COMBINATORICS OF DUMONT DIFFERENTIAL SYSTEM ON THE JACOBI ELLIPTIC FUNCTIONS

SHI-MEI MA, TOUFIK MANSOUR, AND DAVID G.L. WANG

Abstract. In this paper, we relate Jacobi elliptic functions to several combinatorial structures, including the longest alternating subsequences, alternating runs and descents. The Dumont differential system on the Jacobi elliptic functions is defined by $D(x) = yz,\ D(y) = xz,\ D(z) = xy$. As a continuation of the work of Dumont on the connection between this differential system and combinatorics, we give the solutions of several systems of partial differential equations associated with this differential system.

Keywords: Jacobi elliptic functions; Dumont differential system; Combinatorial structures; Longest alternating subsequences

1. Introduction

Let $\mathrm{sn}\,(u,k), \mathrm{cn}\,(u,k)$ and $\mathrm{dn}\,(u,k)$ be three basic *Jacobi elliptic functions*. These functions occur naturally in geometry, analysis, number theory, algebra and combinatorics (see [6, 9, 22, 23]). Finding the connection between Jacobi elliptic functions and combinatorics has been a hot research topic in mathematics for more than forties years (see, e.g., [8, 9, 12, 23, 32] and references therein). As pointed out by Flajolet and Françon [12], the question of possible combinatorial significance of the coefficients appearing in the Maclaurin expansions of the Jacobi elliptic functions has first been raised by Schützenberger. This paper presents several answers related to Dumont's [9] work. In particular, in Section 4, we show that the statistics of permutations, such as the longest alternating subsequences and alternating runs are both closely related to these functions. It should be noted that the study of the distribution of the length of the longest alternating subsequences of permutations was recently initiated by Stanley [28, 29, 30].

The function $\mathrm{sn}\,(u,k)$ is defined as the inverse of the elliptic integral

$$u = F(x,k) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{1}$$

where k is known as the *modulus* and satisfies $0 < k < 1$ (see [15]). Hence $x = \mathrm{sn}\,(u,k)$. The functions $\mathrm{cn}\,(u,k)$ and $\mathrm{dn}\,(u,k)$ are given as follows:

$$\mathrm{cn}\,(u,k) = \sqrt{1-\mathrm{sn}^2(u,k)}, \qquad \mathrm{dn}\,(u,k) = \sqrt{1-k^2\mathrm{sn}^2(u,k)}.$$

Moreover, Jacobi elliptic functions can also be defined in terms of *Jacobi theta functions* (see [14, 17]). Using formal methods, Abel [1] discovered that these functions satisfy the following system of differential equations:

$$\begin{aligned}\frac{d}{du}\mathrm{sn}\,(u,k) &= \mathrm{cn}\,(u,k)\mathrm{dn}\,(u,k),\\ \frac{d}{du}\mathrm{cn}\,(u,k) &= -\mathrm{sn}\,(u,k)\mathrm{dn}\,(u,k),\\ \frac{d}{du}\mathrm{dn}\,(u,k) &= -k^2\mathrm{sn}\,(u,k)\mathrm{cn}\,(u,k).\end{aligned} \tag{2}$$

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]$, where $[n]=\{1,2,\ldots,n\}$. An *interior peak* in π is an index $i\in\{2,3,\ldots,n-1\}$ such that $\pi(i-1)<\pi(i)>\pi(i+1)$. Given a permutation $\pi\in\mathfrak{S}_n$, a value $i\in[n]$ is called a *cycle peak* if $\pi^{-1}(i)<i>\pi(i)$. Let $X(\pi)$ (resp., $Y(\pi)$) be the number of odd (resp., even) cycle peaks of π.

Let D be the derivative operator on the polynomials in three variables. The *Dumont differential system* on the Jacobi elliptic functions is defined by

$$D(x)=yz,\quad D(y)=xz,\quad D(z)=xy. \tag{3}$$

Define the numbers $s_{n,i,j}$ by

$$\begin{aligned}D^{2n}(x) &= \sum_{i,j\geq 0} s_{2n,i,j}x^{2i+1}y^{2j}z^{2n-2i-2j},\\ D^{2n+1}(x) &= \sum_{i,j\geq 0} s_{2n+1,i,j}x^{2i}y^{2j+1}z^{2n-2i-2j+1}.\end{aligned} \tag{4}$$

The study of (3) was initiated by Schett [26] in a slightly different form. Schett found that $\sum_{j\geq 0}s_{n,i,j}=P_{n,\lfloor(n-1)/2\rfloor-i}$, where $P_{n,k}$ is the number of permutations in $\mathfrak{S}_n$ with k interior peaks. Dumont [8] deduced the recurrence relation

$$\begin{aligned}s_{2n,i,j} &= (2j+1)s_{2n-1,i,j}+(2i+2)s_{2n-1,i+1,j-1}+(2n-2i-2j+1)s_{2n-1,i,j-1},\\ s_{2n+1,i,j} &= (2i+1))s_{2n,i,j}+(2j+2)s_{2n,i-1,j+1}+(2n-2i-2j+2)s_{2n,i-1,j},\end{aligned} \tag{5}$$

and established that

$$s_{n,i,j}=|\{\pi\in\mathfrak{S}_n : X(\pi)=i, Y(\pi)=j\}|.$$

Subsequently, Dumont [9] studied the symmetric variants of (2):

$$\begin{aligned}\frac{d}{du}\mathrm{sn}\,(u;a,b) &= \mathrm{cn}\,(u;a,b)\mathrm{dn}\,(u;a,b),\\ \frac{d}{du}\mathrm{cn}\,(u;a,b) &= a^2\mathrm{sn}\,(u;a,b)\mathrm{dn}\,(u;a,b),\\ \frac{d}{du}\mathrm{dn}\,(u;a,b) &= b^2\mathrm{sn}\,(u;a,b)\mathrm{cn}\,(u;a,b),\end{aligned}$$

with the initial conditions $\mathrm{sn}\,(0;a,b)=0$, $\mathrm{cn}\,(0;a,b)=1$ and $\mathrm{dn}\,(0;a,b)=1$. In particular, for (3), Dumont [9, Proposition 2.1] showed the following generating function

$$\sum_{n\geq 0}D^n(x)\frac{u^n}{n!}=\frac{yz\mathrm{sn}\,(u;y',z')+x\mathrm{cn}\,(u;y',z')\mathrm{dn}\,(u;y',z')}{1-x^2\mathrm{sn}^2(u;y',z')}, \tag{6}$$

where $y'=\sqrt{y^2-x^2}$ and $z'=\sqrt{z^2-x^2}$.

This paper is organized as follows. In Section 2, we collect some notation, definitions and results that will be needed in the sequel. In Section 3, we give the solutions of several systems of partial differential equations (PDEs for short) associated with (3), although it has always been a challenging problem to find explicit formulas for solutions of most PDEs, as pointed out by Evans [11]. In Section 4, we present new characterizations for several combinatorial structures as applications.

2. Preliminaries

Recall that $F(x,k)$ is defined in (1). We define

$$\begin{aligned} h_{p,q} &= F\left(\sqrt{\frac{q(1-p)}{q-p}}, \sqrt{\frac{q-p}{1-p}}\right), \\ \ell_{p,q} &= F\left(q\sqrt{\frac{1-p}{q^2-p}}, \sqrt{\frac{q^2-p}{1-p}}\right), \\ k_{p,q} &= \sqrt{\frac{p-1}{q-p}}\arctan\left(\sqrt{\frac{q(p-1)}{q-p}}\right), \\ x_{\pm} &= (p-1)x \pm k_{p,q}. \end{aligned}$$

For any sequence $a_{n,i,j}$, we define the following generating functions

$$\begin{aligned} A = A(x,p,q) &= \sum_{n,i,j\geq 0} a_{n,i,j}\frac{x^n}{n!}p^iq^j, \\ AE = AE(x,p,q) &= \sum_{n,i,j\geq 0} a_{2n,i,j}\frac{x^{2n}}{(2n)!}p^iq^j = \frac{1}{2}\big(A(x,p,q)+A(-x,p,q)\big), \\ AO = AO(x,p,q) &= \sum_{n,i,j\geq 0} a_{2n+1,i,j}\frac{x^{2n+1}}{(2n+1)!}p^iq^j = \frac{1}{2}\big(A(x,p,q)-A(-x,p,q)\big), \end{aligned}$$

where we use the small letters $a,b,c,\ldots$ for sequences, capital letters $A,B,C,\ldots$ for generating functions, and $AE,BE,CE,\ldots,AO,BO,CO,\ldots$ for the even and odd parts of the generating functions, respectively. We denote by H_y the partial derivative of the function H with respect to y.

From (5), we get the following comparable result of (6).

Theorem 1. *We have*

$$\begin{cases} SO(x,p,q) &= \frac{\sqrt{p-1}}{2\sqrt{q}}\left(K\left(\frac{1-q}{1-p}, \sqrt{p-1}x - h_{p,q}\right) - K\left(\frac{1-q}{1-p}, \sqrt{p-1}x + h_{p,q}\right)\right), \\ SE(x,p,q) &= \frac{\sqrt{p-1}}{2\sqrt{p}}\left(K\left(\frac{1-q}{1-p}, \sqrt{p-1}x - h_{p,q}\right) + K\left(\frac{1-q}{1-p}, \sqrt{p-1}x + h_{p,q}\right)\right). \end{cases} \tag{7}$$

where $K(p,x)=\sqrt{1-p}\,\mathrm{cn}\left(\sqrt{p}x, \sqrt{1-1/p}\right), -1<p<1$ *and* $0<q<1$.

Proof. By (5), we have

$$\begin{cases} SO_x &= SE + 2p(1-p)SE_p + 2p(1-q)SE_q + pxSE_x, \\ SE_x &= SO + 2q(1-p)SO_p + 2q(1-q)SO_q + qxSO_x. \end{cases}$$

Equivalently, $(SO', SE') = \frac{1}{\sqrt{p-1}}(\sqrt{q}SO, \sqrt{p}SE)$ satisfies

$$\begin{cases} SO'_x &= 2\sqrt{pq}(1-p)SE'_p + 2\sqrt{pq}(1-q)SE'_q + \sqrt{pq}xSE'_x, \\ SE'_x &= 2\sqrt{pq}(1-p)SO'_p + 2\sqrt{pq}(1-q)SO'_q + \sqrt{pq}xSO'_x. \end{cases} \tag{8}$$

Solving (8) for $SO'_x - SE'_x$ and $SO'_x + SE'_x$ (with the help of maple), we obtain that there exist two (analytical) functions K_1 and K_2 such that

$$\begin{cases} SO' - SE' &= K_1\left(\frac{1-q}{1-p}, \sqrt{p-1}x + h_{p,q}\right), \\ SO' + SE' &= K_2\left(\frac{1-q}{1-p}, \sqrt{p-1}x - h_{p,q}\right). \end{cases} \tag{9}$$

In order to provide explicit formulas for the generating functions SO'_x and SE'_x, we first solve (8) for $q = 0$. In this case, we obtain

$$\begin{cases} SO_x(x,p,0) &= SE(x,p,0) + 2p(1-p)SE_p(x,p,0) + 2pSE_q(x,p,0) + pxSE_x(x,p,0), \\ SE_x(x,p,0) &= SO(x,p,0). \end{cases}$$

Note that the initial conditions are

$$SO(0,p,q) = 0, \quad SE(0,p,q) = 1, \quad SO(x,0,0) = \frac{e^x - e^{-x}}{2}, \quad SE(x,0,0) = \frac{e^x + e^{-x}}{2}.$$

Thus it is easy to see that the solution of this system of PDEs is

$$SO(x,p,0) = -I\,\mathrm{dn}\,(Ix, \sqrt{p})\mathrm{sn}\,(Ix, \sqrt{p}) \quad \text{and} \quad SE(x,p,0) = \mathrm{cn}\,(Ix, \sqrt{p}),$$

with I is the imaginary unit. Therefore, when $q = 0$, (9) gives

$$-\frac{\sqrt{p}}{\sqrt{p-1}}\mathrm{cn}\,(Ix, \sqrt{p}) = K_1\left(\frac{1}{1-p}, \sqrt{p-1}x\right)$$
$$\frac{\sqrt{p}}{\sqrt{p-1}}\mathrm{cn}\,(Ix, \sqrt{p}) = K_2\left(\frac{1}{1-p}, \sqrt{p-1}x\right),$$

which leads to $K_2(p,x) = -K_1(p,x) = K(p,x)$. Hence, by (9) we get (7), as desired. □

In order to provide a unified approach to the sequences discussed in this paper, we introduce the following definitions.

Definition 2. *A pair $(F, G) = (F(x,p,q), G(x,p,q))$ of functions is called a J-pair of the first type if they satisfy the following system of PDEs:*

$$\begin{cases} F_x = 2p\sqrt{q}(1-p)G_p + 2p\sqrt{q}(1-q)G_q + 2p\sqrt{q}xG_x, \\ G_x = 2p\sqrt{q}(1-p)F_p + 2p\sqrt{q}(1-q)F_q + 2p\sqrt{q}xF_x. \end{cases} \tag{10}$$

Remark 3. *By defining*

$$P(x,p,q) = F(x,p,q) - G(x,p,q), \;\; Q(x,p,q) = F(x,p,q) + G(x,p,q),$$

we have

$$\begin{cases} P_x(x,p,q) + 2p\sqrt{q}((1-p)P_p(x,p,q) + (1-q)P_q(x,p,q) + xP_x(x,p,q)) = 0, \\ Q_x(x,p,q) - 2p\sqrt{q}((1-p)Q_p(x,p,q) + (1-q)Q_q(x,p,q) + xQ_x(x,p,q)) = 0. \end{cases}$$

It is not hard to check that the solution (with $p, q \neq 1$ and $q \neq 0$) of these PDEs is given by

$$P(x,p,q) = V\left(\frac{1-q}{1-p},\, x_+\right),$$
$$Q(x,p,q) = V\left(\frac{1-q}{1-p},\, x_-\right),$$

for any two functions V and $\widetilde{V}$.

Definition 4. *A pair $(\widetilde{F},\widetilde{G}) = (\widetilde{F}(x,p,q),\widetilde{G}(x,p,q))$ of functions is called a J-pair of the second type if they satisfy the following system of PDEs:*

$$\begin{cases} \widetilde{F}_x = 2q\sqrt{p}(1-p)\widetilde{G}_p + \sqrt{p}(1-q^2)\widetilde{G}_q + xq\sqrt{p}\widetilde{G}_x, \\ \widetilde{G}_x = 2q\sqrt{p}(1-p)\widetilde{F}_p + \sqrt{p}(1-q^2)\widetilde{F}_q + xq\sqrt{p}\widetilde{F}_x. \end{cases} \tag{11}$$

Remark 5. *We shall give a remark on the solution of* (11). *By defining*

$$\widetilde{P}(x,p,q) = \widetilde{F}(x,p,q) - \widetilde{G}(x,p,q), \qquad \widetilde{Q}(x,p,q) = \widetilde{F}(x,p,q) + \widetilde{G}(x,p,q),$$

we have

$$\begin{cases} \widetilde{P}_x(x,p,q) + \sqrt{q}(2q(1-p)\widetilde{P}_p(x,p,q) + (1-q^2)\widetilde{P}_q(x,p,q) + xq\widetilde{P}_x(x,p,q)) = 0, \\ \widetilde{Q}_x(x,p,q) - \sqrt{q}(2q(1-p)\widetilde{Q}_p(x,p,q) + (1-q^2)\widetilde{Q}_q(x,p,q) + xq\widetilde{Q}_x(x,p,q)) = 0. \end{cases}$$

It is not hard to check that the solution (with $p, q \neq 1$ and $q \neq 0$) of these PDEs is given by

$$\begin{cases} \widetilde{P}(x,p,q) &= W\left(\frac{1-q^2}{1-p}, \sqrt{p-1}x - \ell_{p,q}\right), \\ \widetilde{Q}(x,p,q) &= \widetilde{W}\left(\frac{1-q^2}{1-p}, \sqrt{p-1}x + \ell_{p,q}\right), \end{cases}$$

for any two functions W and $\widetilde{W}$.

3. Solutions of several systems of PDEs

3.1. J-pair of the first type.

There are countless combinatorial structures related to the differential operators xD and Dx; see, e.g., [13, 16, 20]. It is natural to further study (3) via these differential operators. Assume that

$$(xD)^{n+1}(x) = (xD)(xD)^n(x) = xD((xD)^n(x)),$$

$$(Dx)^{n+1}(x) = (Dx)(Dx)^n(x) = D(x(Dx)^n(x)),$$

$$(Dx)^{n+1}(y) = (Dx)(Dx)^n(y) = D(x(Dx)^n(y)).$$

In particular, from (3), we have

$$(xD)(x) = xyz, \quad (xD)^2(x) = xy^2z^2 + x^3y^2 + x^3z^2,$$
$$(Dx)(x) = 2xyz, \quad (Dx)^2(x) = 4xy^2z^2 + 2x^3y^2 + 2x^3z^2,$$
$$(Dx)(y) = y^2z + x^2z, \quad (Dx)^2(y) = y^3z^2 + 5x^2yz^2 + x^2y^3 + x^4y.$$

For $n \geq 0$, we define

$$(xD)^{2n}(x) = \sum_{i,j\geq 0} a_{2n,i,j}x^{2i+1}y^{2j}z^{4n-2i-2j},$$
$$(xD)^{2n+1}(x) = \sum_{i,j\geq 0} a_{2n+1,i,j}x^{2i+1}y^{2j+1}z^{4n-2i-2j+1},$$
$$(Dx)^{2n}(x) = \sum_{i,j\geq 0} c_{2n,i,j}x^{2i+1}y^{2j}z^{4n-2i-2j},$$
$$(Dx)^{2n+1}(x) = \sum_{i,j\geq 0} c_{2n+1,i,j}x^{2i+1}y^{2j+1}z^{4n-2i-2j+1},$$
$$(Dx)^{2n}(y) = \sum_{i,j\geq 0} d_{2n,i,j}x^{2i}y^{2j+1}z^{4n-2i-2j},$$
$$(Dx)^{2n+1}(y) = \sum_{i,j\geq 0} d_{2n+1,i,j}x^{2i}y^{2j}z^{4n-2i-2j+3}.$$

For convenience, we list the first terms of the corresponding generating functions:

$$\begin{aligned}A(x,p,q) = 1 + x + (p(1+q)+q)\frac{x^2}{2!} + (4p^2 + 5p(1+q)+q)\frac{x^3}{3!}\\ + (p^3(4+4q) + p^2(5+50q+5q^2) + p(18q^2+18q) + q^2)\frac{x^4}{4!}\\ + (16p^4 + p^3(148+148q) + p^2(61+394q+61q^2) + p(58q+58q^2) + q^2)\frac{x^5}{5!} + \cdots,\end{aligned}$$

$$\begin{aligned}C(x,p,q) = 1 + 2x + 2(p(1+q)+2q)\frac{x^2}{2!} + 8(p^2+2p(1+q)+q)\frac{x^3}{3!}\\ + 8(p^3(1+q) + 2p^2(1+9q+q^2) + 11pq(1+q) + 2q^2)\frac{x^4}{4!}\\ + 16(2p^4 + 26p^3(1+q) + p^2(17+98q+17q^2) + 26pq(1+q) + 2q^2)\frac{x^5}{5!} + \cdots,\end{aligned}$$

$$\begin{aligned}D(x,p,q) = 1 + (p+q)x + (p^2+p(5+q)+q)\frac{x^2}{2!} + (p^3+p^2(5+18q)+pq(18+5q)+q^2)\frac{x^3}{3!}\\ + (p^4 + p^3(58+18q) + p^2(61+164q+5q^2) + pq(58+18q) + q^2)\frac{x^4}{4!} + \cdots,\end{aligned}$$

Since

$$\begin{aligned}(xD)^{2n+1}(x) &= (xD)(xD)^{2n}(x)\\ &= xD\left(\sum_{i,j\geq 0} a_{2n,i,j}x^{2i+1}y^{2j}z^{4n-2i-2j}\right)\\ &= \sum_{i,j\geq 0}(2i+1)a_{2n,i,j}x^{2i+1}y^{2j+1}z^{4n-2i-2j+1} + \sum_{i,j\geq 0} 2ja_{2n,i,j}x^{2i+3}y^{2j-1}z^{4n-2i-2j+1} +\\ &\quad \sum_{i,j\geq 0}(4n-2i-2j)a_{2n,i,j}x^{2i+3}y^{2j+1}z^{4n-2i-2j-1},\end{aligned}$$

we have

$$a_{2n+1,i,j} = (2i+1)a_{2n,i,j} + (2j+2)a_{2n,i-1,j+1} + (4n-2i-2j+2)a_{2n,i-1,j}. \tag{12}$$

Similarly,

$$a_{2n,i,j} = (2i+1)a_{2n-1,i,j-1} + (2j+1)a_{2n-1,i-1,j} + (4n-2i-2j+1)a_{2n-1,i-1,j-1}. \tag{13}$$

The recurrences (12) and (13) can be written as the following lemma.

Lemma 6. *We have*

$$\begin{cases} AO_x &= AE + 2p(1-p)AE_p + 2p(1-q)AE_q + 2xpAE_x, \\ AE_x &= (q+p-pq)AO + 2pq(1-p)AO_p + 2pq(1-q)AO_q + 2xpqAO_x. \end{cases}$$

Equivalently, (AO', AE') is a J-pair of the first type, where $AO' = \sqrt{\frac{pq}{p-1}}AO$ and $AE' = \sqrt{\frac{p}{p-1}}AE$.

Theorem 7. *Let $y = \frac{1-q}{1-p}$. Define*

$$G(x,p) = \sqrt{\frac{1-p}{\cos^2(x\sqrt{p(1-p)}) - p}} \text{ and } H(x,p) = \frac{(1-p)\sin(2x\sqrt{p(1-p)})}{2\sqrt{p}(\cos^2(x\sqrt{p(1-p)}) - p)^{3/2}}.$$

Then

$$AO(x,p,q) = \frac{1}{2}\sqrt{\frac{p-q}{pq}}(H(yx_-, 1-1/y) - G(yx_+, 1-1/y)),$$
$$AE(x,p,q) = \frac{1}{2}\sqrt{\frac{p-q}{p}}(H(yx_-, 1-1/y) + G(yx_+, 1-1/y)).$$

Proof. By Remark 3 and Lemma 6, we obtain that

$$\sqrt{\frac{pq}{p-1}}AO(x,p,q) - \sqrt{\frac{p}{p-1}}AE(x,p,q) = V(y, x_+),$$

and

$$\sqrt{\frac{pq}{p-1}}AO(x,p,q) + \sqrt{\frac{p}{p-1}}AE(x,p,q) = \tilde{V}(y, x_-),$$

for some functions V and $\tilde{V}$. Moreover, at $q = 0$, then above equations reduce to

$$-V(p,x) = \tilde{V}(p,x) = \sqrt{1-p}AE(-px, 1-1/p, 0).$$

Hence, if we guess that $AE(x,p,0) = G(x,p)$ and $AO(x,p,0) = H(x,p)$, then we get

$$\sqrt{\frac{pq}{p-1}}AO(x,p,q) - \sqrt{\frac{p}{p-1}}AE(x,p,q) = -\sqrt{1-y}G(yx_+, 1-1/y),$$
$$\sqrt{\frac{pq}{p-1}}AO(x,p,q) + \sqrt{\frac{p}{p-1}}AE(x,p,q) = \sqrt{1-y}H(yx_-, 1-1/y),$$

which imply

$$AO(x,p,q) = \frac{1}{2}\sqrt{\frac{(1-y)(p-1)}{pq}}(H(yx_-, 1-1/y) - G(yx_+, 1-1/y)),$$
$$AE(x,p,q) = \frac{1}{2}\sqrt{\frac{(p-1)(1-y)}{p}}(H(yx_-, 1-1/y) + G(yx_+, 1-1/y)).$$

It is routine to check that the functions AO and AE satisfy Lemma 6. This completes the proof. □

Along the same lines, we get

$$\begin{aligned} c_{2n,i,j} &= (2i+2)c_{2n-1,i,j-1} + (2j+1)c_{2n-1,i-1,j} + (4n-2i-2j+1)c_{2n-1,i-1,j-1}, \\ c_{2n+1,i,j} &= (2i+2)c_{2n,i,j} + (2j+2)c_{2n,i-1,j+1} + (4n-2i-2j+2)c_{2n,i-1,j}. \end{aligned} \tag{14}$$

This leads us to the following result.

Lemma 8. *We have*

$$\begin{cases} CO_x = 2CE + 2p(1-p)CE_p + 2p(1-q)CE_q + 2xpCE_x, \\ CE_x = (p+2q-pq)CO + 2pq(1-p)CO_p + 2pq(1-q)CO_q + 2xpqCO_x. \end{cases} \tag{15}$$

Equivalently, (CO', CE') *is a* J*-pair of the first type, where* $CO' = \frac{p\sqrt{q}}{p-1}CO$ *and* $CE' = \frac{p}{p-1}CE$.

Theorem 9. *Define* $y = \frac{1-q}{1-p}$ *and* $G(x,p) = \frac{1-p}{p\cos^2(x\sqrt{p-1})+1-p}$. *Then*

$$\begin{aligned} CO(x,p,q) &= \tfrac{p-1}{2p\sqrt{q}}\big(G(x_-,y) - G(x_+,y)\big), \\ CE(x,p,q) &= \tfrac{p-1}{2p}\big(G(x_-,y) + G(x_+,y)\big), \\ C(x,p,q) &= \tfrac{p-1}{2p\sqrt{q}}\big(G(x_-,y) - G(x_+,y)\big) + \tfrac{p-1}{2p}\big(G(x_-,y) + G(x_+,y)\big). \end{aligned} \tag{16}$$

Proof. By Remark 3 and Lemma 8, we obtain that

$$\frac{p\sqrt{q}}{p-1}CO(x,p,q) - \frac{p}{p-1}CE(x,p,q) = \tilde{V}(y, x_+),$$

and

$$\frac{p\sqrt{q}}{p-1}CO(x,p,q) + \frac{p}{p-1}CE(x,p,q) = V(y, x_-)$$

for some functions V and $\tilde{V}$. Moreover, at $q = 0$, then above equations reduce to

$$V(1/(1-p),(p-1)x) = -\tilde{V}(1/(1-p),(p-1)x).$$

Hence, if we take

$$(1-p)CE(-px, 1-1/p, 0) = G(x,p),$$

$$CO(x,p,q) = \frac{p-1}{2p\sqrt{q}}(G(x_-,y) - G(x_+,y)),$$

$$CE(x,p,q) = \frac{p-1}{2p}(G(x_-,y) + G(x_+,y)),$$

then (16) is a solution for (15), where

$$V\big(1/(1-p),(p-1)x\big) = -\tilde{V}\big(1/(1-p),(p-1)x\big) = (1-p)CE(-px, 1-1/p, 0) = G(x,p).$$

To complete the proof, we need to check that the functions CO and CE satisfy Lemma 8, which is routine. □

Corollary 10. *We have*

$$
\begin{aligned}
C(x,0,q) &= \cosh(2\sqrt{q}x) + \frac{1}{\sqrt{q}}\sinh(2\sqrt{q}x),\\
C(x,1,q) &= \frac{(x^2(q-1)+2x+1)}{(x^2(1-q)-2x+1)(x^2(1-q)+2x+1)},\\
C(x,p,0) &= \frac{(1-p)\sqrt{1-p}\sin(2x\sqrt{p(1-p)})}{\sqrt{p}(\cos^2(x\sqrt{p(1-p)})-p)^2} + \frac{1-p}{\cos^2(x\sqrt{p(1-p)})-p},\\
C(x,p,1) &= \frac{p-1}{p-e^{2x(p-1)}}.
\end{aligned}
$$

Proof. By applying Theorem 9 for $q=0$ or $p=1$, we obtain the formulas of $C(x,p,0)$ and $C(x,1,q)$. Solving (15) for $p=0$ we obtain

$$
\begin{aligned}
CE(x,0,q) &= \alpha_q e^{2\sqrt{q}x} + \beta_q e^{-2\sqrt{q}x},\\
CO(x,0,q) &= \frac{1}{\sqrt{q}}(\alpha_q e^{2\sqrt{q}x} - \beta_q e^{-2\sqrt{q}x}).
\end{aligned}
$$

By using the initial conditions $CE(0,p,q)=1$ and $CO(0,p,q)=0$, we obtain $CE(x,0,p) = \cosh(2\sqrt{q}x)$ and $CO(x,0,q) = \frac{1}{\sqrt{q}}\sinh(2\sqrt{q}x)$, which completes the first part of the proof.

Again, solving (15) with $q=1$ for $CO(x,p,1)-CE(x,p,1)$ and $CO(x,p,1)+CE(x,p,1)$, we obtain

$$
\begin{aligned}
CO(x,p,1)-CE(x,p,1) &= \frac{p-1}{p}V(x(p-1)+\frac{1}{2}\ln p),\\
CO(x,p,1)+CE(x,p,1) &= \frac{p-1}{p}\tilde{V}(x(p-1)-\frac{1}{2}\ln p),
\end{aligned}
$$

where $V,\tilde{V}$ are two fixed functions. By the initial values $CE(0,p,q)=1$ and $CO(0,p,q)=0$, we get

$$V(y) = \frac{e^{2y}}{1-e^{2y}} \text{ and } \tilde{V}(y) = \frac{1}{1-e^{2y}}.$$

Hence,

$$
\begin{aligned}
CO(x,p,1)-CE(x,p,1) &= \frac{(p-1)e^{2x(p-1)}}{1-pe^{2x(p-1)}},\\
CO(x,p,1)+CE(x,p,1) &= \frac{p-1}{p-e^{2x(p-1))}}.
\end{aligned}
$$

This completes the proof. □

Along the same lines, we get

$$
\begin{aligned}
d_{2n,i,j} &= (2i+1)d_{2n-1,i,j} + (2j+2)d_{2n-1,i-1,j+1} + (4n-2i-2j+1)d_{2n-1,i-1,j},\\
d_{2n+1,i,j} &= (2i+1)d_{2n,i,j-1} + (2j+1)d_{2n,i-1,j} + (4n-2i-2j+4)d_{2n,i-1,j-1},
\end{aligned} \tag{17}
$$

which leads to the following result.

Lemma 11. *We have*

$$
\begin{cases}
DO_x = (p+q)DE + 2pq(1-p)DE_p + 2pq(1-q)DE_q + 2pqxDE_x,\\
DE_x = (1+p)DO + 2p(1-p)DO_p + 2p(1-q)DO_q + 2pxDO_x.
\end{cases} \tag{18}
$$

Equivalently, (DO', DE') *is a J-pair of the first type, where* $DO' = \sqrt{\frac{p}{p-1}}DO$ *and* $DE' = \sqrt{\frac{pq}{p-1}}DE$.

By similar arguments as in the proof of Theorem 9 with help of Remark 3 and Lemma 11, we obtain the following result.

Theorem 12. *Define* $y = \frac{1-q}{1-p}$ *and* $G(x,p) = \frac{\sinh(x\sqrt{p-1})}{1-\frac{p}{p-1}\cosh^2(x\sqrt{p-1})}$. *Then*

$$\begin{aligned}
DO(x,p,q) &= \frac{\sqrt{p-1}}{2\sqrt{p}}(G(x_-,y)+G(x_+,y)),\\
DE(x,p,q) &= \frac{\sqrt{p-1}}{2\sqrt{pq}}(G(x_-,y)-G(x_+,y)),\\
D(x,p,q) &= \frac{\sqrt{p-1}}{2\sqrt{pq}}(G(x_-,y)-G(x_+,y))+\frac{\sqrt{p-1}}{2\sqrt{p}}(G(x_-,y)+G(x_+,y)).
\end{aligned}$$

Corollary 13. *Let* $p' = \sqrt{p(p-1)}$. *Then we have*

$$\begin{aligned}
D(x,p,0) &= \frac{(p-1)\cosh(xp')(\cosh^2(xp')-2+p)}{((p-1)\cosh^2(xp')-p\sinh^2(xp'))^2}+\frac{p'\sinh(xp')}{p-\cosh^2(xp')},\\
D(x,1,q) &= \frac{(x^2(q-1)+2x-1)(x^2(1-q)+2x+1)(x^3(q-1)^2+x^2(q-1)-x(q+1)-1)}{(x^2(q-1)-2x\sqrt{q}+1)^2(x^2(q-1)+2x\sqrt{q}+1)^2},\\
D(x,p,1) &= \frac{(1-p)e^{(1-p)x}}{1-pe^{2(1-p)x}}.
\end{aligned}$$

From Corollary 10 and Corollary 13, it is easy to verify that

$$\begin{aligned}
C(x,1,q) &= \sum_{n\geq 0}\sum_{k\geq 0}\binom{2n+1}{2k}q^kx^{2n}+\sum_{n\geq 1}\sum_{k\geq 0}\binom{2n}{2k+1}q^kx^{2n-1},\\
D(x,1,q) &= \sum_{n\geq 0}\sum_{k\geq 0}\binom{2n+1}{2k+1}q^kx^{2n}+\sum_{n\geq 1}\sum_{k\geq 0}\binom{2n}{2k}q^kx^{2n-1}.
\end{aligned}$$

3.2. J-pair of the second type.

In his study [5] of exponential structures in combinatorics, Chen introduced the grammatical method systematically. Let A be an alphabet whose letters are regarded as independent commutative indeterminates. Following Chen, a *context-free grammar* G over A is defined as a set of substitution rules that replace a letter in A by a formal function over A. The formal derivative D is a linear operator defined with respect to a context-free grammar G. Therefore, an equivalent form of (3) is given by the context-free grammar

$$G = \{x \to yz,\ y \to xz,\ z \to xy\}. \tag{19}$$

Dumont [10] considered chains of general substitution rules on words. In particular, he discovered the following result.

Proposition 14 ([10, Section 2.1])**.** *If*

$$G = \{w \to wx,\ x \to wx\}, \tag{20}$$

then

$$D^n(w)=\sum_{k=0}^{n-1}\left\langle {n \atop k} \right\rangle w^{k+1}x^{n-k},$$

where $\left\langle {n \atop k} \right\rangle$ *is the Eulerian number, i.e., the number of permutations in* $\mathfrak{S}_n$ *with* k *descents.*

As a conjunction of (19) and (20), it is natural to consider the context-free grammar

$$G=\{w\rightarrow wx,\ x\rightarrow yz,\ y\rightarrow xz,\ z\rightarrow xy\}. \tag{21}$$

From (21), we have

$$\begin{aligned}
&D(w)=wx,\quad D^2(w)=w(x^2+yz),\quad D^3(x)=w(x^3+xz^2+3xyz+xy^2),\\
&D^4(w)=w(x^4+10x^2yz+4x^2z^2+4x^2y^2+3y^2z^2+y^3z+yz^3),\\
&D(w^2)=2w^2x,\quad D^2(w^2)=w^2(4x^2+2yz),\quad D^3(w^2)=w^2(8x^3+12xyz+2xz^2+2xy^2).
\end{aligned}$$

For $n\geq 0$, we define

$$\begin{aligned}
D^{2n}(w)&=w\sum_{i,j\geq 0}t_{2n,i,j}x^{2i}y^jz^{2n-2i-j},\\
D^{2n+1}(w)&=w\sum_{i,j\geq 0}t_{2n+1,i,j}x^{2i+1}y^jz^{2n-2i-j},\\
D^{2n}(w^2)&=w^2\sum_{i,j\geq 0}r_{2n,i,j}x^{2i}y^jz^{2n-2i-j},\\
D^{2n+1}(w^2)&=w^2\sum_{i,j\geq 0}r_{2n+1,i,j}x^{2i+1}y^jz^{2n-2i-j}.
\end{aligned}$$

The first terms of the corresponding generating functions are given as follows:

$$\begin{aligned}
T(x,p,q)=1&+x+(p+q)\frac{x^2}{2!}+(1+p+3q+q^2)\frac{x^3}{3!}\\
&+(p^2+4p+(10p+1)q+(4p+3)q^2+q^3)\frac{x^4}{4!}\\
&+(p^2+14p+1+(30p+15)q+(14p+29)q^2+15q^3+q^4)\frac{x^5}{5!}+\cdots.\\
R(x,p,q)=1&+2x+(4p+2q)\frac{x^2}{2!}+(2+8p+12q+2q^2)\frac{x^3}{3!}\\
&+(16p+16p^2+(2+56p)q+(12+16p)q^2+2q^3)\frac{x^4}{4!}\\
&+(2+88p+32p^2+(60+240p)q+(148+88p)q^2+60q^3+2q^4)\frac{x^5}{5!}+\cdots.
\end{aligned}$$

Since

$$\begin{aligned}
D^{2n+1}(w) &= D(D^{2n}(w)) \\
&= D\left(w\sum_{i,j\geq 0} t_{2n,i,j}x^{2i}y^jz^{2n-2i-j}\right) \\
&= w\sum_{i,j\geq 0} t_{2n,i,j}x^{2i+1}y^jz^{2n-2i-j} + w\sum_{i,j\geq 0} 2it_{2n,i,j}x^{2i-1}y^{j+1}z^{2n-2i-j+1} \\
&\quad + w\sum_{i,j\geq 0} jt_{2n,i,j}x^{2i+1}y^{j-1}z^{2n-2i-j+1} + w\sum_{i,j\geq 0}(2n-2i-j)t_{2n,i,j}x^{2i+1}y^{j+1}z^{2n-2i-j-1},
\end{aligned}$$

we have

$$t_{2n+1,i,j} = t_{2n,i,j} + (2i+2)t_{2n,i+1,j-1} + (j+1)t_{2n,i,j+1} + (2n-2i-j+1)t_{2n,i,j-1}. \tag{22}$$

Similarly,

$$t_{2n,i,j} = t_{2n-1,i-1,j} + (2i+1)t_{2n-1,i,j-1} + (j+1)t_{2n-1,i-1,j+1} + (2n-2i-j+1)t_{2n-1,i-1,j-1}. \tag{23}$$

By rewriting these recurrence relations in terms of the generating functions TE and TO, we obtain the following result.

Lemma 15. *We have*

$$\begin{cases} TO_x &= TE + 2q(1-p)TE_p + (1-q^2)TE_q + xqTE_x, \\ TE_x &= (p+q-qp)TO + 2pq(1-p)TO_p + p(1-q^2)TO_q + xqpTO_x. \end{cases} \tag{24}$$

Equivalently, (TO', TE') is a J-pair of the second type, where $TO' = \sqrt{\frac{p(1+q)}{1-q}}TO$ and $TE' = \sqrt{\frac{1+q}{1-q}}TE$.

Theorem 16. *Let $\ell'_{p,q} = \sqrt{\frac{1-q^2}{1-p}}\ell_{p,q}$. Then we have*

$$\begin{cases} TO(x,p,q) = \frac{q-1}{\sqrt{p(p-1)}}\mathrm{sn}\,(-\sqrt{q^2-1}x + \ell'_{p,q}, \sqrt{\frac{p-q^2}{1-q^2}}), \\ TE(x,p,q) = \sqrt{\frac{1-q}{1+q}}\mathrm{dn}\,(-\sqrt{q^2-1}x - \ell'_{p,q}, \sqrt{\frac{p-q^2}{1-q^2}}). \end{cases}$$

Proof. By Remark 5, we see that Lemma 15 leads to

$$\begin{cases} \sqrt{\frac{p(1+q)}{1-q}}TO(x,p,q) - \sqrt{\frac{1+q}{1-q}}TE(x,p,q) = W\left(\frac{1-q^2}{1-p}, \sqrt{p-1}x - \ell_{p,q}\right), \\ \sqrt{\frac{p(1+q)}{1-q}}TO(x,p,q) + \sqrt{\frac{1+q}{1-q}}TE(x,p,q) = \widetilde{W}\left(\frac{1-q^2}{1-p}, \sqrt{p-1}x + \ell_{p,q}\right), \end{cases} \tag{25}$$

for some functions W and $\tilde{W}$. Thus, at $q=0$ we have that

$$\begin{cases} \sqrt{p}TO(I\sqrt{p}x, 1-1/p, 0) - TE(I\sqrt{p}x, 1-1/p, 0) &= W\,(p,x), \\ \sqrt{p}TO(I\sqrt{p}x, 1-1/p, 0) + TE(I\sqrt{p}x, 1-1/p, 0) &= \widetilde{W}\,(p,x), \end{cases}$$

Therefore, if we set

$$TE(x,p,0) = \mathrm{dn}\,(Ix, \sqrt{p}) \quad \text{and} \quad TO(x,p,0) = -I\mathrm{sn}\,(Ix, \sqrt{p}),$$

then

$$\begin{cases} -I\sqrt{p}\,\mathrm{sn}\,(-\sqrt{p}x, \sqrt{1-1/p}) - \mathrm{dn}\,(-\sqrt{p}x, \sqrt{1-1/p}) & = W\,(p,x)\,, \\ -I\sqrt{p}\,\mathrm{sn}\,(-\sqrt{p}x, \sqrt{1-1/p}) + \mathrm{dn}\,(-\sqrt{p}x, \sqrt{1-1/p}) & = \widetilde{W}\,(p,x)\,. \end{cases}$$

By (25), we have

$$\begin{cases} \sqrt{\frac{p(1+q)}{1-q}}TO(x,p,q) - \sqrt{\frac{1+q}{1-q}}TE(x,p,q) \\ \qquad = -\sqrt{\frac{q^2-1}{1-p}}\mathrm{sn}\,(-\sqrt{q^2-1}x + \ell'_{p,q}, \sqrt{\frac{p-q^2}{1-q^2}}) - \mathrm{dn}\,(-\sqrt{q^2-1}x + \ell'_{p,q}, \sqrt{\frac{p-q^2}{1-q^2}}), \\ \sqrt{\frac{p(1+q)}{1-q}}TO(x,p,q) + \sqrt{\frac{1+q}{1-q}}TE(x,p,q) \\ \qquad = -\sqrt{\frac{q^2-1}{1-p}}\mathrm{sn}\,(-\sqrt{q^2-1}x - \ell'_{p,q}, \sqrt{\frac{p-q^2}{1-q^2}}) + \mathrm{dn}\,(-\sqrt{q^2-1}x - \ell'_{p,q}, \sqrt{\frac{p-q^2}{1-q^2}}), \end{cases}$$

which implies

$$\begin{cases} TO(x,p,q) = \frac{q-1}{\sqrt{p(p-1)}}\mathrm{sn}\,(-\sqrt{q^2-1}x + \ell'_{p,q}, \sqrt{\frac{p-q^2}{1-q^2}}), \\ TE(x,p,q) = \sqrt{\frac{1-q}{1+q}}\mathrm{dn}\,(-\sqrt{q^2-1}x - \ell'_{p,q}, \sqrt{\frac{p-q^2}{1-q^2}}), \end{cases}$$

which agrees with the case $q = 0$. It suffices and is routine to check that the functions TO and TE satisfy Lemma 15. This completes the proof. □

By the above theorem (or by a direct check using Lemma 15), we realize the following result.

Corollary 17. *Let* $h(x,p) = \frac{\sqrt{p-1}}{\sqrt{p-1}\cosh(x\sqrt{p-1}) - \sqrt{p}\sinh(x\sqrt{p-1})}$. *Then, we have*

$$T(x,p,1) = \frac{1}{2}(h(x,p) + h(-x,p)) + \frac{1}{2\sqrt{p}}(h(x,p) - h(-x,p)),$$
$$T(x,1,q) = \frac{q^2 - 1 + \sqrt{q^2-1}\sinh(x\sqrt{q^2-1})}{(1+q)(q - \cosh(x\sqrt{q^2-1}))}.$$

Along the same lines, we have

$$\begin{aligned} r_{2n+1,i,j} &= 2r_{2n,i,j} + (2i+2)r_{2n,i+1,j-1} + (j+1)r_{2n,i,j+1} + (2n-2i-j+1)r_{2n,i,j-1}, \\ r_{2n,i,j} &= 2r_{2n-1,i-1,j} + (2i+1)r_{2n-1,i,j-1} + (j+1)r_{2n-1,i-1,j+1} \\ &\quad + (2n-2i-j+1)r_{2n-1,i-1,j-1}, \end{aligned} \tag{26}$$

which implies the following result.

Lemma 18. *We have*

$$\begin{cases} RO_x = 2RE + 2q(1-p)RE_p + (1-q^2)RE_q + xqRE_x, \\ RE_x = (2p+q-pq)RO + 2pq(1-p)RO_p + p(1-q^2)RO_q + xpqRO_x. \end{cases} \tag{27}$$

Equivalently, (RO', RE') *is a* J*-pair of the second type, where*

$$RO' = \frac{\sqrt{p}(1+q)}{1-q}RO \quad \text{and} \quad RE' = \frac{1+q}{1-q}RE.$$

Along the line of the proof of Theorem 16 we state the following result.

Theorem 19. *Let*

$$\begin{cases} U\,(p,x) = -2I\sqrt{p}\mathrm{dn}\,(-\sqrt{p}x,p')\mathrm{sn}\,(-\sqrt{p}x,p') - 2p\mathrm{cn}^2(-\sqrt{p}x,p') + 1 - 2/p, \\ \widetilde{U}\,(p,x) = -2I\sqrt{p}\mathrm{dn}\,(-\sqrt{p}x,p')\mathrm{sn}\,(-\sqrt{p}x,p') + 2p\mathrm{cn}^2(-\sqrt{p}x,p') - 1 + 2/p, \end{cases}$$

where $p' = \sqrt{1-1/p}$. *Then*

$$\begin{cases} RO(x,p,q) &= \frac{\sqrt{p}(1-q)}{2(1+q)}\left(U\left(\frac{1-q^2}{1-p},\sqrt{p-1}x-\ell_{p,q}\right)+\widetilde{U}\left(\frac{1-q^2}{1-p},\sqrt{p-1}x+\ell_{p,q}\right)\right), \\ RE(x,p,q) &= \frac{1-q}{2(1+q)}\left(\widetilde{U}\left(\frac{1-q^2}{1-p},\sqrt{p-1}x+\ell_{p,q}\right)-U\left(\frac{1-q^2}{1-p},\sqrt{p-1}x-\ell_{p,q}\right)\right). \end{cases}$$

Proof. By Remark 5, we obtain

$$\begin{cases} \frac{\sqrt{p}(1+q)}{1-q}RO(x,p,q) - \frac{1+q}{1-q}RE(x,p,q) &= W\left(\frac{1-q^2}{1-p},\sqrt{p-1}x-\ell_{p,q}\right), \\ \frac{\sqrt{p}(1+q)}{1-q}RO(x,p,q) + \frac{1+q}{1-q}RE(x,p,q) &= \widetilde{W}\left(\frac{1-q^2}{1-p},\sqrt{p-1}x+\ell_{p,q}\right), \end{cases} \tag{28}$$

for some functions W and $\tilde{W}$. Thus, at $q=0$ we have that

$$\begin{cases} \sqrt{p}RO(I\sqrt{p}x,1-1/p,0) - RE(I\sqrt{p}x,1-1/p,0) &= W\,(p,x)\,, \\ \sqrt{p}RO(I\sqrt{p}x,1-1/p,0) + RE(I\sqrt{p}x,1-1/p,0) &= \widetilde{W}\,(p,x)\,, \end{cases}$$

Therefore, if we set

$$RE(x,p,0) = 2p\mathrm{cn}^2(Ix,\sqrt{p}) - 2p + 1 \quad \text{and} \quad RO(x,p,0) = -2I\mathrm{dn}\,(Ix,\sqrt{p})\mathrm{sn}\,(Ix,\sqrt{p}),$$

then

$$\begin{cases} -2I\sqrt{p}\mathrm{dn}\,(-\sqrt{p}x,p')\mathrm{sn}\,(-\sqrt{p}x,p') - 2p\mathrm{cn}^2(-\sqrt{p}x,p') + 1 - 2/p &= W\,(p,x)\,, \\ -2I\sqrt{p}\mathrm{dn}\,(-\sqrt{p}x,p')\mathrm{sn}\,(-\sqrt{p}x,p') + 2p\mathrm{cn}^2(-\sqrt{p}x,p') - 1 + 2/p &= \widetilde{W}\,(p,x)\,, \end{cases}$$

where $p' = \sqrt{1-1/p}$. By (28), we deduce that

$$\begin{cases} RO(x,p,q) &= \frac{\sqrt{p}(1-q)}{2(1+q)}\left(W\left(\frac{1-q^2}{1-p},\sqrt{p-1}x-\ell_{p,q}\right)+\widetilde{W}\left(\frac{1-q^2}{1-p},\sqrt{p-1}x+\ell_{p,q}\right)\right), \\ RE(x,p,q) &= \frac{1-q}{2(1+q)}\left(\widetilde{W}\left(\frac{1-q^2}{1-p},\sqrt{p-1}x+\ell_{p,q}\right)-W\left(\frac{1-q^2}{1-p},\sqrt{p-1}x-\ell_{p,q}\right)\right), \end{cases}$$

which agrees with the case $q=0$. To complete the proof, we still need to check that the functions RO and RE satisfy Lemma 18, which is routine. □

4. Applications

In this section we apply the results obtained in the previous section to present new characterizations for several combinatorial structures.

4.1. **Peaks, descents and perfect matchings.**

One of the most interesting permutation statistics is the peaks statistic (see, e.g., [2, 7, 18, 19, 21, 25, 31] and the references contained therein). A *left peak* in π is an index $i \in [n-1]$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we take $\pi(0) = 0$. Denote by $\widetilde{P}_{n,k}$ the number of permutations in $\mathfrak{S}_n$ with k left peaks. Recall that $P_{n,k}$ is the number of permutations in $\mathfrak{S}_n$ with k interior peaks. Define

$$P_n(x)=\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}P_{n,k}x^k\quad\text{and}\quad\widetilde{P}_n(x)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\widetilde{P}_{n,k}x^k.$$

The polynomial $P_n(x)$ satisfies recurrence relation

$$P_{n+1}(x)=(nx-x+2)P_n(x)+2x(1-x)\frac{d}{dx}P_n(x),$$

with the initial values $P_1(x)=1$, $P_2(x)=2$, $P_3(x)=4+2x$, and the polynomial $\widetilde{P}_n(x)$ satisfies recurrence relation

$$\widetilde{P}_{n+1}(x)=(nx+1)\widetilde{P}_n(x)+2x(1-x)\frac{d}{dx}\widetilde{P}_n(x), \tag{29}$$

with the initial values $\widetilde{P}_1(x)=1,\widetilde{P}_2(x)=1+x,\widetilde{P}_3(x)=1+5x$ (see [27, A008303, A008971]).

A *descent* of a permutation $\pi\in\mathfrak{S}_n$ is a position i such that $\pi(i)>\pi(i+1)$. Denote by $\operatorname{des}(\pi)$ the number of descents of π. Let

$$A_n(x)=\sum_{\pi\in\mathfrak{S}_n}x^{\operatorname{des}(\pi)}=\sum_{k=0}^{n-1}\left\langle {n \atop k}\right\rangle x^k.$$

The polynomial $A_n(x)$ is called an *Eulerian polynomial*. Let B_n denote the set of signed permutations of $\pm[n]$ such that $\pi(-i)=-\pi(i)$ for all i, where $\pm[n]=\{\pm1,\pm2,\ldots,\pm n\}$. Let

$$B_n(x)=\sum_{k=0}^{n}B(n,k)x^k=\sum_{\pi\in B_n}x^{\operatorname{des}_B(\pi)},$$

where $\operatorname{des}_B=|\{i\in[n]:\omega(i-1)>\omega(i)\}|$ with $\pi(0)=0$. The polynomial $B_n(x)$ is said to be the *Eulerian polynomial of type* B, while $B(n,k)$ is called an *Eulerian number of type* B.

It is well known that Eulerian polynomials are closely retated to peak statistics. In particular, Stembridge [31, Remark 4.8] showed that

$$P_n\left(\frac{4x}{(1+x)^2}\right)=\frac{2^{n-1}}{(1+x)^{n-1}}A_n(x).$$

In [24, Observation 3.1.2], Petersen observed that

$$\widetilde{P}_n\left(\frac{4x}{(1+x)^2}\right)=\frac{1}{(1+x)^n}\left((1-x)^n+\sum_{i=1}^{n}\binom{n}{i}(1-x)^{n-i}2^ixA_i(x)\right).$$

It should be noted that

$$\widetilde{P}_n\left(\frac{4x}{(1+x)^2}\right)=\frac{B_n(x)}{(1+x)^n}.$$

Recall that a *perfect matching* of $[2n]$ is a partition of $[2n]$ into n blocks of size 2. Denote by $N(n,k)$ the number of perfect matchings of $[2n]$ with the restriction that only k matching pairs have odd smaller entries (see [27, A185411]). It is easy to verify that

$$N(n+1,k)=2kN(n,k)+(2n-2k+3)N(n,k-1). \tag{30}$$

Let $N_n(x)=\sum_{k=1}^{n}N(n,k)x^k$. It follows from (30) that

$$N_{n+1}(x)=(2n+1)xN_n(x)+2x(1-x)\frac{d}{dx}N_n(x),$$

with initial values $N_0(x) = 1$, $N_1(x) = x$, $N_2(x) = 2x + x^2$ and $N_3(x) = 4x + 10x^2 + x^3$. There is an expansion of the Eulerian polynomial $A_n(x)$ in terms of $N_n(x)$ (see [20, Theorem 9]):

$$2^n A_n(x) = \sum_{k=0}^{n} \binom{n}{k} N_k(x) N_{n-k}(x).$$

We conclude the following theorem from the discussion above.

Theorem 20. *For $n \geq 1$, we have*

(i) $\sum_{i,j\geq 0} a_{n,i,j} = (2n-1)!!$.
(ii) $\sum_{j\geq 0} a_{n,i,j} = N(n, n-i)$.
(iii) $\sum_{j\geq 0} a_{n,i,\lfloor \frac{n}{2} \rfloor} x^i = \sum_{j\geq 0} a_{n,i,\lfloor \frac{n}{2} \rfloor - i} x^i = \widetilde{P}_n(x)$.
(iv) $\sum_{j\geq 0} c_{n,i,j} = 2^n \left\langle {n \atop i} \right\rangle$.
(v) $\sum_{j\geq 0} d_{n,i,j} = B(n, i)$.
(vi) $\sum_{i\geq 0} c_{n,i,\lfloor \frac{n}{2} \rfloor} x^i = \sum_{i\geq 0} c_{n,i,\lfloor \frac{n}{2} \rfloor - i} x^i = P_{n+1}(x)$.
(vii) $\sum_{i\geq 0} c_{2n-1,i,0} x^{2n-2-i} = \sum_{i\geq 0} c_{2n,i,0} x^{2n-1-i} = P_{2n}(x)$.
(viii) $\sum_{i\geq 0} d_{n,i,\lceil \frac{n}{2} \rceil} x^i = \widetilde{P}_n(x)$ *and* $\sum_{i\geq 0} d_{n,i,\lceil \frac{n}{2} \rceil - i} x^i = \widetilde{P}_{n+1}(x)$.
(ix) $\sum_{i\geq 0} d_{2n,i,0} x^{2n-i} = \sum_{i\geq 0} d_{2n+1,i,0} x^{2n+1-i} = \widetilde{P}_{2n+1}(x)$.

Proof. We only prove the assertion for the sequence $a_{n,i,j}$ and the corresponding assertion for the other sequences follow from similar consideration.

(A) Setting $p, q = 1$ in Lemma 6 gives

$$\begin{cases} AO_x(x,1,1) = AE(x,1,1) + 2xAE_x(x,1,1), \\ AE_x(x,1,1) = AO(x,1,1) + 2xAO_x(x,1,1), \end{cases}$$

which implies $A_x(x,1,1) = A(x,1,1) + 2xA_x(x,1,1)$. Therefore,

$$A(x,1,1) = \frac{A(0,1,1)}{\sqrt{1-2x}} = \frac{1}{\sqrt{1-2x}} = \sum_{n\geq 0} \frac{n!}{2^n} \binom{2n}{n} \frac{x^n}{n!}.$$

Hence, $\sum_{i,j\geq 0} a_{n,i,j} = \frac{n!}{2^n}\binom{2n}{n} = (2n-1)!!$, as required.

(B) Setting $q = 1$ in Lemma 6 gives

$$A_x(x,p,1) = A(x,p,1) + 2p(1-p)A_p(x,p,1) + 2xpA_x(x,1,1).$$

By $A(0,1,p) = 1$, it is a simple routine to check that $A(x,p,1) = \frac{\sqrt{1-p}e^{x(1-p)}}{\sqrt{1-pe^{2x(1-p)}}}$. Therefore, by [20, eq. (25)] we have that

$$A(px, 1/p, 1) = \frac{\sqrt{1-p}}{\sqrt{1-pe^{2x(1-p)}}} = \sum_{n,k\geq 0} N(n,k) x^n p^k,$$

which implies that $A(x,p,1) = \sum_{n,k\geq 0} N(n,n-k)x^n p^k$. Hence $\sum_{j\geq 0} a_{n,k,j} = N(n,n-k)$, as claimed.

(C) Let $f_{n,i} = a_{n,i,\lfloor n/2 \rfloor}$. By (12) and (13), we have

$$f_{n,i} = (2i+1)f_{n-1,i} + (n-2i+1)f_{n-1,i-1}, \quad 0 \leq i \leq \lfloor n/2 \rfloor,$$

with $f_{0,0} = 1$. Define $f_n(x) = \sum_{i\geq 0} f_{n,i} x^i$. Then

$$f_{n+1}(x) = (nx+1)f_n(x) + 2x(1-x)\frac{d}{dx} f_n(x), \tag{31}$$

with the initial condition $f_0(x) = 1$. By comparing (31) with (29), we see that the polynomials $f_n(x)$ satisfy the same recurrence relation and initial conditions as $\widetilde{P}_n(x)$, so they coincide to each other. Similarly, it is easy to verify that

$$\sum_{j\geq 0} a_{n,i,\lfloor \frac{n}{2}\rfloor - i} x^i = \widetilde{P}_n(x),$$

which completes the proof. □

We end this subsection by giving another characterization of the numbers $N(n,k)$ and the polynomials $P_n(x)$. Assume that

$$(xD)^{n+1}(y) = xD(xD)^n(y) = xD\big((xD)^n(y)\big).$$

As a "dual" of $(xD)^n(x)$, we define

$$(xD)^{2n}(y) = \sum_{i,j\geq 0} b_{2n,i,j} x^{2i+2} y^{4n-1-2i-2j} z^{2j},$$
$$(xD)^{2n+1}(y) = \sum_{i,j\geq 0} b_{2n+1,i,j} x^{2i+2} y^{4n-2i-2j} z^{2j+1}.$$

From (3), we have

$$(xD)(y) = x^2 z, \quad (xD)^2(y) = 2x^2yz^2 + x^4 y.$$

The first terms of the generating function $B(x,p,q)$ are given as follows:

$$\begin{aligned} B(x,p,q) = 1 + x + (p+2q)\frac{x^2}{2!} + \big(p^2 + p(2q+8) + 4q\big)\frac{x^3}{3!} \\ + \big(p^3 + p^2(8+28q) + p(16q^2+44q) + 8q^2\big)\frac{x^4}{4!} \\ + \big(p^4 + p^3(88+28q) + p^2(136+364q+16q^2) + p(208q+88q^2) + 16q^2\big)\frac{x^5}{5!} + \cdots . \end{aligned}$$

It is easy to verify that

$$\begin{aligned} b_{2n,i,j} &= (4n-2i-2j)b_{2n-1,i-1,j-1} + (2i+2)b_{2n-1,i,j-1} + (2j+1)b_{2n-1,i-1,j}, \\ b_{2n+1,i,j} &= (4n-2i-2j+1)b_{2n,i-1,j} + (2i+2)b_{2n,i,j} + (2j+2)b_{2n,i-1,j+1}, \end{aligned} \tag{32}$$

which leads to the following result.

Lemma 21. *We have*

$$\begin{cases} BO_x = p - 1 + (2-p)BE + 2p(1-p)BE_p + 2p(1-q)BE_q + 2xpBE_x, \\ BE_x = (p+2q-2pq)BO + 2pq(1-p)BO_p + 2pq(1-q)BO_q + 2xpqBO_x. \end{cases}$$

Equivalently, (BO', BE') is a J-pair of the first type, where

$$BO' = p\sqrt{\frac{q}{p-1}}BO \quad \text{and} \quad BE' = \frac{p-1}{p} + \frac{p}{\sqrt{p-1}}BE.$$

In the same way as the proof of Theorem 20, one can easily show the following result.

Theorem 22. *For $n \geq 1$, we have*

(i) $\sum_{i,j\geq 0} b_{n,i,j} = (2n-1)!!$.

(ii) $\sum_{j\geq 0} b_{n,i,j} = N(n, i+1)$.

(iii) $\sum_{i\geq 0} b_{n,i,\lfloor \frac{n}{2}\rfloor} x^i = P_n(x)$ *and* $\sum_{i\geq 0} b_{n,i,\lfloor \frac{n}{2}\rfloor - i} x^i = \frac{1}{2}P_{n+1}(x)$.

4.2. **Alternating runs and the longest alternating subsequences.**

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. We say that π changes direction at position i if either $\pi(i-1) < \pi(i) > \pi(i+1)$, or $\pi(i-1) > \pi(i) < \pi(i+1)$, where $i \in \{2,3,\ldots,n-1\}$. We say that π has k *alternating runs* if there are $k-1$ indices i such that π changes direction at these positions [27, A059427]. Let $R(n,k)$ denote the number of permutations in $\mathfrak{S}_n$ with k alternating runs. In recent years, Deutsch and Gessel [27, A059427], Stanley [29] also studied the generating function for these numbers. As pointed out by Canfield and Wilf [4, Section 6], the generating function for the numbers $R(n,k)$ can be elusive. Let $R_n(x) = \sum_{k=1}^{n-1} R(n,k)x^k$. The polynomial $R_n(x)$ is closely related to $P_n(x)$:

$$R_n(x) = \frac{x(1+x)^{n-2}}{2^{n-2}} P_n\left(\frac{2x}{1+x}\right).$$

for $n \geq 2$, which was established in [21].

An *alternating subsequence* of π is a subsequence $\pi(i_1)\cdots\pi(i_k)$ satisfying

$$\pi(i_1) > \pi(i_2) < \pi(i_3) > \cdots \pi(i_k).$$

Denote by $\operatorname{as}(\pi)$ the length of the longest alternating subsequence of π. There is a large literature devoted to $\operatorname{as}(\pi)$ (see [3, 30, 33]). Define

$$a_k(n) = \#\{\pi \in \mathfrak{S}_n : \operatorname{as}(\pi) = k\}.$$

It should be noted that $a_k(n)$ is also the number of permutations in $\mathfrak{S}_n$ with k up-down runs. The *up-down runs* of a permutation π are the alternating runs of π endowed with a 0 in the front (see [27, A186370]). For example, the permutation $\pi = 514632$ has 3 alternating runs and 4 up-down runs.

In the same way as the proof of Theorem 20, it is a routine exercise to show the following.

Theorem 23. *For $n \geq 1$, we have*

(i) $\sum_{i,j\geq 0} t_{n,i,j} = \sum_{i,j\geq 0} r_{n-1,i,j} = n!$.
(ii) $\sum_{i\geq 0} t_{n,i,j} = a_{n-j}(n)$.
(iii) $\sum_{j\geq 0} t_{n,i,j} = \widetilde{P}_{n,\lfloor \frac{n}{2}\rfloor - i}$.
(iv) $\sum_{i\geq 0} r_{n,i,j} = R(n+1,n-j)$.
(v) $\sum_{j\geq 0} r_{n,i,j} = P_{n+1,\lfloor \frac{n}{2}\rfloor - i}$.

Recall that a permutation π is *alternating* if

$$\pi(1) > \pi(2) < \pi(3) > \cdots \pi(n).$$

In other words, $\pi(i) < \pi(i+1)$ if i is even and $\pi(i) > \pi(i+1)$ if i is odd. Let E_n denote the number of alternating permutations in $\mathfrak{S}_n$. For instance, $E_4 = 5$, corresponding to the permutations 2143, 3142, 3241, 4132 and 4231. Similarly, define π to be *reverse alternating* if $\pi(1) < \pi(2) > \pi(3) < \cdots \pi(n)$. The bijection $\pi \mapsto \pi^c$ on $\mathfrak{S}_n$ defined by $\pi^c(i) = n+1-\pi(i)$ shows that E_n is also the number of reverse alternating permutations in $\mathfrak{S}_n$. The number E_n is called an *Euler number* (see [30]). By definition, we have

$$P_{2n+1,n} = E_{2n+1},\ \widetilde{P}_{2n,n} = E_{2n}.$$

From (22), (23) and (26), it is easy to verify that $t_{2n,0,0} = r_{2n,0,0} = 0, t_{2n-1,0,0} = 1, r_{2n-1,0,0} = 2$ and for $1 \leq j \leq 2n-1$, we have

$$t_{2n,0,j} = t_{2n,0,2n-j},$$

$$t_{2n,0,j} = t_{2n-1,0,j-1},$$

$$r_{2n,0,j} = r_{2n,0,2n-j},$$

$$r_{2n,0,j} = r_{2n-1,0,j-1}.$$

Therefore, from Theorem 23, we see that the numbers $t_{2n,0,j}$ and $r_{2n,0,j}$ construct a symmetric array with row sums equaling the Euler numbers, as demonstrated in the following array:

$$\begin{array}{ccccc} & & 1 & & \\ & & 2 & & \\ & 1 & 3 & 1 & \\ & 2 & 12 & 2 & \\ 1 & 15 & 29 & 15 & 1 \\ 2 & 60 & 148 & 60 & 2 \\ & & \cdots & & \end{array}$$

For convenience, we list the tables of the values of $t_{n,i,j}$ and $r_{n,i,j}$ for $1 \leq n \leq 4$.

$t_{1,i,j}$	$j=0$
$i=0$	1

$t_{2,i,j}$	$j=0$	$j=1$
$i=0$	0	1
$i=1$	1	0

$t_{3,i,j}$	$j=0$	$j=1$	$j=2$
$i=0$	1	3	1
$i=1$	1	0	0

$t_{4,i,j}$	$j=0$	$j=1$	$j=2$	$j=3$
$i=0$	0	1	3	1
$i=1$	4	10	4	0
$i=2$	1	0	0	0

$r_{1,i,j}$	$j=0$
$i=0$	2

$r_{2,i,j}$	$j=0$	$j=1$
$i=0$	0	2
$i=1$	4	0

$r_{3,i,j}$	$j=0$	$j=1$	$j=2$
$i=0$	2	12	2
$i=1$	8	0	0

$r_{4,i,j}$	$j=0$	$j=1$	$j=2$	$j=3$
$i=0$	0	2	12	2
$i=1$	16	56	16	0
$i=2$	16	0	0	0

Define

$$\mathrm{sn}\,(x,k)=\sum_{n\geq 0}(-1)^n J_{2n+1}(k^2)\frac{x^{2n+1}}{(2n+1)!},$$

$$\mathrm{cn}\,(x,k)=1+\sum_{n\geq 0}(-1)^n J_{2n}(k^2)\frac{x^{2n}}{(2n)!}.$$

Note that

$$J_n(k^2)=\sum_{0\leq 2i\leq n-1} J_{n,2i}k^{2i}.$$

Dumont [8, Corollary 1] found that $s_{2n,i,0}=J_{2n,2i}$ and $s_{2n+1,i,0}=J_{2n+2,2i}$. By comparing (5) with (22) and (23), we get the following result immediately.

Theorem 24. *For $n\geq 1$, we have $J_{n,2i}=t_{n,\lfloor\frac{n}{2}\rfloor-i,0}$.*

It follows from *Leibniz's formula* that

$$\begin{aligned}D^{2n+1}(w)&=D^{2n}(wx)\\&=\sum_{k\geq 0}\binom{2n}{2k}D^{2k}(w)D^{2n-2k}(x)+\sum_{k\geq 0}\binom{2n}{2k+1}D^{2k+1}(w)D^{2n-2k-1}(x),\end{aligned}$$

and similarly,

$$\begin{aligned}D^{2n+2}(w)&=D^{2n+1}(wx)\\&=\sum_{k\geq 0}\binom{2n+1}{2k}D^{2k}(w)D^{2n+1-2k}(x)+\sum_{k\geq 0}\binom{2n+1}{2k+1}D^{2k+1}(w)D^{2n-2k}(x).\end{aligned}$$

Therefore, combining (4), we get

$$t_{2n+1,i,0}=\sum_{k\geq 0}\binom{2n}{2k}\sum_{j=0}^{i}t_{2k,j,0}s_{2n-2k,i-j,0},$$

$$t_{2n+2,i+1,0}=\sum_{k\geq 0}\binom{2n+1}{2k+1}\sum_{j=0}^{i}t_{2k+1,j,0}s_{2n-2k,i-j,0}.$$

Therefore, as a corollary of Theorem 24, we get the following.

Corollary 25 ([32, eq. (20)]). *For $n\geq 0$, we have*

$$J_{2n+1,2n-2i}=\sum_{k\geq 0}\binom{2n}{2k}\sum_{j=0}^{i}J_{2k,2k-2j}J_{2n-2k,2i-2j},$$

$$J_{2n+2,2n-2i}=\sum_{k\geq 0}\binom{2n+1}{2k+1}\sum_{j=0}^{i}J_{2k+1,2k-2j}J_{2n-2k,2i-2j}.$$

We end our paper by giving the following.

Conjecture 26. *Let* $s_{n,i,j}$ *be the numbers defined by* (4). *Set* $\widetilde{s}_{n,i,j} = s_{n,j,i}$, *i.e.,*

$$\widetilde{s}_{n,i,j} = |\{\pi \in \mathfrak{S}_n : X(\pi) = j, Y(\pi) = i\}|,$$

where $X(\pi)$ *(resp.,* $Y(\pi)$*) is the number of odd (resp., even) cycle peaks of* π*. Then*

$$\begin{aligned} \widetilde{s}_{2n+1,i,0} &= t_{2n+1,i,0}, \\ \widetilde{s}_{2n+1,i,j} &= t_{2n+1,i,2j-1} + t_{2n+1,i,2j} \quad \text{for } j \geq 1, \\ \widetilde{s}_{2n,i,j} &= t_{2n,i,2j} + t_{2n,i,2j+1} \quad \text{for } j \geq 0. \end{aligned}$$

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