ON MOMENTS OF CANTOR AND RELATED DISTRIBUTIONS.

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Abstract. We provide several simple recursive formulae for the moment sequence of infinite Bernoulli convolution. We relate moments of one infinite Bernoulli convolution with others having different but related parameters. We give examples relating Euler numbers to the moments of infinite Bernoulli convolutions. One of the examples provides moment interpretation of Pell numbers as well as new identities satisfied by Pell and Lucas numbers.

1. Introduction

The aim of this note is to add a few simple observations to the analysis of the distribution of the so called fatigue symmetric walk (term appearing in [12]). These observations are based on the reformulation of known results scattered through literature. We however pay more attention to the moment sequences and less to the properties of distributions that produce these moment sequences. It seems that the main novelty of the paper lies in the probabilistic interpretation of Pell and Lucas numbers and easy proofs of some identities satisfied by these numbers. However in order to place these results in the proper context we recall definition and basic properties of infinite Bernoulli convolutions. In deriving properties of these convolutions we recall some known, important results.

The paper is organized as follows. After recalling definition and basic facts on the fatigue random walks we concentrate on the moment sequences of infinite Bernoulli convolutions. We formulate a corollary of the results of the paper expressed in terms of moment sequence. This corollary formulated in terms of number sequences provides identities of Pell and Lucas numbers of even order (Remark 8).

2. Infinite Bernoulli Convolutions

Let $\{X_n\}_{n\geq 1}$ be the sequence of i.i.d. random variables such that $P(X_1 = 1) = P(X_1 = -1) = 1/2$. Further let $\{c_n\}_{n\geq 1}$ be a sequence of reals such that $\sum_{n\geq 1} c_n^2 < \infty$. We define random variable:

$$S = \sum_{n\geq 1} c_n X_n.$$

By Kolmogorov 3 series theorem S exists and moreover it is square integrable. $ES^2 = \sum_{n\geq 1} c_n^2$. Obviously $ES = 0$. Let $\varphi(t)$ denote characteristic function of S. By the standard argument we have $\varphi(t) = E\exp(it\sum_{s\geq 1} c_n X_n) = E\prod_{n\geq 1}\exp(itc_n X_n) = \prod_{n\geq 1}(\exp(itc_n)/2 + \exp(-itc_n)/2) = \prod_{n\geq 1}\cos(tc_n)$.

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We will concentrate on the special form of the sequence c_n namely we will assume that $c_n = \lambda^{-n}$ for some $\lambda > 1$.

It is known (see [13]) that for all λ distribution of $S = S(\lambda)$ is continuous that is $P_S(\{x\}) = 0$ for all $x \in \mathbb{R}$. Moreover it is also known (see [16], [17]) that if for almost $\lambda \in (1, 2]$ this distribution is absolutely continuous and for almost all $\lambda \in (1, \sqrt{2}]$ it has square integrable density. Garsia in [7], Theorem 1.8 showed examples of such λ leading to absolutely continuous distribution. Namely such $\lambda \in (1, 2)$ are the roots of monic polynomials P with integer coefficients such that $|P(0)| = 2$ and $\lambda \prod_{|\alpha_i|>1} |\alpha_i| = 2$, where $\{\alpha_i\}$ are the remaining roots of P.

There are known (see [4], [5]) countable instances of $\lambda \in (1, 2]$ that this distribution is singular. We will denote by φ_λ the characteristic function of $S(\lambda)$. Following [4] we know that the values λ such that $\varphi_\lambda(t)$ does not tend to zero as $t \longrightarrow \infty$ consequently related distribution is singular (by Riemann–Lebesgue Lemma) are the so called Pisot or PV- numbers i.e. sole roots of such monic irreducible polynomials P with integer coefficients having the property that all other roots have absolute values less than 1. We must then have $P(0) = 1$. Examples of such numbers are the so called 'golden ratio' $\left(1 + \sqrt{5}\right)/2$ or the so called 'silver ratio' $1 + \sqrt{2}$. Moreover following [14] one knows that PV numbers are the only numbers $\lambda \in (1, 2]$ for which φ_λ does not tend to zero. Of course singularity of the distribution of $S(\lambda)$ can occur for λ not being PV numbers.

For $\lambda > 2$ it is known that the distribution of S is singular [10].

To simplify notation we will write $\operatorname{supp} X$, where X is a random variable meaning $\operatorname{supp} P_X$, where P_X denotes distribution of X. Similarly $X * Y$ denotes random variable whose distribution is a convolution of distributions of X and Y.

We have simple Lemma.

Lemma 1. *i)* $\operatorname{supp}(S(\lambda)) \subset \left[-\frac{1}{\lambda-1}, \frac{1}{\lambda-1}\right]$.

In particular:

ia) if $\lambda = 2$ *then* $S \sim U([-1, 1])$ *and*

ib) if $\lambda = 3$ *then* $\operatorname{supp}(S + 1/2)$ *is equal to Cantor set.*

In general if λ *is a positive integer then* $\operatorname{supp}(S + 1/(\lambda - 1))$ *consist of all numbers of the form* $\sum_{j\geq 1} r_j \lambda^{-j}$ *where* $r_j \in \{0, 2\}$. *Moreover the distribution of* $(S(\lambda) + 1/(\lambda - 1))$ *is 'uniform' on this a set.*

ii) $\forall k \geq 1$:

$$\varphi_\lambda(\lambda^k t) = \varphi_\lambda(t) \prod_{j=0}^{k-1} \cos\left(\lambda^j t\right). \tag{2.1}$$

iii) $\forall k \geq 1 : S(\lambda) \sim \sum_{i=1}^{k} \lambda^{i-1} S_i(\lambda^k)$, *where* $S_i(\tau)$ *(* $i = 1, \ldots, k$*) are i.i.d. random variables each having distribution* $S(\tau)$. *Consequently* $\varphi_\lambda(t) = \prod_{j=1}^{k} \varphi_{\lambda^k}\left(\lambda^{j-1} t\right)$

iv) Let us denote $m_n(\lambda) = ES(\lambda)^n$. *Then* $\forall n \geq 1 : m_{2n-1}(\lambda) = 0$ *and*

$$m_{2n}(\lambda) = \frac{1}{\lambda^{2kn} - 1} \sum_{j=0}^{n-1} \binom{2n}{2j} m_{2j}(\lambda) W_{2(n-j)}^{(k)}(\lambda),$$

with $m_0 = 1$, *where* $W_n^{(1)} = 1$, $W_n^{(k)}(\lambda) = \frac{d^n}{dt^n}(\prod_{j=1}^{k} \cosh(\lambda^{j-1} t))\Big|_{t=0}$
$= \frac{1}{2^{k-1}} \sum_{i_1=0,\ldots,i_{k-1}=0}^{1} (1 + \sum_{j=1}^{k} (2i_j - 1)\lambda^j)^{2n}$.

In particular we have:

$$m_{2n}(\lambda) = \frac{1}{\lambda^{2n}-1}\sum_{j=0}^{n-1} m_{2j}(\lambda)\binom{2n}{2j}, \tag{2.2}$$

$$m_{2n}\left(\lambda\right) = \frac{1}{\lambda^{4n}-1}\sum_{j=0}^{n-1}\binom{2n}{2j} m_{2j}(\lambda)\sum_{l=0}^{2(n-j)}\binom{2(n-j)}{2l}\lambda^{2l}. \tag{2.3}$$

v) $\forall k\geq 1: m_{2k}(\lambda)=\frac{-1}{\lambda^{2k}-1}\sum_{j=0}^{k-1}\binom{2k}{2j}\lambda^{2j}E_{2(k-j)}m_{2j}(\lambda)$, *where* E_k *denotes* $k-th$ *Euler number.*

Proof. i) First of all notice that $\frac{1}{\lambda-1}=\sum_{n\geq 1}1/\lambda^n$, hence $S+\frac{1}{\lambda-1}=\sum_{n\geq 1}\frac{1}{\lambda^n}(X_n+1)$. Now since $P(X_n+1=0)=P(X_n+1=2)=1/2$ we see that $\text{supp}(S+\frac{1}{\lambda-1})\subset[0,\frac{2}{\lambda-1}]$. Notice also that if $\lambda=2$ then $S+1=2\sum_{n\geq 1}\frac{1}{2^n}Y_n$, where $P(Y_n=0)=P(Y_n=1)=1/2$. In other words $(S+1)/2$ is any number chosen from $[0,1]$ with equal chances that is $(S+1)/2$ has uniform distribution on $[0,1]$.

When $\lambda=3$ we see that $S+1/2$ is a number that can be written with the help of $'0'$ and $'2'$ in ternary expansion. In other words $S+1/2$ is number drawn from Cantor set with equal chances. For λ integer we argue in the similar way.

ii) We have $\varphi_S(\lambda^k t)=\prod_{n\geq 1}\cos(\lambda^k t\frac{1}{\lambda^n})=\varphi_S\left(t\right)\prod_{j=0}^{k-1}\cos\left(\lambda^j t\right)$.

iii) Fix integer k. Notice that we have:

$$S\left(\lambda\right)=\sum_{n\geq 1}\lambda^{-n}X_n=$$
$$\sum_{j\geq 1}\lambda^{-kj}X_{kj}+\sum_{j\geq 1}\lambda^{-kj+1}X_{kj-1}+\ldots+\sum_{j\geq 1}\lambda^{-kj+k-1}X_{kj-k+1}=$$
$$\sum_{m=1}^{k}\lambda^{m-1}\sum_{j\geq 1}\left(\lambda^k\right)^{-j}X_{kj-m+1}.$$

Now since by assumption all X_i are i.i.d. we deduce that $S_i(\lambda^k)$ are i.i.d. random variables with distribution defined by $\varphi_{\lambda^k}(t)$. Hence we have $\varphi_\lambda(t)=\prod_{j=1}^{k}\varphi_{\lambda^k}(\lambda^{j-1}t)$.

iv) First of all we notice that $\varphi_S(t)$ is an even function hence all derivatives of odd order at zero are equal 0. Secondly let $\psi_\lambda(t)$ denote moment generating function of $S(\lambda)$. It is easy to notice that $\psi_\lambda(s)=\varphi_{S(\lambda)}(-is)$. Let us denote $m_{2n}\left(\lambda\right)=\psi_\lambda^{(2n)}(0)$. Basing on the elementary formula

$$\cosh(\alpha)\cosh(\beta)=\frac{1}{2}(\cosh(\alpha+\beta)+\cosh\left(\alpha-\beta\right)),$$

we can easily obtain by induction the following identity:

$$\prod_{j=0}^{k-1}\cosh(\lambda^j t)=\frac{1}{2^{k-1}}\sum_{i_1=0,\ldots,i_{k-1}=0}^{1}\cosh(t(1+\sum_{j=1}^{k}(2i_j-1)\lambda^j)).$$

Since $\left.(\cosh \alpha t)^{(2n)}\right|_{t=0}=\alpha^{2n}$, we have

$$\left.\left(\prod_{j=0}^{k-1}\cosh(\lambda^{j}t)\right)^{(2n)}\right|_{t=0}=$$

$$\frac{1}{2^{k-1}}\sum_{i_1=0,\ldots,i_{k-1}=0}^{1}\left.\left(\cosh(t(1+\sum_{j=1}^{k}(2i_j-1)\lambda^{j}))\right)^{(2n)}\right|_{t=0}=W_{2n}^{(k)}\left(\lambda\right).$$

Now using Leibnitz formula for differentiation applied to (2.1) we get

$$f^{(2n)}(\lambda^{k}t)\lambda^{2kn}=\sum_{j=0}^{2n}\binom{2n}{j}(\prod_{i=0}^{k-1}\cosh\lambda^{i}t)^{(j)}f^{(2n-j)}\left(t\right).$$

Setting $t=0$ and using the fact that all derivatives of both f and $\cosh t$ of odd order at zero are zeros we get the desired formula.

v) We use result of [18] that states that for each N inverse of lower triangular matrix of degree $N\times N$ with (i,j) entry $\binom{2i}{2j}$ is the lower triangular matrix with $(i,j)-th$ entry equal to $\binom{2i}{2j}E_{2(i-j)}$. □

Remark 1. *Formula (2.2) is known in a slightly different form it appeared in* [8], [6] *and* [1].

Remark 2. *Notice that polynomials* $\left\{W_{n}^{(k)}(\lambda)\right\}_{k,n\geq 1}$ *satisfy the following recursive relationship for* $k>1$ *:*

$$W_{n}^{(k)}\left(\lambda\right)=\sum_{j=0}^{n}\binom{2n}{2j}\lambda^{2j}W_{j}^{(k-1)}\left(\lambda\right),$$

with $W_{n}^{(1)}(\lambda)=1$. *Hence its generating functions satisfy* $\Theta_{k}(t)$ *the following relationship*

$$\Theta_{k}(t,\lambda)=\Theta_{k-1}(\lambda t,\lambda)\cosh t,$$

where we have have denoted: $\Theta_{k}(t,\lambda)=\sum_{n\geq 0}\frac{t^{2n}}{(2n)!}W_{n}^{(k)}(\lambda)$.

Remark 3. *Notice that the above mentioned lemma provides an example of two singular distributions whose convolution is a uniform distribution. Namely we have* $S(4)*2S(4)=S(2)$. *Similarly we have* $S(2)=S\left(8\right)*2S(8)*4S(8)$ *or* $S(2)=S(2^{k})*\ldots*2^{k-1}S(2^{k})$. *The first example was already noticed by Kersher and Wintner in* [10],*(22a).*

Remark 4. *We can deduce even more from theses examples namely following the result of Kersher* [9], *p.451 that characteristic functions* $\varphi_{n}(t)$ *of* $S\left(n\right)$ *(where n is an integer* >2*) do not tend to zero as* $t\longrightarrow\infty$. *Thus since we have* $\varphi_{4}(t)\varphi_{4}(2t)$ $=\sin t/t$ *and* $\varphi_{8}(t)\varphi_{8}(2t)\varphi_{8}(4t)=\sin t/t$ *we deduce that if* $t_{k}\longrightarrow\infty$ *is a sequence such that* $|\varphi_{4}(t_{k})|>\varepsilon>0$ *for suitable* ε *then* $\varphi_{4}(2t_{k})\longrightarrow 0$. *Similarly if* $t_{k}\longrightarrow\infty$ *such that* $|\varphi_{8}(t_{k})|>\varepsilon>0$ *then* $\varphi_{8}(2t_{k})\varphi_{8}(4t_{k})\longrightarrow 0$. *Similar observations can be made can be made in more general situation. As it is known from the papers of Erdős* [4], [5] *the situation that* $|\varphi_{\lambda}(t_{k})|>\varepsilon>0$ *for some sequence* $t_{k}\longrightarrow\infty$ *occurs when* λ *is a Pisot number (briefly PV-number). On the other hand as it is known roots of Pisot numbers are not Pisot, hence using above mentioned result of Salem*

$|\varphi_{\lambda^{1/k}}(t)| \longrightarrow 0$ as $t \longrightarrow \infty$, where λ is some PV number and $k > 1$ any integer. But we have $\varphi_{\lambda^{1/k}}(t_n) = \prod_{j=1}^{k} \varphi_\lambda \left(\lambda^{(j-1)/k} t_n\right) \longrightarrow 0$, where $t_n \longrightarrow \infty$ is such a sequence that $|\varphi_\lambda(t_n)| > \varepsilon > 0$.

Remark 5. *One knows that if $\lambda = q/p$ where p and q are relatively prime integers and $p > 1$ then $\varphi_\lambda(t) = O((\log|t|)^{-\gamma})$ where $\gamma = \gamma(p, q) > 0$ as $t \longrightarrow \infty$ (see* [9]*,(3)). Besides we know that then distribution of $S(\lambda)$ is singular. Hence from our considerations it follows that if $\lambda = (q/p)^{1/k}$ for some integer k then $\varphi_\lambda(t) = O((\log|t|)^{-k\gamma})$. Is it also singular?*

3. Moment sequences

To give connection of certain moment sequences with some known integer sequences let us remark the following:

Remark 6. *i)*

$$\begin{aligned} 9^n m_{2n}(3) &= \sum_{j=0}^{n} \binom{2n}{2j} m_{2j}(3), \\ 81^n m_{2n}(3) &= \sum_{j=0}^{n} \binom{2n}{2j} m_{2j}(3)(2^{4(n-j)-1} + 2^{2(n-j)-1}). \end{aligned}$$

ii)

$$\begin{aligned} 5^n m_{2n}\left(\sqrt{5}\right) &= \sum_{j=0}^{n} \binom{2n}{2j} m_{2j}(\sqrt{5}), \\ 25^n m_{2n}\left(\sqrt{5}\right) &= \sum_{j=0}^{n} \binom{2n}{2j} m_{2j}(\sqrt{5}) 4^{n-j} L_{2(n-j)}/2, \end{aligned}$$

where L_n denotes $n-$th Lucas number defined below.

Proof. i) The first assertion is a direct application of (2.2) while in proving the second one we use (2.3) and the fact that $\sum_{j=0}^{n}\binom{2n}{2j} 9^j = 4^n(4^n + 1)/2$ as shown by [21], (seq. no. A026244). ii) Again the first statement follows (2.2) while the second follows (2.3) and the fact that $\sum_{j=0}^{n}\binom{2n}{2j} 5^j = 4^n T_n(3/2)$, where T_n denotes Chebyshev polynomial of the first kind. ([21], seq. no. A099140). Further we use the fact that $T_n(3/2) = L_{2n}/2$. ([21], seq. no. A005248). □

We also have the following Lemma.

Lemma 2. $\forall n \geq 1, k \geq 2:$

$$m_{2n}(\lambda) = \sum_{i_1,\ldots,i_k=0}^{n} \frac{(2n)!}{(2i_1)!\ldots(2i_k)!} \lambda^{2(i_2+2i_3\ldots(k-1)i_k)} \prod_{j=1}^{k} m_{2i_j}\left(\lambda^k\right). \tag{3.1}$$

In particular:

$$m_{2n}(\lambda) = \sum_{j=0}^{n}\binom{2n}{2j}\lambda^{2j}m_{2j}(\lambda^{2})m_{2n-2j}(\lambda^{2}), \tag{3.2}$$

$$m_{2n}(\lambda) = \sum_{\substack{i,j=0\\ i+j\leq n}}\frac{(2n)!}{(2i)!(2j)!(2(n-i-j))!}\times \lambda^{2i}\lambda^{4j}m_{2j}(\lambda^{3})m_{2i}(\lambda^{3})m_{2n-2i-2j}(\lambda^{3}). \tag{3.3}$$

Proof. (3.1) follows directly Lemma 1, iii). □

As a corollary we get the following four observations:

Corollary 1. *i)* $\forall n\geq 1:4^{n}=\sum_{j=0}^{n}\binom{2n+1}{2j+1}$ *and* $1=\sum_{j=0}^{n}\binom{2n+1}{2j+1}4^{j}E_{2(n-j)}$.

ii) $S\left(\sqrt{2}\right)$ *has density*

$$g(x)=\left\{\begin{array}{ccc}\sqrt{2}/4 & if & |x|\leq\sqrt{2}-1,\\ \sqrt{2}(\sqrt{2}+1-|x|)/8 & if & \sqrt{2}-1\leq|x|\leq\sqrt{2}+1,\\ 0 & if & |x|>1+\sqrt{2}.\end{array}\right.$$

iii) Let us denote $\delta_{n}=(\sqrt{2}+1)^{n}$*, then*

$$m_{2n}(\sqrt{2})=\left(\delta_{2n+2}-\delta_{2n+2}^{-1}\right)/(4\sqrt{2}(n+1)(2n+1)).$$

Proof. Since for $\lambda=2$ random variable $S\sim U[-1,1]$ its moments are equal to $ES^{2n}=\frac{1}{2n+1}$. Now we use Lemma 1 iii) and iv).

ii) From the proof of Lemma 2 it follows that $S\left(\sqrt{2}\right)\sim S\left(2\right)+\sqrt{2}S\left(2\right).$ Now keeping in mind that $S\left(2\right)\sim U(-1,1)$ we deduce that $g(x)=\frac{\sqrt{2}}{8}\int_{-1}^{1}h(x-t)dt$, where $h\left(x\right)=\left\{\begin{array}{ccc}\frac{\sqrt{2}}{4} & if & |x|\leq\sqrt{2},\\ 0 & if & otherwise.\end{array}\right.$. iii) By straightforward calculations we get $m_{2n}(\sqrt{2})=2\int_{0}^{\sqrt{2}+1}x^{2n}g(x)dx=\frac{\sqrt{2}}{2}\int_{0}^{\sqrt{2}-1}x^{2n}dx+\frac{\sqrt{2}}{4}\int_{\sqrt{2}-1}^{\sqrt{2}+1}x^{2n}(\sqrt{2}+1-x)dx$. □

Remark 7. *Let us apply formulae: (3.2), (2.2), (2.3) and observe by direct calculation that* $2\sum_{j=0}^{n}\binom{2n}{2j}2^{j}=(1+\sqrt{2})^{2n}+(1-\sqrt{2})^{2n}$. *We get the following identities:* $\forall n\geq 1:$

$$\begin{aligned}m_{2n}(\sqrt{2}) &= \sum_{j=0}^{n}\binom{2n}{2j}\frac{2^{j}}{(2n-2j+1)(2j+1)}\\ &= \frac{1}{2^{n}-1}\sum_{j=0}^{n-1}\binom{2n}{2j}m_{2j}(\sqrt{2})\\ &= \frac{1}{4^{n}-1}\sum_{j=0}^{n-1}\binom{2n}{2j}m_{2j}(\sqrt{2})\tau_{2(n-j)},\end{aligned}$$

where $\tau_{n}=(1+(-1)^{n})(\delta_{n}+\delta_{n}^{-1})/4.$

Now let us recall the definition of the so called Pell and Pell–Lucas numbers. Using sequence δ_n Pell numbers $\{P_n\}$ and Pell–Lucas numbers $\{Q_n\}$ are defined

$$P_n = (\delta_n + (-1)^{n+1}\delta_n^{-1})/(2\sqrt{2}), \tag{3.4}$$

$$Q_n = \delta_n + (-1)^n \delta_n^{-1}, \tag{3.5}$$

where δ_n is defined in 1,iii).

Using these definitions we can rephrase assertions of Corollary 1 and Remark 7 adding to recently discovered ([15], [2]) new identities satisfied by Pell and Pell-Lucas numbers and of course probabilistic interpretation of Pell numbers.

Remark 8. *i)* $m_{2n}(\sqrt{2}) = \frac{P_{2n+2}}{(2n+2)(2n+1)}$, $\tau_{2n} = Q_{2n}/2$.

ii) $\forall n \geq 1:$

$$P_{2n+2} = \sum_{j=0}^{n} \binom{2n+2}{2j+1} 2^j, \tag{3.6}$$

$$Q_{2n} = 2\sum_{j=0}^{n} \binom{2n}{2j} 2^j, \tag{3.7}$$

$$2^{n-1}P_{2n} = \sum_{j=0}^{n} \binom{2n}{2j} P_{2j}, \tag{3.8}$$

$$2^{2n-1}P_{2n} = \sum_{j=0}^{n} \binom{2n}{2j} P_{2j}Q_{2(n-j)}, \tag{3.9}$$

$$\sum_{j=0}^{n} \binom{2n}{2j} (1+\sqrt{2})^{2j} = 2^{n-1} + 2^{n-2}Q_{2n} + 2^{n-1}\sqrt{2}P_{2n}. \tag{3.10}$$

Proof. Only last statement requires justification. First we find that $\sum_{j=0}^{n}\binom{2n}{2j}Q_{2j} = 2^n(1 + Q_{2n}/2)$ using (3.7). Then we use (3.4), (3.5) and (3.8). □

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