

ON ENUMERATION IN CLASSICAL INVARIANT THEORY

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ABSTRACT. In this paper we study the Frobenius characters of the invariant subspaces of the tensor powers of a representation V . The main result is a formula for these characters for a polynomial functor of V involving the characters for V . The main application is to representations V for which these characters are known. The best understood case is for V the vector representation of a symplectic group or special linear group. Other cases where there are some related results are the defining representations of the symmetric groups and the adjoint representations of the general linear groups. This formula uses methods of enumerative combinatorics and in particular is similar to the cycle index series of regular graphs.

1. INTRODUCTION

For Cayley and Sylvester the central problem of invariant theory was to determine the structure of the ring of invariants of a quantic. In the language of representation theory, the problem is to determine the structure of the ring of invariant polynomials on the symmetric power of a complex vector space. A preliminary problem is to determine the Hilbert series of the ring of invariant polynomials. Following the development of quantum mechanics, attention moved on to invariant tensors. The new feature is that the symmetric group acts naturally on tensor powers by permuting tensor indices. This leads to the general problem. Let G be a reductive complex algebraic group and V a finite dimensional rational representation. Then each isotypic subspace of $\otimes^r V$ has a natural action of the symmetric group, \mathfrak{S}_r , and the problem is to determine the Frobenius characters of these representations. There are several cases of this problem that have been studied but there are remarkably few cases for which the problem has been solved. The introduction of [Han85] states that the problem of determining the characters for the adjoint representation of a classical group is “certainly intractable”.

The main result of this paper is a contribution to finding these characters. Assume we have a representation V for which the characters of the W -isotypic subspaces are known. Then Theorem 2 is a formula for the characters of the W -isotypic subspaces for any representation of the form $P(V)$ where P is a polynomial functor, for example a symmetric or exterior power. This is not a simple formula as it expresses the multiplicity in terms of plethysms of symmetric functions.

The groups which have been most intensively studied are the eponymous groups studied in [Wey39], namely the general and special linear groups, the symplectic groups, the orthogonal (and related) groups and the symmetric groups. Each of these has a vector or defining representation V . In this paper we do not consider the orthogonal groups. This gives three sequences of groups with a representation: Furthermore for each sequence we have a diagram category \mathcal{D} and for each (G, V) an evaluation functor from \mathcal{D} to the tensor subcategory of the category of representations of G generated by V and its dual. The first fundamental theorems state that each of these evaluation functors is full. The diagram categories are, respectively, the Brauer category, the partition category and the directed Brauer category.

Date: 21 February 2014.

These diagram categories have a tensor product and the symmetric groups acts on tensor powers. This means that the basic problem of determining the characters of isotypic components in tensor powers can be formulated in this setting. A straightforward account of this is given in section 4. A similar point of view with unrelated results is given in [CEF12].

There have been two lines of applications of the theory of symmetric functions; one is the application to enumeration pioneered by MacMahon, Redfield, Polya and described in [BLL98]; the other is the application to the character theory of classical groups pioneered by Littlewood and is described in [Mac95, Chapter I]. The work in this paper connects these two lines.

1.1. Symbolic method. The original motivation for this work was to understand the symbolic method in the invariant theory of binary forms. Our reference for this is [GY10]. Originally the symbolic method was used to understand invariant polynomials and later, after the development of quantum mechanics, was used to understand invariant tensors.

The starting point for the symbolic method is a construction of elements of $\otimes^r([n])^{\text{SL}(2, \mathbb{C})}$ for $r, n \geq 0$. An element of this vector space is called an invariant tensor of degree r of a binary quantic. The basic invariant is the determinant in $\otimes^2([1])^{\text{SL}(2, \mathbb{C})}$. Then this basic invariant is used to associate an invariant tensor to a bracket monomial. A *bracket monomial* which represents an element of $\otimes^r([n])^{\text{SL}(2, \mathbb{C})}$ consists of a labelled set V with r elements, an unlabelled set E together with an n -to-1 function $E \rightarrow V$ and a partition of E into ordered pairs. The elements of V are traditionally labelled $\alpha, \beta, \gamma, \delta, \dots$. Then, for each ordered pair, we record the ordered pair of elements of V . Thus, the basic invariant is denoted by the ordered pair (α, β) . If the ordered pair (α, β) is repeated k times then this is abbreviated to $(\alpha, \beta)^k$.

Example. The following are examples of bracket monomials

- $(\alpha, \beta)^n \in \otimes^2([n])^{\text{SL}(2, \mathbb{C})}$
- $(\alpha, \beta)(\beta, \gamma)(\gamma, \alpha) \in \otimes^3([2])^{\text{SL}(2, \mathbb{C})}$
- $(\alpha, \beta)^2(\beta, \gamma)^2(\gamma, \alpha)^2 \in \otimes^3([4])^{\text{SL}(2, \mathbb{C})}$
- $(\alpha, \beta)^2(\alpha, \gamma)(\beta, \delta)(\gamma, \delta)^3 \in \otimes^4([4])^{\text{SL}(2, \mathbb{C})}$

There is also a graphical representation of a bracket monomial where each element of V is a node and each ordered pair is represented by a directed edge.

The theorem that is called the Fundamental Theorem in [GY10, Chapter II] is that the bracket monomials span the space of invariant polynomials. The theorem that is called the First Fundamental Theorem in [Wey39] is that the bracket monomials span the space of invariant tensors.

The bracket monomials are a permutation representation of the symmetric group \mathfrak{S}_r with action given by permutations of V . The cycle index series of this permutation representation is

$$(1) \quad \langle h_r[X.h_n[Y]], h_{nr/2}[h_1^2[Y]] \rangle_Y$$

However the bracket monomials satisfy relations. the first relation is $(\alpha, \beta) + (\beta, \alpha) = 0$. If we take this relation into account we obtain the theorem that $\otimes^r([n])^{\text{SL}(2, \mathbb{C})}$ as a representation of \mathfrak{S}_r is a quotient of the representation whose Frobenius character is

$$(2) \quad \langle h_r[X.h_n[Y]], h_{nr/2}[e_2[Y]] \rangle_Y$$

The bracket monomials also satisfy a further identity. This identity is given in [GY10, §22]. The quantum group version of this identity is fundamental for the definition of the Jones polynomial of a link. The graphical interpretation is that we

can eliminate crossings and in terms of Frobenius characters we replace $h_{nr/2}[e_2]$ by the Schur function $s_{nr/2, nr/2}$. The Second Fundamental Theorem in [Wey39] is that no more relations are required so $s_{r/2, r/2}$ is the Frobenius character of $\otimes^r([1])^{\text{SL}(2, \mathbb{C})}$ and more generally, the Frobenius character of $\otimes^r([n])^{\text{SL}(2, \mathbb{C})}$ is

$$(3) \quad \langle h_r[X.h_n[Y]], s_{nr/2, nr/2}[Y] \rangle_Y$$

2. EXAMPLES

Next we give examples which illustrate the methods used in the paper. The notation for symmetric functions is given in section 3.2.

The starting point for this investigation was the observation that the methods of graphical enumeration could be applied to classical invariant theory. The simplest example starts with the enumeration of trivalent graphs (where loops and multiple edges are permitted). This is sequence A005638. A table of connected cubic graphs with n vertices, for $n \leq 14$, is given in [RW98, 3. regular graphs]. If the graph has v vertices and e edges then the trivalent condition implies that $3v = 2e$. In particular v must be even. Then the formula for the number of unlabeled graphs with $2n$ vertices and $3n$ edges given in [Rea59, (5.8)] is

$$\langle h_{2n}[h_3], h_{3n}[h_2] \rangle$$

This formula uses plethysm, or plethystic substitution of symmetric functions and the inner product of symmetric functions.

This result can be strengthened. Consider the species of structure which associates to a set U with $2n$ elements the set of trivalent graphs with vertices U . The cycle index series for this species is

$$(4) \quad \langle h_{2n}[X.h_3[Y]], h_{3n}[h_2[Y]] \rangle_Y$$

The interpretation of this is that $h_3[X.h_{2n}[Y]]$ is the cycle index series of the two-sorted species which associates to a set V with $6n$ elements and a set U with $2n$ elements the set of $3-1$ functions $V \rightarrow U$. Also $h_2[h_{3n}]$ is the cycle index series which associates to a set V with $6n$ elements the set of perfect matchings on V . This means that (4) is the cycle index series of the species where a structure on a set U with $2n$ elements is a set V with $6n$ elements with both a $3-1$ map to U and a perfect matching. This is precisely a trivalent graph with vertices U where the set V is the set of half-edges of the graph.

These symmetric functions can be written in the basis of power sum symmetric functions. For $n = 1$ this symmetric function is $p_{1^2} + p_2$, for $n = 2$ it is

$$\frac{47}{24}p_{1^4} + \frac{9}{4}p_{21^2} + \frac{19}{8}p_{2^2} + \frac{2}{3}p_{31} + \frac{3}{4}p_4$$

and for $n = 3$ this symmetric function is

$$\begin{aligned} \frac{59}{9}p_{1^6} + \frac{151}{24}p_{21^4} + 7p_{2^21^2} + \frac{131}{24}p_{2^3} + \frac{13}{18}p_{31^3} \\ + \frac{5}{6}p_{321} + \frac{11}{9}p_{33} + \frac{3}{4}p_{41^2} + \frac{3}{2}p_{42} + \frac{2}{3}p_6 \end{aligned}$$

In this basis the number of unlabelled trivalent graphs on $2n$ vertices is the sum of the coefficients and the number of labelled trivalent graphs on $2n$ vertices is $(2n)!$ times the coefficient of $p_{1^{2n}}$.

These symmetric functions can also be written in the basis of Schur functions. The coefficients are then non-negative integers. For $n = 1$ this gives $2s_2$, for $n = 2$ this gives

$$2s_{1^4} + 2s_{21^2} + 8s_{2^2} + 5s_{31} + 8s_4$$

and for $n = 3$ this symmetric function is

$$3s_{1^6} + 25s_{21^4} + 33s_{2^21^2} + 50s_{2^3} + 52s_{31^3} \\ + 101s_{321} + 30s_{33} + 55s_{41^2} + 102s_{42} + 52s_{51} + 31s_6$$

In this basis the number of unlabelled trivalent graphs on $2n$ vertices is the coefficient of s_{2n} and the number of labelled trivalent graphs on $2n$ vertices is given by substituting f_λ , the number of standard Young tableaux of shape λ , for s_λ .

Compare this with the classical problem of finding the Hilbert series of the polynomial invariants of the binary quartic. Let $[3]$ be the four dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{C})$. Then the problem is to determine $\dim S^r([3])^{\mathrm{SL}(2, \mathbb{C})}$, the dimension of the subspace of invariant tensors in the r -th symmetric power of $[3]$, for all $r \geq 0$. A variation of this problem is to determine $\dim \otimes^r([3])^{\mathrm{SL}(2, \mathbb{C})}$, the dimension of the subspace of invariant tensors in the r -th tensor power of $[3]$, for all $r \geq 0$ and a refinement of this problem is to determine the character of $\otimes^r([3])^{\mathrm{SL}(2, \mathbb{C})}$ as a representation of \mathfrak{S}_r .

For each $r \geq 0$ there is a basis of $\otimes^r([3])^{\mathrm{SL}(2, \mathbb{C})}$ given by Temperley-Lieb diagrams. We have that $\otimes^r([3])^{\mathrm{SL}(2, \mathbb{C})} = 0$ if r is odd, so we assume r is even and then put $r = 2n$. Then the basis of Temperley-Lieb diagrams is described informally as a non-crossing trivalent graph on $2n$ vertices. At first sight a formal description of the species is that a structure on a set U with $2n$ elements is a set V with $6n$ elements with both a $3 - 1$ map to U and a non-crossing perfect matching. The cycle index series of a set of size $2k$ with a non-crossing perfect matching is $s_{k,k}$. The cycle index of this species is

$$(5) \quad \langle h_3[X.h_{2n}[Y]], s_{3n,3n}[Y] \rangle_Y$$

This is not the correct cycle index series as the action of the symmetric group involves a sign. There are two conventions for the sign leading to the conjugate pair of symmetric functions

$$(6) \quad \langle h_3[X.h_{2n}[Y]], s_{2^{3n}}[Y] \rangle_Y \quad \langle h_3[X.e_{2n}[Y]], s_{3n,3n}[Y] \rangle_Y$$

The first formula follows the sign convention in the symbolic method approach to classical invariant theory and which is followed in the mathematical physics literature. In this convention each edge of the perfect matching is oriented with the understanding that reversing the orientation of a single edge multiplies by -1 . The effect of this is to tensor the representation on non-crossing perfect matchings by the sign representation. The second formula follows the sign convention in the quantum group literature. In this convention we take the action of the permutation group on the vertices to be tensored by the sign representation. The origin of these two conventions are that in the first formula we consider the defining representation of $\mathrm{SL}(2, \mathbb{C})$ as a two dimensional vector space with a symplectic form and in the second convention we consider it to be an odd vector space with a symmetric bilinear form.

Note that in neither case do we have the cycle index series of a species. This is because we now have representations of the symmetric group which do not have a basis preserved by the symmetric group action.

Next we discuss another example. This example is taken from [Mac17] which is the paper that inspired Redfield. This is the earliest example of plethysm being used for an enumeration.

A pack of cards consists of m identical sets of n cards. The nm cards are dealt into n hands each with m cards. The hands are unordered and the cards in each hand are also unordered. Let $f(m, n)$ be the number of possible deals. Equivalently, $f(m, n)$ is the number of $n \times n$ matrices with entries in $\{0, 1, \dots, m\}$, all row sums n and all column sums m . Two matrices are equivalent if one can be obtained from the other by permuting the rows.

MacMahon shows that $f(m, n)$ is the coefficient of the monomial symmetric function associated to the partition m^n in the expansion of $h_n[h_m]$. This is equivalent to

$$f(m, n) = \langle h_n[h_m], h_m^n \rangle$$

Consider all matrices with entries in $\{0, 1, \dots, m\}$, all row sums n and all column sums m . Then \mathfrak{S}_n acts by permuting the rows and $f(m, n)$ is the number of orbits. The cycle index series of this action is

$$\langle h_n[X.h_m[Y]], h_m^n[Y] \rangle_Y$$

The meaning of this is that we have a set U with n elements and a set W with nm elements. Then a structure is a m -1 function $W \rightarrow U$ and a partition of W into a list of n subsets each of size m .

Example. For example, $f(2, 3) = 5$. If we take these matrices with the symmetric group acting by permuting the rows then we have 21 matrices in 5 orbits. The cycle index series is

$$5s_3 + 7s_{2,1} + 2s_{1,1,1}$$

3. POLYNOMIAL FUNCTORS

First we review the theory of polynomial functors. This approach to Schur-Weyl duality is given in [Mac95, Appendix A] and [Gre07, §6].

3.1. Polynomial functors. Let \mathbf{Vect} be the category of finite dimensional vector spaces and linear maps. A *polynomial functor* is a functor $P: \mathbf{Vect} \rightarrow \mathbf{Vect}$ such that for any two finite dimensional vector spaces, U and V , the map $P(U, V): \text{Hom}(U, V) \rightarrow \text{Hom}(P(U), P(V))$, $f \mapsto P(f)$, is polynomial.

The polynomial functor is *homogeneous* of degree r if $P(U, V)$ is homogeneous of degree r for all U and V . Every polynomial functor is a sum of homogeneous polynomial functors.

The fundamental result due to Schur is that, for each $r \geq 0$, the category of finite dimensional representations of \mathfrak{S}_r is equivalent to the category of homogeneous polynomial functors of degree r .

Definition 1. Let A be a finite dimensional representation of \mathfrak{S}_r then the associated polynomial functor is defined on objects by $V \mapsto A \otimes_{K[\mathfrak{S}_r]} (\otimes^r V)$.

This construction defines a functor from the category of finite dimensional representations of \mathfrak{S}_r to the category of homogeneous polynomial functors of degree r . This functor is an equivalence where the inverse equivalence is given by polarisation.

The basic examples are that the r -th symmetric power functor corresponds to the the trivial representation, the r -th exterior power functor corresponds to the the sign representation and the r -th tensor power functor corresponds to the the regular representation. The notation we use for the tensor algebra of a vector space V is

$$(7) \quad T^\bullet(V) = \bigoplus_{r \geq 0} \otimes^r V$$

The irreducible representations of \mathfrak{S}_r are indexed by partitions of size r . Hence for each $\lambda \vdash r$ we have $S(\lambda)$, an irreducible representation of \mathfrak{S}_r , and \mathbb{S}^λ , the associated homogeneous polynomial functor of degree r . In particular, \mathbb{S}^r is the r -th symmetric power functor and \mathbb{S}^{1^r} is the r -th exterior power functor.

Let G be an affine algebraic group. Let $\text{Rep}(G)$ be the category of finite dimensional rational representations. Then a polynomial functor P also gives a functor

$P: \text{Rep}(G) \rightarrow \text{Rep}(G)$. The functors $\mathbb{S}^\lambda: \text{Rep}(G) \rightarrow \text{Rep}(G)$ are called Schur functors. Equivalently, put $\dim V = N$ and consider the representation V as a homomorphism $\rho: G \rightarrow \text{GL}(N)$. Then $\mathbb{S}^\lambda(V)$ is the restriction along ρ of the irreducible representation of $\text{GL}(N)$ corresponding to λ .

The problem of determining the characters of the isotypic subspaces of $\otimes^r V$ and the problem of decomposing the representation $\mathbb{S}^\lambda(V)$ are equivalent. This equivalence is [Lit47, Theorem III].

Example. The linear invariants of ten quaternary quadrics were first studied in [TY26]. This article contained an error that was corrected in [Lit47]. This is the character of the space of tensors invariant under $\text{SL}(4)$ in the representation $\otimes^{10} S^2(\mathbb{C}^4)$. This vector space has dimension 3396 and the character is

$$s_{6,2^2} + s_{6,1^4} + s_{5,3,1^2} + s_{5,2^2,1} + s_{4^2,2} + s_{4,2^3} + s_{3^2,2,1^2} + s_{4,3,1^3} + s_{4,2,1^4} + s_{2^3,1^4} + s_{1^{10}}$$

3.2. Symmetric functions. Our notation for symmetric functions follows [Mac95, Chapter I] and [Sta99, Chapter 7]. These text books give the necessary background on symmetric functions. The graded ring of symmetric functions is denoted Sym . The standard symmetric functions which are polynomial generators for the ring of symmetric functions are the homogeneous symmetric functions, $\{h_r: r \geq 0\}$ and the elementary symmetric functions $\{e_r: r \geq 0\}$. The power sum symmetric functions, $\{p_r: r \geq 0\}$, are polynomial generators for the \mathbb{Q} -algebra $\text{Sym} \otimes_{\mathbb{Z}} \mathbb{Q}$.

The *character* of a homogeneous polynomial functor of degree r is a symmetric function of degree r . The character of P is denoted by $\mathbf{ch}(P)$ and is determined by the property that for any invertible diagonal matrix A with diagonal entries a_1, \dots, a_n we have

$$\mathbf{ch}(P)(a_1, \dots, a_n, 0, 0, \dots) = \text{Tr } P(A)$$

For example, $\mathbf{ch} \mathbb{S}^{(n)} = h_n$ and $\mathbf{ch} \mathbb{S}^{1^n} = e_n$. The Schur function, s_λ , is defined by $s_\lambda = \mathbf{ch} \mathbb{S}^\lambda$.

Let $\rho: \mathfrak{S}_r \rightarrow \text{End}(A)$ be a representation of \mathfrak{S}_r and P the associated polynomial functor. Then we have

$$\mathbf{ch} P = \frac{1}{r!} \sum_{\pi \in \mathfrak{S}_r} \text{Tr } \rho(\pi) p_{\lambda(\pi)}$$

where $\lambda(\pi)$ is the cycle type of π and Tr is the matrix trace.

We will make extensive use of the series H , defined by

$$H = \prod_{i \geq 1} \frac{1}{1 - x_i} = \sum_{n \geq 0} h_n = \sum_{\lambda} \frac{1}{z_\lambda} p_\lambda = \exp \left(\sum_{k \geq 1} \frac{p_k}{k} \right)$$

This is an element of the completion $\widehat{\text{Sym}} = \prod_{r \geq 0} \text{Sym}_r$.

3.3. Products. Symmetric functions have a remarkable number of operations. These operations have been given a number of different names by different authors. One reason for this is that a product can mean a multiplication or a scalar product. The operations can all be computed in $\text{Sym}_{\mathbb{Q}}$ using explicit formulae given in terms of the power sum symmetric functions.

First Sym is a graded Hopf algebra and a graded λ -ring. We will refer to these operations as external. Second, for $r \geq 0$, the Grothendieck group of representations of \mathfrak{S}_r is a Hopf algebra and a λ -ring. We will refer to these operations as internal.

There is an inner product introduced by Redfield who called it the cap product. This is an inner product on symmetric functions characterised by the property that if P_1, P_2 are homogeneous polynomial functors both of degree r associated to representations A_1, A_2 of \mathfrak{S}_r then

$$\langle \mathbf{ch} P_1, \mathbf{ch} P_2 \rangle = \dim \text{Hom}(P_1, P_2) = \dim \text{Hom}_{\mathfrak{S}_r}(A_1, A_2)$$

The operations that occur throughout this paper are the external product, external plethysm and the inner product. The internal plethysm appears implicitly in section 5.3 and the internal product is used in section 5.2.

The external product corresponds to tensor product of polynomial functors. Let P_1, P_2 be a homogeneous polynomial functors of degrees r_1 and r_2 , respectively. Then we have a homogeneous polynomial functor $P_1 \otimes P_2$ of degree $r_1 + r_2$ given by $P_1 \otimes P_2: V \mapsto P_1(V) \otimes P_2(V)$. If P_1, P_2 correspond to the representations A_1 and A_2 then $P_1 \otimes P_2$ corresponds to the representation $(A_1 \otimes A_2) \uparrow_{\mathfrak{S}_{r_1} \times \mathfrak{S}_{r_2}}^{\mathfrak{S}_{r_1+r_2}}$. Then the external product is determined by

$$\mathbf{ch} P_1 \otimes P_2 = \mathbf{ch} P_1 \cdot \mathbf{ch} P_2$$

This is also determined by the property that the maps $\text{Sym} \rightarrow \mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ are homomorphisms of graded rings.

The external plethysm corresponds to composition of polynomial functors. This operation was introduced in [Lit35, §4]. Let P, Q be a homogeneous polynomial functors of degrees r and s , respectively. Then we have a homogeneous polynomial functor $P \circ Q$ of degree rs given by $P \circ Q: V \mapsto P(Q(V))$. If P, Q correspond to the representations A and B then $P \circ Q$ corresponds to the representation constructed by first taking $(\otimes^r A) \otimes B$ as a representation of the wreath product $\mathfrak{S}_r \wr \mathfrak{S}_s$ and then inducing to \mathfrak{S}_{rs} . Then the external plethysm is determined by

$$\mathbf{ch} P \circ Q = \mathbf{ch} P[\mathbf{ch} Q]$$

A theme running through this paper is the following symmetric function constructed from the two symmetric functions F and G ,

$$\langle H[X.F[Y]], G[Y] \rangle_Y$$

These can be computed in terms of the power sum symmetric functions by

$$\langle H[X.F[Y]], G[Y] \rangle_Y = \sum_{\lambda} \frac{1}{z_{\lambda}} \langle p_{\lambda}[F], G \rangle p_{\lambda}$$

and can also be computed in terms of the Schur symmetric functions by

$$\langle H[X.F[Y]], G[Y] \rangle_Y = \sum_{\lambda} \langle s_{\lambda}[F], G \rangle s_{\lambda}$$

There are two integers associated to a homogeneous symmetric function which are of interest both in combinatorics and in representation theory. One integer is defined as the sum of the coefficients in the basis of power sum symmetric functions or as the coefficient of s_r in the basis of Schur functions. In combinatorics this counts the number of orbits in a permutation representation or the number of unlabelled structures. In representation theory this is the dimension of the subspace of invariant tensors. In the above this is given by

$$\sum_{\lambda \vdash r} \frac{1}{z_{\lambda}} \langle p_{\lambda}[F], G \rangle = \langle s_r[F], G \rangle$$

The other integer is the order of the permutation representation or the dimension of the linear representation. This is given by $r!$ times the coefficient of p_{1^r} in the basis of power sum symmetric functions. In the basis of Schur functions it is given by the substitution $s_{\lambda} \mapsto \dim S(\lambda)$. In the above this is given by

$$\langle F^r, G \rangle = \langle p_{1^r}[F], G \rangle = \sum_{\lambda \vdash r} \langle s_{\lambda}[F], G \rangle \dim S(\lambda)$$

4. DIAGRAM CATEGORIES

In this section we give a conceptual and uniform explanation of the phenomenon of stability. This phenomenon originates in the work of Littlewood for the symplectic and general linear groups and in work of Murnaghan for the symmetric groups.

4.1. Schur functors. In this section we adapt the diagram categories and define Schur functors. The three diagram categories that we consider in this paper are: the Brauer category, partition category and the directed Brauer category. Each of these categories is a $K[\delta]$ -linear rigid strict symmetric monoidal category where K is a field of characteristic zero.

There are two obstructions to applying definition 1 to a diagram category. One is that we cannot form the direct sum of objects in a diagram category and the other is that we cannot take invariant tensors. In order to remedy this we enlarge the diagram category by replacing D by $\text{Proj}(D)$, the category of finitely generated projective right D -modules. This is an enlargement since we have an inclusion $D \rightarrow \text{Proj}(D)$ which sends the object $[n]$ to the functor $[m] \mapsto \text{Hom}_D([m], [n])$. Furthermore every object of $\text{Proj}(D)$ is a summand of a finite direct sum of these objects.

Example. Let $K\mathfrak{S}_\bullet$ be the K -linear category with objects $[n]$ for $n \geq 0$ and

$$\text{Hom}([n], [m]) = \begin{cases} K\mathfrak{S}_n & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$$

Then $\text{Proj}(K\mathfrak{S}_\bullet)$ is equivalent to the category of polynomial functors.

The category $\text{Proj}(D)$ has two properties. One is that we have a direct sum which is both a product and a coproduct. The other is that every idempotent morphism has an image, a kernel and a cokernel. In order to apply the definition (1) it remains to show that $\text{Proj}(D)$ is a symmetric monoidal category.

Consider the tensor functor $\otimes: D \otimes D \rightarrow D$. This gives a restriction functor from representations of D to representations of $D \otimes D$. This functor has a left adjoint which is an extension of the induction functor for finite dimensional algebras and an example of a Kan extension. Explicitly, if F, G are representations of D then the representation $F \otimes G$ is given by

$$(F \otimes G)([m]) = \bigoplus_{n_1, n_2} F([n_1]) \otimes G([n_2]) \otimes \text{Hom}_D([n_1] \otimes [n_2], [m]) / \sim$$

where \sim is the subspace spanned by all elements of the form

$$a_1 f_1 \otimes a_2 f_2 \otimes \psi - a_1 \otimes a_2 \otimes (f_1 \otimes f_2) \psi$$

This functor restricts to $\otimes: \text{Proj}(D) \otimes \text{Proj}(D) \rightarrow \text{Proj}(D)$. This makes $\text{Proj}(D)$ a monoidal category.

The action of the symmetric groups on tensor powers in D induces an action on tensor powers in $\text{Proj}(D)$. Similarly, the antiinvolution on D induces an antiinvolution on $\text{Proj}(D)$. These structures then make $\text{Proj}(D)$ a rigid symmetric monoidal category. Furthermore, (1) now defines polynomial functors on $\text{Proj}(D)$.

5. INVARIANT THEORY

In this section we prove the main theorem and present applications.

Proposition 1. *Let P be a polynomial functor and V, W representations of G . Then*

$$\dim \text{Hom}_G(P(V), W) = \langle \mathbf{ch} P, \mathbf{ch} \text{Hom}_{\mathfrak{S}}(T^\bullet(V), W) \rangle$$

Proof. Assume, without loss of generality, that P is homogeneous of degree r and that P corresponds to the representation A . Then, by the \otimes -Hom adjunction applied to the bimodule $\otimes^r V$, there is a natural isomorphism of vector spaces

$$\mathrm{Hom}_G(A \otimes_{\mathfrak{S}_r} (\otimes^r V), W) \cong \mathrm{Hom}_{\mathfrak{S}_r}(A, \mathrm{Hom}_G(\otimes^r V, W))$$

Taking dimensions gives the proposition. \square

Theorem 2. *Let G be a reductive algebraic group, $V, W \in \mathrm{Rep}(G)$, and P a virtual polynomial functor. Then*

$$\mathbf{ch} \mathrm{Hom}_G(T^\bullet(P(V)), W) = \langle H[X.(\mathbf{ch} P)[Y]], \mathbf{ch} \mathrm{Hom}_G(T^\bullet(V), W)[Y] \rangle_Y$$

Proof. The proof consists of expanding both sides to get the same expression in both cases.

The left hand side expands to

$$\sum_{\lambda} \dim \mathrm{Hom}_G(\mathbb{S}^\lambda(P(V)), W) \cdot s_\lambda$$

The Cauchy identity is

$$H[X.Y] = \sum_{\lambda} s_\lambda[X] \cdot s_\lambda[Y]$$

and so

$$H[X.(\mathbf{ch} P)[Y]] = \sum_{\lambda} s_\lambda[X] \cdot s_\lambda[\mathbf{ch} P[Y]]$$

Hence the right hand side expands to

$$\sum_{\lambda} \langle s_\lambda[\mathbf{ch} P], \mathbf{ch} \mathrm{Hom}_G(T^\bullet(V), W) \rangle \cdot s_\lambda$$

The coefficients of s_λ in these two equations are equal by Proposition 1 applied to the polynomial functor $\mathbb{S}^\lambda \circ P$. \square

5.1. Symplectic group. Take G to be the symplectic group, $\mathrm{Sp}(2n)$, of rank n and V to be the defining representation of dimension $2n$. These plethysms are studied in [Lit58, §3].

The irreducible rational representations are indexed by partitions λ with $\ell(\lambda) \leq n$. For each partition, $\alpha = (\alpha_1, \alpha_2, \dots)$ we write 2α for the partition $(2\alpha_1, 2\alpha_2, \dots)$ and $\alpha \cup \alpha$ for the partition $(\alpha_1, \alpha_1, \alpha_2, \alpha_2, \dots)$.

Proposition 3.

$$\mathbf{ch} \mathrm{Hom}_{\mathrm{Sp}(2n)}(T^\bullet(V), K) = \sum_{\beta, \ell(\beta) \leq n} s_{2\beta}$$

For the stable case, which is given by the Brauer category

$$\mathbf{ch} \mathrm{Hom}_{\mathbb{B}}(T^\bullet(V), K) = \sum_{\beta} s_{2\beta} = H[e_2]$$

For non-trivial representations the stable result is due to Littlewood. The unstable result is given in [Tok87], [Sun90a], [Sun90b].

Associated to each partition, λ , is a symmetric function sp_λ . These were introduced in [Lit06]; they are defined in §11.8 Theorem by a Jacobi-Trudi determinant and expressed in terms of the Schur functions in §11.9 Theorem II. They have been studied in [Koi89] and [FK97]. In the unstable case these correspond to an irreducible character, to zero or to the negative of an irreducible character. The precise rules are the Newell-Littlewood modification rules given in [New51].

Definition 2.

$$\text{sp}_\lambda = \sum_{\alpha \cup \alpha \subseteq \lambda} (-1)^{|\alpha|} s_{\lambda/\alpha \cup \alpha}$$

Theorem 4. For any n and any partition λ with $\ell(\lambda) \leq n$,

$$\mathbf{ch} \text{Hom}_{\text{Sp}(2n)}(T^\bullet(V(\lambda)), K) = \left\langle H[X.\text{sp}_\lambda[Y]], \sum_{\beta, \ell(\beta) \leq n} s_{2\beta}[Y] \right\rangle_Y$$

and for the Brauer category

$$\mathbf{ch} \text{Hom}_B(T^\bullet(V(\lambda)), K) = \langle H[X.\text{sp}_\lambda[Y], H[e_2]] \rangle_Y$$

Putting $\lambda = (p)$ and using $\text{sp}_p = s_p$ gives

Corollary 5. For any n and any p ,

$$\mathbf{ch} \text{Hom}_{\text{Sp}(2n)}(T^\bullet(S^p(V)), K) = \left\langle H[X.s_p[Y]], \sum_{\beta, \ell(\beta) \leq n} s_{2\beta}[Y] \right\rangle_Y$$

and for the Brauer category

$$\mathbf{ch} \text{Hom}_B(T^\bullet(S^p(V)), K) = \langle H[X.s_p[Y], H[e_2]] \rangle_Y$$

The case $p = 2$ is studied in [Han85]. The analogue of this result for the orthogonal groups gives $\langle H[X.s_p[Y]], H[h_2] \rangle_Y$ which is the cycle index series of the species of p -regular graphs.

The groups $\text{Sp}(2)$ and $\text{SL}(2)$ are isomorphic. The characters $S^p(S^k(V))$ are given by the Cayley-Sylvester formula. The generalisation to the characters of $\mathbb{S}^\lambda(S^k V)$ for all partitions λ and all k is given in [Bri94]. Putting $n = 1$ gives

Corollary 6. For any $p > 0$,

$$\mathbf{ch} \text{Hom}_{\text{SL}(2)}(T^\bullet(\mathbb{S}^p(V)), K) = \left\langle H[X.\text{sp}_\lambda[Y]], \sum_{k \geq 0} s_{k,k}[Y] \right\rangle_Y$$

5.2. General linear group. Take G to be the general linear group $\text{GL}(n)$, V to be the defining representation and V^* the dual representation.

Definition 3. Define C_+ by

$$C_+ = \sum_{r \geq 1} \frac{1}{r} \sum_{d|r} \phi(d) p_{d^{r/d}}$$

Definition 4.

$$Z_P = H[C_+] = \prod_{k \geq 1} \frac{1}{1 - p_k} = \sum_{\lambda} p_\lambda$$

The following expression for Z_P which uses the internal product, $*$, is given in [ST94]

$$Z_P = \sum_{\lambda} s_\lambda * s_\lambda$$

Then the stable formula is given in [Sta84] and [Ste87].

Proposition 7. For $n > 0$,

$$\mathbf{ch} \text{Hom}_D(T^\bullet(V \otimes V^*), K) = Z_P$$

This generalises to

Proposition 8. For $n > 0$,

$$\mathbf{ch} \text{Hom}_D(T^\bullet(V \otimes V^*), V(\alpha, \beta)) = \sum_{\lambda} (s_\alpha \cdot s_\lambda) * (s_\beta \cdot s_\lambda) = (s_\alpha * s_\beta) \left[\frac{s_1}{1 - s_1} \right] \cdot Z_P$$

The unstable versions of these results are not known.

5.3. Symmetric group. Let G be the symmetric group \mathfrak{S}_n and V the defining permutation representation. The decomposition of V into irreducible representations is $V \cong S(n-1, 1) \oplus S(n)$. These plethysms are studied in [Lit58, §5].

The following proposition is a restatement of [ST94, Theorem 5.1] and [Sta99, Exercise 7.74].

Proposition 9. *For any $n > 0$ and any $\mu \vdash n$.*

$$\mathbf{ch} \operatorname{Hom}_{\mathfrak{S}_n}(T^\bullet V, S(\mu)) = s_\mu[H]$$

Then applying Theorem 2 gives

Theorem 10.

$$\mathbf{ch} \operatorname{Hom}_{\mathfrak{S}_n}(T^\bullet P(V), S(\mu)) = \langle H[X.\mathbf{ch} P[Y]], s_\mu[H] \rangle_Y$$

Since the k -th exterior power of V is $S(n-k, k) \oplus S(n-k+1, k-1)$ for $1 \leq k \leq n-1$, a particular case of this is:

Corollary 11. *For $n > 0$ and for $0 \leq k \leq n-1$,*

$$\mathbf{ch} \operatorname{Hom}_{\mathfrak{S}_n}(T^\bullet(S(n-k, k) \oplus S(n-k+1, k-1)), S(\mu)) = \langle H[X.e_k[Y]], s_\mu[H[Y]] \rangle_Y$$

In the stable case, which is given by the partition category,

$$\mathbf{ch} \operatorname{Hom}_{\Pi}(T^\bullet V, S(\mu)) = s_\mu[H_+].H[H_+] = (s_\mu.H)[H_+]$$

and for the irreducible representation $S(n-1, 1)$,

$$\mathbf{ch} \operatorname{Hom}_{\Pi}(T^\bullet S(n-1, 1), S(\mu)) = s_\mu[H_+].H_+[H_+] = (s_\mu.H_+)[H_+]$$

Proposition 12. *For all $\mu \vdash n$, and all polynomial functors P ,*

$$\mathbf{ch} \operatorname{Hom}_{\Pi}(T^\bullet(P(V)), S(\mu)) = \langle H[X.(\mathbf{ch} P)[Y]], (s_\mu.H)[H_+[Y]] \rangle_Y$$

$$\mathbf{ch} \operatorname{Hom}_{\Pi}(T^\bullet P(S(n-1, 1)), S(\mu)) = \langle H[X.(\mathbf{ch} P)[Y]], (s_\mu.H_+)[H_+[Y]] \rangle_Y$$

Since the k -th exterior power of $S(n-1, 1)$ is $S(n-k, k)$ for $0 \leq k \leq n-1$, a particular case of this is:

Corollary 13. *For $n > 0$ and for $0 \leq k \leq n-1$,*

$$\mathbf{ch} \operatorname{Hom}_{\Pi}(T^\bullet S(n-k, k), S(\mu)) = \langle H[X.e_k[Y]], (s_\mu.H_+)[H_+[Y]] \rangle_Y$$

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