square. Each set of transformations defined an equivalence relation that sorted the squares into different equivalence classes. This applies to any Latin board, although the set of transformations may vary. For example in a Latin square of order 9 there is freedom to permute any two columns, but in sudoku boards the set of legitimate permutations only includes permutations of columns 1-3, permutations of columns 4-6, permutations of columns 7-9 and permutation of blocks of columns 1-3, 4-6 and 7-9.

Each set of transformations will generate different classes, and the more elements in the set the less equivalence classes there will be and the more boards per class will result. Equivalence classes are important when enumerating Latin boards that comply with a particular warp class specification.

When several boards are found, they are much more valuable if they belong to different equivalence classes because if they are in the same one, from one of them we can easily find the others using the transformations in the available set. Algorithms that enumerate or count boards can be designed to avoid looking for boards in the same class of those already found, so as to not incur in unnecessary calculations [cd755]. This needs as a key element the set of transformations that generates the classes, that should be as large as possible for the reasons just exposed.

We also saw that the elements of D_4 (and those in the larger symmetry group of the tiled torus for that matter) were elements of the set of paratopic transformations of the Latin square. This is true also for the symmetry group and set of transformations of any Latin board. So for them, and in absence of knowledge of a larger set, one can always use for the purpose the elements in the symmetry group of the contained symmetric board as the set of transformations.

Finding Latin boards is tantamount to finding the corresponding woven boards. Back to the example, it is clear that D_4 does not act on the lines of woven board B_2 (Figure 1.2.3): a 90º counterclockwise rotation for example takes the diagonal line / to diagonal \, which is not a line in B_2. But it is also clear that the rotated board complies too with the warp class specification, i.e.: it is also a woven board. In fact it is immediate to see that all elements in the orbit of the set of lines in B_2 under D_4 are compliant, isomorphic Latin boards.

4.2 Critical sets

Latin squares, sudoku boards and Helios boards are Latin boards. From the concept of Latin board we can derive others. The definitions that follow are generalizations of [cd146] for Latin boards with orthogonal warp classes

Definition 4.2.1 A Latin board is a *partial (Latin) board* if it has missing symbols and no symbol occurs more than once in any symmetric line.

A partial board poses an interesting problem: can it be completed to a full board? This is in general a hard question. For Latin squares the problem is known to be NP-complete, so no worst-case polynomial time algorithm for the task is known to date. Again, for tractable inputs, dancing links [k] or backtrack algorithms [cd757] can be used.

Definition 4.2.2 A partial board is *completable* if it can be completed to one or several full boards.

Definition 4.2.3 A completable partial board is a *subcritical set* if it can be completed to exactly one full board.

The symmetry and balance of Latin boards make them redundant structures: they contain more information than the strictly necessary to describe them, hence the possibility of subcritical sets.

Definition 4.2.4 A *Latin game* consists of completing a subcritical set to a full board.

Definition 4.2.5 A subcritical set is a *critical set* if the removal of any of the symbols destroys the uniqueness of the completion.

Figure 4.2 (right) shows a critical set for the board on the left. An interesting object of research is the size of *minimal critical sets*, i.e.: those with the minimum number of symbols (see [cd147] for sizes of critical sets of Latin squares). The current quantity for sudoku is 17, a figure claimed to be ultimate in [mtc]. A minimal critical set taken from [rt201] is shown in Figure 4.1 (right), with the corresponding full board shown on the left.

4.3 Five steps to a Latin board

We summarize here the steps to obtain a Latin board:

1. Choose a symmetry group T
2. Choose or construct a source of points with symmetry T
3. Define design points and lines so that a woof or weft board is obtained
4. Find a k-warp class through the symmetric board. This one plus the warp class is a woven board
5. If there are n warp lines in the woven board, choose an n-set S of symbols and label (the points on) each warp line with a different element of S. Different sets of symbols or permutations thereof will produce different Latin boards

5. LATIN POLYTOPES

Definition 5.1 A *Latin polytope* is a Latin board that contains a symmetric board with the symmetry group of a finite regular polytope.

The idea behind the definition is that symmetry goes hand in hand with regularity. Finite regular polytopes can be real or complex. Among the real, there are convex and non-convex ones. Among the convex there are closed intervals in one dimension, regular polygons in two dimensions, 5 platonic polyhedra in three, 6 polychora in four and n-dimensional ($n > 4$) simplexes, cubes and cross-polytopes.

Non-convex polytopes include the regular star-polytopes: the 4 Kepler-Poinsot polyhedra and the 10 regular 4-polytopes. For a comprehensive list of the regular polytopes see [p]. Properly truncated versions of the mentioned polytopes are sources of points too, as are symmetric orthogonal projections of high dimensional polytopes onto lower dimension spaces (bidimensional projections for example feature dihedral symmetry). Figure 7.1 shows the symmetry groups of finite real convex regular polyhedra. For the symmetry group of the general regular polytope see [c130].

To obtain a Latin polytope we follow the steps in Section 4.3. As pointed out in Section 2.3, a way to obtain a symmetric board is to use sources. In the examples that follow we use sources derived from regular polytopes and a backtrack algorithm optimized with the techniques exposed in Section 4.1. The examples are proof that Latin polytopes exist.

6. LATIN POLYGONS

Definition 6.1 A *Latin polygon* is a Latin polytope that contains a symmetric board with the symmetry of a regular polygon. The Latin polygon is called *Latin triangle*, *Latin square*, etc.

The symmetry groups of regular polygons are the D_n dihedral groups, with order $2{\cdot}n$ (n reflections and n rotations). Some dihedral groups properly contain other dihedral groups. When a Latin polygon has more than one symmetry group we will name it after the polygon with the group of highest order.

Examples 6.1 A Latin square is a Latin board that contains a weft board with D_4 symmetry, so Latin squares are Latin polygons. Sudoku and helios boards are also Latin polygons as they contain woof boards with dihedral symmetry.

A way to obtain symmetric boards for Latin polygons is to look for sources of points that share characteristics with the tiled square, like the ones defined next

Definition 6.2 A *biregular polygon* is a regular polygon tiled with more than one regular polygon. The number of tiling polygons's sides on each side of the tiled polygon is the *order* of the biregular polygon. The tiling polygons are called *faces*.

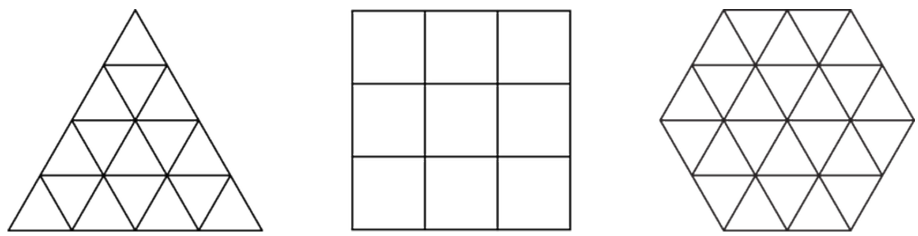


FIGURE 6.1

Constraining the number of faces so that the order is always greater than 1 avoids trivial tilings. Given that there are only 3 regular tilings in $\mathbf{E}^2$ [ma] it is not difficult to prove that in this space there are only 3 families of biregular polygons: equilateral triangles tiled with equilateral triangles (*biregular triangles*), squares tiled with squares (*biregular squares*) and regular hexagons tiled with equilateral triangles (*biregular hexagons*), see Figure 6.1.

biregular polygon	order	tiling polygon (face)	tiled polygon	symmetry group	#vertices	#edges	#faces
triangle	n	triangle	triangle	D_3	(n+1)·(n+2)/2	3·n·(n+1)/2	n^2
square	n	square	square	D_4	$(n+1)^2$	2·n·(n+1)	n^2
hexagon	n	triangle	hexagon	D_6	$3·n^2+3·n+1$	3·n·(3·n+1)	$6·n^2$

FIGURE 6.2

Biregular polygons feature dihedral symmetry and so they are eligible sources for woof and weft boards as we will see next. Figure 6.2 show the relevant parameters of each family. The following sections show a few examples of Latin polygons based on biregular polygons.

6.1 Latin triangles

Example 6.1.1 Figure 6.1.1 shows a biregular triangle of order 6 on which we have chosen face centers as design points (red dots). A design line is specified as the set of points pointed to by the same letter. Each line has size 12 and belongs to one of 3 parallel classes: {a, b, c}, {d, e, f} and {g, h, i}. We have then a uniform resolution.

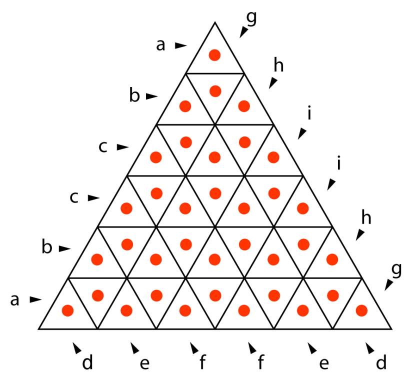


FIGURE 6.1.1

Furthermore, each line in a class intersects any other in another classes in 4 points. To see this, take the smaller upper 3-row triangle and turn it clockwise around the center of the larger triangle's right side until both halves of this side coincide. We see then that every line in the first class intersects in 4 points any other line in the second class. The symmetry of the board allow us to conclude that this applies to any two lines belonging in different classes, so the SIN is {4}. It is immediate to verify also that D_3 acts transitively on the classes, so we have a weft board. In fact, provided that the order n of the biregular triangle is even, we have a whole family of weft boards likewise built, each with n^2 points and 3 uniform parallel classes, each with $n/2$ lines of size $2·n$ each.

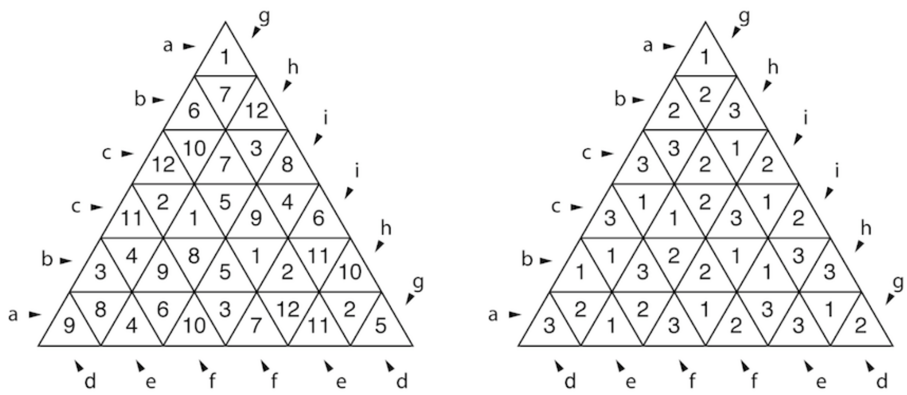


FIGURE 6.1.2

Figure 6.1.2 (left) shows a board with all numbers from 1 to 12 on every line line of the previous weft board, i.e.: it is a Latin board with an orthogonal warp class. The SIN in the resulting Latin board is {4, 1}. The board is then a Latin triangle, accompanied by its source in the picture. This kind of Latin triangle is called *Monthai* [mm] (see [mc] for larger Latin triangles of this kind). Figure 6.1.2 (right) shows a Latin triangle with the same symmetric board but this time with a 4-warp class: each of its lines intersects in 4 points every symmetric line. The SIN is then {4}. These examples show the usefulness of parameter k in the warp class (Definition 3.1): it allows to establish a compromise between balance and orthogonality.

Example 6.1.2 Figure 6.1.3 shows a biregular triangle of order 7 on which we have chosen vertices as design points (red dots). A design line is specified as the set of points pointed to by the same letter. Each line has size 9 and belongs to one of 3 parallel classes: {a, b, c, d}, {e, f, g, h} and {i, j, k, l}. We have then a uniform resolution.

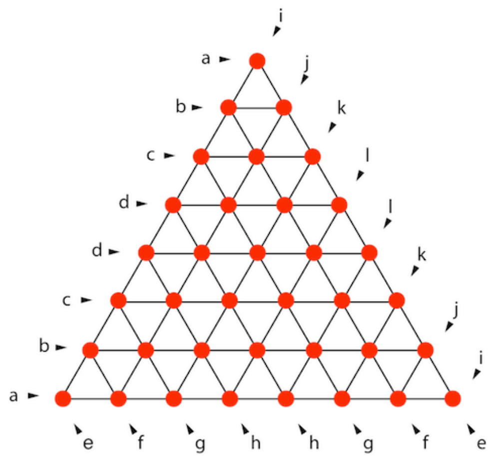


FIGURE 6.1.3

It is immediate to verify that D_3 acts transitively on the classes, and also that the SIN here is not a singleton, so we have a woof board, not a weft one. In fact, provided that the order n of the biregular triangle is odd we have a whole family of woof boards, each with $(n+1)\cdot(n+2)/2$ points and 3 uniform parallel classes, each with $(n+1)/2$ lines of size $n+2$ each.

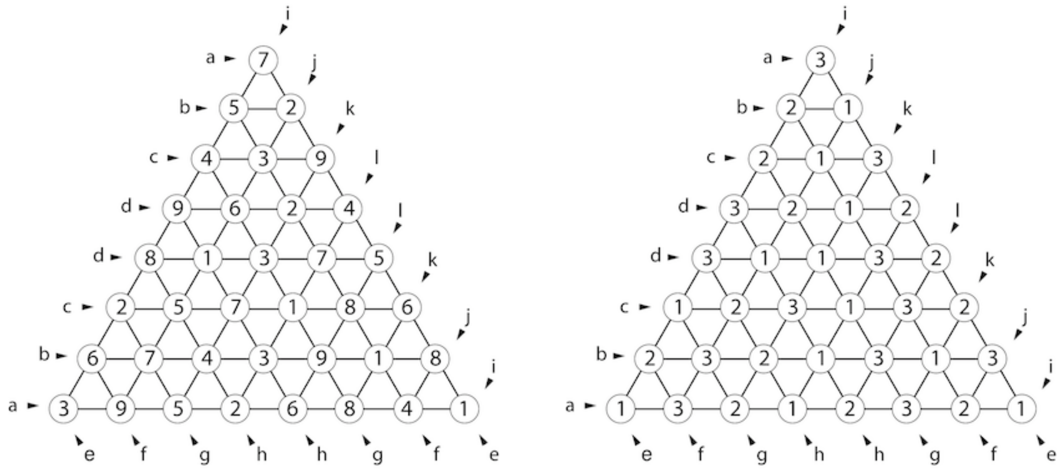


FIGURE 6.1.4

Figure 6.1.4 (left) shows a board with all numbers from 1 to 9 on every line of the previous woof board, i.e.: it is a Latin board with an orthogonal warp class. The board is then a Latin triangle, accompanied by its source in the picture. Figure 6.1.4 (right) shows a Latin triangle with the same symmetric board but this time with a 4-warp class: each of its lines intersects in 4 points every symmetric line.

Example 6.1.3 Figure 6.1.5 shows a biregular triangle of order 5 on which we have chosen edges' centers as design points (red dots). As before, a design line is specified as the set of points pointed to by the same letter. Each line has size 10 and belongs to one of 3 parallel classes: $\{a, b, c\}$, $\{d, e, f\}$ and $\{g, h, i\}$. We have then a uniform resolution.

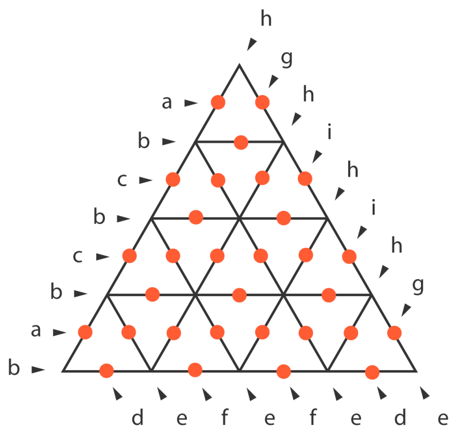


FIGURE 6.1.5

It is immediate to verify that D_3 acts transitively on the classes, and also that the SIN here is not a singleton, so we have a woof board, not a weft one.

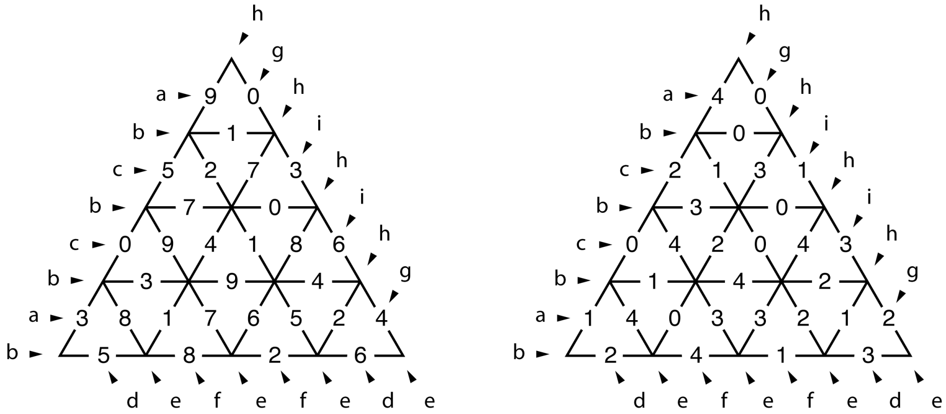


FIGURE 6.1.6

Figure 6.1.6 (left) shows a board with all numbers from 0 to 9 on every line of the previous woof board, i.e.: it is a Latin board with an orthogonal warp class. The board is a Latin triangle, featured in the picture with its source. Figure 6.1.6 (right) shows a Latin triangle with the same symmetric board but this time with a 4-warp class: each of its lines intersects in 4 points every symmetric line.

6.2 Generalized Latin squares

Example 6.2.1 Figure 6.2.1 shows a biregular square of order 8 with face centers as design points (red dots). A design line is made up of sets of points pointed to by the same letter. Each one has size 16 and belongs in either parallel class $\{a, b, c, d\}$ or $\{e, f, g, h\}$. We have then a 16-uniform resolution in which each line in a class intersects any other in the other classes in 4 points, so the SIN is $\{4\}$. It is immediate to verify that D_4 acts transitively on the classes, so we have a weft board.

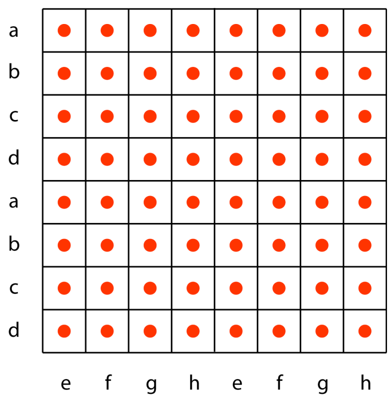


FIGURE 6.2.1

We can build symmetric boards on a biregular square of any order n (n^2 points). If the order is prime we can only have symmetric (weft) boards that will originate conventional Latin squares. For composite orders we have several options to group rows –and columns– into design lines. If n is even and lines come in two pieces like in the example there are 2 uniform parallel classes, each with $n/2$ symmetric lines each of size $2{\cdot}n$ each. The particular pairing used will determine if the board is woof, weft or non-symmetric.

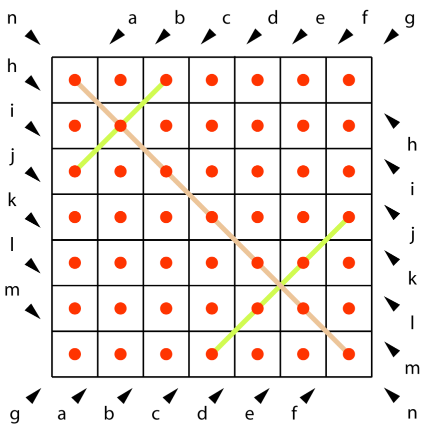
a	9	2	15	7	8	1	12	14
b	11	8	5	12	2	13	10	3
c	14	4	1	16	15	9	7	6
d	13	6	3	10	4	16	11	5
a	10	3	16	13	5	11	6	4
b	6	7	4	9	16	15	14	1
c	12	5	2	11	3	10	13	8
d	1	14	8	15	7	12	9	2
	e	f	g	h	e	f	g	h

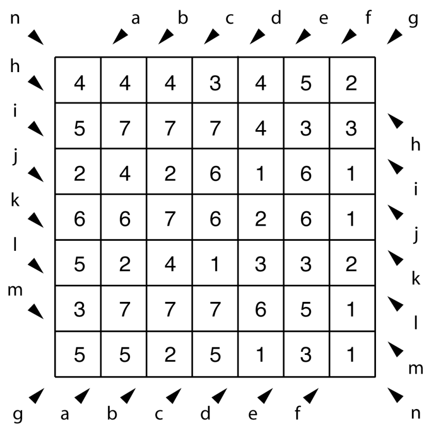
a	4	1	3	1	3	3	1	3
b	3	3	1	4	4	1	2	3
c	2	2	2	4	4	4	4	3
d	4	1	2	3	2	2	4	1
a	1	2	2	2	2	4	4	4
b	2	2	4	2	1	4	3	1
c	1	1	3	1	3	3	1	2
d	1	4	3	2	3	3	1	4
	e	f	g	h	e	f	g	h

FIGURE 6.2.2

Figure 6.2.2 (left) shows a board with all numbers from 1 to 16 on every symmetric line, i.e.: it is a Latin board with an orthogonal warp class and SIN {4}. The board is a (generalized) Latin square accompanied by its source in the picture (for a generalized Latin square with a woof board and 3 parallel classes see [mt]). Figure 6.2.2 (right) shows another generalized Latin square but this time with a 4-warp class: each of its lines intersects in 4 points every other symmetric line. The SIN is then {4}.

Example 6.2.4 A Latin square is a Latin polygon with SIN {1}, a weft board with D_4 symmetry and a biregular square as source with face centers as design points. This description does not univocally determines the conventional Latin square as the next example shows. Figure 6.2.3 (left) shows a biregular polygon of order 7 whose faces have been chosen as design points. 14 lines of size 7 have been formed, each one coming from either one or two slanted sets of points pointed to by the same letter (for example line c in green or line n in brown). There are 2 parallel classes here: {a, b, c, d, e, f, g} and {h, i, j, k, l, m, n}. As D_4 acts transitively on the classes the design is a weft board with SIN {1}.





4	4	4	3	4	5	2
5	7	7	7	4	3	3
2	4	2	6	1	6	1
6	6	7	6	2	6	1
5	2	4	1	3	3	2
3	7	7	7	6	5	1
5	5	2	5	1	3	1

FIGURE 6.2.3

Figure 6.2.3 (right) shows a board with all numbers from 1 to 7 on every symmetric line, i.e.: it is a Latin board with an orthogonal warp class. The SIN in the Latin board is {1}. The board is then a (generalized) Latin square, accompanied by its source in the picture.

6.2.1 **Biregular square's vertices as design points**. We can also choose the biregular square's vertices as design points, but the resulting boards are essentially the same as those built with face centers. The reason is the natural isomorphim beetween boards based on face centers and those based on vertices shown in Figure 6.2.4.

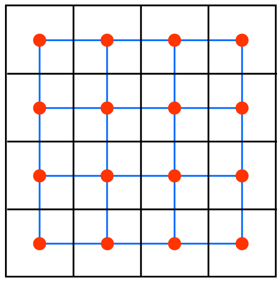


FIGURE 6.2.4

6.3 Latin hexagons

Example 6.3.1 Figure 6.3.1 shows a biregular hexagon of order 3 on which we have chosen face centers as design points (red dots). A design line is made up of sets of points pointed to by the same letter. Each line has size 18 and belongs to one of 3 parallel classes: {a, b, c}, {d, e, f}, {g, h, i}. We have then a 18-uniform resolution. It is easy to verify that the SIN is not a singleton and that D_6 acts on the lines, so we have a woof board (D_3 acts also on the lines but we choose to focus on the largest symmetry group).

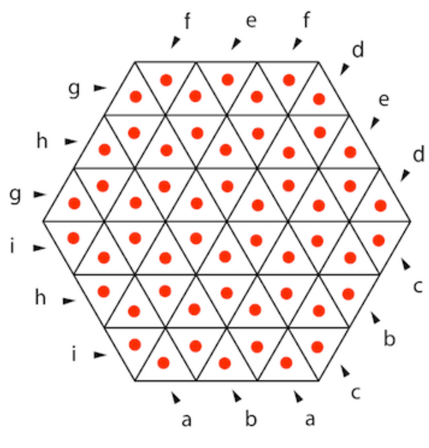


FIGURE 6.3.1

If lines come in two pieces like in the example we can have symmetric boards for any order n of the biregular hexagon ($6{\cdot}n^2$ points) with 3 uniform parallel classes, each with n symmetric lines of size $6{\cdot}n$ each.

If, starting from the bottom-left letter and going counterclockwise, we change the sequence of letters around the hexagon to a, b, c, c, b, a, d, e, f, f, e, d, g, h, i, i, h, g we obtain a weft board, as now all the condition for it are fulfilled. To see that the SIN is a singleton in this new board split the hexagon in half horizontally and join the upper half's left side to the lower half's right side. Now it is easy to check that each line in {g, h, i} intersect each other in {a, b, c} in 6 points. Due to the symmetry of the board this happens with any two lines belonging to different classes, hence the SIN is {6} and the board is a weft one.

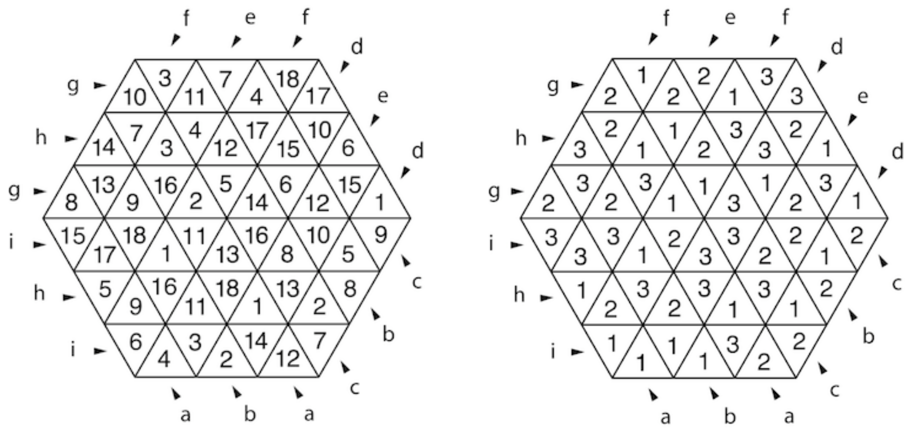


FIGURE 6.3.2

Figure 6.3.2 (left) shows a board with all numbers from 1 to 18 on every symmetric line, i.e.: it is a Latin board with an orthogonal warp class. The board is a Latin hexagon, accompanied by its source in the picture. This type of Latin hexagon is called *Douze France* [md]. Figure 6.3.2 (right) shows another Latin hexagon with the same symmetric board but this time with a 6-warp class: each of its lines intersects in 6 points every other symmetric line.

Example 6.3.4 Figure 6.3.3 shows a biregular hexagon of order 3 on which we have chosen edge centers as design points (red dots). A design line is made up of sets of points pointed to by the same letter. Each line has size 14 and belongs to one of 3 parallel classes: {a, b, c}, {d, e, f} and {g, h, i}. We have then a 14-uniform resolution but the SIN is not a singleton. It is immediate to verify that D_6 acts on the lines, so we have a woof board.

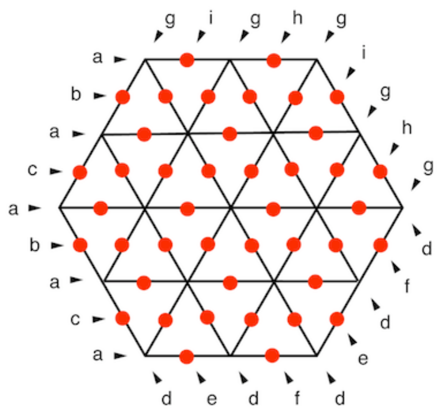


FIGURE 6.3.3

Figure 6.3.4 (left) shows a board with all numbers from 1 to 14 on every symmetric line, i.e.: it is a Latin board with an orthogonal warp class. The board is a Latin hexagon accompanied by its source in the picture. Figure 6.3.4 (right) shows another Latin hexagon with the same symmetric board but this time with a 2-warp class: each of its lines intersects in 2 points every other symmetric line.

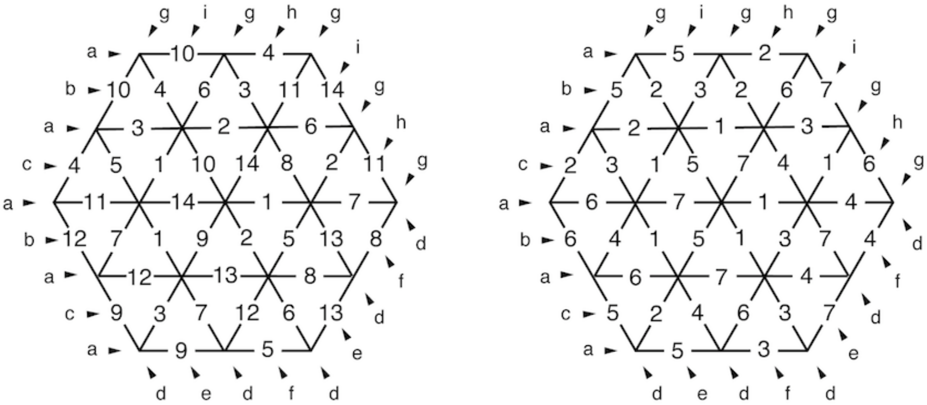


FIGURE 6.3.4

7. LATIN POLYHEDRA

Definition 7.1 A *Latin polyhedron* is a Latin polytope that contains a symmetric board with the symmetry of a regular polyhedron.

There are five finite, real, convex, regular polyhedra and three polyhedral symmetry groups (Figure 7.1). We may name the Latin polyhedron either after its polyhedral group or after the polyhedron that is closer to the source used. In any case the underlying symmetry group will be clear. In the examples that follow we will adopt the second naming convention.

symmetry group	finite real convex regular polyhedron	group order
tetrahedral	tetrahedron	12
octahedral	cube, octahedron	24
icosahedral	dodecahedron, icosahedron	60

FIGURE 7.1

A few examples of Latin polyhedra follow. Each one starts with a source of points followed by a symmetric woof or weft board. Afterwards a Latin polyhedron with an orthogonal warp class is shown and finally a critical set for the polyhedron.

7.1 A Latin tetrahedron

The picture in Figure 7.1.1 (left) is a source meant to be folded along the inner black lines into a tetrahedron. On the right we have selected the triangular faces' orthocenters as design points. Each design line is made up of points enclosed between two drawn lines of the same color. Once folded, the two portions of each line become one across the tetrahedron's surface. The design is resolvable as there are 3 parallel classes that are acted upon by the tetrahedral group. The SIN is a singleton, as once unfolded this board is a Latin triangle like the one we saw in Example 6.1.1 which has a singleton SIN. This design is then a weft board.

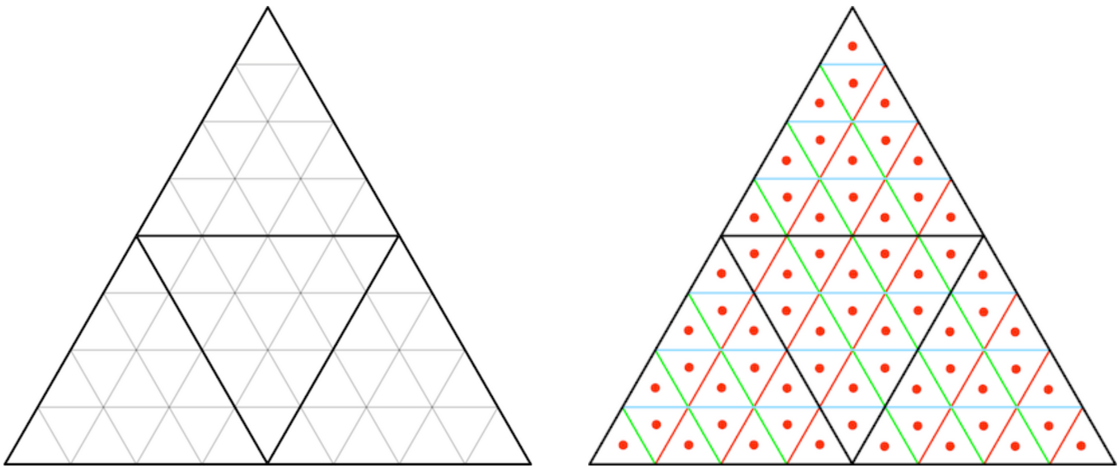


FIGURE 7.1.1

Figure 7.1.2 (left) shows that each symmetric line has all numbers from 1 to 16, i.e.: there is an orthogonal warp class with 16 lines with 4 points each: the board is a Latin tetrahedron. On the right there is a critical set for it. Note that if we fold similarly the Latin triangle in Example 6.1.1 the two portions that make up each symmetric line become one. The resulting object is another –smaller– Latin tetrahedron.

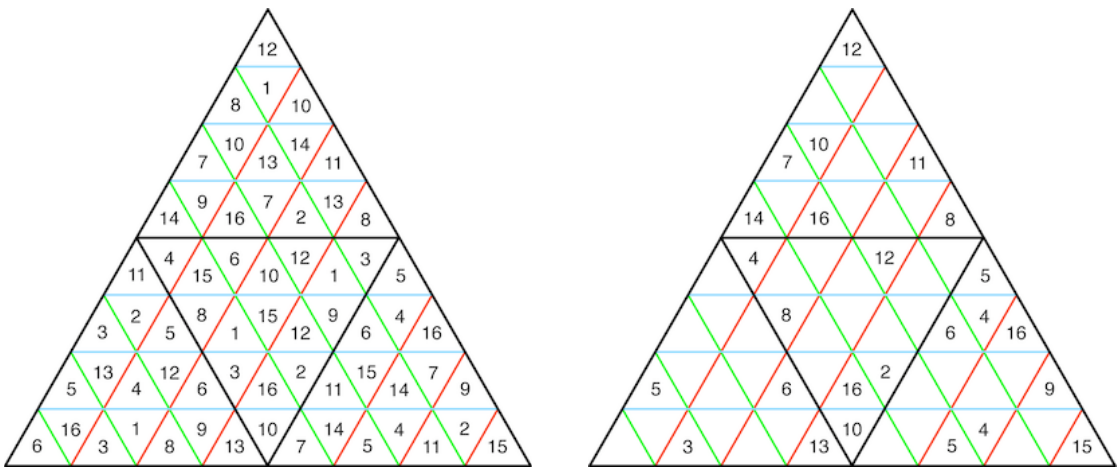


FIGURE 7.1.2

7.2 A Latin cube

The picture in Figure 7.2.1 is a source meant to be folded along the inner black lines into a cube. In Figure 7.2.2 we have chosen the square faces' centers as design points. Each design line is made up of points enclosed between two drawn lines of the same color. Once folded, the portions of each line become one across the cube's surface. The design has no parallel classes and thus it is not resolvable, but the lines are acted upon by the octahedral group. The design is then a woof board.

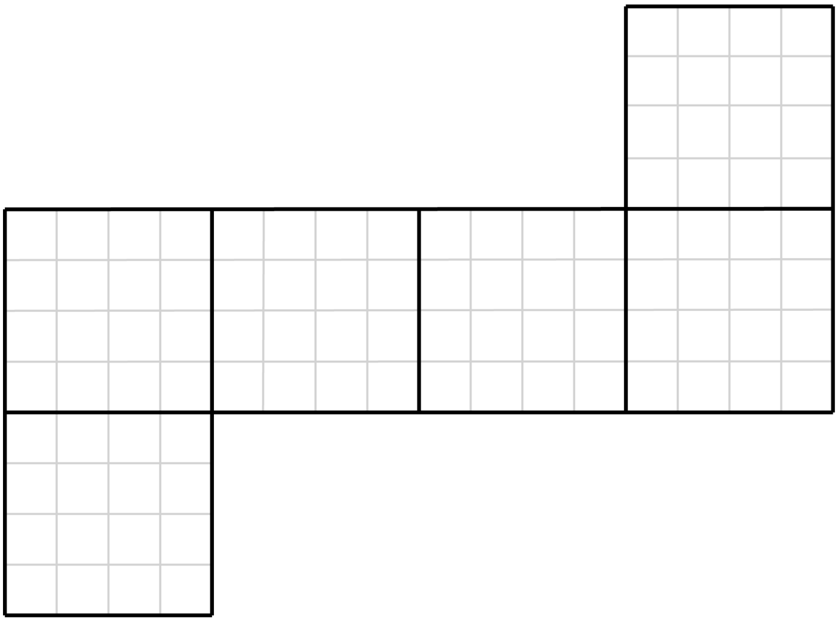


FIGURE 7.2.1

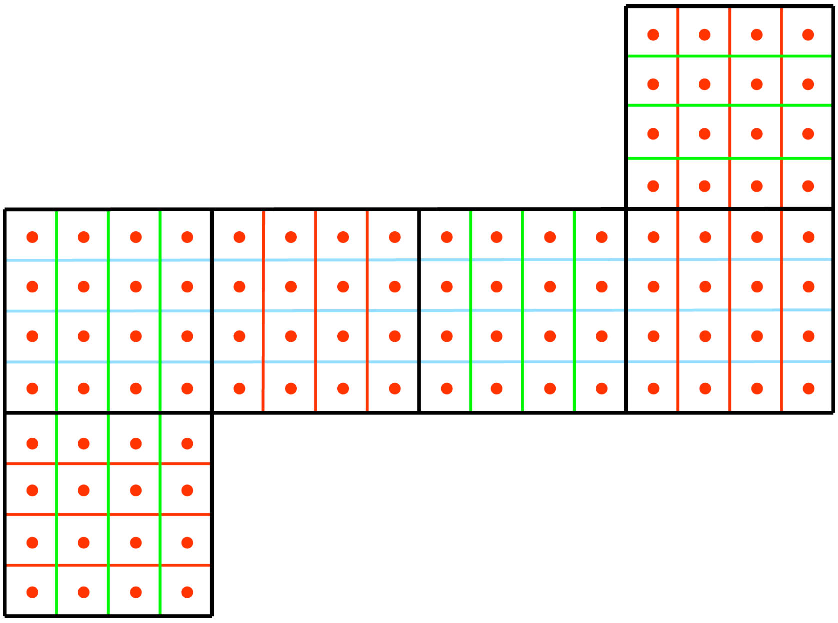


FIGURE 7.2.2

Figure 7.2.3 shows that each symmetric line has all numbers from 1 to 16, i.e.: there is an orthogonal warp class with 16 lines with 6 points each: the board is a Latin cube (or octahedron if we follow the first naming criterion mentioned above). Figure 7.2.4 shows a critical set for it. (note that there is another unrelated combinatorial object also called Latin cube [dk187]).

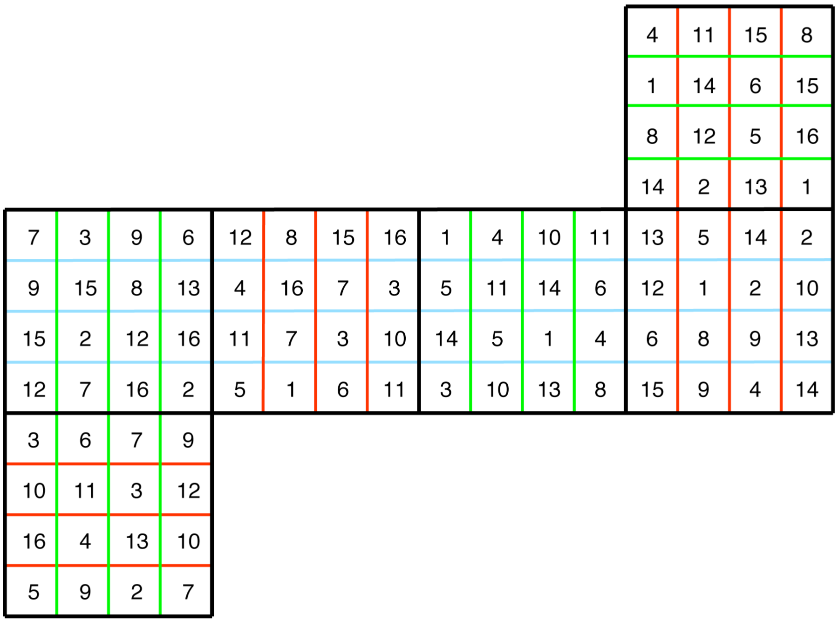


FIGURE 7.2.3

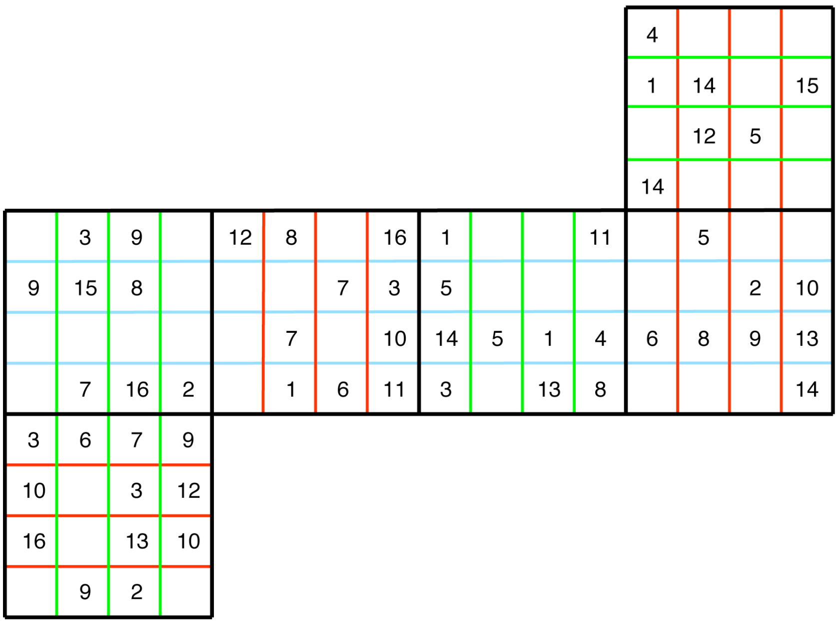


FIGURE 7.2.4

7.3 A Latin octahedron

The picture in Figure 7.3.1 is a source meant to be folded along the inner black lines into an octahedron. In Figure 7.3.2 we have chosen the triangular faces' orthocenters as design points. Each design line is made up of points enclosed between two drawn lines of the same color. Once folded, the portions of each line become one across the octahedron's surface. The design has no parallel classes and hence it is not resolvable, but the lines are acted upon by the octahedral group. The design is then a woof board.

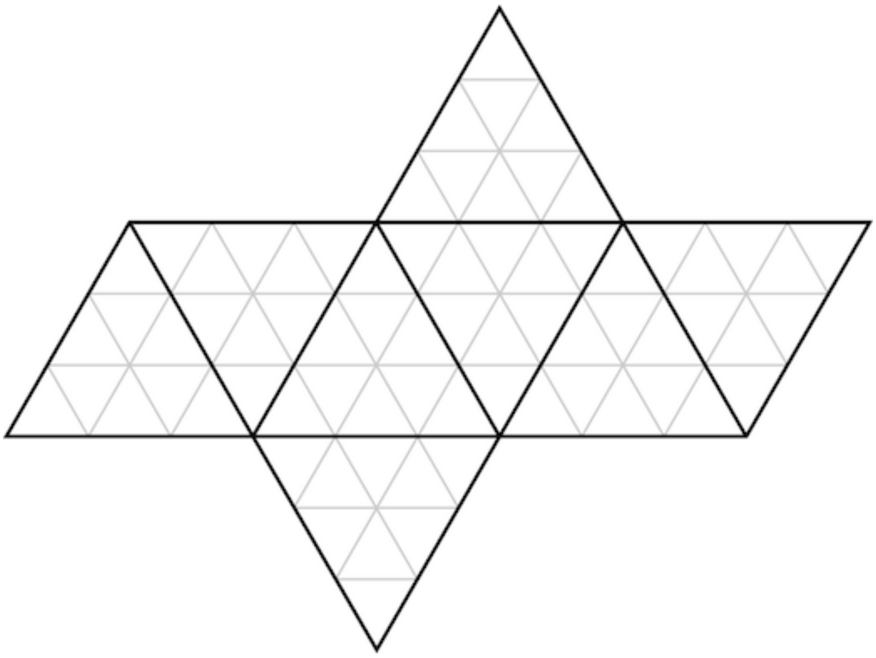


FIGURE 7.3.1

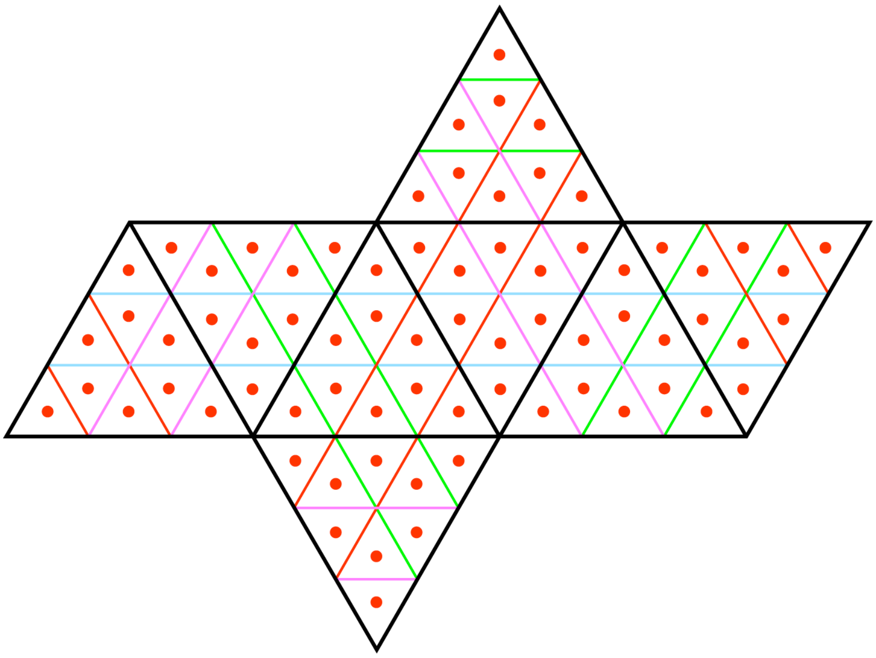


FIGURE 7.3.2

Figure 7.3.3 shows that each symmetric line has all numbers from 1 to 18, i.e.: there is an orthogonal warp class with 18 lines with 4 points each. The board is a Latin octahedron. Figure 7.3.4 shows a critical set for it.

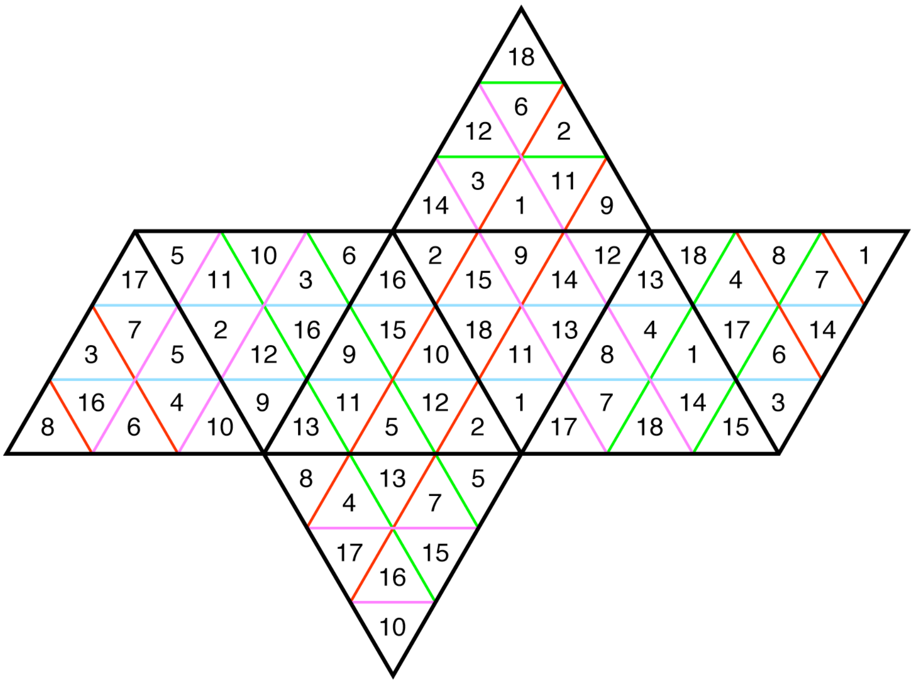


FIGURE 7.3.3

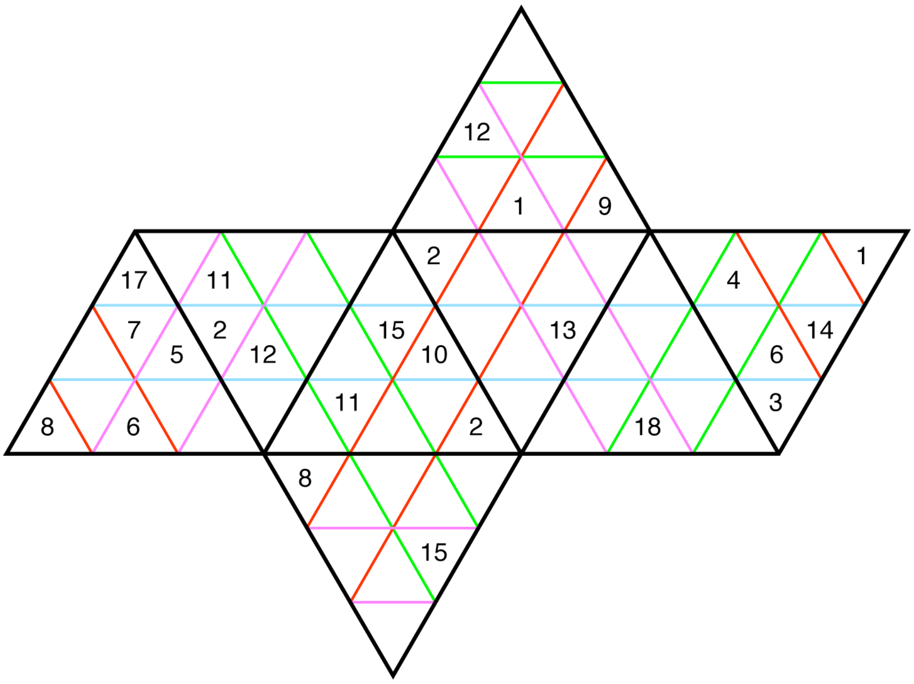


FIGURE 7.3.4

7.4 A Latin icosahedron

The picture in Figure 7.4.1 is a source meant to be folded along the inner black lines into an icosahedron. In Figure 7.4.2 we have chosen the triangular faces' orthocenters as design points. Each design line is made up of points enclosed between two drawn lines of the same color. Once folded the portions of each line become one across the icosahedron's surface. The design has no parallel classes and thus it is not resolvable, but the lines are acted upon by the icosahedral group, the design is then a woof board.

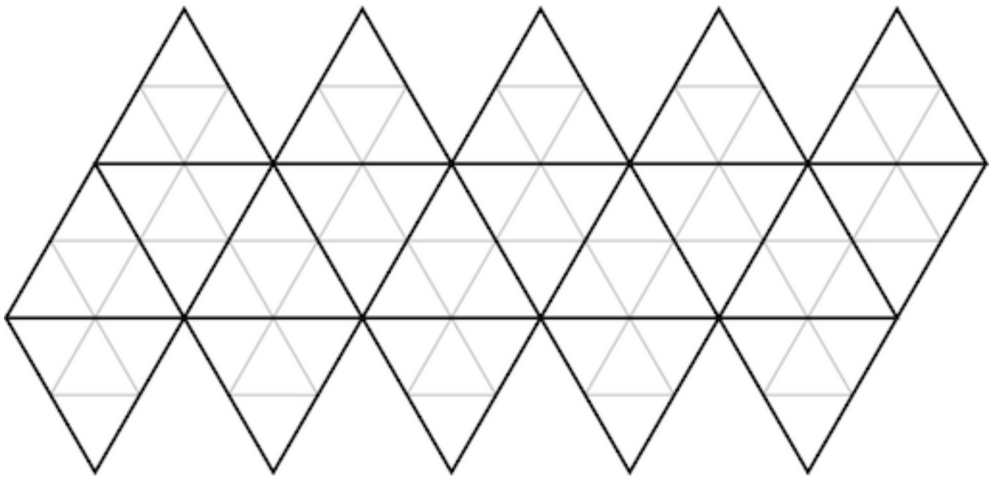


FIGURE 7.4.1

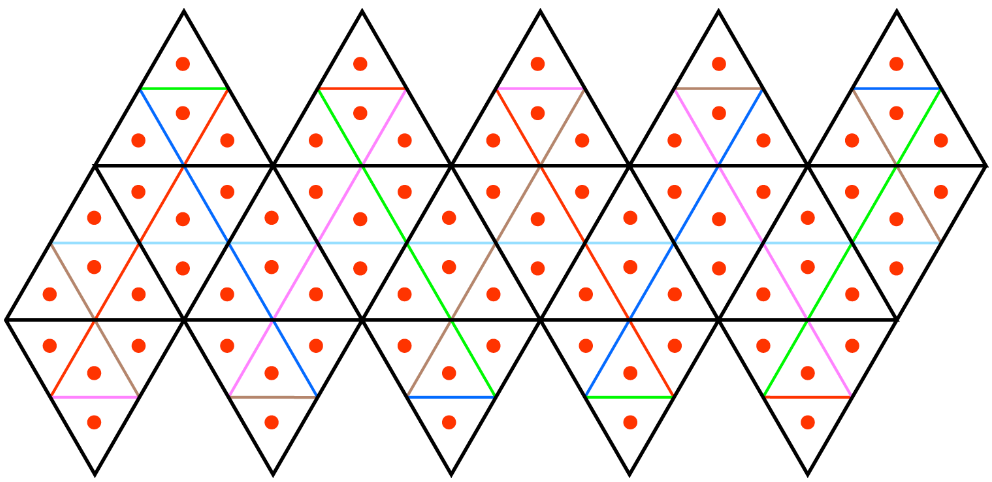


FIGURE 7.4.2

Figure 7.4.3 shows that each symmetric line has all numbers from 1 to 20, i.e.: there is an orthogonal warp class with 20 lines with 4 points each. The board is a Latin icosahedron. Figure 7.4.4 shows a critical set for it.

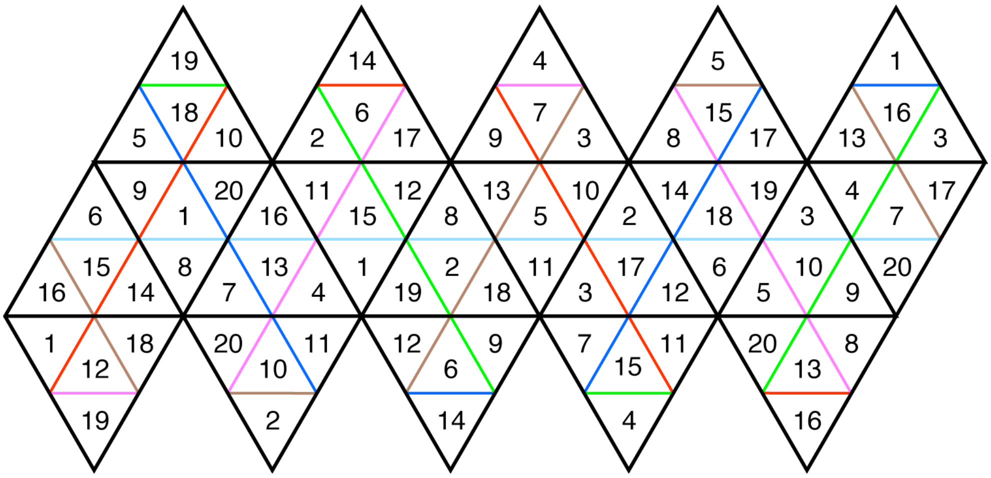


FIGURE 7.4.3

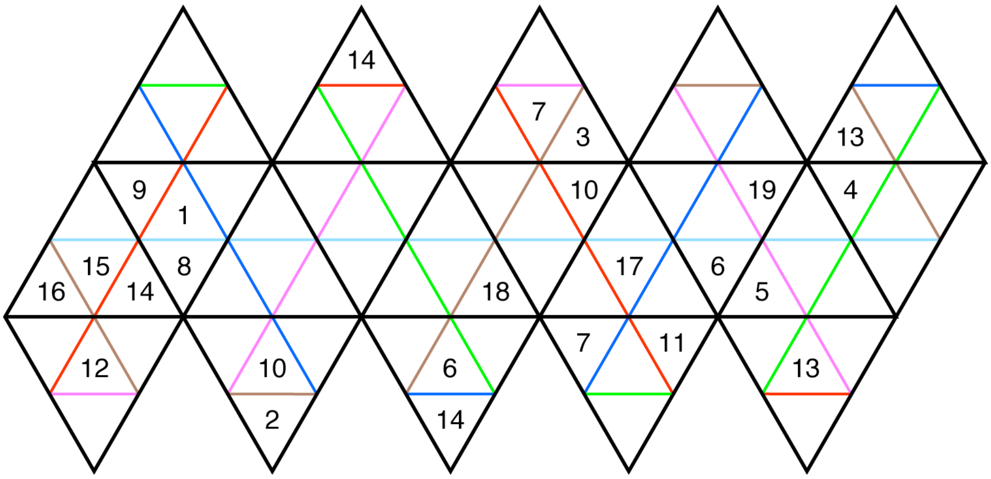


FIGURE 7.4.4

7.5 A Latin dodecahedron

The picture in Figure 7.5.1 is a source meant to be folded along the inner black lines into a dodecahedron. In Figure 7.5.2 we have chosen the edges' centers as design points. Each design line is made up of points lying on a drawn line of a particular color. Once folded the portions of each line become one running across the dodecahedron's surface. The design has no parallel classes and thus it is not resolvable, but the lines are acted upon by the icosahedral group: the design is a woof board.

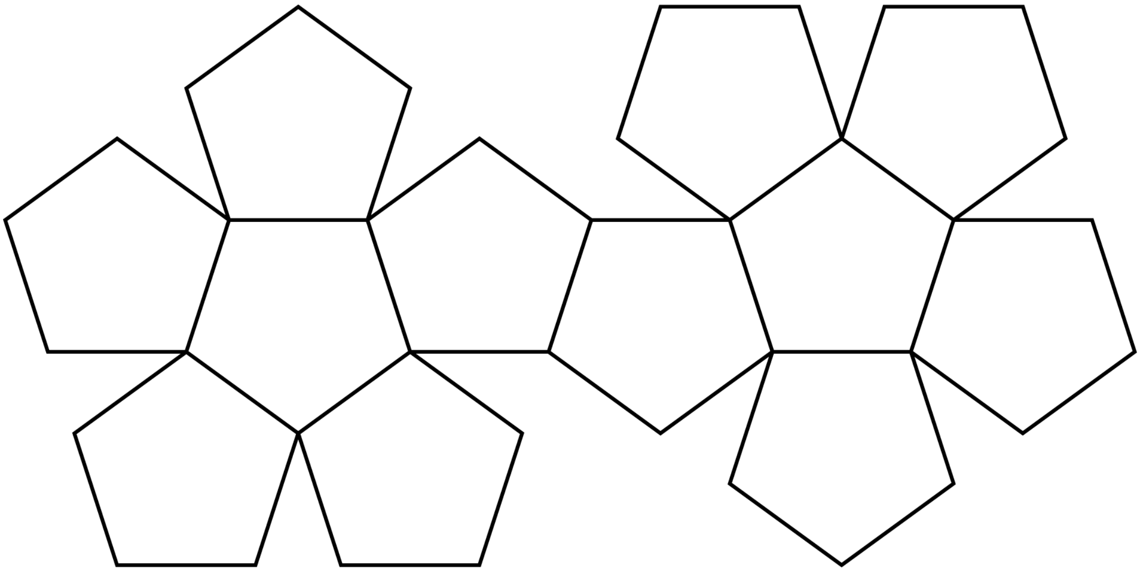


FIGURE 7.5.1

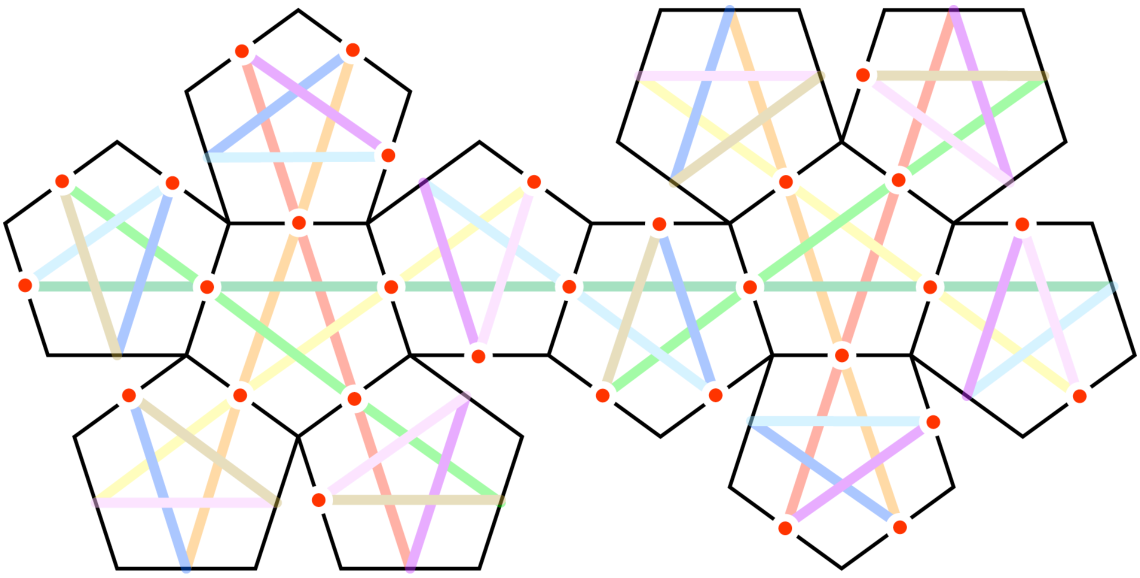


FIGURE 7.5.2

Figure 7.5.3 shows that each symmetric line has all numbers from 1 to 6, i.e.: there is an orthogonal warp class with 6 lines with 5 points each. The design is a Latin dodecahedron (or icosahedron if we follow the first naming criterion mentioned above). Figure 7.5.4 shows a critical set for it.

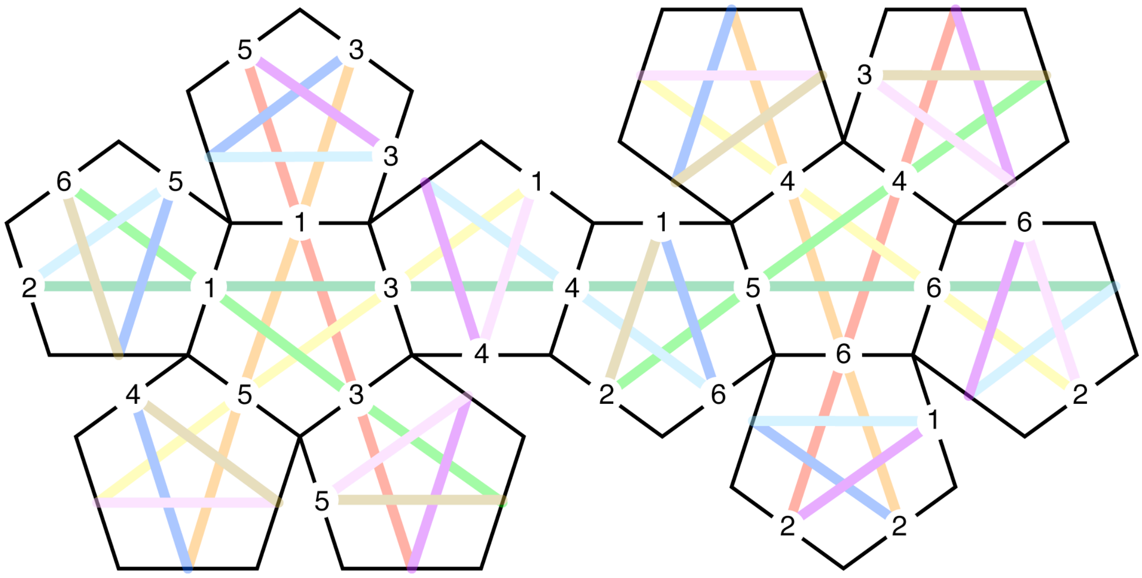


FIGURE 7.5.3

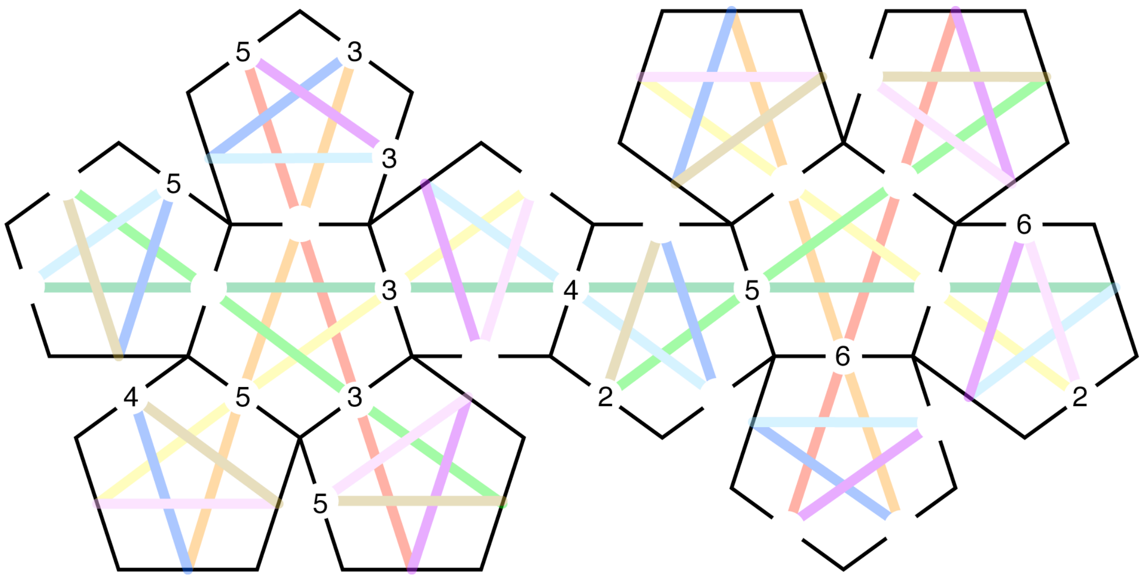


FIGURE 7.5.4

7.6 A woof dodecahedral board

The picture in Figure 7.6.1 is a source meant to be folded along the inner black lines into a dodecahedron. In Figure 7.6.2 we have chosen the orthocenters of the triangles inside the pentagons as design points. Each design line is made up of points lying on drawn lines like the sample one in yellow. All design lines, each denoted by a drawn dashed line of a different color, are showed in Figure 7.6.3. Once folded the portions of each line become one running across the dodecahedron's surface. The design has no parallel classes and thus it is not resolvable, but the lines are acted upon by the icosahedral group. The design is then a woof board. If the calculations carried out are not wrong, this board does not admit any orthogonal warp class through it.

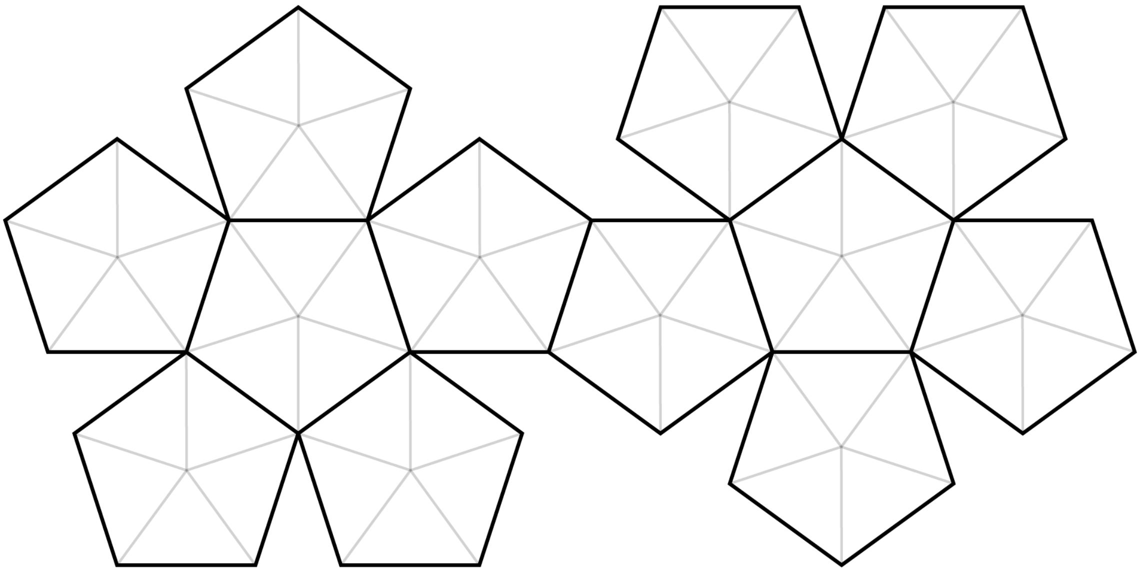


FIGURE 7.6.1

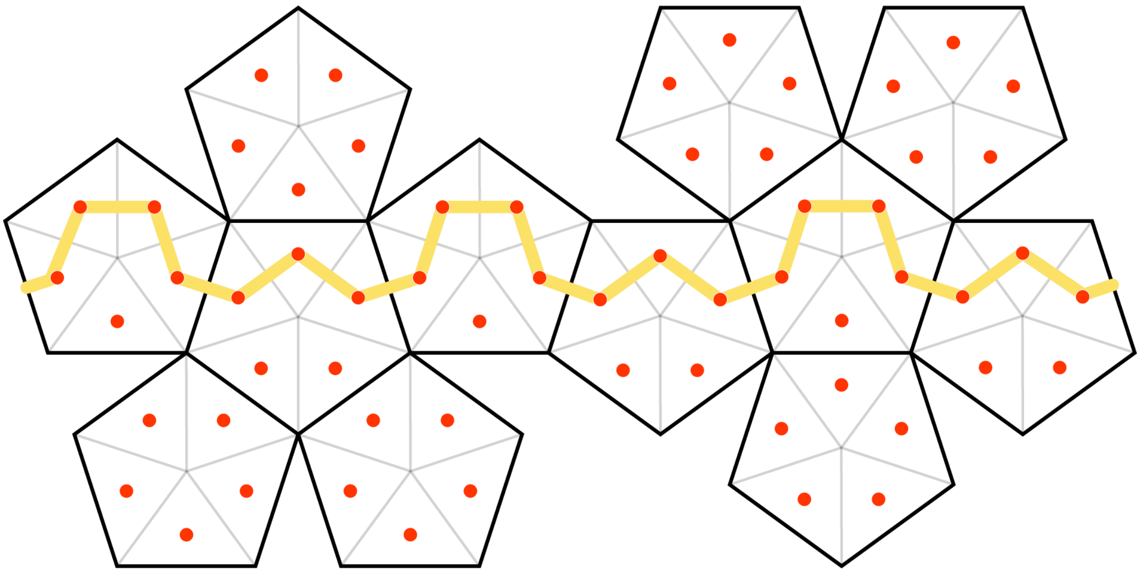


FIGURE 7.6.2

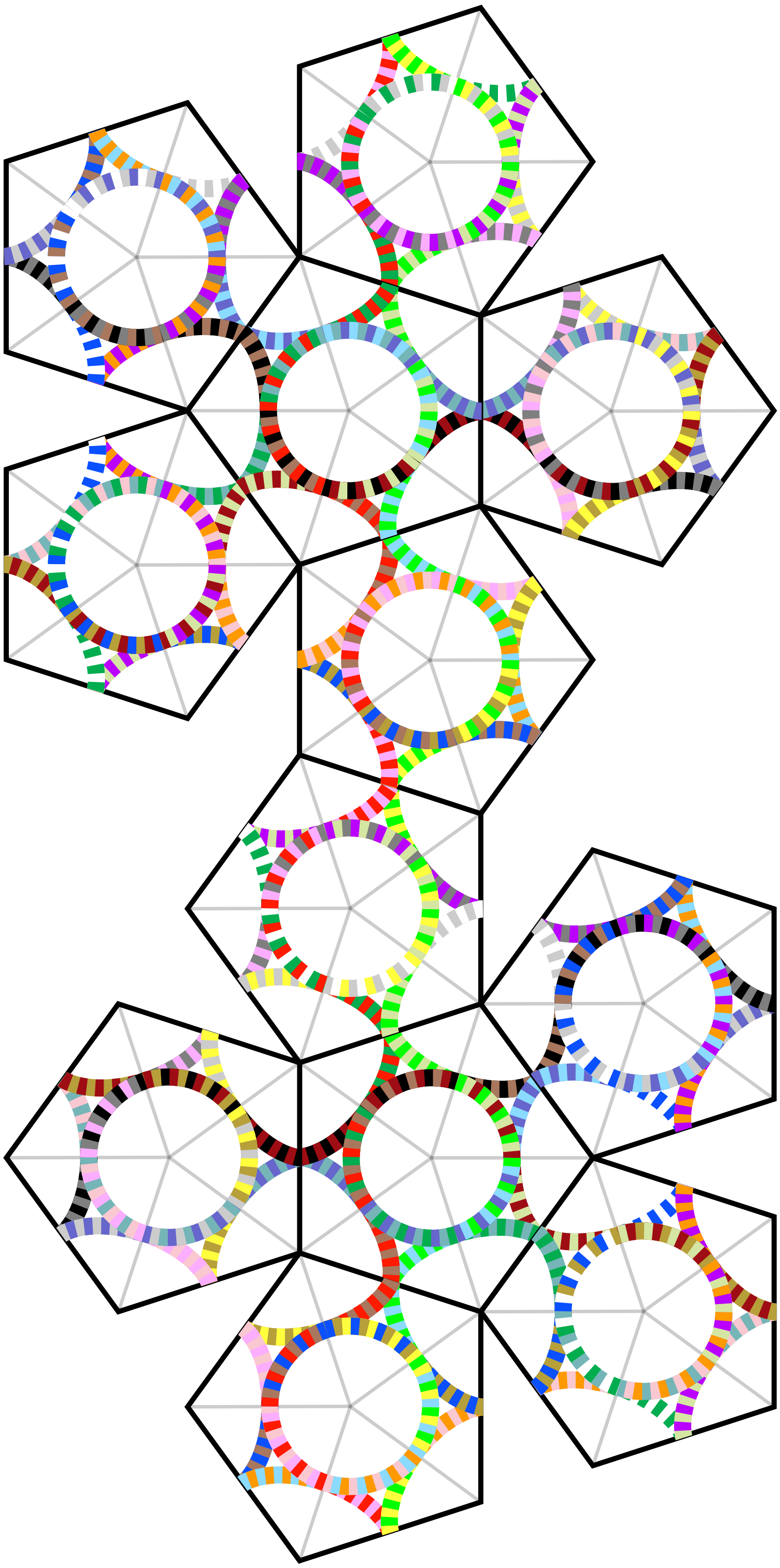


FIGURE 7.6.3

8. CONCLUSION

Designs with geometric points, symmetric lines with balanced intersections, additional labelled lines intersecting the former in a balanced way: this is the generalization of Latin squares that in our view justifies the resulting objects to be called Latin boards or Latin combinatorial designs.

In a Latin board geometric symmetry readily becomes design symmetry in the form of automorphisms in the contained symmetric board and isomorphisms among Latin boards. This initial, immediately available symmetry can be used to speed up backtrack algorithms that count or enumerate boards. Future work may look for new Latin boards such as:

- Latin polygons based on biregular polygons with more symmetric lines than the examples
- Latin polygons with a set of points based on subsets of vertices, edges or faces of biregular polygons, instead of the full set. Or on unions of these subsets
- Latin polygons based on biregular polygons some of whose tiles are themselves biregular polygons
- Latin polygons based on biregular polygons with more than one point per face or edge
- Latin polygons that, like the Helios board, are not based on biregular polygons
- Latin "surface" polyhedra from sources that are polyhedrons with biregular polygons as faces
- Latin "volume" polyhedra, from sources that are partitions of regular polyhedra. Here we have more choices for the set of design points: vertices, edges, faces and parts
- Latin polychora and higher dimensional Latin polytopes
- Latin polyhedra containing weft boards (in the examples only the Latin tetrahedron had one)
- in general, Latin polytopes from sources recursively defined in which any element with dimension above 0 (edges, faces, parts, hyperparts) is subjected to further partition
- infinite Latin polytopes (after removing the restriction for the finitude of the set of design points)
- Latin polytopes in spaces other than the Euclidean ones

As usual, the study of properties goes along with the discovery of new designs: conditions for existence, full automorphism group, equivalence classes, critical sets, SIN, relationship with other designs, derived permutation groups, applications, etc.

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Reference format: [cd130] means: page 130 in reference [cd]; [cdVII] means: page VII in reference [cd]

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