

Generalizing Wallis' formula

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Abstract. The present note generalizes Wallis' formula, $\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \dots$, using the Euler-Mascheroni constant g and the Glaisher-Kinkelin constant A :

$$\frac{4^\gamma}{2^{\ln 2}} = \frac{\sqrt{2}}{1} \cdot \frac{\sqrt{2}}{\sqrt[3]{3}} \cdot \frac{\sqrt[4]{4}}{\sqrt[3]{3}} \cdot \frac{\sqrt[4]{4}}{\sqrt[5]{5}} \cdot \frac{\sqrt[6]{6}}{\sqrt[5]{5}} \cdot \frac{\sqrt[6]{6}}{\sqrt[7]{7}} \cdot \dots \quad \text{and} \quad \left(\frac{2^2 \pi e^\gamma}{A^{12}} \right)^{\frac{\pi^2}{6}} = \frac{\sqrt[4]{2}}{1} \cdot \frac{\sqrt[4]{2}}{\sqrt[3]{3}} \cdot \frac{\sqrt[4]{4}}{\sqrt[3]{3}} \cdot \frac{\sqrt[4]{4}}{\sqrt[5]{5}} \cdot \frac{\sqrt[6]{6}}{\sqrt[5]{5}} \cdot \frac{\sqrt[6]{6}}{\sqrt[7]{7}} \cdot \dots$$

Wallis' formula, named after the English mathematician John Wallis (1616 –1703), is popular in many calculus courses (see [2], [1] p. 338]). It is a slowly convergent product, but its importance is historic and aesthetic. The present paper proposes two similar equally pleasing formulas, in a rather straightforward way, without using the gamma function for generating these product formulas. Perhaps some readers will take up the challenge of finding even easier proofs on the level of a calculus course, similar to those for Wallis' formula.

The Dirichlet eta function is defined for any complex number s with real part > 0 by:

$$\eta(s) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} + \dots$$

For $s = 0$ it lead Y. L. Yung and J. Sondow to a remarkably elegant proof for Wallis' formula $\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \dots$ (see [2]), and their proof will be adapted here to other values of s .

Theorem. For appropriate values of s (and if $n^0\sqrt{2n}$ is interpreted as $2n$),

$$e^{2\eta(s)} = \frac{2^{\sqrt[2]{2}}}{\sqrt[1]{1}} \cdot \frac{2^{\sqrt[2]{2}}}{\sqrt[3]{3}} \cdot \frac{4^{\sqrt[4]{4}}}{\sqrt[3]{3}} \cdot \frac{4^{\sqrt[4]{4}}}{\sqrt[5]{5}} \cdot \frac{6^{\sqrt[6]{6}}}{\sqrt[5]{5}} \cdot \frac{6^{\sqrt[6]{6}}}{\sqrt[7]{7}} \cdot \dots \quad (*)$$

Proof. By definition,

$$\eta(s) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} + \dots \quad (\text{for } \text{Re}(s) > 0)$$

$$= \frac{1}{2} + \frac{1}{2} \left[\left(\frac{1}{1^s} - \frac{1}{2^s} \right) + \left(-\frac{1}{2^s} + \frac{1}{3^s} \right) + \left(\frac{1}{3^s} - \frac{1}{4^s} \right) + \left(-\frac{1}{4^s} + \frac{1}{5^s} \right) + \dots \right] \quad (\text{for } \text{Re}(s) > -1)$$

$$\text{Thus: } \eta'(s) = \frac{1}{2} \left[\left(-\frac{\ln(1)}{1^s} + \frac{\ln(2)}{2^s} \right) + \left(\frac{\ln(2)}{2^s} - \frac{\ln(3)}{3^s} \right) + \left(-\frac{\ln(3)}{3^s} + \frac{\ln(4)}{4^s} \right) + \left(\frac{\ln(4)}{4^s} - \frac{\ln(5)}{5^s} \right) + \dots \right]$$

$$= \frac{1}{2} \ln \left(\frac{2^{1/2^s}}{1^{1/1^s}} \cdot \frac{2^{1/2^s}}{3^{1/3^s}} \cdot \frac{4^{1/4^s}}{3^{1/3^s}} \cdot \frac{4^{1/4^s}}{5^{1/5^s}} \dots \right),$$

$$\text{and so } e^{2\eta(s)} = \frac{2^{1/2^s}}{1^{1/1^s}} \cdot \frac{2^{1/2^s}}{3^{1/3^s}} \cdot \frac{4^{1/4^s}}{3^{1/3^s}} \cdot \frac{4^{1/4^s}}{5^{1/5^s}} \dots \quad \blacksquare$$

In the case $s = 0$, the literature tells us $\eta'(0) = \frac{1}{2} \ln\left(\frac{\pi}{2}\right)$, and this leads to the familiar Wallis formula $\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \dots$

In the case $s = 1$, $\eta'(1)$ can be obtained from the property $\sum_{k=1}^{\infty} (-1)^k \frac{\ln(k)+\gamma}{k} = -\frac{1}{2} (\ln 2)^2$ (see [5]). Indeed, using the Euler-Mascheroni constant γ , it follows that

$$\begin{aligned} \eta'(1) &= \sum_{k=1}^{\infty} (-1)^k \frac{\ln(k)}{k} = -\gamma \sum_{k=1}^{\infty} (-1)^k \frac{1}{k} - \frac{1}{2} (\ln 2)^2 \\ &= -\gamma(-\ln 2) - \frac{1}{2} (\ln 2)^2 = \frac{1}{2} (2\gamma - \ln 2) \ln 2 = \frac{1}{2} \ln 2^{(2\gamma - \ln 2)}, \end{aligned}$$

Substitution in (*) leads to:

$$2^{(2\gamma - \ln 2)} = \frac{2^{1/2}}{1^{1/1}} \cdot \frac{2^{1/2}}{3^{1/3}} \cdot \frac{4^{1/4}}{3^{1/3}} \cdot \frac{4^{1/4}}{5^{1/5}} \dots,$$

and this is the formula given in the abstract.

In the case $s = 2$, we need the Glaisher-Kinkelin constant \mathcal{A} , given by

$$\mathcal{A} = \lim_{n \rightarrow +\infty} \frac{H(n)}{n^{\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}} \cdot e^{-\frac{n^2}{4}}} = 1.28242\dots$$

where the 'hyperfactorial' $H(n)$ is defined by $H(n) = \prod_{k=1}^n k^k$.

Now combining the expression for $\eta'(s)$ (see [3]) for $s = 2$ and a well-known result using \mathcal{A} (see [4]), implies that

$$\eta'(2) = \frac{1}{2} \frac{\pi^2}{6} \ln 2 + \frac{1}{2} \frac{\pi^2}{6} [\gamma + \ln(2\pi) - 12 \ln \mathcal{A}] = \frac{1}{2} \frac{\pi^2}{6} \ln \left(\frac{2^2 \pi e^\gamma}{\mathcal{A}^{12}} \right).$$

Substitution in (*) leads to:

$$\left(\frac{2^2 \pi e^\gamma}{\mathcal{A}^{12}} \right)^{\frac{\pi^2}{6}} = \frac{2^{1/2^2}}{1^{1/1^2}} \cdot \frac{2^{1/2^2}}{3^{1/3^2}} \cdot \frac{4^{1/4^2}}{3^{1/3^2}} \cdot \frac{4^{1/4^2}}{5^{1/5^2}} \dots$$

and this is the formula given in the abstract.

References

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