Enumerating Permutations by their Run Structure

Christopher J. Fewster[1], Daniel Siemssen[2]

[1] Department of Mathematics, University of York, Heslington, York YO10 5DD, UK.
   E-Mail: chris.fewster@york.ac.uk
[2] Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, 16146 Genova, Italy.
   E-Mail: siemssen@dima.unige.it

Abstract. Motivated by a problem in quantum field theory, we study the up and down structure of circular and linear permutations. In particular, we count the length of the (alternating) runs of permutations by representing them as monomials and find that they can always be decomposed into so-called 'atomic' permutations introduced in this work. This decomposition allows us to enumerate the (circular) permutations of a subset of $\mathbb{N}$ by the length of their runs. Furthermore, we rederive, in an elementary way and using the methods developed here, a result due to Kitaev on the enumeration of valleys.

1. Introduction

Let us adopt the following notation for integer intervals: $[a \mathinner{.\,.} b] \doteq [a, b] \cap \mathbb{N} = \{a, a+1, \ldots, b\}$ with the special case $[n] \doteq [1 \mathinner{.\,.} n]$.

Let $S \subset \mathbb{N}$ contain n elements. A *permutation* of S is a linear ordering $\sigma_1, \sigma_2, \ldots, \sigma_n$ of the elements of S. We denote the set of all $n!$ permutations of S by $\mathfrak{S}_S$ and by $\mathfrak{S}_n$ if $S = [n]$. Almost all statements and proofs below are given for the special case $S = [n]$ and the obvious generalization to generic finite subsets of $\mathbb{N}$ is implicit. It is customary to write the permutation as the word $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ and we can identify the permutation σ with the bijection $\sigma(i) = \sigma_i$.

Replacing the linear ordering by a circular one, we arrive at the notion of *circular permutations* of S, *i.e.*, the arrangements of the elements $\sigma_1, \sigma_2, \ldots, \sigma_n$ of S around an oriented circle (turning the circle over generally produces a different permutation). In this case one can always choose σ_1 to be the smallest element of S so that $\sigma_1 = 1$ if $S = [n]$. It is clear that the set $\mathfrak{C}_S$ of all circular permutations of S contains $(n-1)!$ elements; if $S = [n]$, we denote it by $\mathfrak{C}_n$. Circular permutations may be written as words of the form $\sigma = \dot{\sigma}_1 \sigma_2 \cdots \dot{\sigma}_n$, where we emphasize the circular symmetry by a dot above outermost characters of the word.[1] Moreover, due to the circular symmetry we can define $\sigma(0) \doteq \sigma(n)$ and $\sigma(n+1) \doteq \sigma(1)$.

Let us introduce yet another class of permutations. We denote by $\mathfrak{A}_S^\pm \subset \mathfrak{S}_S$ the rising and falling *'atomic'* permutations[2] of an n-element subset $S \subset \mathbb{N}$. We define $\mathfrak{A}_S^+$ as those permutations that, when written as a word, begin with the smallest and end with the largest element of S. The falling atomic permutations $\mathfrak{A}_S^-$ are reversed rising atomic permutations, *i.e.*, they begin with the largest element of S and end with the smallest. Naturally, the cardinality of $\mathfrak{A}_S^\pm$ is $(n-2)!$. If $S = [n]$, we write $\mathfrak{A}_n^\pm$ and see that it is the set of permutations of the form $1 \cdots n$ (*i.e.*, $\sigma_1 = 1$ and $\sigma_n = n$) or $n \cdots 1$ (*i.e.*, $\sigma_1 = n$ and $\sigma_n = 1$) respectively.

We say that i is a *descent* of σ if $\sigma(i) > \sigma(i+1)$ and, conversely, that it is an *ascent* of a (circular) permutation σ if $\sigma(i) < \sigma(i+1)$. For example, the permutation 52364178 has the descents $1, 4, 5$ and the ascents $2, 3, 6, 7$ whereas the circular permutation $\dot{1}4536 78\dot{2}$ has the descents $3, 7, 8$ and the ascents $1, 2, 4, 5, 6$. We can collect all the descents of a (circular) permutation σ in the *descent set*

$$D(\sigma) \doteq \{i \mid \sigma(i) > \sigma(i+1)\}.$$

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[1] This is in analogy with the notation for repeating decimals when representing rational numbers.
[2] The rationale for this naming should become clear later when we see that arbitrary permutations can be decomposed into atomic permutations, but no further.
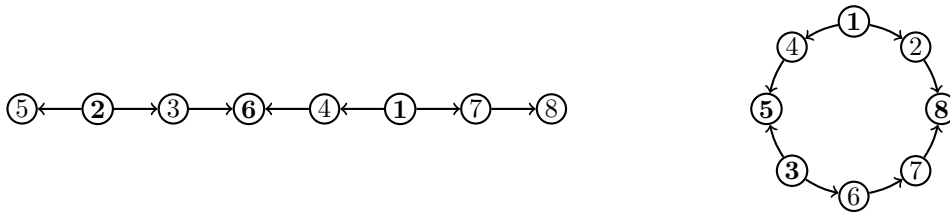
Figure 1. The two directed graphs representing the runs of the permutation 52364178 (left) and the circular permutation $\dot{1}453678\dot{2}$ (right). Peaks and valleys are indicated by boldface numbers.

It is an elementary exercise in enumerative combinatorics to count the number of permutations of $[n]$ whose descent set is given by a fixed $S \subseteq [n-1]$. Let $S = \{s_1, s_2, \ldots, s_k\}$ be an ordered subset of $[n-1]$, then

$$\beta(S) \doteq \left|\{\sigma \in \mathfrak{S}_n \mid D(\sigma) = S\}\right| = \sum_{T \subseteq S} (-1)^{|S-T|} \binom{n}{s_1, s_2 - s_1, s_3 - s_2, \ldots, n - s_k},$$
(1.1)

see, for example, [3, Theorem 1.4]. This result is easily adapted to circular permutations.

Related to the notions of *ascents* and *descents* are the concepts of *peaks* and *valleys*: A peak occurs at position i if $\sigma(i-1) < \sigma(i) > \sigma(i+1)$, whereas a valley occurs in the opposite situation $\sigma(i-1) > \sigma(i) < \sigma(i+1)$. The peaks and valleys of a (circular) permutation σ split it into *runs*, also called *sequences* or *alternating runs*.

Definition 1. A *run* r of a (linear or circular) permutation σ is an interval $[i \, . \, . \, j]$ such that $\sigma(i) \gtreqless \sigma(i+1) \gtreqless \cdots \gtreqless \sigma(j)$ is a monotone sequence, either increasing or decreasing, and so that it cannot be extended in either direction; its length is defined to be $j - i$. If σ is a permutation of an n-element set, the collection of the lengths of all runs gives a partition p of $n - 1$ (linear permutations) or n (circular permutations). The partition p is called the *run structure* of σ.

It follows that a run starts and ends at peaks, valleys or at the outermost elements of a permutation. For example, the permutation 52364178 has runs $[1 \, . \, . \, 2]$, $[2 \, . \, . \, 4]$, $[4 \, . \, . \, 6]$, $[6 \, . \, . \, 8]$ with lengths $1, 2, 2, 2$, whereas the circular permutation $\dot{1}453678\dot{2}$ has runs $[1 \, . \, . \, 3]$, $[3 \, . \, . \, 4]$, $[4 \, . \, . \, 7]$, $[7 \, . \, . \, 9]$, of lengths $2, 1, 3, 2$. Representing these runs by their image under the permutation, they are more transparently written as 52, 236, 641, 178 and 145, 53, 3678, 821 respectively. The runs of (circular) permutations can also be neatly represented as directed graphs as shown in Figure 1. In these graphs the peaks and valleys correspond to double sinks and double sources.

Motivated by a problem in mathematical physics [5] (see also section 6), we are interested in the following issue, which we have not found discussed in the literature. By definition, the run structure associates each permutation $\sigma \in \mathfrak{C}_n$ with a partition p of n. For example, $\dot{1}453678\dot{2}$ and $\dot{1}345278\dot{6}$ both correspond to the same partition $1 + 2 + 2 + 3$ of 8. Our interest is in the inverse problem: given a partition p of n, we ask for the number $Z_{\mathfrak{C}}(p)$ of circular permutations whose run structure is given by p. One may consider similar questions for other classes of permutations, with slight changes; for example, note that the run structure of a permutation $\sigma \in \mathfrak{S}_n$ is a partition of $n - 1$.

The literature we have found focuses on issues such as the number of permutations in which all runs have length one, *e.g.*, André's original study [2], or on the enumeration question where the order of run lengths is preserved, so obtaining a map to compositions, rather than partitions, of n. See, for example, [4], which studies this for the ordinary (noncircular) permutations. In principle our question could be obtained by specialising this more difficult problem, but here we take a direct approach. Another alternative approach not followed here, would be to take (1.1) as a starting point and to sum over all

descent sets corresponding to a particular run structure. However, this would neither be straightforward from the theoretical side, nor would it make for efficient computation. By contrast, our direct method was designed to facilitate computation; for the application in [5] calculations were taken up to $n = 65$ using exact integer arithmetic in Maple™ [8].

Sections 2, 3 and 4 deal, respectively, with the enumeration of the run structure of atomic, circular and linear permutations. Using a suitable decomposition, this is accomplished in each case by reducing the enumeration problem to that for atomic permutations. In section 5 we apply and extend the methods developed in the preceding sections to enumerate the valleys of permutations, thereby reproducing a result of Kitaev [6]. Finally, in section 6, we discuss the original motivation for our work and other applications.

2. Atomic permutations

Let us first discuss the significance of the atomic permutations. We say that a permutation $\sigma \in \mathfrak{S}_S$ of S *contains an atomic permutation* $\pi \in \mathfrak{A}_T$, $T \subset S$, if π can be considered a subword of σ. The atomic permutation π in σ is called *inextendible* if σ contains no other atomic permutation $\pi' \in \mathfrak{A}_{T'}$, $T' \subset S$, such that $T \subsetneq T'$.

In particular, any permutation $\sigma \in \mathfrak{S}_S$ of S with $|S| \geq 2$ contains an inextendible atomic permutation $\pi \in \mathfrak{S}_T$ of a subset $T \subset S$ that contains both the smallest and the largest element of S. That is, if $S = [n]$ and we consider σ as a word, it contains a subword π of the form $1 \cdots n$ or $n \cdots 1$. The permutation π will be called the *principal atom* of σ.

Proposition 2. *Any permutation $\sigma \in \mathfrak{S}_S$ of a finite set $S \subset \mathbb{N}$ can be uniquely decomposed into a tuple $(\pi^1, \ldots, \pi^k)$ of inextendible atomic permutations $\pi^i \in \mathfrak{A}_{T_i}$, $T_i \subset S$ (non-empty) such that $\pi^i_{|T_i|} = \pi^{i+1}_1$ for all $i < k$ and $\cup_i T_i = S$. We call π^i the atoms of σ.*

Proof. Existence: It is clear that any permutation of a set of 1 or 2 elements is an atomic permutation. Suppose, for some $n \geq 3$, that all permutations of $n - 1$ elements or less can be decomposed into inextendible atomic permutations. Without loss of generality, we show that any non-atomic permutation $\sigma \in \mathfrak{S}_n$ also has a decomposition into inextendible atomic permutations. Regarding σ as a word, we can write $\sigma = \alpha \cdot n \cdot \omega$, where α and ω are non-empty subwords. Notice that the permutations $\alpha \cdot n$ and $n \cdot \omega$ have a unique decomposition by assumption. Since an atomic permutation begins or ends with the largest element, we find that a decomposition of σ into inextendible atomic permutations is given by the combination of the decompositions of $\alpha \cdot n$ and $n \cdot \omega$.

Uniqueness: This is clear from the definition of inextendibility. □

We now begin the enumeration of atomic permutations according to their run structure. That is, for every partition p of $n - 1$ we aim to find the number $Z_{\mathfrak{A}}(p)$ of atomic permutations $\mathfrak{A}_n^{\pm}$ of length n.

Observe that any $\sigma \in \mathfrak{A}_n^+$ can be extended to a permutation in $\mathfrak{A}_{n+1}^+$ by replacing n by $n + 1$ and reinserting n in any position after the first and before the last. Thus, 13425 can be extended to 153426, 135426, 134526 or 134256. Every permutation in $\mathfrak{A}_{n+1}^+$ arises in this way, as can be seen by reversing the procedure. The effect on the run lengths can be described as follows.

Case 1: The length of one of the runs can be increased by one by inserting n either at

a) the end of an increasing run if it does not end in $n + 1$, thereby increasing its length (*e.g.*, $13425 \rightarrow 134526$)

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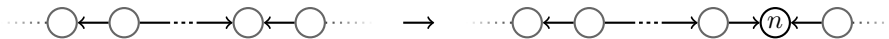
b) the penultimate position of an increasing run, thereby increasing its own length if it ends in $n+1$ (*e.g.*, $13425 \to 135426$) or increasing the length of the following decreasing run otherwise (*e.g.*, $13425 \to 134256$)
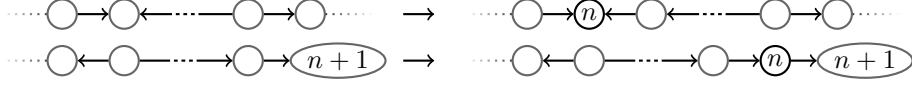
Case 2: Any run of length $i+j \geq 2$ becomes three run of lengths 1, i and j if we insert n either after

a) i elements of an increasing run (*e.g.*, $13425 \to 153426$ exemplifies $i=1$, $j=1$)

b) $i+1$ elements of a decreasing run (*e.g.*, $14325 \to 143526$ for $i=1$, $j=1$)

An analogous argument can be made for the falling atomic permutations $\mathfrak{A}_n^-$.

Notice that partitions of positive integers can be represented by the monomials in the ring of polynomials[3] $\mathbb{Z}[x_1, x_2, \dots]$ in infinitely many variables $x_1, x_2, \dots$ and with integer coefficients. That is, we express a partition $p = p_1 + p_2 + \cdots + p_m$ as $x_{p_1} x_{p_2} \cdots x_{p_m}$ (*e.g.*, the partition $1+2+2+3$ of 8 is written as $x_1 x_2^2 x_3$).

Now, let p be a partition of $n-1$ and X the corresponding monomial. To this permutation there correspond $Z_{\mathfrak{A}}(p)$ permutations in $\mathfrak{A}_n^{\pm}$ which can be extended to permutations in $\mathfrak{A}_{n+1}^{\pm}$ in the manner described above. Introducing the (formally defined) differential operator

$$\mathcal{D} \doteq \mathcal{D}_0 + \mathcal{D}_+ \quad \text{with} \quad \mathcal{D}_0 \doteq \sum_{i=1}^{\infty} x_{i+1} \frac{\partial}{\partial x_i}, \quad \mathcal{D}_+ \doteq \sum_{i,j \geq 1} x_1 x_i x_j \frac{\partial}{\partial x_{i+j}}, \tag{2.1}$$

we can describe this extension in terms of the action of $\mathcal{D}$ on X. We say that $\mathcal{D}_0$ is the *degree-preserving* part of $\mathcal{D}$; it represents the case 1 of increasing the length of a run: the differentiation $\partial/\partial x_i$ removes one of the runs of length i and replaces it by a run of length $i+1$, keeping account of the number of ways in which this can be done. Similarly, case 2 of splitting a run into 3 parts is represented by the *degree-increasing* part $\mathcal{D}_+$. For example, each of the 7 atomic permutations corresponding to the partition $1+1+3$ can be extended as

$$\mathcal{D} x_1^2 x_3 = 2 x_1 x_2 x_3 + x_1^2 x_4 + x_1^4 x_2,$$

i.e., each can be extended to two atomic permutations corresponding to the partitions $1+2+3$, one corresponding to $1+1+4$ and one to $1+1+1+1+2$.

Therefore, starting from the trivial partition 1 of 1, represented as x_1, we can construct a recurrence relation for polynomials $A_n = A_n(x_1, x_2, \dots, x_n)$ which, at every step $n \geq 1$, encode the number of atomic permutations $Z_{\mathfrak{A}}(p)$ of length $n+1$ with run structure given by a partition p of n as the coefficients of the corresponding monomial in A_n. The polynomial A_n, accordingly defined by

$$A_n = \sum_{p \vdash n} Z_{\mathfrak{A}}(p) \prod_{i=1}^{n} x_i^{p(i)}, \tag{2.2}$$

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[3]If one wants to encode also the order of the run (*e.g.*, to obtain a map from permutations of length n to the compositions of n), one can exchange the polynomial ring with a noncommutative ring. Alternatively, if one wants to encode the direction of a run, one could study instead the ring $\mathbb{Z}[x_1, y_1, x_2, y_2, \dots]$, where x_i denotes an increasing run of length i and y_j encodes a decreasing run of length j.

where the sum is over all partitions p of n and $p(i)$ denotes the multiplicity of i in the partition p, can thus be computed from the recurrence relation

$$A_1 \doteq x_1, \tag{2.3a}$$

$$A_n \doteq \mathcal{D}A_{n-1}, \qquad (n \geq 2). \tag{2.3b}$$

We say that the polynomials A_n enumerate the run structure of the atomic permutations.

We summarize these results in the following proposition:

Proposition 3. *The number $Z_{\mathfrak{A}}(p)$ of rising or falling atomic permutations of length $n-1$ corresponding to a given run structure (i.e., a partition p of n), is determined by the polynomial A_n via (2.2). The polynomials A_n satisfy the recurrence relation (2.3).*

Note that atomic permutations always contain an odd number of runs and thus $Z_{\mathfrak{A}}(p)$ is zero for even partitions p.

It will prove useful to combine all generating functions A_n into the formal series

$$\mathcal{A}(\lambda) \doteq \sum_{n=0}^{\infty} A_{n+1} \frac{\lambda^n}{n!} = \sum_{n=0}^{\infty} \mathcal{D}^n A_1 \frac{\lambda^n}{n!},$$

which can be expressed compactly as the exponential

$$\mathcal{A}(\lambda) = \exp(\lambda\mathcal{D})A_1.$$

The first few A_n are given by

$$A_2 = x_2$$
$$A_3 = x_3 + x_1^3$$
$$A_4 = x_4 + 5x_2x_1^2$$
$$A_5 = x_5 + 7x_3x_1^2 + 11x_2^2x_1 + 5x_1^5$$
$$A_6 = x_6 + 9x_4x_1^2 + 11x_2^3 + 38x_3x_2x_1 + 61x_2x_1^4.$$

For example, we can read off from A_5 that there is 1 permutation in $\mathfrak{A}_6^{\pm}$ corresponding to the trivial partition $5 = 5$, 7 corresponding to the partition $5 = 1+1+3$, 11 corresponding to $5 = 1+2+2$ and 5 corresponding to $5 = 1+1+1+1+1$. As a check, we note that $1 + 7 + 11 + 5 = 24$, which is the total number of elements of $\mathfrak{A}_6^{\pm}$; similarly, the coefficients in the expression for A_6 sum to 120, the cardinality of $\mathfrak{A}_7^{\pm}$. A direct check that the coefficients in A_n sum to $(n-1)!$ for all n will be given in the last paragraph of section 5.

We now consider what can be said concerning the degree k part, $A_n^{(k)}$, of the polynomials A_n. The first degree term $A_n^{(1)}$ of A_n is x_n as can be seen by a trivial induction using $A_n^{(1)} = \mathcal{D}_0 A_{n-1}^{(1)}$, which follows from the recurrence relation (2.3). Therefore $Z_{\mathfrak{A}}(n) = 1$.

For $A_n^{(k)}$ with $k > 1$ the effect of $\mathcal{D}_+$ has to be taken into account as well, complicating matters considerably. Nevertheless, the general procedure is clear: once $A_m^{(k-2)}$ is known for all $m < n$, $A_n^{(k)}$ can be obtained as

$$A_n^{(k)} = \mathcal{D}_0 A_{n-1}^{(k)} + \mathcal{D}_+ A_{n-1}^{(k-2)} = \sum_{m=k-1}^{n-1} \mathcal{D}_0^{n-m-1} \mathcal{D}_+ A_m^{(k-2)}.$$

Here one can make use of the following relation. Applying $\mathcal{D}_0$ repeatedly to any monomial $x_{i_1} x_{i_2} \cdots x_{i_k}$ of degree k yields, as a consequence of the Leibniz rule,

$$\mathcal{D}_0^n x_{i_1} x_{i_2} \cdots x_{i_k} = \sum_{\substack{j_1, j_2, \ldots, j_k \geq 0 \\ j_1 + j_2 + \cdots + j_k = n}} \binom{n}{j_1, j_2, \ldots, j_k} x_{i_1+j_1} x_{i_2+j_2} \cdots x_{i_k+j_k}. \tag{2.4}$$

This observation provides the means to determine the third degree term $A_n^{(3)}$. Applying $\mathcal{D}_+$ to any $A_m^{(1)} = x_m$ with $m \geq 2$ produces $x_1 x_p x_q$ with $p + q = m$ and $p, q \geq 1$. Moreover, the repeated action of $\mathcal{D}_0$ on $x_1 x_p x_q$ is described by (2.4) and thus

$$A_n^{(3)} = \sum_{\substack{p,q,r,s,t \geq 0 \\ 1+p+q+r+s+t=n}} \binom{n-p-q-1}{r,s,t} x_{1+r} x_{p+s} x_{q+t}.$$

After some algebra this yields

Proposition 4. *The third degree term $A_n^{(3)}$ of the polynomial $A_n, n \geq 3$, is given by*

$$A_n^{(3)} = \sum_{\substack{i,j,k \geq 1 \\ i+j+k=n}} \sum_{q=1}^{k} \frac{n-q-1}{n-q-j} \binom{n-q-2}{i-1, j-1, k-q} x_i x_j x_k. \tag{2.5}$$

The equation (2.5) for the third degree term $A_n^{(3)}$ can be rewritten into a formula for $Z_{\mathfrak{A}}(p_1 + p_2 + p_3)$, *i.e.*, the number of permutations of $[n+1]$ that start with 1, end with $n+1$ and have three runs of lengths p_1, p_2, p_3, by changing the first sum to a sum over $i, j, k \in \{p_1, p_2, p_3\}$. In particular, this gives rise to three integer series for the special cases

$$Z_{\mathfrak{A}}(n + n + n), \quad Z_{\mathfrak{A}}(1 + n + n), \quad Z_{\mathfrak{A}}(1 + 1 + n),$$

with $n \in \mathbb{N}$.

The first series

$$Z_{\mathfrak{A}}(n + n + n) = \sum_{q=1}^{n} \frac{3n-q-1}{2n-q} \binom{3n-q-2}{n-1, n-1, n-q}$$

$$= 1, 11, 181, 3499, 73501, 1623467, \ldots \qquad (n \geq 1)$$

gives the number of atomic permutations with three runs of equal length n. It does not appear to be known in the literature nor can it be found in the OEIS [9] and the authors were not able to express it in a closed form. For the second series, however, a simple closed form can be found:

$$Z_{\mathfrak{A}}(1 + n + n) = \sum_{q=1}^{n} \left(\binom{2n-q}{n-1} + \binom{2n-q-1}{n-1} \right) + \frac{1}{2} \binom{2n}{n}$$

$$= 2 \binom{2n}{n} - 1 = 11, 39, 139, 503, 1847, \ldots, \qquad (n \geq 2)$$

is the number of atomic permutations in $\mathfrak{A}_{2n+2}^{\pm}$ with two runs of length n. One may understand this directly: there are $\binom{2n}{n}$ permutations in which the length 1 run is between the others and $\binom{2n}{n} - 1$ in which it is either first or last. The third series, $Z_{\mathfrak{A}}(1 + 1 + n)$, *i.e.*, the number of atomic permutations in $\mathfrak{A}_{n+3}^{\pm}$ with two runs of length 1, is given by the odd numbers bigger than 3:

$$Z_{\mathfrak{A}}(1 + 1 + n) = 2n + 1 = 5, 7, 9, 11, 13, 15, \ldots, \qquad (n \geq 2).$$

Observe that terms of the form x_1^n in A_n encode alternating permutations, which were already investigated by André in the 1880's [1]. As a consequence of his results, we find that the alternating atomic permutations are enumerated by the *secant numbers* S_n, the coefficients of the Maclaurin series of $\sec x = S_0 + S_1 x^2/2! + S_2 x^4/4! + \cdots$,

$$Z_{\mathfrak{A}}\left(\sum_{i=1}^{2n+1} 1 \right) = S_n = 1, 1, 5, 61, 1385, 50521, \ldots \qquad (n \geq 0, \text{ OEIS series A000364}).$$

This is due to the fact that all alternating atomic permutations of $[2n]$ can be understood as the reverse alternating permutations of $[2 . . 2n - 1]$ with a prepended 1 and an appended $2n$. Moreover, since any x_1^{2n+1} can only be produced through an application of $\mathcal{D}$ on $x_2 x_1^{2(n-1)}$, we also have $Z_\mathfrak{A}\big(2 + \sum_{i=1}^{2(n-1)} 1\big) = S_n$.

3. Circular permutations

The methods developed in the last section to enumerate atomic permutations can also be applied to find the number of circular permutations $Z_\mathfrak{C}(p)$ with a given run structure p. Indeed, any circular permutation in $\mathfrak{C}_{n-1}$ can be extended to a permutation in $\mathfrak{C}_n$ by inserting n at any position after the first (*e.g.*, $\dot{1}453\dot{2}$ can be extended to $\dot{1}6453\dot{2}$, $\dot{1}46 53\dot{2}$, $\dot{1}4563\dot{2}$, $\dot{1}4536\dot{2}$ or $\dot{1}4532\dot{6}$). As in the case of atomic permutations, this extension either increases the length of a run or splits a run into three runs. Namely, we can increase the length of one run by inserting n at the end or the penultimate position of an increasing run or we can split a run of length $i + j \geq 2$ into three runs of lengths i, j and 1 by inserting n after i elements of an increasing run or after $i + 1$ elements of a decreasing run.

We introduce polynomials C_n representing the run structures of all elements of $\mathfrak{C}_n$, by analogy with the polynomials A_n in the previous section:

$$C_n = \sum_{p \vdash n} Z_\mathfrak{C}(p) \prod_{i=1}^{n} x_i^{p(i)} \tag{3.1}$$

and we say that the polynomials C_n enumerate the run structure of the circular permutations. In the last paragraph we saw that we can use the differential operator $\mathcal{D}$ introduced in (2.1) to find a recurrence relation similar to (2.3). Namely,

$$C_2 \doteq x_1^2, \tag{3.2a}$$
$$C_n \doteq \mathcal{D}C_{n-1}, \qquad (n \geq 3) \tag{3.2b}$$

giving in particular

$$C_3 = 2x_2 x_1$$
$$C_4 = 2x_2^2 + 2x_3 x_1 + 2x_1^4$$
$$C_5 = 2x_4 x_1 + 6x_3 x_2 + 16x_1^3 x_2$$
$$C_6 = 2x_5 x_1 + 8x_4 x_2 + 6x_3^2 + 62x_1^2 x_2^2 + 26x_1^3 x_3 + 16x_1^6$$

from which we can read off that there are 2 permutations in $\mathfrak{C}_5$ corresponding to $5 = 4+1$, 6 corresponding to the partition $5 = 3 + 2$ and 16 corresponding to $5 = 2 + 1 + 1$. As a check, we note that $6 + 16 + 2 = 24$, which is the total number of elements of $\mathfrak{C}_5$; similarly, the coefficients in the expression for C_6 sum to 120, the cardinality of $\mathfrak{C}_6$. More on this can be found in the last paragraph of section 5.

In summary, we have a result analogous to proposition 3:

Proposition 5. *The number $Z_\mathfrak{C}(p)$ of circular permutations of length n corresponding to a given run structure p is determined by the polynomial C_n via (3.1). The polynomials C_n satisfy the recurrence relation (3.2).*

Note that circular permutations, exactly opposite to atomic permutations, always contain an even number of runs and thus $Z_\mathfrak{C}(p)$ is zero for odd partitions p.

The enumeration of circular and atomic permutations is closely related. In fact, introducing a generating function $\mathcal{C}$ as the formal series

$$\mathcal{C}(\lambda) \doteq \sum_{n=0}^{\infty} C_{n+2} \frac{\lambda^n}{n!} = \sum_{n=0}^{\infty} \mathcal{D}^n C_2 \frac{\lambda^n}{n!} = \exp(\lambda \mathcal{D}) C_2,$$

one can show the following:

Proposition 6. *The formal power series $\mathcal{C}$ is the square of a formal series $\mathcal{A}$; namely,*

$$\mathcal{C}(\lambda) = \mathcal{A}(\lambda)^2 = \big(\exp(\lambda\mathcal{D})A_1 \big)^2, \tag{3.3}$$

where $A_1 \doteq x_1$.

Proof. This may be seen in various ways, but the most convenient is to study the first-order partial differential equation (in infinitely many variables)

$$\frac{\partial \mathcal{C}}{\partial \lambda} - \mathcal{D}\mathcal{C} = 0, \quad \mathcal{C}(0) = C_2 \tag{3.4}$$

satisfied by $\mathcal{C}$.

We can now apply the method of characteristics to this problem. Since it has no inhomogeneous part, the p.d.e. (3.4) asserts that $\mathcal{C}$ is constant along its characteristics. So, given λ and $x_1, x_2, \ldots$, let $\chi_1(\mu), \chi_2(\mu), \ldots$ be solutions to the characteristic equations with $\chi_r(\lambda) = x_r$, *i.e.*, $\chi_1(\mu), \chi_2(\mu), \ldots$ are the characteristic curves which emanate from the point $(\lambda, x_1, x_2, \ldots)$. Then,

$$\mathcal{C}(\lambda)|_{x_\bullet} = \mathcal{C}(0)|_{\chi_\bullet(0)} = C_2\big(\chi_1(0)\big) = \chi_1(0)^2.$$

Applying the same reasoning again to $\mathcal{A}$, which obeys the same p.d.e. as $\mathcal{C}$ but with initial condition $\mathcal{A}(0) = A_1$,

$$\mathcal{A}(\lambda)|_{x_\bullet} = \mathcal{A}(0)|_{\chi_\bullet(0)} = A_1\big(\chi_1(0)\big) = \chi_1(0).$$

Therefore, proposition 6 follows by patching these two equations together. $\qquad\square$

As a consequence also the polynomials A_n and C_n are related via

$$C_n = \sum_{m=1}^{n-1} \binom{n-2}{m-1} A_m A_{n-m}. \tag{3.5}$$

It then follows that the second degree part of C_n is given by

$$C_n^{(2)} = \sum_{m=1}^{n-1} \binom{n-2}{m-1} x_m x_{n-m}$$

and, applying (2.5), that the fourth degree part can be written as

$$C_n^{(4)} = \sum_{\substack{i,j,k,l \geq 1 \\ i+j+k+l=m}} \sum_{q=1}^{k} 2\,\frac{n-l-q-1}{n-l-q-j} \binom{n-2}{n-l-1}\binom{n-l-q-2}{i-1,j-1,k-q} x_i x_j x_k x_l.$$

Similar to the atomic permutations, we find that the alternating circular permutations satisfy (*cf.* [2, §41])

$$Z_{\mathfrak{C}}\bigg(\sum_{i=1}^{2n} 1 \bigg) = T_n = 1, 2, 16, 272, 7936, 353792, \ldots \quad (n \geq 1, \text{ OEIS series A000182})$$

and also $Z_{\mathfrak{C}}\big(2 + \sum_{i=1}^{2n-3} 1\big) = T_n$, where T_n are the *tangent numbers*, the coefficients of the Maclaurin series of $\tan x = T_1 x_1 + T_2 x_3/3! + T_3 x_5/5! + \cdots$. Furthermore, from (3.5) we find the relation

$$T_{n+1} = \sum_{m=0}^{n} \binom{2n}{2m} S_m S_{n-m},$$

which can be traced back to $\tan' x = \sec^2 x$.

To conclude this section, we note that the argument of proposition 6 proves rather more: namely, that $\exp(\lambda\mathcal{D})$ defines a ring homomorphism from $\mathbb{C}[x_1, x_2, \dots]$ to the ring of formal power series $\mathbb{C}[[x_1, x_2, \dots]]$. This observation can be used to accelerate computations: for example, the fact that $A_3 = x_3 + x_1^3$ implies that

$$\mathcal{A}''(\lambda) = \mathcal{A}(\lambda)^3 + \exp(\lambda\mathcal{D})x_3,$$

which reduces computation of $A_{n+3} = \mathcal{D}^{n+2}x_1$ to the computation of $\mathcal{D}^n x_3$. Once $\mathcal{A}$ is obtained, we may of course determine $\mathcal{C}$ by squaring.

4. Linear permutations

In the last section we studied the run structures of circular permutations $\mathfrak{C}_n$ and discovered that their run structures can be enumerated by the polynomials A_n. One might ask, what the underlying reason for this is. Circular permutations of $[n]$ have the same run structure as the linear permutations of the multiset $\{1, 1, 2, \dots, n\}$ which begin and end with 1. These permutations can then be split into two atomic permutations at the occurrence of their maximal element. For example, the circular permutation $\dot{1}453\dot{2}$ can be split into the two atomic permutations 145 of $\{1, 4, 5\}$ and 5321 of $\{1, 2, 3, 5\}$. This also gives us the basis of a combinatorial argument for the fact that $\mathcal{C} = \mathcal{A}^2$. Similarly it is in principle possible to encode the run structures of any subset of permutations using the polynomials A_n. The goal of this section is to show how this may be accomplished for $\mathfrak{S}_S$ for any $S \subset \mathbb{N}$.

As in sections 2 and 3, we want to find polynomials

$$L_n = \sum_{p \vdash n} Z_{\mathfrak{S}}(p) \prod_{i=1}^{n} x_i^{p(i)}$$

that enumerate the run structure of the permutations $\mathfrak{S}_{n+1}$. This may be achieved in a two step procedure. Since every permutation has a unique decomposition into inextendible atomic permutations, we can enumerate the set of permutations according to this decomposition. The enumeration of permutations by their run structure follows because the enumeration of atomic permutations has already been achieved in section 2.

The key to our procedure is to understand the factorisation of the run structure into those of atomic permutations. Considering $\sigma \in \mathfrak{S}_n$ as a word, we can write it as the concatenation $\sigma = \alpha \cdot \pi \cdot \omega$, where π is the principal atom of σ (see beginning of section 2) and α, ω are (possibly empty) subwords of σ. Since the decomposition of σ into its atoms also decomposes its run structure, the complete runs of σ are determined by the runs of $\alpha \cdot 1$, π and $n \cdot \omega$ if π is rising, or of $\alpha \cdot n$, π and $1 \cdot \omega$ if π is falling.

Let S_ω be the set of letters in ω and define $\rho : S_\omega \to S_\omega$ to be the involution mapping the i'th smallest element of S_ω to the i'th largest, for all $1 \leq i \leq |S_\omega|$. Then the run structure of $n \cdot \omega$ is identical to that of $1 \cdot \rho(\omega)$, where $\rho(\omega)$ is obtained by applying ρ letterwise to ω. Furthermore, in the case $\pi = 1 \cdots n$, the combined run structures of $\alpha \cdot 1$ and $n \cdot \omega$ are precisely the run structure of $\alpha \cdot 1 \cdot \rho(\omega)$, while, if $\pi = n \cdots 1$, the combined run structures of $\alpha \cdot n$ and $1 \cdot \omega$ precisely form the run structure of $\alpha \cdot n \cdot \rho(\omega)$. We refer to $\alpha \cdot 1 \cdot \rho(\omega)$ or $\alpha \cdot n \cdot \rho(\omega)$ as the *residual permutation.*

Summarising, the run structure of σ may be partitioned into that of π and either $\alpha \cdot 1 \cdot \rho(\omega)$ or $\alpha \cdot n \cdot \rho(\omega)$; accordingly, the monomial for σ factorises into that for the principal atom π and that for the residual permutation. Therefore, the polynomial enumerating linear permutations by run structure can be given in terms of the those enumerating atomic permutations of the same or shorter length and of linear permutations of strictly shorter length.

This argument can be used to give a recursion relation for L_n, which enumerates permutations of $[n + 1]$ by their run structure. Taking into account that the principal

atom consists of $m + 1$ letters, where $1 \leq m \leq n$, of which $m - 1$ may be chosen freely from the set $[2 \ldotp\ldotp n]$, and that it might be rising or falling, and that the residual permutation may be any linear permutation on a set of cardinality $n - m + 1$, we obtain the recursion relation

$$L_n = 2 \sum_{m=1}^{n} \binom{n-1}{m-1} A_m L_{n-m}, \qquad L_0 = 1.$$

Passing to the generating function,

$$\mathcal{L}(\lambda) \doteq \sum_{n=0}^{\infty} L_n \frac{\lambda^n}{n!},$$

we may deduce that

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 2 \mathcal{A}(\lambda) \mathcal{L}(\lambda). \tag{4.1}$$

Our main result in this section is:

Proposition 7. *The run structure of all permutations in $\mathfrak{S}_{n+1}$ is enumerated by*

$$L_n = \sum_{p \vdash n} \frac{2^{|p|}}{\operatorname{ord} p} \binom{n}{p} \prod_{i=1}^{|p|} A_{p_i}, \quad L_0 = 1, \tag{4.2}$$

where the sum is over all partitions $p = p_1 + p_2 + \cdots$ of n, $|p|$ is the number of parts of partition, $\operatorname{ord} p$ is the symmetry order of the parts of p (e.g., for $p = 1 + 1 + 2 + 3 + 3$ we have $\operatorname{ord} p = 2!2!$) and $\binom{n}{p}$ is the multinomial with respect to the parts of p. The generating function for the L_n is

$$\mathcal{L}(\lambda) \doteq \sum_{n=0}^{\infty} L_n \frac{\lambda^n}{n!} = \exp\left(2 \int_0^\lambda \mathcal{A}(\mu) \, \mathrm{d}\mu \right). \tag{4.3}$$

Proof. Equation (4.3) follows immediately from (4.1), as $\mathcal{L}(0) = 1$, whereupon Faà di Bruno's formula [10, Eq. (1.4.13)] yields (4.2). □

To conclude this section, we remark that the first few L_n are given by

$L_1 = 2A_1$

$L_2 = 4A_1^2 + 2A_2$

$L_3 = 8A_1^3 + 12A_1 A_2 + 2A_3$

$L_4 = 16A_1^4 + 48A_1^2 A_2 + 12A_2^2 + 16A_1 A_3 + 2A_4$

$L_5 = 32A_1^5 + 160A_1^3 A_2 + 120A_1 A_2^2 + 80A_1^2 A_3 + 40A_2 A_3 + 20A_1 A_4 + 2A_5$

$L_6 = 64A_1^6 + 480A_1^4 A_2 + 320A_1^3 A_3 + 720A_1^2 A_2^2 + 120A_1^2 A_4 + 480A_1 A_2 A_3 + 120A_2^3$
$\qquad + 24A_1 A_5 + 60A_2 A_4 + 40A_3^2 + 2A_6.$

Expanding the A_k and writing the L_n instead in terms of x_i, we obtain from these

$L_1 = 2x_1$

$L_2 = 4x_1^2 + 2x_2$

$L_3 = 10x_1^3 + 12x_1 x_2 + 2x_3$

$L_4 = 32x_1^4 + 58x_1^2 x_2 + 12x_2^2 + 16x_1 x_3 + 2x_4$

$L_5 = 122x_1^5 + 300x_1^3 x_2 + 142x_1 x_2^2 + 94x_1^2 x_3 + 40x_2 x_3 + 20x_1 x_4 + 2x_5$

$L_6 = 544x_1^6 + 1682x_1^4 x_2 + 568x_1^3 x_3 + 1284x_1^2 x_2^2 + 138x_1^2 x_4 + 556x_1 x_2 x_3 + 142x_2^3$
$\qquad + 24x_1 x_5 + 60x_2 x_4 + 40x_3^2 + 2x_6,$

which show no obvious structure, thereby making proposition 7 that much more remarkable.

5. Counting valleys

Instead of enumerating permutations by their run structure, we can count the number of valleys of a given (circular) permutation. Taken together, the terms C_n involving a product of $2k$ of the x_i relate precisely to the circular permutations $\mathfrak{C}_n$ with k valleys. Since any circular permutation in $\mathfrak{C}_n$ can be understood as a permutation of $[3 \mathbin{.\,.} n+1]$ with a prepended 1 and an appended 2 (*cf.* beginning of section 4), C_n may also be used to enumerate the valleys of ordinary permutations of $[n-1]$. Namely, terms of C_{n+1} with a product of $2(k+1)$ variables x_i relate to the permutations of $\mathfrak{S}_n$ with k valleys (*i.e.*, terms of L_{n+1} which are a product of $2k$ of the x_i).

Let $V(n,k)$ count the number of permutations of n elements with k valleys. Then we see that the generating function for $V(n,k)$ for each fixed $n \geq 1$ is

$$K_n(\kappa) \doteq \sum_{k=1}^{n} \kappa^k V(n,k) = \frac{1}{\kappa} C_{n+1}(\sqrt{\kappa}, \dots, \sqrt{\kappa})$$

and we define $K_0(\kappa) \doteq 1$. The first few K_n are

$$K_1(\kappa) = 1$$
$$K_2(\kappa) = 2$$
$$K_3(\kappa) = 4 + 2\kappa$$
$$K_4(\kappa) = 8 + 16\kappa$$
$$K_5(\kappa) = 16 + 88\kappa + 16\kappa^2$$
$$K_6(\kappa) = 32 + 416\kappa + 272\kappa^2,$$

which coincide with the results in [11]. In particular, the constants are clearly the powers of 2, the coefficients of κ give the sequence A000431 of the OEIS [9] and the coefficients of κ^2 are given by the sequence A000487. Likewise, the coefficients of κ^3 may be checked against the sequence A000517. In fact, the same polynomials appear in André's work, in which he obtained a generating function closely related to (5.1) below; see [2, §158] (his final formula contains a number of sign errors, and is given in a form in which all quantities are real for κ near 0; there is also an offset, because his polynomial $A_n(\kappa)$ is our $K_{n-1}(\kappa)$).

Proposition 8. *The bivariate generating function, i.e., the generating function for arbitrary n, is*

$$\mathcal{K}(\nu, \kappa) = \sum_{n=0}^{\infty} K_n(\kappa) \frac{\nu^n}{n!} = 1 + \frac{1}{\kappa} \int_0^{\nu} \mathcal{C}(\mu)|_{x_1 = x_2 = \dots = \sqrt{\kappa}} \, \mathrm{d}\mu$$

and is given in closed form by

$$\mathcal{K}(\nu, \kappa) = 1 - \frac{1}{\kappa} + \frac{\sqrt{\kappa - 1}}{\kappa} \tan\left(\nu\sqrt{\kappa - 1} + \arctan(1/\sqrt{\kappa - 1})\right). \tag{5.1}$$

This result was found by Kitaev [6] and in the remainder of this section we will show how it may be derived from the recurrence relation (3.2) of C_n.

To this end, we first note that C_{n+1} satisfies the useful scaling relation

$$\lambda^{n+1} C_{n+1}(x_1, x_2, \dots, x_n) = C_{n+1}(\lambda x_1, \lambda^2 x_2, \dots, \lambda^n x_n).$$

Setting $x_i = x/\lambda = \sqrt{\kappa}$ for all i, this implies

$$\lambda^{n+1} C_{n+1}(\sqrt{\kappa}, \dots, \sqrt{\kappa}) = C_{n+1}(x, \lambda x, \dots, \lambda^{n-1} x)$$

and we find, by inserting the recurrence relations (3.2) and applying the chain rule, that with this choice of variables

$$\frac{1}{x^2}C_{n+1}(x, \lambda x, \ldots, \lambda^{n-1}x) = \lambda x \frac{\partial C_n}{\partial x} + x^2 \frac{\partial C_n}{\partial \lambda} + 2\lambda C_n.$$

Hence, in turn, $K_n(\kappa) = \kappa^{-1}C_{n+1}(\sqrt{\kappa}, \ldots, \sqrt{\kappa})$ satisfies the recurrence relation

$$K_n(\kappa) = 2\kappa(1-\kappa)K'_{n-1}(\kappa) + \big(2 + (n-2)\kappa\big)K_{n-1}(\kappa) \qquad (5.2)$$

for $n \geq 2$. For the bivariate generating function $\mathcal{K}$ this, together with $K_0 = K_1 = 1$, implies the p.d.e.

$$(1 - \nu\kappa)\frac{\partial \mathcal{K}}{\partial \nu} + 2\kappa(\kappa - 1)\frac{\partial \mathcal{K}}{\partial \kappa} + (\kappa - 2)\mathcal{K} = \kappa - 1,$$

which is to be solved subject to the initial condition $\mathcal{K}(0, \kappa) = 1$.

The above equation may be solved as follows: first, we note that there is a particular integral $1 - 1/\kappa$, so it remains to solve the homogeneous equation. In turn, using an integrating factor, the latter may be rewritten as

$$(1 - \nu\kappa)\frac{\partial}{\partial \nu}\frac{\kappa\mathcal{K}}{\sqrt{\kappa - 1}} + 2\kappa(\kappa - 1)\frac{\partial}{\partial \kappa}\frac{\kappa\mathcal{K}}{\sqrt{\kappa - 1}} = 0, \qquad (5.3)$$

for which the characteristics obey

$$\frac{\mathrm{d}\nu}{\mathrm{d}\kappa} = \frac{1 - \nu\kappa}{2\kappa(\kappa - 1)}.$$

Solving this equation, we find that

$$\nu\sqrt{\kappa - 1} + \arctan\frac{1}{\sqrt{\kappa - 1}} = \text{const}$$

along characteristics; as (5.3) asserts that $\kappa\mathcal{K}/\sqrt{\kappa - 1}$ is constant on characteristics, this gives

$$\mathcal{K}(\nu, \kappa) = 1 - \frac{1}{\kappa} + \frac{\sqrt{\kappa - 1}}{\kappa}f\big(\nu\sqrt{\kappa - 1} + \arctan(1/\sqrt{\kappa - 1})\big)$$

for some function f. Imposing the condition $\mathcal{K}(0, \kappa) = 1$, it is plain that $f = \tan$, and we recover Kitaev's generating function (5.1).

To close this section, we note that (5.2) has the consequence that $K_n(1) = nK_{n-1}(1)$ for all $n \geq 2$ and hence that $C_{n+1}(1, \ldots, 1) = K_n(1) = n!$ for such n, and indeed all $n \geq 1$, because $C_2(1, 1) = K_1(1) = 1$. The generating function obeys

$$\mathcal{C}(\lambda)|_{x_\bullet=1} = \sum_{n=0}^{\infty}(n+1)!\frac{\lambda^n}{n!} = (1 - \lambda)^{-2}$$

for all non-negative $\lambda < 1$ from which it also follows that

$$\mathcal{A}(\lambda)|_{x_\bullet=1} = (1 - \lambda)^{-1} \qquad (5.4)$$

(as $A_1(1) = 1$, we must take the *positive* square root) and hence $A_n|_{x_\bullet=1} = (n-1)!$ for all $n \geq 1$. This gives a consistency check on our results: the coefficients in the expression for A_n sum to $(n-1)!$, the cardinality of $\mathfrak{A}_{n+1}^{\pm}$, while those in C_n sum to the cardinality of $\mathfrak{C}_n$. Furthermore, inserting (5.4) into the generating function $\mathcal{L}(\lambda)$ in (4.3), we find

$$\mathcal{L}(\lambda)|_{x_\bullet=1} = \sum_{n=0}^{\infty} L_n(1, \ldots, 1)\frac{\lambda^n}{n!} = (1 - \lambda)^{-2},$$

and thus $L_{n+1}(1, \ldots, 1) = n!$, which is the cardinality of $\mathfrak{S}_n$.

6. Other applications

The original motivation for this work arose in quantum field theory, in computations related to the probability distribution of measurement outcomes for quantities such as averaged energy densities [5]. One actually computes the cumulants κ_n ($n \in \mathbb{N}$) of the distribution: $\kappa_1 = 0$, while for each $n \geq 2$, κ_n is given as a sum indexed by circular permutations σ of $[n]$ such that $\sigma(1) = 1$ and $\sigma(2) < \sigma(n)$, in which each permutation contributes a term that is a multiplicative function of its run structure:

$$\kappa_n = \sum_{\sigma} \Phi(\sigma)$$

where $\Phi(\sigma)$ is a product over the runs of σ, with each run of length r contributing a factor y_r. Owing to the restriction $\sigma(2) < \sigma(n)$, precisely half of the circular permutations are admitted, and so $\kappa_n = \frac{1}{2} C_n(y_1, y_2, \ldots, y_n)$. Thus the cumulant generating function is

$$W(\lambda) \doteq \sum_{n=2}^{\infty} \kappa_n \frac{\lambda^n}{n!} = \frac{1}{2} \int_0^{\lambda} d\mu \, (\lambda - \mu) \mathcal{C}(\mu)|_{x_\bullet = y_\bullet}$$

$$= \frac{1}{2} \int_0^{\lambda} d\mu \, (\lambda - \mu) \mathcal{A}(\mu)|_{x_\bullet = y_\bullet}^2$$

and the moment generating function is $\exp W(\lambda)$ in the usual way. This expression makes sense a formal power series, but also as a convergent series within an appropriate radius of convergence. The values of y_n depend on the physical quantity involved and the way it is averaged. In one case of interest

$$y_n = 8^n \int_{(\mathbb{R}^+)^{\times n}} dk_1 \, dk_2 \, \cdots \, dk_n \, k_1 k_2 \, \cdots \, k_n \exp\left[-k_1 - \left(\sum_{i=1}^{n-1} |k_{i+1} - k_i|\right) - k_n\right]$$

$$= 2^n \sum_{r_{n-1}=0}^{2} \sum_{r_{n-2}=0}^{2+r_{n-1}} \cdots \sum_{r_1=0}^{2+r_2} \prod_{k=1}^{n-1} (1 + r_k)$$

$$= 2, 24, 568, 20256, 966592, \ldots \qquad (n \geq 1)$$

(the sums of products must be interpreted as an overall factor of unity in the case $n = 1$). Numerical investigation leads to a remarkable identity

$$\mathcal{A}(\lambda)|_{x_\bullet = y_\bullet} = \frac{2}{1 - 12\lambda} \qquad \text{(conjectured)}$$

with exact agreement for all terms so far computed (checked up to $n = 65$). We do not have a proof for this statement, but the conjecture seems fairly secure. For example, we have shown above that $A_5 = x_5 + 7x_3 x_1^2 + 11x_2^2 x_1 + 5x_1^5$; substituting for x_n the values of y_n obtained above, we find $A_5 = 995328$ which coincides with the fourth order coefficient in the expansion

$$\frac{2}{1 - 12\lambda} = 2 + 24\lambda + 576 \frac{\lambda^2}{2!} + 20736 \frac{\lambda^3}{3!} + 995328 \frac{\lambda^4}{4!} + O(\lambda^5).$$

In [5], this conjecture was used to deduce

$$\exp\left(W(\lambda)\right) = e^{-\lambda/6}(1 - 12\lambda)^{-1/72} \qquad \text{(conjectured)},$$

which is the moment generating function of a shifted Gamma distribution. The other generating functions of interest, with these values for the x_k are

$$\mathcal{C}(\lambda)|_{x_\bullet = y_\bullet} = \frac{4}{(1 - 12\lambda)^2}, \qquad \mathcal{L}(\lambda)|_{x_\bullet = y_\bullet} = (1 - 12\lambda)^{-1/3} \qquad \text{(conjectured)}.$$

For example, we have $C_5 = 2x_4x_1 + 6x_3x_2 + 16x_1^3x_2 = 165888$ and $L_5 = 122x_1^5 + 300x_1^3x_2 + 142x_1x_2^2 + 94x_1^2x_3 + 40x_2x_3 + 20x_1x_4 + 2x_5 = 3727360$, to be compared with the terms of order λ^3 and λ^5, respectively, in the expansions

$$\frac{4}{(1-12\lambda)^2} = 4 + 96\lambda + 3456\frac{\lambda^2}{2!} + 165888\frac{\lambda^3}{3!} + 995328\frac{\lambda^4}{4!} + O(\lambda^5),$$

$$(1-12\lambda)^{-1/3} = 1 + 4\lambda + 64\frac{\lambda^2}{2!} + 1792\frac{\lambda^3}{3!} + 71680\frac{\lambda^4}{4!} + 3727360\frac{\lambda^5}{5!} + O(\lambda^6).$$

A natural question is whether there are other sequences that can be substituted for the x_k to produce generating functions with simple closed forms. To close, we give three further examples, with the corresponding generating functions computed. The first has already been encountered in section 5 and corresponds to the case $x_k = 1$ for all $k \in \mathbb{N}$.

The second utilizes the alternating Catalan numbers: setting

$$x_{2k+1} = \frac{(-1)^k}{k+1}\binom{2k}{k}, \quad (k \geq 0), \qquad x_{2k} = 0, \quad (k \geq 1)$$

and thus $A_{2k} = 0$, we obtain, again experimentally,

$$\mathcal{C}(\lambda) = \mathcal{A}(\lambda) = 1, \quad \mathcal{L}(\lambda) = \mathrm{e}^{2\lambda} \qquad \text{(conjectured)}$$

with exact agreement checked up to permutations of length $n = 65$. For example, one sees easily that with $x_1 = 1$, $x_3 = -1$, $x_5 = 2$ and $x_2 = x_4 = 0$, the expressions A_k and C_k given in sections 2 and 3 vanish for $2 \leq k \leq 6$, and have $A_1 = C_1 = 1$, likewise, $L_k = 2^k$ for $1 \leq k \leq 6$.

Third, André's classical result on alternating permutations (*cf.* last and penultimate paragraph of section 2 and 3 respectively) gives the following: setting

$$x_1 = 1, \qquad x_k = 0, \quad (k \geq 2)$$

we have, using (3.3) and (4.3),

$$\mathcal{A}(\lambda) = \sec\lambda, \quad \mathcal{C}(\lambda) = \sec^2\lambda, \quad \mathcal{L}(\lambda) = (\sec\lambda + \tan\lambda)^2.$$

It seems highly likely to us that many other examples can be extracted from the structures we have described.

Moreover, we remark that it is possible to implement a merge-type sorting algorithm, called *natural merge sort* [7, Chap. 5.2.4], based upon splitting permutations of an ordered set S into its runs, which are ordered (alternatingly in ascending and descending order) sequences $S_i \subset S$. Repeatedly merging these subsequences, one ultimately obtains an ordered sequence. For example, first, we split the permutation 542368719 into 542, 368, 71 and 9. Then, we reverse every second sequence (depending on whether the first or the second sequence is in ascending order): $542 \mapsto 245$ and $71 \mapsto 17$. Depending on the implementation of the merging in the following step, this 'reversal' step can be avoided. Last, we merge similarly to the standard merge sort: $245 \vee 368 \mapsto 234568$, $17 \vee 9 \mapsto 179$ and finally $234568 \vee 179 \mapsto 123456789$. Natural merge sort is a fast sorting algorithm for data with preexisting order. Using the methods developed above to enumerate permutations by their run structure, it is in principle possible to give average (instead of best- and worst-case) complexity estimates for such an algorithm.

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