A q-QUEENS PROBLEM
III. PARTIAL QUEENS

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ABSTRACT. Parts I and II showed that the number of ways to place q nonattacking queens or similar chess pieces on an $n \times n$ square chessboard is a quasipolynomial function of n in which the coefficients are essentially polynomials in q. We explore this function for partial queens, which are pieces like the rook and bishop whose moves are a subset of those of the queen. We compute the five highest-order coefficients of the counting quasipolynomial, which are constant (independent of n), and find the periodicity of the next two coefficients, which depend on the move set. For two and three pieces we derive the complete counting functions and the number of combinatorially distinct nonattacking configurations. The method, as in Parts I and II, is geometrical, using the lattice of subspaces of an inside-out polytope.

CONTENTS

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1. Introduction

The well known n-Queens Problem asks for the number of ways to place n nonattacking queens on an $n \times n$ chessboard. A more general question separates the number of pieces from the size of the board; that is the q-Queens Problem, which asks for the number of ways to place q nonattacking queens on an $n \times n$ board. This paper is Part III of a series [1] in which we develop a general method for solving such questions and apply it not only to queens but to other pieces of the type called "riders", whose moves have unlimited distance. We convert the chess problem into a geometry problem: moves become hyperplanes in $\mathbb{R}^{2q}$; the $n \times n$ board becomes the set of $1/(n+1)$-fractional lattice points in the unit square; and the number of nonattacking configurations becomes a linear combination of the numbers of q-tuples of these lattice points that lie in subspaces determined by the move hyperplanes. The ultimate goal is to produce a usable general formula for any such problem, but that is probably impossibly difficult because it requires complete knowledge of configuration theorems in the real plane. Still, we provide important information about the nature of such a formula.

We proved in Part I that in each such problem the number of solutions is a quasipolynomial function of n—that means it is given by a cyclically repeating sequence of polynomials as n varies—and that the coefficient of each power of n is (up to a factor) a polynomial function of q. In Part II we narrowed our focus to the square board and found, for instance, that the coefficients of the very highest powers of n do not vary periodically with n, or at most have period 2. We briefly review the essentials from Parts I and II in Section 2.

This part is the capstone of the series. In it we repeatedly apply the theory from Parts I and II to gain detailed information on the counting functions for a well-behaved family of pieces that we name "partial queens"—those rider pieces whose moves are a subset of those of the queen. First, we explicitly calculate the coefficients of the four highest powers of n for partial queens. Surprisingly, we are also able to prove formulas for the *periodic* parts of the coefficients of the next two highest powers of n (but not for the nonperiodic parts). We further establish the highest and lowest powers of q and their coefficients within the coefficient of each power of n. These results combine to form our main theorem, Theorem 3.1. The proof of this theorem, which comprises the remainder of Section 3, requires us to determine the counting quasipolynomials for all low-codimensional subspaces in the lattice of intersections of move hyperplanes. In Section 4, we further apply our theory to calculate the number of nonattacking placements of two and three partial queens on a square board of any size and the number of combinatorially distinct nonattacking configurations of three partial queens, which turns out to be determined by the number, not the kind, of moves.

A deeper reason we study this set of pieces is that our ultimate object is to find out what factors control properties of the counting formula, such as the period (the length of a repeating cycle in the cyclically repeating polynomials), the periods of the individual coefficients of powers of n, formulas for the coefficients in terms of the set of moves of the piece, or anything that will let us predict aspects of the counting functions by knowing the moves of the piece under consideration. For this partial queens can be a valuable test set of pieces, not as hard as general riders but varied enough to suggest patterns for counting functions. Indeed, it was the formulas and their proofs for partial queens that led us to several of the general properties proved in Parts I and II.

In Part IV, next in the series, counting function properties that we establish here (supplemented by a theorem in Part V) will let us prove some of the specific formulas for bishops and queens derived empirically through extensive computation, or brilliantly intuited, by Kotěšovec in [2, 3].

We conclude Part III with questions related to our results and methods, and we append a dictionary of notation.

2. Essentials

2.1. **Review.**

We assume acquaintance with the notation and methods of Parts I and II as they apply to the square board. For easy reference we review the most important here.

The square board $[n]^2$, where $[n] := \{1, 2, \ldots, n\}$, consists of the integral points in the interior of the integral multiple $(n+1)[0,1]^2$ of the unit square. We write $[n] := \{1, \ldots, n\}$, so the set of points of the board is

$$[n]^2 = (n+1)(0,1)^2 \cap \mathbb{Z}^2.$$

We write δ_{ij} for the Kronecker delta.

A *move* of a piece $\mathbb{P}$ is the difference between two positions on the board; it may be any integral multiple of a nonempty set $\mathbf{M}$ of *basic moves*. The latter are non-zero, non-parallel integral vectors $m_r = (c_r, d_r)$ in lowest terms, i.e., c_r and d_r are relatively prime. (The slope d_r/c_r contains all necessary information and can be specified instead of m_r itself.) One piece *attacks* another if the former can reach the latter by a move. The constraint is that no two pieces may attack one another, or to say it mathematically, if there are pieces at positions z_i and z_j, then $z_j - z_i$ is not a multiple of any m_r. For a move $m = (c, d)$, we define

$$\hat{c} := \min(|c|, |d|), \quad \hat{d} := \max(|c|, |d|)$$

We assume that $q > 0$. We treat configurations of q pieces as $1/(n+1)$-fractional lattice points in the interior of the $2q$-dimensional inside-out polytope $([0,1]^{2q}, \mathscr{A}_\mathbb{P})$, where $\mathscr{A}_\mathbb{P}$ is the *move arrangement* whose members are the *move hyperplanes* or *attack hyperplanes*

$$\mathcal{H}^{d/c}_{ij} := \{\mathbf{z} \in \mathbb{R}^{2q} : (z_j - z_i) \cdot (d, -c) = 0\}.$$

(Inside-out polytopes are explained in Section I.2.) A coordinate vector in $\mathbb{R}^{2q}$ is $\mathbf{z} = (z_1, z_2, \ldots, z_q)$ where $z_i = (x_i, y_i) \in \mathbb{R}^2$. The *intersection lattice* $\mathscr{L}(\mathscr{A}_\mathbb{P})$ is the lattice of all intersections of subsets of the move arrangement, ordered by reverse inclusion; its Möbius function is μ. Of the $1/(n+1)$-fractional points in $[0,1]^{2q}$, those in the move hyperplanes represent attacking configurations; the others represent nonattacking configurations. The number of nonattacking configurations of q unlabelled pieces on an $n \times n$ board is $u_\mathbb{P}(q;n)$, whose full expression is

$$u_\mathbb{P}(q;n) = \gamma_0(n)n^{2q} + \gamma_1(n)n^{2q-1} + \gamma_2(n)n^{2q-2} + \cdots + \gamma_{2q}(n)n^0.$$

What we actually compute is the number of nonattacking labelled configurations, $o_\mathbb{P}(q;n)$, which equals $q!u_\mathbb{P}(q;n)$. The Ehrhart theory of inside-out polytopes implies that these counting functions are quasipolynomials in n.

In Part II we defined $\alpha(\mathcal{U};n)$ as the number of integral points in the intersection of the essential part of an intersection subspace $\mathcal{U} \in \mathscr{L}(\mathscr{A}_\mathbb{P})$ with the hypercube $[n]^{2\kappa}$, where κ is the number of pieces involved in the equations of $\mathcal{U}$. (The *essential part* is the restriction of $\mathcal{U}$ to the coordinates of pieces that appear in those equations.) The formula

$$q!u_{\mathbb{Q}^{hk}}(q;n) = \sum_{\mathcal{U} \in \mathscr{L}(\mathscr{A}_\mathbb{P})} \mu(\hat{0}, \mathcal{U})\alpha(\mathcal{U};n)n^{2q-2\kappa}, \tag{2.1}$$

from Equation (I.2.1) with $t = n+1$ and $E_{\mathcal{U} \cap \mathcal{P}^\circ}(t) = E_{\mathcal{U} \cap (0,1)^{2q}}(t) = \alpha(\mathcal{U};n)n^{2q-2\kappa}$, is the foundation stone of this paper. We also defined the abbreviations

$$\alpha^{d/c}(n) := \alpha(\mathcal{H}^{d/c}_{12};n),$$

the number of ordered pairs of positions that attack each other along slope d/c (they may occupy the same position; that is considered attacking). Similarly,

$$\beta^{d/c}(n) := \alpha(\mathcal{W}^{d/c}_{123}; n),$$

the number of ordered triples that are collinear along slope d/c; $\mathcal{W}^{d/c}_{123} := \mathcal{H}^{d/c}_{12} \cap \mathcal{H}^{d/c}_{23}$. Proposition II.3.2 gives general formulas for α and β. We need only a few examples in Part III:

$$\begin{aligned} \alpha^{0/1}(n) &= \alpha^{1/0}(n) = n^3, &\qquad \alpha^{\pm 1/1}(n) &= \frac{2n^3+n}{3}, \\ \beta^{0/1}(n) &= \beta^{1/0}(n) = n^4, &\qquad \beta^{\pm 1/1}(n) &= \frac{n^4+n^2}{2}. \end{aligned} \tag{2.2}$$

2.2. **Partial queens.**

A *partial queen* is a piece $\mathbb{Q}^{hk}$, whose moves are h horizontal and vertical moves and k diagonal moves of slopes ± 1, where $h, k \in \{0, 1, 2\}$ and (to avoid the trivial case $\mathbf{M} = \varnothing$) we assume $h + k \geq 1$. This includes the cases of the bishop ($h = 0$ and $k = 2$) and the queen ($h = k = 2$), and allows for pieces such as a *one-armed queen* ($h = 2$ and $k = 1$) and a *semiqueen* ($h = k = 1$). By restricting to partial queens it is possible to explicitly calculate the contributions to $q! u_{\mathbb{Q}^{hk}}(q; n)$ of intersection subspaces up to codimension 3. From this, we can calculate the coefficients γ_1, γ_2, and γ_3 and the counting quasipolynomials $u_{\mathbb{Q}^{hk}}(2; n)$ and $u_{\mathbb{Q}^{hk}}(3; n)$.

3. Coefficients

Kotěšovec proposed formulas for the coefficients γ_1 and γ_2 of the counting quasipolynomials for queens and bishops and other riders [3, third ed., pp. 13, 210, 223, 249, 652, 663; also in later eds.]. Our main theorem proves the generalization of his conjectures to partial queens and to γ_3.

Theorem 3.1. (I) *For a partial queen $\mathbb{Q}^{hk}$, the coefficient $q!\gamma_i$ of n^{2q-i} in $o_{\mathbb{Q}^{hk}}(q;n)$ is a polynomial in q, periodic in n, with leading term*

$$\Big(-\frac{3h+2k}{6}\Big)^i \frac{q^{2i}}{i!}.$$

(II) *The coefficients γ_i for $i \le 4$ of the five highest powers of n in the quasipolynomial $u_{\mathbb{Q}^{hk}}(q;n)$ are independent of n.*

The coefficients γ_i for $i = 1, 2, 3$ are given by

$$\gamma_1 = -\frac{1}{(q-2)!}\frac{3h+2k}{6}, \tag{3.1}$$

$$\gamma_2 = \frac{1}{2!(q-2)!}\Big\{(q-2)_2\Big(\frac{3h+2k}{6}\Big)^2 + (q-2)\frac{4h+2k+8hk+12\delta_{h2}+5\delta_{k2}}{6} + (h+k-1)\Big\} \tag{3.2}$$

and

$$\begin{aligned}\gamma_3 = -\frac{1}{3!(q-2)!}\Big\{&(q-2)_4\Big(\frac{3h+2k}{6}\Big)^3 \\ &+ (q-2)_3\frac{(3h+2k)(4h+8hk+2k+12\delta_{h2}+5\delta_{k2})}{12} \\ &+ (q-2)_2\frac{30h^2+20k^2-8k+257hk+160(2k+3)\delta_{h2}+68(3h+2)\delta_{k2}}{20} \\ &+ (q-2)\big[6h(h-1)+10kh+4k(k-1)+8k\delta_{h2}+5h\delta_{k2}\big]+k\Big\}.\end{aligned} \tag{3.3}$$

(For the individual quasipolynomials, see Table 3.)

(III) *The next coefficient, γ_5, has period 2 if $k = 2$ and $h \neq 0$ (and $q \ge 3$), with periodic part $-(-1)^n h\delta_{k2}/8(q-3)!$, but otherwise is independent of n.*

(IV) *The following coefficient, γ_6, is constant except that it has period 2 if $k = 2$ (and $q \ge 4$), with periodic part $-(-1)^n\delta_{k2}/8(q-3)!$.*

We write the falling factorials in terms of $q-2$ instead of q because every nontrivial coefficient γ_i ($\gamma_0 = 1/q!$ being "trivial") has a numerator factor $(q)_k$ with $k \ge 2$ and a denominator factor $q!$ (since $u_{\mathbb{P}} = o_{\mathbb{P}}/q!$). Therefore $u_{\mathbb{P}}(q;n)$ as a whole looks like

$$\frac{n^{2q}}{q!} + \frac{(q)_2(\text{nontrivial quasipolynomial in } n \text{ and } q)}{q!}.$$

It seems natural to cancel the repetitious factor $(q)_2$ in every coefficient other than γ_0.

Proof. Theorem II.4.2 says that $(q)_{2i}$ gives the highest power of q and its coefficient is $(-a_{10}/2)^i/i!$, where $a_{10} = \sum_{(c,d)\in\mathbf{M}}(3\hat{d}-\hat{c})/3\hat{d}^2 = h\frac{3}{3} + k\frac{2}{3}$ since there are h moves with $(\hat{c},\hat{d}) = (0,1)$ and k with $(\hat{c},\hat{d}) = (1,1)$.

	$k=0$	$k=1$	$k=2$
$h=0$	—	$\frac{4q^2-8q}{72(q-2)!}$	$\frac{16q^2-26q+24}{72(q-2)!}$
$h=1$	$\frac{9q^2-21q+6}{72(q-2)!}$	$\frac{25q^2-41q+18}{72(q-2)!}$	$\frac{49q^2-71q+18}{72(q-2)!}$
$h=2$	$\frac{36q^2-60q+12}{72(q-2)!}$	$\frac{64q^2-92q}{72(q-2)!}$	$\frac{100q^2-134q-24}{72(q-2)!}$

	$k=0$	$k=1$	$k=2$
$h=0$	—	$-\frac{5q^4-25q^3+31q^2-5q+141}{810(q-2)!}$	$-\frac{40q^4-155q^3+329q^2-220q-6}{810(q-2)!}$
$h=1$	$-\frac{(q^2-q)(q-2)(q-3)}{48(q-2)!}$	$-\frac{625q^4-2450q^3+3821q^2-2380q+156}{6480(q-2)!}$	$-\frac{1715q^4-5740q^3+6799q^2-3470q+384}{6480(q-2)!}$
$h=2$	$-\frac{(q^3-2q^2+q)(q-2)}{6(q-2)!}$	$-\frac{640q^4-2120q^3+1781q^2-505q+876}{1620(q-2)!}$	$-\frac{1250q^4-3775q^3+1999q^2+190q+2364}{1620(q-2)!}$

Table 3.1. The high-order coefficients of $u_{\mathbb{Q}^{hk}}(q;n)$ for the individual partial queens.

The coefficient γ_1 is from Theorem II.4.2. For the other coefficients we prove two lemmas that state the total contributions to $u_{\mathbb{Q}^{hk}}(q;n)$ from subspaces of all codimensions $k \leq 3$.

We use notation of the form $\mathcal{U}^\nu_\kappa$ or $\mathcal{U}^\nu_{\kappa\mathrm{a}}$ to represent a subspace of codimension ν in the intersection semilattice $\mathscr{L}(\mathscr{A})$ that involves κ pieces, with a letter index to differentiate between distinct types of subspace with these same numbers. In addition, we wish to differentiate between those subspaces that are indecomposable and those that decompose into subspaces of smaller codimension; for the latter we write an asterisk after the number of pieces and we specify the exact constituent subspaces. For example, we will have a subspace $\mathcal{U}^3_{5^*\mathrm{a}}{:}\mathcal{U}^1_2\mathcal{U}^2_{3\mathrm{a}}$.

We total up the contributions to $q!u_{\mathbb{Q}^{hk}}(q;n) = o_{\mathbb{Q}^{hk}}(q;n)$ in Equation 2.1 from all subspaces of codimension ν. To do that we break down those subspaces into types. For each type we determine the Möbius function $\mu(\hat{0},\mathcal{U})$ and count the number of lattice points in the intersection $\mathcal{U}\cap(0,1)^{2q}$. To perform this count in type $\mathcal{U}^\nu_{\kappa\mathrm{a}}$, we count the number of ways to place κ attacking pieces in the designated way, and then multiply by $n^{2(q-\kappa)}$ for the number of ways to place the remaining pieces whose positions are not constrained.

The subspaces $\mathcal{U}$ that contribute to γ_2 are those of codimension two, because neither the whole space nor any hyperplane has an n^{2q-2} term. To get exact formulas we need both the Ehrhart quasipolynomial of $\mathcal{U}\cap(0,1)^{2q}$ for each codimension-2 subspace in $\mathscr{L}\big(\mathscr{A}_{\mathbb{Q}^{hk}}\big)$ and the Möbius function $\mu(\hat{0},\mathcal{U})$. That means we have to analyze the different kinds of codimension-2 subspaces $\mathcal{U}$, whose defining equations may involve two, three, or four pieces. We summarize the results in a lemma (including codimension zero for completeness).

Lemma 3.2. *The contributions to $u_{\mathbb{Q}^{hk}}(q;n)$ from subspaces of codimension $\nu \leq 2$ are as follows.*

(I) *From* $\operatorname{codim}\mathcal{U}=0$:

$$\frac{1}{q!}n^{2q}. \tag{3.4}$$

(II) *From* $\operatorname{codim}\mathcal{U} = 1$:

$$-\frac{1}{q!}\left\{(q)_2\frac{3h+2k}{6}n^{2q-1} + (q)_2\frac{k}{6}n^{2q-3}\right\}. \tag{3.5}$$

(III) *From* $\operatorname{codim}\mathcal{U} = 2$:

$$\begin{aligned} &\frac{1}{q!}\bigg\{\left[(q)_4\frac{1}{2}\Big(\frac{3h+2k}{6}\Big)^2 + (q)_3\frac{4h+2k+8hk+12\delta_{h2}+5\delta_{k2}}{12} + (q)_2\frac{h+k-1}{2}\right]n^{2q-2} \\ &\quad + \left[(q)_4\frac{k(3h+2k)}{36} + (q)_3\frac{k(2h+1)+2\delta_{k2}}{6}\right]n^{2q-4} \\ &\quad + \left[(q)_4\frac{k^2}{72} + (q)_3\big[1-(-1)^n\big]\frac{\delta_{k2}}{8}\right]n^{2q-6}\bigg\}. \end{aligned} \tag{3.6}$$

Proof. The case $\nu = 0$ is from $\alpha((0,1)^{2q};n) = n^{2q}$. The case $\nu = 1$ is that of hyperplanes:

Type $\mathcal{U}^1_2$: The hyperplanes contribute

$$-\binom{q}{2}\sum_{(c,d)\in\mathbf{M}}\alpha^{d/c}(n)\cdot n^{2q-4} = -\binom{q}{2}\Big[\frac{3h+2k}{3}n^{2q-1} + \frac{k}{3}n^{2q-3}\Big]$$

to $o_{\mathbb{Q}^{hk}}(q;n)$ since we choose an unordered pair of pieces and a single slope, and the Möbius function is -1.

It remains to solve $\nu = 2$. We break the subspaces down into four types.

Type $\mathcal{U}^2_2$: The subspace $\mathcal{U}$ is defined by two hyperplane equations involving the same two pieces, $\mathcal{U} = \mathcal{H}^{d/c}_{ij} \cap \mathcal{H}^{d'/c'}_{ij}$ where $d/c \neq d'/c'$ and $i < j$. Thus, $\mathcal{U} = \mathcal{W}^{=}_{ij}$, the subspace corresponding to the equation $z_i = z_j$, i.e., to two pieces in the same location.

There is one such subspace for each of the $\binom{q}{2}$ unordered pairs of pieces. There are n^2 ways to place the two attacking pieces in $\mathcal{U}$. The Möbius function is $\mu(\hat{0},\mathcal{U}) = h+k-1$, by Lemma I.3.1.

The total contribution to $o_{\mathbb{Q}^{hk}}(q;n)$ is

$$\binom{q}{2}(h+k-1)n^{2q-2}.$$

Type $\mathcal{U}^2_{3\mathrm{a}}$: The subspace $\mathcal{U}$ is defined by two hyperplane equations of the same slope involving three pieces, say $\mathcal{U} = \mathcal{H}^{d/c}_{12} \cap \mathcal{H}^{d/c}_{23}$. This subspace is $\mathcal{W}^{d/c}_{123} = \mathcal{H}^{d/c}_{12} \cap \mathcal{H}^{d/c}_{13} \cap \mathcal{H}^{d/c}_{23}$. There is one such subspace for each of the $\binom{q}{3}$ unordered triples of pieces. The number of ways to place the three pieces is $\beta^{d/c}(n)$ in Equation (2.2). Summing over $(c,d) \in \mathbf{M}$ gives $\big[h + \frac{1}{2}k\big]n^4 + \frac{1}{2}kn^2$.

The Möbius function is $\mu(\hat{0},\mathcal{U}) = 2$ by Lemma I.3.1. The total contribution of this type is

$$\binom{q}{3}\big\{(2h+k)n^{2q-2} + kn^{2q-4}\big\}.$$

Type $\mathcal{U}^2_{3\mathrm{b}}$: The subspace $\mathcal{U}$ is defined by two hyperplane equations of different slopes involving three pieces, say $\mathcal{U} = \mathcal{H}^{d/c}_{12} \cap \mathcal{H}^{d'/c'}_{23}$.

First, we count the number of ways in which we can place three pieces ($\mathbb{P}_1$, $\mathbb{P}_2$, and $\mathbb{P}_3$) so that $\mathbb{P}_1$ and $\mathbb{P}_2$ are on a line of slope d/c and $\mathbb{P}_2$ and $\mathbb{P}_3$ are on a line of slope

d'/c'. Depending on d/c and d'/c', we have the following numbers of choices for the placements of the chosen pieces in the given attacking configuration:

Case VH. If $\{d/c, d'/c'\} = \{0/1, 1/0\}$, we have n^2 choices for $\mathbb{P}_2$; then we place $\mathbb{P}_1$ in one of n positions in the same column as $\mathbb{P}_2$ and place $\mathbb{P}_3$ in one of n positions in the same row as $\mathbb{P}_2$. This gives a total of n^4 placements of the three pieces. This case contributes only when $h = 2$.

Case DV. If one slope is diagonal and the other vertical or horizontal, we first choose the positions of $\mathbb{P}_1$ and $\mathbb{P}_2$, which we specify are attacking each other diagonally. This can be done in $\alpha^{1/1}(n)$ ways. Then we place $\mathbb{P}_3$ in line with $\mathbb{P}_2$ in n ways. This gives a total of $\frac{2}{3}n^4 + \frac{1}{3}n^2$ placements of the three pieces, contributing hk times.

Case DD. If $\{d/c, d'/c'\} = \{1/1, -1/1\}$, then the number of possibilities for placing $\mathbb{P}_1$ on the diagonal of slope $+1$ and $\mathbb{P}_3$ on the diagonal of slope -1 depends on the position (x, y) where we place $\mathbb{P}_2$. Consider the positions (x, y) satisfying $x \geq y$ and $x + y \leq n$; if we rotate this triangle of positions about the center of the square, we see that there are four points with the same number of possibilities for each position in the triangle, except when n is odd, in which case we must consider the position $(\frac{n+1}{2}, \frac{n+1}{2})$ independently. (See Figure 3.1.)

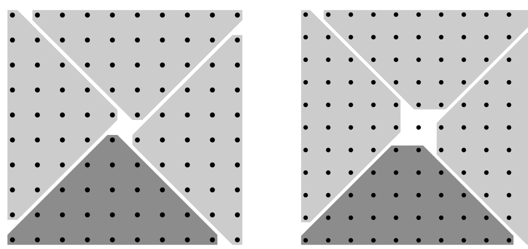

FIGURE 3.1. The triangle of positions that we consider in Case DD, along with its rotations. The left figure shows that all positions are covered when n is even; the right figure shows that position $(\frac{n+1}{2}, \frac{n+1}{2})$ is considered independently.

For a position (x, y) of $\mathbb{P}_2$ in this triangle, the number of choices for $\mathbb{P}_1$ is $n - x + y$ and the number of choices for $\mathbb{P}_3$ is $x + y - 1$. This gives the following number of placements in Case DD:

$$\begin{cases} 4\displaystyle\sum_{y=1}^{n/2}\sum_{x=y}^{n-y}(n-x+y)(x+y-1) & \text{if } n \text{ is even,} \\ n^2 + 4\displaystyle\sum_{y=1}^{(n-1)/2}\sum_{x=y}^{n-y}(n-x+y)(x+y-1) & \text{if } n \text{ is odd,} \end{cases}$$

$$= \begin{cases} \frac{5}{12}n^4 + \frac{1}{3}n^2 & \text{if } n \text{ is even,} \\ \frac{5}{12}n^4 + \frac{1}{3}n^2 + \frac{1}{4} & \text{if } n \text{ is odd,} \end{cases}$$

$$= \left[\frac{5}{12}n^4 + \frac{1}{3}n^2 + \frac{1}{8}\right] - (-1)^n\frac{1}{8}.$$

This quantity contributes only when $k = 2$.

In Type $\mathcal{U}^2_{3\mathrm{b}}$, $\mu(\hat{0},\mathcal{U})=1$. There are $(q)_3$ ways to choose the three pieces. The total contribution to $o_{\mathbb{Q}^{hk}}(q;n)$ depends on h and k; it is

$$(q)_3\bigg\{\Big[\delta_{h2}+\frac{2}{3}hk+\frac{5}{12}\delta_{k2}\Big]n^{2q-2}+\Big[\frac{1}{3}hk+\frac{1}{3}\delta_{k2}\Big]n^{2q-4}+\frac{1}{8}\delta_{k2}n^{2q-6}-(-1)^n\frac{1}{8}\delta_{k2}n^{2q-6}\bigg\}.$$

Type $\mathcal{U}^2_{4*}{:}\mathcal{U}^1_2\mathcal{U}^1_2$: The subspace $\mathcal{U}$ is defined by two hyperplane equations involving four distinct pieces. Hence, $\mathcal{U}=\mathcal{H}^{d/c}_{12}\cap\mathcal{H}^{d'/c'}_{34}$, which decomposes into the two hyperplanes $\mathcal{H}^{d/c}_{12}$ and $\mathcal{H}^{d'/c'}_{34}$, where d'/c' may equal d/c. The Möbius function is $\mu(\hat{0},\mathcal{U})=1$.

There are $2!\binom{q}{2,2,q-4}=(q)_4/4$ ways to choose an ordered pair of unordered pairs of pieces. Assign any slope d/c to the first pair and d'/c' to the second. Each pair of slopes, distinct or equal, appears twice, once for each ordering of the unordered pairs, so we divide by 2. The number of attacking configurations in each case is $\alpha^{d/c}(n)\cdot\alpha^{d'/c'}(n)$. The total contribution of all cases (before multiplication by n^{2q-8}) is

$$\frac{(q)_4}{8}\sum_{(c,d),(c',d')\in\mathbf{M}}\alpha^{d/c}(n)\cdot\alpha^{d'/c'}(n)=\frac{(q)_4}{8}\Big[\sum_{(c,d)\in\mathbf{M}}\alpha^{d/c}(n)\Big]^2=\frac{(q)_4}{8}\Big[\frac{3h+2k}{3}n^3+\frac{k}{3}n\Big]^2.$$

Thus, the contribution of Type $\mathcal{U}^2_{4*}$ to $o_{\mathbb{Q}^{hk}}(q;n)$, after multiplication by the n^{2q-8} ways to place the remaining pieces, is

$$\frac{1}{8}(q)_4\bigg\{\Big[h^2+\frac{4}{3}hk+\frac{4}{9}k^2\Big]n^{2q-2}+\Big[\frac{2}{3}hk+\frac{4}{9}k^2\Big]n^{2q-4}+\frac{1}{9}k^2n^{2q-6}\bigg\}.$$

Adding up the various types gives the total contribution to $o_{\mathbb{Q}^{hk}}(q;n)$; dividing by $q!$ concludes the proof of Lemma 3.2. □

Our next task is to find the contributions to γ_3 of subspaces of codimension three. We solve that by combining the Ehrhart quasipolynomials of all those subspaces.

Lemma 3.3. *The total contribution to $u_{\mathbb{Q}^{hk}}(q;n)=\frac{1}{q!}o_{\mathbb{P}}(q;n)$ from subspaces of codimension 3 is*

$$\begin{aligned}-\frac{1}{q!}\bigg\{\bigg[\ & n^{2q-3}\bigg((q)_3\frac{12h(h-1)+20kh+8k(k-1)+8k\delta_{h2}+5h\delta_{k2}}{12}\\
&\quad+(q)_4\frac{30h^2+20k^2-8k+257hk+160(2k+3)\delta_{h2}+68(3h+2)\delta_{k2}}{120}\\
&\quad+(q)_5\frac{(3h+2k)(4h+8hk+2k+12\delta_{h2}+5\delta_{k2})}{72}+(q)_6\frac{(3h+2k)^3}{1296}\bigg)\\
&+n^{2q-5}\bigg((q)_3\frac{8k(h+k-1)+8k\delta_{h2}+11h\delta_{k2}}{24}\\
&\quad+(q)_4\frac{k(31h+k+1)+32k\delta_{h2}+(34h+24)\delta_{k2}}{24}\\
&\quad+(q)_5\frac{2k\,(6h^2+8hk+5h+3k)+12k\delta_{h2}+(12h+13k)\delta_{k2}}{72}\\
&\quad+(q)_6\frac{k(3h+2k)^2}{432}\bigg)\end{aligned}$$

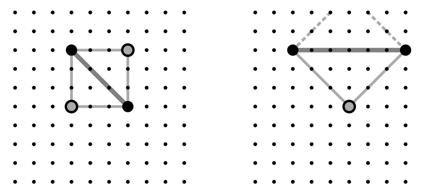

FIGURE 3.2. Given two attacking queens on the hypotenuse of a right triangle, there may be one or two options for a third mutually attacking queen, as explained in Type $\mathcal{U}^3_{3\mathrm{a}}$.

$$
\begin{aligned}
&+ n^{2q-7}\bigg((q)_4 \frac{2k(4h-1)+(61h+76)\delta_{k2}}{120} \\
&\qquad + (q)_5 \frac{4(2h+1)k^2+(14k+9h)\delta_{k2}}{144} + (q)_6 \frac{k^2(3h+2k)}{432} \bigg) \\
&+ n^{2q-9}\bigg((q)_5 \frac{k\delta_{k2}}{48} + (q)_6 \frac{k^3}{1296} \bigg) \bigg] \\
&- (-1)^n \bigg[n^{2q-5}(q)_3 \frac{h\delta_{k2}}{8} + n^{2q-7}\bigg((q)_4 \frac{(h+2)\delta_{k2}}{4} + (q)_5 \frac{(3h+2k)\delta_{k2}}{48} \bigg) + n^{2q-9}(q)_5 \frac{k\delta_{k2}}{48} \bigg] \bigg\}.
\end{aligned}
$$

Proof. The subspaces $\mathcal{U}$ defined by three hyperplane equations may involve three, four, five, or six pieces. We treat each number of pieces in turn.

Type $\mathcal{U}^3_{3\mathrm{a}}$: The subspace $\mathcal{U}$ is defined by three hyperplane equations of distinct slopes involving the same three pieces, say $\mathcal{U} = \mathcal{H}^{d/c}_{12} \cap \mathcal{H}^{d'/c'}_{13} \cap \mathcal{H}^{d''/c''}_{23}$ where d/c, d'/c', and d''/c'' are distinct.

There is one subspace $\mathcal{U}$ for every valid choice of three slopes and each of the $(q)_3/2$ ways to choose pieces and assign pairs to the slopes.

As exhibited in Figure 3.2, there are two kinds of subspace $\mathcal{U}$, with the hypotenuse of the right triangle either on a diagonal (Case $\triangle_1$) or on a vertical or horizontal line (Case $\triangle_2$).

Case $\triangle_1$. Once we have chosen the positions of the two pieces on a diagonal, there are always two ways to place a third piece to complete the triangular configuration. We conclude that the number of such configurations is $2k\alpha^{1/1}(n) = k\frac{4n^3+2n}{3}$, but only when $h = 2$.

Case $\triangle_2$. There are h horizontal and vertical moves, so h orientations for the hypotenuse. We take the case of a horizontal hypotenuse.

First we choose the vertical coordinate y of the hypotenuse. Form the two triangles with vertices $(1, y), (n, y)$, and either $(\frac{n+1}{2}, y + \frac{n}{2})$ (the upper triangle) or $(\frac{n+1}{2}, y - \frac{n}{2})$ (the lower triangle). $\mathbb{P}_3$ may have any (integral) location in these triangles that is in the board $[1, n]^2$, and once it is positioned the locations of $\mathbb{P}_1$ and $\mathbb{P}_2$ are determined. Thus, we need only count the valid locations for $\mathbb{P}_3$ for each height y. We do so by counting the integral points in both triangles and subtracting those outside the board.

The number of integral points in one triangle with hypotenuse n (the number of points) is $T(n) := (n^2 + 2n + \varepsilon)/4$ where $\varepsilon := \frac{1}{2}[1 - (-1)^n] \equiv n \bmod 2$. The number in both triangles together is $T(n)+T(n-2)=\frac{1}{2}(n^2+\varepsilon)$ (note that ε is the same for both triangles).

The number of triangle points outside the board depends on y. For $y=\frac{n+1}{2}$ (the midline, which exists when n is odd), there are no such points. Thus, we total the count for all $y \le n/2$ and double it. The excluded part of the triangle has upper edge extending from $(y+1,0)$ to $(n-y,0)$, with $n-2y$ points, so the number of excluded points is $T(n-2y)=\frac{1}{4}[(n-2y+1)^2-1+\varepsilon]$. Summing over y and doubling to include $y>(n+1)/2$,

$$2\sum_{y=1}^{(n-\varepsilon)/2} T(n-2y)=\frac{n^3-4n+3n\varepsilon}{12}.$$

This is subtracted from the total of triangle areas and the result multiplied by h, giving

$$h\left\{n\frac{n^2+\varepsilon}{2}-\frac{n^3-4n+3n\varepsilon}{12}\right\}=\left\{\frac{5h}{6}n^3+\frac{11h}{12}n\right\}-(-1)^n\frac{h}{4}n$$

as the number of configurations. This case applies only when $k=2$.

We have $\mu(\hat{0},\mathcal{U})=-1$ because the number of hyperplanes that contain $\mathcal{U}$ is $3=\operatorname{codim}\mathcal{U}$. The total contribution of this type to $o_{\mathbb{Q}^{hk}}(q;n)$ is

$$-(q)_3\left\{\left[\delta_{h2}\frac{2k}{3}+\delta_{k2}\frac{5h}{12}\right]n^{2q-3}+\left[\delta_{h2}\frac{k}{3}+\delta_{k2}\frac{11h}{24}\right]n^{2q-5}-(-1)^n\delta_{k2}\frac{h}{8}n^{2q-5}\right\}.$$

Type $\mathcal{U}^3_{3\mathrm{b}}$: The subspace $\mathcal{U}$ is defined by three hyperplane equations involving three pieces and two or three slopes, of the form $\mathcal{U}=\mathcal{H}^{d/c}_{12}\cap\mathcal{H}^{d'/c'}_{12}\cap\mathcal{H}^{d''/c''}_{23}$ where d''/c'' is any chosen slope, and $d/c, d'/c'$ are arbitrary distinct slopes. This subspace equals $\mathcal{W}^{=}_{12}\cap\mathcal{W}^{d''/c''}_{123}$; thus, it does not depend on the choice of d/c and d'/c', and $\mathcal{H}^{d''/c''}_{23}$ can be replaced by $\mathcal{H}^{d''/c''}_{13}$ in the definition of $\mathcal{U}$. Moreover, the number of ways to place the three pieces equals the number of ways to place an ordered pair of pieces in a line of slope d''/c'', i.e., $\alpha^{d''/c''}(n)$ from Equation (2.2). This should be multiplied by n^{2q-6} for the remaining $q-3$ pieces.

By Lemma I.3.1 the Möbius function is $\mu(\hat{0},\mathcal{U})=-2(h+k-1)$. We can specify the pieces involved in $(q)_3/2$ ways. The contribution to $o_{\mathbb{Q}^{hk}}(q;n)$ is therefore

$$-(q)_3(h+k-1)\left\{\left[h+\frac{2k}{3}\right]n^{2q-3}+\frac{k}{3}n^{2q-5}\right\}.$$

Type $\mathcal{U}^3_{4\mathrm{a}}$: The subspace $\mathcal{U}$ is defined by three hyperplane equations of the same slope involving four pieces, say $\mathcal{U}=\mathcal{W}^{d/c}_{1234}=$ (for instance) $\mathcal{H}^{d/c}_{12}\cap\mathcal{H}^{d/c}_{23}\cap\mathcal{H}^{d/c}_{34}$. There are $\binom{q}{4}$ ways to choose the four pieces.

The number of ways to place four attacking pieces in $\mathcal{U}$ is $\sum_{l\in\mathbf{L}^{d/c}(n)} l^4$ (see Section II.2), which depends on d/c. When $d/c\in\{0/1,1/0\}$, the number is $\sum_{l\in\mathbf{L}^{d/c}(n)} l^4=n^5$. When $d/c\in\{1/1,-1/1\}$, the number is $\sum_{l=1}^{n} l^4+\sum_{l=1}^{n-1} l^4=\frac{1}{15}(6n^5+10n^3-n)$.

We have $\mu(\hat{0},\mathcal{U})=-6$ because $\mathcal{U}$ is contained in six hyperplanes $\mathcal{H}^{d/c}_{ij}$, four codimension-2 subspaces of type $\mathcal{U}^2_{3\mathrm{a}}$, and three codimension-2 subspaces of type

$\mathcal{U}^2_{4*}$. The total contribution to $o_{\mathbb{Q}^{hk}}(q;n)$ is

$$-\binom{q}{4}\left\{\left[6h+\frac{12k}{5}\right]n^{2q-3}+4kn^{2q-5}-\frac{2k}{5}n^{2q-7}\right\}.$$

Type $\mathcal{U}^3_{4\mathrm{b}}$: The subspace $\mathcal{U}$ is defined by three hyperplane equations, two having the same slope and involving the same piece, say $\mathcal{U} = \mathcal{W}^{d/c}_{123} \cap \mathcal{H}^{d'/c'}_{34} =$ (for example) $\mathcal{H}^{d/c}_{12} \cap \mathcal{H}^{d/c}_{23} \cap \mathcal{H}^{d'/c'}_{34}$, where $d'/c' \neq d/c$.

There is a subspace for each of $(q)_4/2!$ choices of pieces (since $\mathbb{P}_1$ and $\mathbb{P}_2$ are unordered) and for each ordered pair of slopes d/c and d'/c'.

Just as with subspaces of type $\mathcal{U}^2_{3\mathrm{b}}$, we have three cases.

Case VH. Choosing $\mathbb{P}_3$'s position in n^2 ways, place $\mathbb{P}_1$ and $\mathbb{P}_2$ in the same column in n^2 ways, and place $\mathbb{P}_4$ in $\mathbb{P}_3$'s row in n ways. Multiply by two for interchanging slopes, for a total of $2n^5$ placements when $h = 2$.

Case DV. As in Type $\mathcal{U}^2_{3\mathrm{a}}$, the number of ways to place $\mathbb{P}_1$, $\mathbb{P}_2$, and $\mathbb{P}_3$ in the same diagonal is given by Equation (2.2); multiply by the n ways to place $\mathbb{P}_4$ in the same column as $\mathbb{P}_3$. Considering the choice of diagonal and that of column or row, we get $\frac{hk}{2}(n^5 + n^3)$.

Or, place $\mathbb{P}_3$ and $\mathbb{P}_4$ in the same diagonal in $\alpha^{1/1}(n)$ ways and multiply by n^2 placements of $\mathbb{P}_1$ and $\mathbb{P}_2$ in $\mathbb{P}_3$'s column; we get $\frac{hk}{3}(2n^5 + n^3)$.

The total is $hk(\frac{7}{6}n^5 + \frac{5}{6}n^3)$.

Case DD. Here $k = 2$ and $\{d/c, d'/c'\} = \{1/1, -1/1\}$. We reduce the computation by symmetry, as in Type $\mathcal{U}^2_{3\mathrm{b}}$, but here the symmetry in Figure 3.1 is broken by having two pieces in one of the diagonals. Thus, we count the placements where $\mathbb{P}_3$ is on one of the two main diagonals separately from the other placements. See Figure 3.3 for a visual representation.

For $\mathbb{P}_3$ at a point (x, y) in the bottom triangle $y + 1 \leq x \leq n - y$, the number of placements with $\mathbb{P}_1$ and $\mathbb{P}_2$ on the diagonal of slope $+1$ and $\mathbb{P}_4$ on the diagonal of slope -1 through (x, y) equals the number with $\mathbb{P}_1$ and $\mathbb{P}_2$ on the diagonal of slope -1 and $\mathbb{P}_4$ on the diagonal of slope $+1$ through $(n + 1 - x, y)$, which is also in the bottom triangle. Therefore, if we double the number of the former kind we get the total number with $\mathbb{P}_3$ in the bottom triangle. (Note that we are combining the counts of two different [but isomorphic] subspaces $\mathcal{U}$. In particular, the configuration with all pieces in the same place is counted twice, but only once for each subspace.) Multiplying this by 4 for the four triangles, we have the number of configurations where $\mathbb{P}_3$ is off the two main diagonals. To get the actual number note that when (x, y) is in the bottom triangle, its positive diagonal has $n - x - y$ points and its negative diagonal has $x + y - 1$ points.

Similarly, if we count the configurations with $\mathbb{P}_3$ in the lower left or lower right half-diagonal and $\mathbb{P}_1, \mathbb{P}_2$ on the diagonal with positive slope, double the result. We double this again to account for the upper half-diagonals.

When n is odd, the center point contributes n^3 for each choice of the diagonal of $\mathbb{P}_1, \mathbb{P}_2$.

Thus we have the number of placements in Case DD (which exists only when $k = 2$):

$$\begin{cases} 8\sum_{y=1}^{n/2}\sum_{x=y+1}^{n-y}(n-x+y)^2(x+y-1) \\ \qquad + 4\sum_{y=1}^{n/2}[n(2y-1)^2 + n^2(2y-1)] & \text{if } n \text{ is even,} \\ 8\sum_{y=1}^{(n-1)/2}\sum_{x=y+1}^{n-y}(n-x+y)^2(x+y-1) \\ \qquad + 4\sum_{y=1}^{(n-1)/2}[n(2y-1)^2 + n^2(2y-1)] + 2n^3 & \text{if } n \text{ is odd,} \end{cases}$$

$$= \begin{cases} \frac{3}{5}n^5 + \frac{2}{3}n^3 - \frac{4}{15}n & \text{if } n \text{ is even,} \\ \frac{3}{5}n^5 + \frac{2}{3}n^3 + \frac{11}{15}n & \text{if } n \text{ is odd,} \end{cases}$$

$$= \left[\frac{3}{5}n^5 + \frac{2}{3}n^3 + \frac{7}{30}n\right] - (-1)^n\frac{1}{2}n.$$

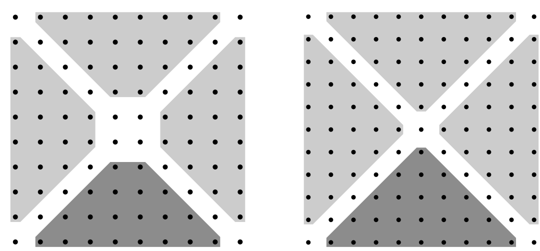

FIGURE 3.3. The triangle of positions that we consider in Case DD of Type $\mathcal{U}^3_{4\mathrm{b}}$, for even n (left) and odd n (right).

In all cases of Type $\mathcal{U}^3_{3\mathrm{b}}$, $\mu(\hat{0}, \mathcal{U}) = -2$ because $\mathcal{U}$ is contained in four hyperplanes, $\mathcal{H}^{d/c}_{ij}$ with $i, j \in \{1, 2, 3\}$ and $\mathcal{H}^{d'/c'}_{34}$, and the four subspaces $\mathcal{W}^{d/c}_{123}$ and $\mathcal{H}^{d/c}_{ij} \cap \mathcal{H}^{d'/c'}_{34}$ of codimension 2.

Therefore, the total contribution to $o_{\mathbb{Q}^{hk}}(q; n)$ from Type $\mathcal{U}^3_{4\mathrm{b}}$ is

$$-(q)_4 \left\{ \left[2\delta_{h2} + \frac{3}{5}\delta_{k2} + \frac{7}{6}hk\right]n^{2q-3} + \left[\frac{2}{3}\delta_{k2} + \frac{5}{6}hk\right]n^{2q-5} + \frac{7}{30}\delta_{k2}n^{2q-7} \right.$$
$$\left. - (-1)^n\frac{1}{2}\delta_{k2}n^{2q-7} \right\}.$$

Type $\mathcal{U}^3_{4\mathrm{c}}$: The subspace $\mathcal{U}$ is defined by three hyperplane equations, two having the same slope but not involving the same piece, say $\mathcal{U} = \mathcal{H}^{d/c}_{12} \cap \mathcal{H}^{d'/c'}_{23} \cap \mathcal{H}^{d/c}_{34}$. There are $(q)_4/2$ choices for $\mathbb{P}_1$ through $\mathbb{P}_4$ because of the symmetry.

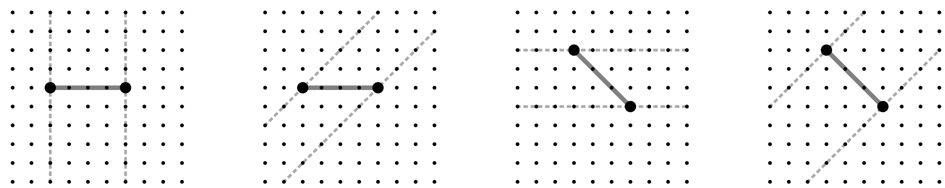

Figure 3.4. The possible attacking configurations in Type $\mathcal{U}^3_{4c}$. From left to right are cases VHV, DHD, HDH, and DDD.

We have the following cases (see Figure 3.4):

Case VHV. If $\{d/c, d'/c'\} = \{0/1, 1/0\}$, we can choose the pieces and positions for $\mathbb{P}_2$ and $\mathbb{P}_3$ in a row in n^3 ways, and then place $\mathbb{P}_1$ in $\mathbb{P}_2$'s column and $\mathbb{P}_4$ in $\mathbb{P}_3$'s column in n^2 ways. With two possible orientations (VHV or HVH), the number of attacking configurations is $2n^5$ when $h = 2$.

Case DHD. We consider the case where the outer attacking move is diagonal and the inner attacking move is horizontal or vertical. Without loss of generality, suppose $d/c = +1/1$ and $d'/c' = 0/1$. We investigate the possibilities for $\mathbb{P}_1$ and $\mathbb{P}_4$ based on choosing the row for $\mathbb{P}_2$ and $\mathbb{P}_3$.

Suppose that $\mathbb{P}_2$ and $\mathbb{P}_3$ are in row y, where $1 \le y \le n$. The positions that do not diagonally attack a position in row y are those in two right triangles, one in the upper left and the other in the lower right, with legs having, respectively, $n - y$ and $y - 1$ points. Placing $\mathbb{P}_1$ and $\mathbb{P}_4$ in any attacking positions determines where $\mathbb{P}_2$ and $\mathbb{P}_3$ are. Thus, the number of configurations is $\sum_{y=1}^{n} \left[n^2 - \binom{n-y+1}{2} - \binom{y}{2}\right]^2 = \frac{9}{20}n^5 + \frac{5}{12}n^3 + \frac{2}{15}n$, which contributes hk times.

Case HDH. When the inner attacking move is diagonal and the outer attacking move is horizontal or vertical, we first choose the positions of $\mathbb{P}_2$ and $\mathbb{P}_3$, in one of $\alpha^{1/1}(n)$ ways. There are n^2 ways to place $\mathbb{P}_1$ in relation to $\mathbb{P}_2$ and $\mathbb{P}_4$ in relation to $\mathbb{P}_3$, giving a total contribution of $\frac{2}{3}hkn^5 + \frac{1}{3}hkn^3$.

Case DDD. Here $\{d/c, d'/c'\} = \{1/1, -1/1\}$; say $d/c = 1/1$. We first determine the number of positions diagonally attacking a piece placed in a diagonal D_y of slope -1 passing through $(1, y)$ for a fixed $y \in [2n-1]$. As y varies, the multiset of the number of positions attacking the positions on it along each opposite diagonal has the following pattern:

$$
\begin{aligned}
&D_1, D_{2n-1}\colon\ \{n\},\\
&D_2, D_{2n-2}\colon\ \{n-1, n-1\},\\
&D_3, D_{2n-3}\colon\ \{n-2, n, n-2\},\\
&D_4, D_{2n-4}\colon\ \{n-3, n-1, n-1, n-3\},\\
&\quad\cdots \qquad \ldots,\\
&D_{n-1}, D_{n+1}\colon\ \begin{cases} \{3, \ldots, n-1, n-1, \ldots, 3\} & \text{if } n \text{ is even},\\ \{3, \ldots, n-2, n, n-2, \ldots, 3\} & \text{if } n \text{ is odd}, \end{cases}
\end{aligned}
$$

$$D_n\colon\ \begin{cases} \{1,3,\dots,n-1,n-1,\dots,3,1\} & \text{if } n \text{ is even,} \\ \{1,3,\dots,n-2,n,n-2,\dots,3,1\} & \text{if } n \text{ is odd.} \end{cases}$$

Then $\mathbb{P}_1$ and $\mathbb{P}_4$ can each be placed arbitrarily and independently in any of the opposite diagonals that attack D_y. The choice of the opposite diagonal determines the locations of $\mathbb{P}_2$ and $\mathbb{P}_3$, respectively. Given y, the number of placements of $\mathbb{P}_1$ and $\mathbb{P}_4$ is the square of the sum of all lengths in D_y; thus, the total number of ways to place the four pieces is

$$\begin{cases} 2\sum_{j=0}^{(n/2)-1}\left[n+2\sum_{i=1}^{j}(n-2i)\right]^2 + 2\sum_{j=0}^{(n/2)-2}\left[2\sum_{i=0}^{j}(n-2i-1)\right]^2 \\ \qquad + \left[2\sum_{i=0}^{(n/2)-1}(n-2i-1)\right]^2 & \text{if } n \text{ is even,} \\ 2\sum_{j=0}^{(n-3)/2}\left[n+2\sum_{i=1}^{j}(n-2i)\right]^2 + 2\sum_{j=0}^{(n-3)/2}\left[2\sum_{i=0}^{j}(n-2i-1)\right]^2 \\ \qquad + \left[n+2\sum_{i=1}^{(n-1)/2}(n-2i)\right]^2 & \text{if } n \text{ is odd,} \end{cases}$$

which simplifies for both parities to $\frac{4}{15}n^5 + \frac{1}{3}n^3 + \frac{2}{5}n$.

We double this quantity for the second subspace resulting from choosing slope $d/c = -1$. The result is $\frac{8}{15}n^5 + \frac{2}{3}n^3 + \frac{4}{5}n$, valid when $k = 2$.

In this type, once again, $\mu(\hat{0}, \mathcal{U}) = -1$. The total contribution to $o_{\mathbb{Q}^{hk}}(q;n)$ is

$$-(q)_4\left\{\left[\delta_{h2} + \frac{67}{120}hk + \frac{4}{15}\delta_{k2}\right]n^{2q-3} + \left[\frac{3}{8}hk + \frac{1}{3}\delta_{k2}\right]n^{2q-5} + \left[\frac{1}{15}hk + \frac{2}{5}\delta_{k2}\right]n^{2q-7}\right\}.$$

Type $\mathcal{U}^3_{4\mathrm{d}}$: The subspace $\mathcal{U}$ is defined by three hyperplane equations having distinct slopes, say $\mathcal{U} = \mathcal{H}_{12}^{d/c} \cap \mathcal{H}_{23}^{d'/c'} \cap \mathcal{H}_{34}^{d''/c''}$. The arguments here are similar to those for Type $\mathcal{U}^3_{4\mathrm{d}}$; however, because of the lack of symmetry, there are now $(q)_4$ choices for the pieces $\mathbb{P}_1$ through $\mathbb{P}_4$, provided we fix d/c and d''/c''.

We place pieces $\mathbb{P}_2$ and $\mathbb{P}_3$ first, and then pieces $\mathbb{P}_1$ and $\mathbb{P}_4$.

Figure 3.5 shows the four cases we consider.

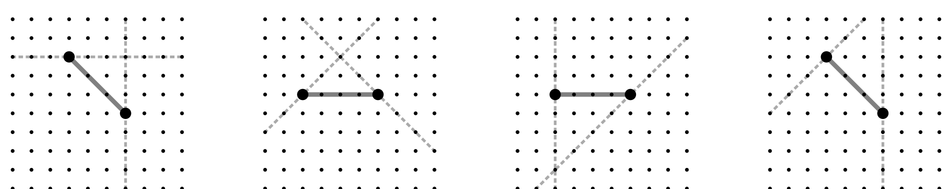

FIGURE 3.5. The possible attacking configurations in Type $\mathcal{U}^3_{4\mathrm{d}}$. From left to right are cases HDV, DHD, VHD, and DDV.

Case HDV. We assume $d/c = 0/1$ and $d''/c'' = 1/0$. The argument is the same as in case HDH of Type $\mathcal{U}^3_{4\mathrm{c}}$. The contribution is $\frac{2}{3}kn^5 + \frac{1}{3}kn^3$ when $h = 2$.

Case DHD. We assume $d/c = 1/1$ and $d''/c'' = -1/1$. This case has the same contribution as case DHD of Type $\mathcal{U}^3_{4\mathrm{c}}$, namely, $\frac{9}{20}hn^5 + \frac{5}{12}hn^3 + \frac{2}{15}hn$ when $k = 2$.

Case VHD. We choose $d/c = 0/1$ and assume $d''/c'' = 1/1$. We first place $\mathbb{P}_3$ and $\mathbb{P}_4$ on the diagonal in $\alpha^{1/1}(n)$ ways, then place $\mathbb{P}_2$ and $\mathbb{P}_1$ from $\mathbb{P}_3$ in n^2 ways.

We double for the two orderings of the slopes $0/1$ and $1/0$ and multiply by h for the possible diagonal slopes d''/c''. The contribution here is $\frac{4}{3}kn^5 + \frac{2}{3}kn^3$, applicable when $h = 2$.

Case DDV. We choose $d/c = 1/1$ and assume $d''/c'' = 1/0$. Case DD in Type $\mathcal{U}^2_{3\mathrm{b}}$ counts configurations of $\mathbb{P}_1$, $\mathbb{P}_2$, and $\mathbb{P}_3$ in two attacking moves along diagonals of slopes $+1$ and -1. Then we place $\mathbb{P}_4$ in relation to $\mathbb{P}_3$ in n ways. Accounting for the two different orderings of the slopes $1/1$ and $-1/1$, the contribution when $k = 2$ is $\left[\frac{5}{6}hn^5 + \frac{2}{3}hn^3 + \frac{1}{4}hn\right] - (-1)^n\frac{1}{4}hn$.

In all cases, $\mu(\hat{0}, \mathcal{U}) = -1$. The total contribution to $o_{\mathbb{Q}^{hk}}(q; n)$ is

$$-(q)_4\left\{\left[2k\delta_{h2} + \frac{77h}{60}\delta_{k2}\right]n^{2q-3} + \left[k\delta_{h2} + \frac{13h}{12}\delta_{k2}\right]n^{2q-5} + \frac{23h}{60}\delta_{k2}n^{2q-7} - (-1)^n\frac{h}{4}\delta_{k2}n^{2q-7}\right\}.$$

Type $\mathcal{U}^3_{4\mathrm{e}}$: The subspace $\mathcal{U}$ is defined by three hyperplane equations of different slope, all involving the same piece, say $\mathcal{U} = \mathcal{H}^{d/c}_{12} \cap \mathcal{H}^{d'/c'}_{13} \cap \mathcal{H}^{d''/c''}_{14}$. Given the set of slopes, there are $(q)_4$ ways to choose the pieces.

The number of ways to place four attacking pieces in $\mathcal{U}$ depends on the slopes. When $\{1/1, -1/1\} \subset \{d/c, d'/c', d''/c''\}$, then first place $\mathbb{P}_1$ and the two pieces defined along diagonals as in Case DD from Type $\mathcal{U}^2_{3\mathrm{b}}$ and subsequently the last piece horizontally or vertically in n ways, giving $hn\left\{\left[\frac{5}{12}n^4 + \frac{1}{3}n^2 + \frac{1}{8}\right] - (-1)^n\frac{1}{8}\right\}$ ways for the four pieces (contributing only when $k = 2$).

When $\{0/1, 1/0\} \subset \{d/c, d'/c', d''/c''\}$, then place $\mathbb{P}_1$ and the piece aligned diagonally in $k\alpha^{1/1}$ ways and place the other two pieces in n^2 ways, giving $kn^2\left\{\frac{2}{3}n^3 + \frac{1}{3}n\right\}$ placements (that contribute only when $h = 2$).

Once more, $\mu(\hat{0}, \mathcal{U}) = -1$. The total contribution to $o_{\mathbb{Q}^{hk}}(q; n)$ is

$$-(q)_4\left\{\left[\frac{5h}{12}\delta_{k2} + \frac{2k}{3}\delta_{h2}\right]n^{2q-3} + \left[\frac{h}{3}\delta_{k2} + \frac{k}{3}\delta_{h2}\right]n^{2q-5} + \frac{h}{8}\delta_{k2}n^{2q-7} - (-1)^n\frac{h}{8}\delta_{k2}n^{2q-7}\right\}.$$

Type $\mathcal{U}^3_{4*}{:}\mathcal{U}^1_2\mathcal{U}^2_2$: The subspace $\mathcal{U}$ decomposes into a hyperplane $\mathcal{H}^{d/c}_{12}$ and a codimension-2 subspace $\mathcal{W}^{=}_{34}$ of type $\mathcal{U}^2_2$. We write $\mathcal{W}^{=}_{34} = \mathcal{H}^{d'/c'}_{34} \cap \mathcal{H}^{d''/c''}_{34}$, where $d'/c' \neq d''/c''$. There is no restriction on d/c.

There are $(q)_4/4$ ways to choose the ordered pair of pairs of pieces, $\{\mathbb{P}_1, \mathbb{P}_2\}$ and $\{\mathbb{P}_3, \mathbb{P}_4\}$.

Since $\mathbb{P}_4$ is essentially merged with $\mathbb{P}_3$, the number of attacking configurations is $\sum_{(c,d)\in\mathbf{M}} \alpha^{d/c}(n) = \big(h + \frac{2}{3}k\big)n^3 + \frac{1}{3}kn$.

The Möbius function is a product, $\mu(\hat{0}, \mathcal{U}) = \mu(\hat{0}, \mathcal{H}^{d/c}_{12})\mu(\hat{0}, \mathcal{W}^{=}_{34}) = 1 - |\mathbf{M}|$ (see Section I.3.2). The total contribution to $o_{\mathbb{Q}^{hk}}(q; n)$ is

$$-(q)_4(h + k - 1)\left\{\left[\frac{h}{4} + \frac{k}{6}\right]n^{2q-3} + \frac{k}{12}n^{2q-5}\right\}.$$

Type $\mathcal{U}^3_{5^*\mathrm{a}}{:}\mathcal{U}^1_2\mathcal{U}^2_{3\mathrm{a}}$: The subspace $\mathcal{U}$ decomposes into a hyperplane and a codimension-2 subspace of type $\mathcal{U}^2_{3\mathrm{a}}$, say $\mathcal{U} = \mathcal{H}^{d/c}_{12} \cap \mathcal{W}^{d'/c'}_{345}$, where d/c may equal d'/c'. We can choose the pieces in $(q)_5/2!3!$ ways.

The number of attacking configurations is $\sum_{(c,d)\in\mathbf{M}} \alpha^{d/c}(n) = \big(h + \frac{2}{3}k\big)n^3 + \frac{1}{3}kn$ times the count from Type $\mathcal{U}^2_{3\mathrm{a}}$, $\big(h + \frac{1}{2}k\big)n^4 + \frac{1}{2}kn^2$.

As for the Möbius function, $\mu(\hat{0},\mathcal{U}) = \mu(\hat{0},\mathcal{H}_{12}^{d/c})\mu(\hat{0},\mathcal{U}_{3a}^2) = -2$. The contribution to $o_{\mathbb{Q}^{hk}}(q;n)$ is therefore

$$-(q)_5\Big\{\Big[\frac{h^2}{6}+\frac{7hk}{36}+\frac{k^2}{18}\Big]n^{2q-3}+\Big[\frac{5hk}{36}+\frac{k^2}{12}\Big]n^{2q-5}+\frac{k^2}{36}n^{2q-7}\Big\}.$$

Type $\mathcal{U}_{5^*\mathrm{b}}^3$:$\mathcal{U}_2^1\mathcal{U}_{3\mathrm{b}}^2$: The subspace $\mathcal{U}$ decomposes into a hyperplane and a codimension-2 subspace of type $\mathcal{U}_{3\mathrm{b}}^2$. We write $\mathcal{U} = \mathcal{H}_{12}^{d/c} \cap \mathcal{H}_{34}^{d'/c'} \cap \mathcal{H}_{45}^{d''/c''}$, with $d'/c' \neq d''/c''$ and arbitrary d/c. We can choose the five pieces in $(q)_5/2!$ ways.

The number of attacking configurations is $\sum_{(c,d)\in\mathbf{M}} \alpha^{d/c}(n)$ times the count from Type $\mathcal{U}_{3\mathrm{b}}^2$, thus $\big[\delta_{h2}+\frac{2}{3}hk+\frac{5}{12}\delta_{k2}\big]n^4+\big[\frac{1}{3}hk+\frac{1}{3}\delta_{k2}\big]n^2+\frac{1}{8}\delta_{k2}-(-1)^n\frac{1}{8}\delta_{k2}$.

Here again $\mu(\hat{0},\mathcal{U}) = -1$. Consequently, the total contribution to $o_{\mathbb{Q}^{hk}}(q;n)$ is

$$\begin{aligned}
-\frac{1}{2}(q)_5\Big\{&\Big[\frac{3h+2k}{3}\delta_{h2}+\frac{2}{3}h^2k+\frac{4}{9}hk^2+\frac{5(3h+2k)}{36}\delta_{k2}\Big]n^{2q-3}\\
&+\Big[\frac{1}{3}k\delta_{h2}+\frac{1}{3}h^2k+\frac{4}{9}hk^2+\Big(\frac{1}{3}h+\frac{13}{36}k\Big)\delta_{k2}\Big]n^{2q-5}\\
&+\Big[\frac{1}{9}hk^2+\Big(\frac{1}{8}h+\frac{7}{36}k\Big)\delta_{k2}\Big]n^{2q-7}+\frac{1}{24}k\delta_{k2}n^{2q-9}\\
&-(-1)^n\left(\frac{3h+2k}{24}\delta_{k2}n^{2q-7}+\frac{1}{24}k\delta_{k2}n^{2q-9}\right)\Big\}.
\end{aligned}$$

Type $\mathcal{U}_{6^*}^3$:$\mathcal{U}_2^1\mathcal{U}_2^1\mathcal{U}_2^1$: The subspace $\mathcal{U}$ is defined by three hyperplane equations involving six distinct pieces. Thus, $\mathcal{U} = \mathcal{H}_{12}^{d/c} \cap \mathcal{H}_{34}^{d'/c'} \cap \mathcal{H}_{56}^{d''/c''}$ is decomposable into the three indicated hyperplanes, whose slopes are not necessarily distinct. The Möbius function is $\mu(\hat{0},\mathcal{U}) = -1$.

There are $\binom{q}{2,2,2,q-6} = (q)_6/48$ ways to choose an unordered triple of unordered pairs of pieces. Then we fix an arbitrary ordering of the three pairs and assign any slope d/c to the first pair, d'/c' to the second, and d''/c'' to the third. The number of attacking configurations in each case is $\alpha^{d/c}(n)\cdot\alpha^{d'/c'}(n)\cdot\alpha^{d''/c''}(n)$. The total contribution of all cases (before multiplication by n^{2q-12}) is

$$\begin{aligned}
&-\frac{(q)_6}{48}\sum_{(c,d),(c',d'),(c'',d'')\in\mathbf{M}}\alpha^{d/c}(n)\cdot\alpha^{d'/c'}(n)\cdot\alpha^{d''/c''}(n)\\
&=-\frac{(q)_6}{48}\Big[\sum_{(c,d)\in\mathbf{M}}\alpha^{d/c}(n)\Big]^3=-\frac{(q)_6}{48}\Big[\frac{3h+2k}{3}n^3+\frac{k}{3}n\Big]^3.
\end{aligned}$$

Thus, the contribution of Type $\mathcal{U}_{6^*}^3$ to $o_{\mathbb{Q}^{hk}}(q;n)$, after multiplication by the n^{2q-12} ways to place the remaining pieces, is

$$-(q)_6\left\{\frac{(3h+2k)^3}{1296}n^{2q-3}+\frac{3k(3h+2k)^2}{1296}n^{2q-5}+\frac{3k^2(3h+2k)}{1296}n^{2q-7}+\frac{k^3}{1296}n^{2q-9}\right\}.$$

Summing the contributions of each type completes the proof of Lemma 3.3. (We verified the sum via Mathematica.) □

We resume the proof of Theorem 3.1.

No contributions to γ_2 come from subspaces of codimension 0 or 1. To find γ_2 we extract the coefficient of n^{2q-2} from (3.6).

Summarizing the analysis for γ_3: When calculating γ_3, there are contributions from the subspaces of codimensions 3, 2, and 1. Combining their contributions implies that the coefficients γ_3 are as in (3.3).

The contribution to γ_4 from any subspace of codimension four or greater is necessarily constant, and by our calculations the contribution is constant for every subspace of lesser codimension. This implies that γ_4 is constant for all partial queens.

A periodic contribution to γ_5 can arise only from subspaces of codimension 1 through 4, and by Lemma 3.2 only from codimensions 3 and 4. The coefficient of n^{2q-5} in $\alpha(\mathcal{U}; n)$ for $\operatorname{codim} \mathcal{U} = 4$ is zero, by Theorem II.3.4. The periodic parts of all codimension-3 subspaces are collected in Lemma 3.3, in which the periodic coefficient of n^{2q-5} is $-(-1)^n (q)_3 h \delta_{k2}/8$, so that is the periodic part of $q!\gamma_5$.

A periodic contribution to γ_6 can arise only from subspaces of codimension 1 through 5, and by Lemmas 3.2 and 3.3 only from codimensions 2, 4, and 5. Theorem II.3.4 shows that codimension 5 contributes nothing, while codimension 4 contributes with a period of $\operatorname{lcm}\{1\} = 1$; that is, a constant. It follows that γ_6 has periodic part $-(q)_3(-1)^n \frac{\delta_{k2}}{8}/q!$ from codimension 2. $\square$

If we hold q fixed, the counting quasipolynomials for the queen and the partial queen $\mathbb{Q}^{12}$ are the only ones of partial queens that have non-constant coefficient γ_5 with period 2. As for γ_6, it will have period 2 when the piece has two diagonal moves, as is true of both bishops and queens, but otherwise it is constant.

4. Two and Three Partial Queens

These observations are on display when we use our theory to calculate the counting quasipolynomial $u_{\mathbb{Q}^{hk}}(3;n)$. The results agree with formulas proposed by Kotěšovec, who supplemented his formulas for bishops and queens by independently calculating (but, as is his practice, not proving) the other cases in his fifth edition [3] after we suggested studying partial queens.

Complete formulas for two or three partial queens are in Theorems 4.1 and 4.2.

Theorem 4.1. *The counting quasipolynomial for two partial queens* $\mathbb{Q}^{hk}$ *is*

$$u_{\mathbb{Q}^{hk}}(2;n) = \frac{1}{2}n^4 - \frac{3h+2k}{6}n^3 + \frac{h+k-1}{2}n^2 - \frac{k}{6}n.$$

Proof. In Theorem II.2.3 there are h moves with $(\hat{c},\hat{d}) = (0,1)$ and k with $(\hat{c},\hat{d}) = (1,1)$. So, all $\hat{d}_r = 1$ and $n \bmod \hat{d}_r = 0$. □

Theorem 4.2. *The counting quasipolynomial for three partial queens* $\mathbb{Q}^{hk}$ *is a polynomial when* $k < 2$ *and has period* 2 *when* $k = 2$. *The formula is*

$$\begin{aligned} u_{\mathbb{Q}^{hk}}(3;n) = {} & \frac{1}{6}n^6 - \frac{3h+2k}{6}n^5 \\ & + \left[\frac{3h+2k}{6} + \frac{h(k+1)+(h+1)k}{3} - \frac{1}{2} + \delta_{h2} + \frac{5\delta_{k2}}{12}\right]n^4 \\ & - \left[\frac{(h+k-1)(3h+2k)}{3} + \frac{k}{6} + \frac{2k\delta_{h2}}{3} + \frac{5h\delta_{k2}}{12}\right]n^3 \\ & + \left[\frac{(h+k-1)^2(h+k+2)}{6} + \frac{hk}{3} + \frac{k}{6} + \frac{\delta_{k2}}{3}\right]n^2 \\ & - \left[\frac{(h+k-1)k}{3} + \frac{k\delta_{h2}}{3} + \frac{11h\delta_{k2}}{24}\right]n + \frac{\delta_{k2}}{8} \\ & + (-1)^n\frac{\delta_{k2}}{8}\big(hn-1\big). \end{aligned}$$

Table 4.1 lists the quasipolynomials for the various partial queens.

Note that $u_{\mathbb{Q}^{00}}(3;n) = \binom{n^2}{3}$, $u_{\mathbb{Q}^{10}}(3;n) = n^3\binom{n^3}{3}$, and $u_{\mathbb{Q}^{20}}(3;n) = [(n)_3]^2$, as one would expect from elementary counting.

In all instances, these equations agree with Kotěšovec's conjectures and data. The formulas for γ_3 for the queen, $\mathbb{Q}^{22}$, and the partial queens $\mathbb{Q}^{11}$, $\mathbb{Q}^{21}$, and $\mathbb{Q}^{12}$ are new. (After we suggested partial queens, Kotěšovec computed many values of the counting functions and inferred formulas which we employed to correct and verify our theoretical calculations.)

Proof. The only subspaces that contribute to $o_{\mathbb{Q}^{hk}}(3;n) = 3!u_{\mathbb{Q}^{hk}}(3;n)$ are those that involve three pieces or fewer. The subspace $\mathbb{R}^{2q}$ of codimension 0 contributes n^{2q}. The contribution from codimension 1 is given in Equation (3.5). In the proof of Theorem 3.1, we already have calculated the contributions from subspaces of types $\mathcal{U}^2_2$, $\mathcal{U}^2_{3\text{a}}$, $\mathcal{U}^2_{3\text{b}}$, $\mathcal{U}^3_{3\text{a}}$, and $\mathcal{U}^3_{3\text{b}}$. There is one final type of subspace, involving three pieces.

Type $\mathcal{U}^4_3$: The subspace $\mathcal{U}$ is defined by four hyperplane equations on three pieces that specify that the pieces all occupy one position on the board; that is, $\mathcal{U} = \mathcal{W}^{=}_{ijl}$.

(h,k)	$u_{\mathbb{Q}^{hk}}(3;n)$
$(0,0)$	$\frac{n^6}{6} - \frac{n^4}{2} + \frac{n^2}{3}$
$(1,0)$	$\frac{n^6}{6} - \frac{n^5}{2} + \frac{n^4}{3}$
$(2,0)$	$\frac{n^6}{6} - n^5 + \frac{13n^4}{6} - 2n^3 + \frac{2n^2}{3}$
$(0,1)$	$\frac{n^6}{6} - \frac{n^5}{3} + \frac{n^4}{6} - \frac{n^3}{6} + \frac{n^2}{6}$
$(1,1)$	$\frac{n^6}{6} - \frac{5n^5}{6} + \frac{5n^4}{3} - \frac{11n^3}{6} + \frac{7n^2}{6} - \frac{n}{3}$
$(2,1)$	$\frac{n^6}{6} - \frac{4n^5}{3} + \frac{25n^4}{6} - \frac{37n^3}{6} + \frac{25n^2}{6} - n$
$(0,2)$	$\frac{n^6}{6} - \frac{2n^5}{3} + \frac{5n^4}{4} - \frac{5n^3}{3} + \frac{4n^2}{3} - \frac{2n}{3} + \frac{1}{8} - (-1)^n\frac{1}{8}$
$(1,2)$	$\frac{n^6}{6} - \frac{7n^5}{6} + \frac{41n^4}{12} - \frac{65n^3}{12} + \frac{14n^2}{3} - \frac{43n}{24} + \frac{1}{8} + (-1)^n\left\{\frac{n}{8} - \frac{1}{8}\right\}$
$(2,2)$	$\frac{n^6}{6} - \frac{5n^5}{3} + \frac{79n^4}{12} - \frac{25n^3}{2} + 11n^2 - \frac{43n}{12} + \frac{1}{8} + (-1)^n\left\{\frac{n}{4} - \frac{1}{8}\right\}$

TABLE 4.1. The quasipolynomials that count nonattacking configurations of three partial queens.

There is one subspace for each of the $\binom{q}{3}$ unordered triples of pieces. The number of points in a subspace is n^2, the size of the board.

According to Lemma I.3.1, $\mu(\hat{0}, \mathcal{U}) = (h+k-1)^2(h+k+2)$, which happily gives 0 when $h+k=1$.

Consequently, the contribution to $o_{\mathbb{Q}^{hk}}(q;n)$ is

$$\binom{q}{3}(h+k-1)^2(h+k+2)n^{2q-4}.$$

Combining all contributions and dividing by $q! = 6$ gives the formula of the theorem. □

We can now calculate the number of combinatorial types for two and three partial queens.

Corollary 4.3. *The number of combinatorial types of nonattacking configuration of q partial queens Q^{hk} is $h+k$ when $q=2$ and when $q=3$ is given by Table 4.2.*

$h \setminus k$	0	1	2
0	–	1	6
1	1	6	17
2	6	17	36

TABLE 4.2. The number of combinatorial types of nonattacking configuration for three partial queens.

Proof. Set $n = -1$ in $u_{Q^{hk}}(q; n)$ and apply Theorem I.5.3. $\square$

For $q = 2$ we get the number of basic moves, in accord with Proposition I.5.6. For $q = 3$ the number of types depends only on the number of moves, just as when we compared three queens to three nightriders in the end of Section I.5. The numbers match [4, Sequence A084990], whose formula is $s(s^2 + 3s - 1)/3$ with $s := |\mathbf{M}|$.

Conjecture 4.4. The number of combinatorial configuration types of three pieces is

$$|\mathbf{M}|\big(|\mathbf{M}|^2 + 3|\mathbf{M}| - 1\big)/3.$$

5. VOLUMES AND EVALUATIONS

Here are an observation and a related problem suggested by our calculation of partial queen coefficients and similar computations for the nightrider in Part IV.

5.1. A quasipolynomial observation.

It is striking that with Theorems 3.1 and IV.3.3 we can know the period and periodic part of a quasipolynomial coefficient without knowing anything about the rest of the coefficient.

5.2. A problem of volumes. It should be possible to find the volume of $\mathcal{U} \cap [0,1]^{2q}$ without the trouble of finding its complete Ehrhart quasipolynomial. Doing so would provide the leading term of $\alpha(\mathcal{U}; n)$ and thereby the exact contribution of $\mathcal{U}$ to $\gamma_{\operatorname{codim} \mathcal{U}}$. This would be helpful for all pieces, not only partial queens.

The advantage would be that, if $\alpha(\mathcal{U}; n)$ were known for all subspaces of lesser codimension than i and if $\operatorname{vol}(\mathcal{U} \cap [0,1]^{2q})$ were known for all subspaces of codimension i, then γ_i would be completely known. Thus we could complete the evaluations of γ_5 and γ_6 in Theorem 3.1 and of γ_3 for nightriders in Part IV.

DICTIONARY OF NOTATION

$(c,d),(c_r,d_r)$ – coords of move vector (pp. 4)
$(\hat{c},\hat{d})$ – $(\min,\max)$ of c,d (p. 4)
d/c – slope of line or move (p. 4)
h – # horiz, vert moves of partial queen (p. 5)
k – # diagonal moves of partial queen (p. 5)
$m=(c,d)$, $m_r=(c_r,d_r)$ – basic move (p. 4)
n – size of square board (p. 2)
$n+1$ – dilation factor for board (p. 4)
$[n]=\{1,\ldots,n\}$ (p. 4)
$[n]^2$ – square board (p. 4)
$o_{\mathbb{P}}(q;n)$ – # nonattacking lab configs (p. 4)
q – # pieces on a board (p. 2)
r – move index (p. 4)
$u_{\mathbb{P}}(q;n)$ – # nonattacking unlab configs (p. 4)
$z=(x,y)$, $z_i=(x_i,y_i)$ – piece position (p. 4)

$\mathbf{z}=(z_1,\ldots,z_q)$ – vector in $\mathbb{R}^{2q}$
$\mathbf{z}=(z_1,\ldots,z_q)$ – configuration (p. 4)

$\alpha(\mathcal{U};n)$ – # attacking configs in $\mathcal{U}$ (p. 4)
$\alpha^{d/c}(n)$ – # 2-piece collinear attacks (p. 4)
$\beta^{d/c}(n)$ – # 3-piece collinear attacks (p. 4)
γ_i – coefficient of $u_{\mathbb{P}}$ (p. 4)
δ_{ij} – Kronecker delta (p. 4)
$\varepsilon=\frac{1}{2}[1-(-1)^n]\equiv n \bmod 2$ (p. 11)
ν – $\operatorname{codim}\mathcal{U}$ (p. 7)
μ – Möbius function of $\mathscr{L}$ (p. 4)
κ – # of pieces in eqns of $\mathcal{U}$ (p. 4)

$\mathbf{M}$ – set of basic moves (p. 4)

$\mathscr{A}_{\mathbb{P}}$ – move arr of piece $\mathbb{P}$ (p. 4)
$\mathcal{H}_{ij}^{d/c}$ – hyperplane for move (c,d) (p. 4)
$\mathscr{L}$ – intersection semilattice (p. 4)
$[0,1]^{2q}$ – polytope (p. 4)
$([0,1]^{2q},\mathscr{A}_{\mathbb{P}})$ – inside-out polytope (p. 4)
$\mathcal{U}$ – subspace in intersection semilatt (p. 4)
$\mathcal{U}_{\kappa\mathrm{a}}^{\nu}$ – subsp of codim ν with κ moves (p. 7)
$\mathcal{W}_{i\ldots}^{d/c}$ – subspace of collinearity (p. 5)
$\mathcal{W}_{i\ldots}^{=}$ – subspace of equal position (p. 8)

$\mathbb{R}$ – real numbers
$\mathbb{Z}$ – integers

$\mathbb{P}$ – piece (p. 4)
$\mathbb{P}_i$ – i-th labelled copy of $\mathbb{P}$ (p. 4)
$\mathbb{Q}^{hk}$ – partial queen (p. 5)

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