The numbers of support-tilting modules for a Dynkin algebra

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Abstract. The Dynkin algebras are the hereditary artin algebras of finite representation type. The paper exhibits the number of support-tilting modules for any Dynkin algebra. Since the support-tilting modules for a Dynkin algebra of Dynkin type Δ correspond bijectively to the non-crossing partitions of type Δ, the calculations presented here may also be considered as a categorification of results concerning non-crossing partitions. An appendix is included with a proof of the Ingalls-Thomas bijections for hereditary artin algebras in general: these bijections show that the number of support-tilting modules is the same as the number of antichains and the number of normal or of conormal modules without self-extensions.

1. Introduction. Let Λ be a hereditary artin algebra. We will consider here left Λ-modules of finite length and call them just modules. The category of all modules will be denoted by $\operatorname{mod}\Lambda$. We will denote by $n = n(\Lambda)$ the *rank* of Λ, this is by definition the number of simple modules (always, when counting numbers of modules of a certain kind, we actually mean the number of isomorphism classes). Following earlier considerations of Brenner and Butler, tilting modules have been defined in [HR1]. In the present setting, a *tilting module* is a module without self-extensions with precisely n isomorphism classes of indecomposable direct summands. The endomorphism ring of a tilting module is said to be a tilted algebra. There is a wealth of papers devoted to tilted algebras, and the Handbook of Tilting Theory [AHK] can be consulted for references.

The present paper deals with the Dynkin algebras, these are the connected hereditary artin algebras which are representation-finite, thus their valued quivers are of Dynkin type $\Delta_n = \mathbb{A}_n, \mathbb{B}_n, \dots, \mathbb{G}_2$, see [DR1]. Its aim is to discuss the number of tilting modules for such an algebra. The corresponding tilted algebras have been classified by various authors in the eighties. It seems to be clear that a first step of such a classification result was the determination of all tilting modules, however there are only few traces in the literature (also the Handbook [AHK] is of no help). Apparently, the relevance of the number of tilting modules was seen at that time only in special cases. The tilting modules for a linearly

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ordered quiver of type $\mathbb{A}_n$ were exhibited in [HR2] and Gabriel [G] pointed out that here we encounter one of the numerous appearances of the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$. For the cases $\mathbb{D}_n$, the number of tilting modules was determined by Bretscher-Läser-Riedtmann [BLR] in their study of self-injective representation-finite algebras.

Given a module M, we denote by $\Lambda(M)$ its *support algebra,* this is the factor algebra of Λ modulo the ideal which is generated by all idempotents e with $eM = 0$, this is again a hereditary artin algebra (but usually not connected, even if Λ is connected). The rank of the support algebra of M will be called the *support-rank* of M. A module T is said to be *support-tilting* provided M considered as a $\Lambda(M)$-module is a tilting module. It may be well-known that the number of tilting modules of a Dynkin algebra only depends on its Dynkin type, but we could not find this result in the literature, thus section 2 provides a direct proof. It follows that also the number of support-tilting modules of a Dynkin algebra with support-rank s only depends on the type Δ_n, we denote this number by $a_s(\Delta_n)$. Of course, $a_n(\Delta_n)$ is just the number of tilting modules, and we denote by $a(\Delta_n)$ the number of all support-tilting modules, thus $a(\Delta_n) = \sum_{s=0}^{n} a_s(\Delta_n)$.

The present paper presents the numbers $a(\Delta_n)$ and $a_s(\Delta_n)$ for $0 \le s \le n$ in a unified way. Of course, the exceptional cases $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2$ can be treated with a computer (but actually, also by hand), thus our main interest lies in the series $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$. In the case $\mathbb{A}$, we obtain in this way the **Catalan triangle*** in the case $\mathbb{B}$ and $\mathbb{C}$ the increasing part of the **Pascal triangle,** and finally in the case $\mathbb{D}$ an **expansion** of the increasing part of the **Lucas triangle** (see section 6, an outline will be given later in the introduction).

Results. All the numbers which are presented here for the cases $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$ are related to the binomial coefficients $\binom{s}{t}$ and they coincide for $\mathbb{B}_n$ and $\mathbb{C}_n$ (as we will show in section 2), thus it is sufficient to deal with the cases $\mathbb{A}, \mathbb{B}, \mathbb{D}$. For $\mathbb{B}$, the binomial coefficients themselves will play a dominant role, for the cases $\mathbb{A}$ and $\mathbb{D}$, suitable multiples are relevant. In case $\mathbb{A}$, these are the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, as well as related numbers. For the case $\mathbb{D}$, it will be convenient to use the notation $\left[{t \atop s}\right] = \frac{s+t}{t}\binom{t}{s}$ as proposed by Bailey [B], since the relevant numbers in case $\mathbb{D}$ can be written in this way.

The numbers $a(\Delta_n)$ and $a_s(\Delta_n)$ for $0 \le s \le n$:

Δ_n	$\mathbb{A}_n$	$\mathbb{B}_n, \mathbb{C}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\mathbb{F}_4$	$\mathbb{G}_2$
$a_n(\Delta_n)$	$\frac{1}{n+1}\binom{2n}{n}$	$\binom{2n-1}{n-1}$	$\left[{2n-2 \atop n-2}\right]$	418	2 431	17 342	66	5
$a_s(\Delta_n)$ $0 \le s < n$	$\frac{n-s+1}{n+1}\binom{n+s}{s}$	$\binom{n+s-1}{s}$	$\left[{n+s-2 \atop s}\right]$	………	………	see section 7	………	………
$a(\Delta_n)$	$\frac{1}{n+2}\binom{2n+2}{n+1}$	$\binom{2n}{n}$	$\left[{2n-1 \atop n-1}\right]$	833	4 160	25 080	105	8

* The reader should be aware that there are several triangles of numbers which are named Catalan triangles in the literature. Here we refer to the triangle of numbers $a_s(n) = \frac{n-s+1}{n+1}\binom{n+s}{s}$, see A009766 in [S].

Remark 1. In parallelity to the Bailey notation $\left[{t \atop s} \right]$ one may be tempted to introduce the following notation for the Catalan triangle: $\left] {t \atop s} \right[= \frac{t-2s+1}{t-s+1}\binom{t}{s}$. Then the numbers for the case $\mathbb{A}$ are written as follows:

$$a_n(\mathbb{A}_n) = \left] {2n \atop n} \right[,\ a_s(\mathbb{A}_n) = \left] {n+s \atop s} \right[,\ a(\mathbb{A}) = \left] {2n+2 \atop n+1} \right[.$$

Remark 2. The reader should observe that for $\mathbb{A}_n$ and $\mathbb{B}_n$, the formula given for $a_s(\Delta_n)$ and $0 \le s < n$ works also for $s = n$. This is not the case for $\mathbb{D}_n$: Whereas $\binom{2n-2}{n-2} = \binom{2n-2}{n}$, the numbers $\left[{2n-2 \atop n-2} \right]$ and $\left[{2n-2 \atop n} \right]$ are different (the difference will be highlighted at the end of section 6). The Lucas triangle consists of the numbers $\left[{t \atop s} \right]$ for all $0 \le s \le t$, it uses the numbers $\left[{2n-2 \atop n} \right]$ at the positions, where the $\mathbb{D}$-triangle (which we now will consider) uses the numbers $\left[{2n-2 \atop n-2} \right]$.

The triangles $\mathbb{A}, \mathbb{B}, \mathbb{D}$**.** The non-zero numbers $a_s(\Delta_n)$ with $\Delta = \mathbb{A}, \mathbb{B}, \mathbb{D}$ yield three triangles which have similar properties, we will exhibit them in 6.1, 6.2, 6.3. The triangle $\mathbb{A}$ is the Catalan triangle itself, this is A009766 in Sloane's OEIS [S]. The triangle $\mathbb{B}$ is the triangle A059481, it corresponds to the increasing part of the Pascal triangle (thus it consists of the binomial coefficients $\binom{t}{s}$ with $2s \le t+1$). The triangle $\mathbb{D}$ is an expansion of the increasing part of the Lucas triangle A029635: taking the increasing parts of the rows in the Lucas triangle (thus the numbers $\left[{t \atop s} \right]$ with $2s \le t+1$), we obtain numbers which occur in the triangle $\mathbb{D}$, namely the numbers $a_s(\mathbb{D}_n)$ with $0 \le s < n$. But then the numbers $a_n(\mathbb{D}_n)$ on the diagonal are still missing — as we have seen above, these numbers are given by a similar, however deviating formula (they are listed as the sequence A129869). The Lucas triangle is A029635, but the triangle $\mathbb{D}$ itself was at the time of the writing of the paper not yet recorded in OEIS.

We see that the entries $a_s(n)$ of the triangles $\mathbb{A}$ and $\mathbb{B}$, as well as those of the lower triangular part of the triangle $\mathbb{D}$ can be obtained in a unified way from three triangles with entries $z_s(t)$ which satisfy the following recursion formula

$$z_s(t) = z_{s-1}(t-1) + z_s(t-1)$$

(they are exhibited in 6.1′, 6.2′, 6.3′) using the shearing $a_s(n) = z_s(n+s-1)$. The recursion formula can be rewritten as $z_s(t) = \sum_{i=0}^{s} z_i(t-s+i+1)$ (sometimes called the hockey stick formula). A consequence of the hockey stick formula is the fact that summing up the rows of any of the three triangles $\mathbb{A}, \mathbb{B}, \mathbb{D}$, we obtain again numbers which appear in the triangle.

Let us provide further details on the triangles to be sheared. Consider first the case $\mathbb{B}$. Here we start with the Pascal triangle, thus we deal with the triangle with numbers $z_s(t) = \binom{t}{s}$ and the initial conditions are $z_0(t) = z_t(t) = 1$ for all $t \ge 0$. In case $\mathbb{D}$, we start with the Lucas triangle with numbers $z_s(t) = \left[{t \atop s} \right]$, the initial conditions are $z_0(t) = 1,\ z_t(t) = 2$ for all $t \ge 1$ (these initial conditions are the reason for calling the Lucas triangle also the $(1,2)$-triangle). In the case $\mathbb{A}$ we start with a sheared Catalan triangle, here the initial conditions are $z_0(t) = 1$ and $z_{t+1}(2t) = 0$ for all $t \ge 0$.

As we have mentioned, the relevance of the numbers of tilting and support-tilting modules for the Dynkin algebras was not realized in the eighties. It became apparent through the work of Fomin and Zelevinksy when dealing with cluster algebras and the corresponding cluster complexes (see in particular [FZ] and [FR]): the support-tilting modules correspond bijectively to the cluster-tilting objects of a cluster category. There is an independent development, namely the theory of generalized non-crossing partitions (see for example [A]) which has to be mentioned. The basic setting for using the representation theory of hereditary artin algebras in order to deal with non-crossing partitions was pointed out by Ingalls and Thomas [IT], they have shown (at least for path algebras of quivers) that there are many counting problems for $\operatorname{mod}\Lambda$ which yield the same answer, namely the numbers $a(\Delta_n)$ and $a_s(\Delta_n)$. Since we will need this result also in the case $\Delta = \mathbb{B}$ and $\mathbb{C}$ (where Λ is not the path algebra of a quiver), we include an appendix with a proof of the Ingalls-Thomas bijections for hereditary artin algebras in general. Our proof draws attention to three additional counting problems with the same answer: to count the number of antichains in $\operatorname{mod}\Lambda$ as well as the number of normal or of conormal modules without self-extensions.

Here are the definitions: An *antichain* $A = \{A_1, \ldots, A_t\}$ in $\operatorname{mod}\Lambda$ is a set of pairwise orthogonal bricks (a brick is a module whose endomorphism ring is a division ring, two bricks A_1, A_2 are said to be orthogonal, provided $\operatorname{Hom}(A_1, A_2) = 0 = \operatorname{Hom}(A_2, A_1)$).

A module X is said to *generate* a module Y provided Y is a factor module of a direct sum of copies of X. Dually, a module X *cogenerates* a module Y provided Y is a submodule of a direct sum of copies of X (since the modules considered here are of finite length, it is sufficient to look at direct sums of copies of X, for general modules one would have to use products). A module M is defined to be *normal* provided given a direct decomposition $M = M' \oplus M''$ such that M' generates M'', we have $M'' = 0$. And M is *conormal* provided given a direct decomposition $M = M' \oplus M''$ such that M' cogenerates M'', we have $M'' = 0$.

Since the support-tilting modules for a Dynkin algebra of Dynkin type Δ correspond bijectively to the non-crossing partitions of type Δ, the calculations presented here may be considered as a categorification of results concerning non-crossing partitions (for a general outline see [HK]). For the corresponding discussion of the number of ad-nilpotent ideals of a Borel subalgebra of a simple Lie algebra see Panyushev [P].

The case of the Coxeter diagrams $\mathbb{H}_3$ and $\mathbb{H}_4$ can be treated in a similar way, using hereditary artinian rings which are not artin algebras, see [FR].

Outline of the paper: We repeat that there is an inductive procedure using the hook formula (and a modified hook formula), in order to obtain the numbers $a_s(\mathbb{A}_n)$ for $0 \le s \le n$, as well as the numbers $a_s(\Delta_n)$ for $\Delta = \mathbb{B}, \mathbb{D}$ for $0 \le s < n$, provided we know the numbers $a_n(\Delta_n)$, see section 4. As we have mentioned, for the numbers $a_n(\mathbb{D}_n)$ we may refer to [BLR]. In section 2, we will show that the numbers $a_n(\mathbb{B}_n)$ and $a_n(\mathbb{C}_n)$ coincide, thus it remains to determine the numbers $a_n(\mathbb{B}_n)$, this will be done in section 3. In section 5, we calculate $a(\Delta_n)$ for $\Delta = \mathbb{A}, \mathbb{B}, \mathbb{D}$. Section 6 presents the triangles $\mathbb{A}, \mathbb{B}, \mathbb{D}$ as well as the corresponding Catalan, Pascal, and Lucas triangles, and some observation concerning repetition of numbers in the triangles are recorded. The final section 7 provides the numbers $a_s(\Delta_n)$ for the exceptional cases $\Delta_n = \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2$.

2. The valued quiver of a hereditary artin algebra.

Let Λ be a hereditary artin algebra. Since by assumption $\operatorname{Ext}^i_\Lambda = 0$ for $i \ge 2$, we write $\operatorname{Ext}(M, M')$ instead of $\operatorname{Ext}^1_\Lambda(M, M')$. The quiver $Q(\Lambda)$ has as vertices the isomorphism classes $[S]$ of the simple Λ-modules and there is an arrow $[S] \to [S']$ provided $\operatorname{Ext}(S, S') \neq 0$. Note that $Q(\Lambda)$ is finite and directed (the latter means that the simple modules can be labeled $S(i)$ such that the existence of an arrow $[S(i)] \to [S(j)]$ implies that $i > j$). We endow $Q(\Lambda)$ with a valuation as follows: Given an arrow $[S] \to [S']$, consider $\operatorname{Ext}(S, S')$ as a left $\operatorname{End}(S)^{\mathrm{op}}$-module and also as a left $\operatorname{End}(S')$-module and put

$$v([S],[S']) = (\dim_{\operatorname{End}(S)} \operatorname{Ext}(S, S'))(\dim_{\operatorname{End}(S')^{\mathrm{op}}} \operatorname{Ext}(S, S'))$$

provided $v([S],[S']) > 1$. Given a vertex i of $Q(\Lambda)$, we denote by $S(i) = S_\Lambda(i), P(i) = P_\Lambda(i), I(i) = I_\Lambda(i)$ a simple, a projective or an injective module corresponding to the vertex i, respectively.

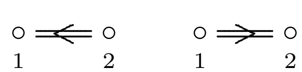
If M is a module, the set of vertices of the quiver $Q(\Lambda(M))$ will be called the *support* of M and M is said to be *sincere* provided any vertex of $Q(\Lambda)$ belongs to the support of M (thus provided the only idempotent $e \in \Lambda$ with $eM = 0$ is $e = 0$).

We also will be interested in the corresponding valued graph $\overline{Q}(\Lambda)$ which is obtained from the valued quiver $Q(\Lambda)$ by replacing the arrows by edges, one says that one *forgets the orientation* of the quiver.

In the special case where $v([S],[S']) = v$ with $v = 2$ or $v = 3$, it is usual to replace the arrow $[S] \longrightarrow [S']$ by a double arrow $[S] \Longrightarrow [S']$ (if $v = 2$) or a similar triple arrow (if $v = 3$). Using the bimodule $\operatorname{Ext}(S, S')$ one obtains an embedding either of $\operatorname{End} S$ into $\operatorname{End} S'$, or of $\operatorname{End} S'$ into $\operatorname{End} S$, thus one of the division rings is a subring of the other, with index equal to v. One marks the relative size of the endomorphism rings by an additional arrowhead drawn in the middle of the edge, pointing from the larger endomorphism ring to the smaller one (it should be stressed that these inner arrowheads must not be confused with the outer ones). For example, in case there are two simple modules labeled 1 and 2 with an arrow $1 \leftarrow 2$ and $v(1, 2) = 2$, there are the following two possibilities:

On the left, we see that $\operatorname{End} S(1)$ is a division subring of $\operatorname{End} S(2)$, on the right, $\operatorname{End} S(2)$ is a division subring of $\operatorname{End} S(1)$. (Let us exhibit corresponding algebras: Let $K : k$ be a field extension of degree 2 and consider the algebras $\Lambda = \begin{bmatrix} k & K \\ 0 & K \end{bmatrix}$, and $\Lambda' = \begin{bmatrix} K & K \\ 0 & k \end{bmatrix}$, the left quiver shown above is $Q(\Lambda)$, the right quiver is $Q(\Lambda')$.)

Here are for both valued quivers the valued graphs which are obtained by forgetting the orientation (thus deleting the outer arrowheads, but not the inner ones)



they are called $\mathbb{B}_2$ and $\mathbb{C}_2$, respectively (observe that there is a difference between $\mathbb{B}_2$ and $\mathbb{C}_2$ only if they occur as subgraphs of larger graphs).

We recall the following [DR1]: *A connected hereditary artin algebra Λ is representation-finite if and only if $\overline{Q}(\Lambda)$ is one of the Dynkin diagrams $\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2$ and in this case the indecomposable Λ-modules correspond bijectively to the positive roots.*

We want to show that the number of basic tilting modules is independent of the orientation We recall that a module is said to be *basic* provided it is a direct sum of pairwise non-isomorphic indecomposable modules; an artin algebra Λ is *basic* provided the regular representation ${}_\Lambda\Lambda$ is basic. In the case of the tensor algebra of a species (in particular in the case of the path algebra of a quiver), any change of orientation is obtained by applying a sequence of BGP-reflection functors, see [DR2]. For a general hereditary artin algebra Λ, we have to deal with APR-tilting functors as defined by Auslander, Platzeck and Reiten [APR]. In order to do so, we may assume that Λ is basic. We start with a simple projective module S, write ${}_\Lambda\Lambda = S \oplus P$ with a projective module P and consider $W = P \oplus \tau^- S$ (where $\tau = \tau_\Lambda$ is the Auslander-Reiten translation in $\operatorname{mod}\Lambda$) and $\Lambda' = (\operatorname{End} W)^{\mathrm{op}}$. Note that W is a tilting module (called an APR-tilting module) and the quiver $Q(\Lambda')$ is obtained from the quiver $Q(\Lambda)$ by changing the orientation of all the arrows which involve the vertex $\omega = [S]$. Let $\Lambda'' = (\operatorname{End} P)^{\mathrm{op}}$, this is the restriction of Λ to the quiver Q'' obtained from $Q(\Lambda)$ by deleting the vertex ω and the arrows ending in ω. Of course, Q'' is also a subquiver of $Q(\Lambda')$ and Λ'' is the restriction of Λ' to Q'' (thus Λ is a one-point coextension of Λ'', whereas Λ' is a one-point extension of Λ''). We denote by S' the simple Λ'-module with support ω.

Proposition 1. *Let Λ be a hereditary artin algebra and S a simple projective module. Let W be the APR-tilting module defined by S and $\Lambda' = (\operatorname{End} W)^{\mathrm{op}}$. Then there is a canonical bijection η between the basic tilting Λ-modules and the basic tilting Λ'-modules.*

Proof: In order to define η, we distinguish two cases.

First, if T is a basic tilting module such that S is not a direct summand of T, let $\eta(T) = \operatorname{Hom}(W, T)$, this is a basic tilting Λ'-module and S' is not a direct summand of $\eta(T)$.

Second, consider a basic tilting Λ'-module such that S is a direct summand, say let $S \oplus T$ be a basic tilting Λ'-module. Let $T'' = T/U$, where U is the sum of the images of all the maps $S \to T$. Obviously, T'' is a basic tilting Λ''-module which we may consider as a Λ'-module. We form the universal extension T' of T'' using copies of S'. Then $T' \oplus S'$ is a basic tilting Λ'-module (and S' is a direct summand). $\square$

Remark. We may identify the Grothendieck groups $K_0(\Lambda)$ and $K_0(\Lambda')$, using the common factor algebra Λ'' and identifying the dimension vectors of S and S'. Then, in the first case, the dimension vector of $\eta(T)$ is obtained from the dimension vector of T by applying the reflection σ defined by S. In the second case, the dimension vectors of T and $\eta(T)$ coincide. Actually, here we use twice the internal reflection defined by S in [R2], first in the category $\operatorname{mod}\Lambda$, second in the category $\operatorname{mod}\Lambda'$.

Let Λ be a Dynkin algebra and assume that the vertices of both $Q(\Lambda)$ are labeled $1 \le i \le n$. Let $P(i) = P_\Lambda(i)$ be indecomposable projective. Since we assume that

Λ is a Dynkin algebra, there is a natural number $q(i) = q(P(i))$ such that $\tau^{-q(i)}P(i)$ is indecomposable injective; the modules $M(i,u) = \tau^{-u}P(i)$ with $0 \le u \le q(i)$ and $1 \le i \le n$ furnish a complete list of the indecomposable Λ-modules.

Proposition 2. *Let Λ, Λ' be Dynkin algebras and assume that the simple modules of both algebras are indexed by $1 \le i \le n$. Assume that $q(P_\Lambda(i)) = q(P_{\Lambda'}(i)) = q(i)$ for all $1 \le i \le n$. If the support of $M(u,i) = \tau_\Lambda^{-u}P_\Lambda(i)$ and $M'(i,u) = \tau_{\Lambda'}^{-u}P_\Lambda(i)$ coincide for all $0 \le u \le q(i)$ and $1 \le i \le n$, then $a_s(\Lambda) = a_s(\Lambda')$ for all s.*

Proof. We may interpret the numbers $a_s(\Lambda)$ and $a_s(\Lambda')$ as the number of antichains in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda'$, respectively, which have support-rank s. Note that the support of a module M is the set of numbers $1 \le i \le n$ such that $\operatorname{Hom}(P(i), M) \ne 0$.

We show that $\operatorname{Hom}(M(i,u), M(j,v)) = 0$ if and only if $\operatorname{Hom}(M'(i,u), M'(j,v)) = 0$. If $u \le v$, the Auslander-Reiten translation (see for example [ARS]) furnishes a group isomorphism

$$\operatorname{Hom}(M(i,u), M(j,v)) \simeq \operatorname{Hom}(M(i,0), M(j,v-u)) = \operatorname{Hom}(P_\Lambda(i), M(j,v-u)),$$

and similarly, we have $\operatorname{Hom}(M'(i,u), M'(j,v)) \simeq \operatorname{Hom}(P_{\Lambda'}(i), M'(j,v-u))$. It follows that $\operatorname{Hom}(M(i,u), M(j,v)) = 0$ if and only if i is not in the support of $M(j,v-u)$ if and only if i is not in the support of $M'(j,v-u)$ if and only if $\operatorname{Hom}(M'(i,u), M'(j,v)) = 0$.

If $u > v$, then

$$\operatorname{Hom}(M(i,0), M(j,v)) \simeq \operatorname{Hom}(M(i,u-v), M(j,0)) = 0,$$

since $M(i,u-v)$ is indecomposable and non-projective, whereas $M(j,0)$ is projective. Similarly, we also have $\operatorname{Hom}(M'(i,0), M'(j,v)) = 0$.

As a consequence we see that given an antichain $A = \{A_1, \ldots, A_t\}$ in $\operatorname{mod}\Lambda$, the function $M(i,u) \mapsto M'(i,u)$ yields an antichain $A' = \{A'_1, \ldots, A'_t\}$ in $\operatorname{mod}\Lambda'$, Of course, the support-rank of A and A' are the same. This completes the proof. $\square$

Corollary. *For all $0 \le s \le n$, we have $a_s(\mathbb{B}_n) = a_s(\mathbb{C}_n)$.*

Proof: Apply the Proposition to the algebras Λ and Λ' with valued quiver

$$\underset{1}{\circ} \longleftarrow \underset{2}{\circ} \longleftarrow \underset{3}{\circ} \leftarrow \cdots - \underset{n-2}{\circ} \longleftarrow \underset{n-1}{\circ} \Leftarrow\!\!\!\!\!= \Rightarrow \underset{n}{\circ}$$

$$\underset{1}{\circ} \longleftarrow \underset{2}{\circ} \longleftarrow \underset{3}{\circ} \leftarrow \cdots - \underset{n-2}{\circ} \longleftarrow \underset{n-1}{\circ} \Leftarrow\!\!\!\Longleftarrow \underset{n}{\circ}$$

respectively; the first valued quiver is of type $\mathbb{B}_n$, the second of type $\mathbb{C}_n$. It is well-known (and easy to see) that $q(P_\Lambda(i)) = n-1 = q(P_{\Lambda'}(i))$ for all $1 \le i \le n$ and that the modules $M(i,u)$ and $M'(i,u)$ for $1 \le i \le n$ and $0 \le u \le n-1$ have the same support.

3. The tilting modules for $\mathbb{B}_n$.

We are going to determine the number of tilting modules for the Dynkin algebras of type $\mathbb{B}_n$, namely we will show that $a_n(\mathbb{B}_n) = \binom{2n-1}{n-1}$. By induction, we assume knowledge about the representation theory of $\mathbb{B}_i$ with $i < n$ and the calculation of $a_s(\mathbb{B}_n)$ for $s < n$ as shown in section 4. We consider a Dynkin algebra Λ with quiver

$$\underset{1}{\circ} \longleftarrow \underset{2}{\circ} \longleftarrow \underset{3}{\circ} \leftarrow \cdots - \underset{n-2}{\circ} \longleftarrow \underset{n-1}{\circ} \Longleftrightarrow \underset{n}{\circ}$$

We interpret $a_n(\mathbb{B}_n)$ as the number of sincere antichains (by definition, an antichain $A = \{A_1, \ldots, A_t\}$ is *sincere* provided the module $\bigoplus A_i$ is sincere) and write it as the sum

$$a_n(\mathbb{B}_n) = u(\mathbb{B}_n) + v(\mathbb{B}_n)$$

where $u(\mathbb{B}_n)$ is the number of antichains with a sincere element, whereas $v(\mathbb{B}_n)$ is the number of sincere antichains without a sincere element. These two numbers will be calculated separately.

Let us denote by $w(\mathbb{B}_n)$ the number of antichains which do not contain any injective module. We claim that

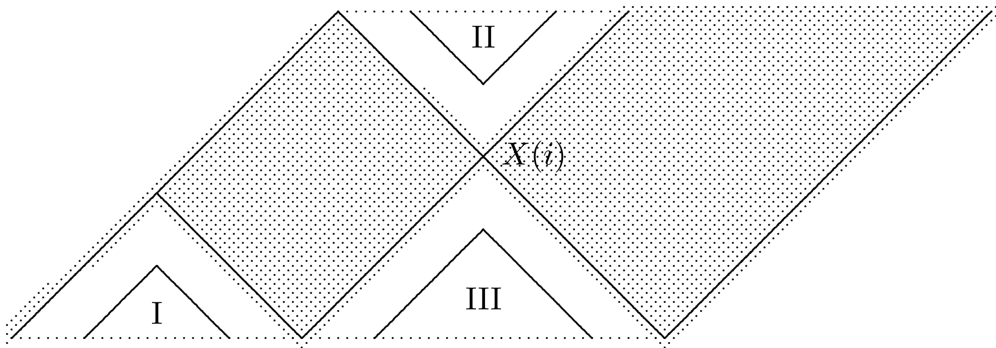
$$w(\mathbb{B}_n) = a_n(\mathbb{B}_n).$$

Proof: Let $\mathcal{W}$ be the set of antichains without injective modules and $\mathcal{S}$ the set of sincere antichains. We want to construct a bijection $\eta\colon \mathcal{S} \to \mathcal{W}$. Note that an element of $\mathcal{S}$ contains at most one injective modules, since the injective modules are pairwise comparable with respect to Hom. If $A \in \mathcal{S}$ contains no injective module, then let $\eta(A) = A$. If $A \in \mathcal{S}$ contains the injective module $I(i)$, let $\eta(A)$ be obtained from A by deleting $I(i)$. Note that $\eta(A)$ is no longer sincere, since all the modules $A^{(i)}$ in $\eta(A)$ satisfy $(A^{(i)})_i = 0$. Conversely, assume that B is an antichain in $\mathcal{W}$. If B is sincere, then it belongs to $\mathcal{S}$ and by definition $\eta(B) = B$. If B is non sincere, let i be the smallest number such that i is not in the support of B. Let A be obtained from B by adding $I(i)$. Then clearly A is sincere and $\eta(A) = B$. This completes the proof.

Now let us determine $u(\mathbb{B}_n)$. Note that the sincere indecomposable representations of Λ are the modules $X(i) = \tau^{-n+i}P(i)$ with $1 \le i \le n$, the dimension vector of $X(n)$ is $(1, \ldots, 1)$, whereas for $1 \le i < n$, it has height $n+i$ and is of the form $(1, \ldots, 1) + (0, \ldots, 0, 1, \ldots, 1)$. Let $u_i(\mathbb{B}_n)$ be the antichains which include $X(i)$, thus

$$u(\mathbb{B}_n) = \sum\nolimits_{i=1}^{n} u_i(\mathbb{B}_n).$$

Let $\mathcal{X}_i$ be the set of indecomposable modules M such that $\operatorname{Hom}(X(i), M) = 0 = \operatorname{Hom}(M, X(i))$. Thus, the antichains which contain $X(i)$ correspond bijectively to the antichains in $\mathcal{X}_i$. In general, the set $\mathcal{X}_i$ consists of three triangles I, II, III:



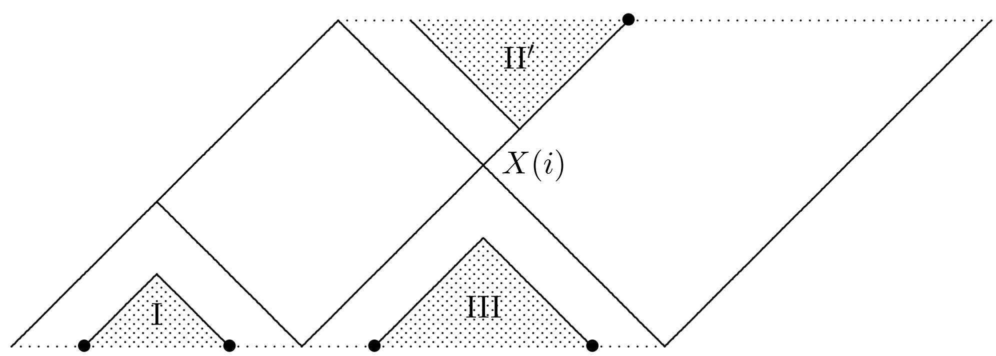
The triangle I is the wing at the vertex $\tau^{-1}P(n-i-1)$, the triangle II is the wing at the vertex $\tau^{-n+i-1}P(i+2)$, and the triangle III is the wing at the vertex $\tau^{-n+i+1}P(i-2)$.

We also are interested in a larger triangle II′ which contains the triangle II as well as $n-i$ additional modules (all being successors of $X(i)$), namely the wing at the vertex $\tau^{-n+i}P(i+1)$.

The full subcategory $\mathcal{X}'$ with the indecomposables in the wings I, II′, III is an abelian subcategory, namely the thick subcategory with simple objects

$$S(2), S(3), \ldots, S(n-i+1); \quad \tau^{n-i}P(n); \quad S(n-i+3), \ldots, S(n-1)$$

(the definition of a thick subcategory will be recalled in the appendix).



It is of type $\mathbb{B}_{n-i} \cup \mathbb{A}_{i-2}$ (the $\mathbb{A}_{i-2}$-part is given by the triangle III, whereas the $\mathbb{B}_{n-i}$-part is given by the triangle I and II′, these are the representations M of Λ such that $M_0 = 0$ and such that the restriction to $[i, n]$ is a direct sum of copies of its thin sincere indecomposable representation). Note that the indecomposables in I and II just correspond to the non-injective indecomposables in the $\mathbb{B}_{n-i}$-part. This shows that

$$u_i(\mathbb{B}_n) = w(\mathbb{B}_{n-i})a(\mathbb{A}_{i-2}) = a_{n-i}(\mathbb{B}_{n-i})a_{i-1}(\mathbb{A}_{i-1})$$

In the special cases $i = 1, 2, n-1, n$, the same formula holds: For $i = 1$ and $i = 2$, the triangle III is empty, whereas the triangles I and II′ together yield a category of type $\mathbb{B}_{n-i}$.

In the cases $i = n-1$ and $i = n$, the triangles I and II are empty, whereas the triangle III yields a category of type $\mathbb{A}_{i-2}$.

Thus we see:

$$u(\mathbb{B}_n) = \sum\nolimits_{i=1}^{n} u_i(\mathbb{B}_n) = \sum\nolimits_{i=1}^{n} a_{i-1}(\mathbb{A}_{i-1})a_{n-i}(\mathbb{B}_{n-i}).$$

But the latter expression is the recursion formula for $a_{n-1}(\mathbb{B}_n)$, since the number of support-tilting modules T with support $\{1, 2, \ldots, n\} \setminus \{i\}$ is just $a_{i-1}(\mathbb{A}_{i-1})a_{n-i}(\mathbb{B}_{n-i})$.

This shows that

$$\boxed{u(\mathbb{B}_n) = a_{n-1}(\mathbb{B}_n) = \binom{2n-2}{n-1}.}$$

Second, let us determine $v(\mathbb{B}_n)$. Let $\mathcal{V}$ be the set of sincere antichains without a sincere element. Let $A = (A^{(1)}, \ldots, A^{(r)})$ be in $\mathcal{V}$. We may assume that $(A^{(1)})_1 \neq 0$ and that $A^{(1)}$ is maximal with this property. Since $A^{(1)}$ is not sincere, we must have $(A^{(1)})_n = 0$, thus $A^{(1)}$ is a representation of a Dynkin algebra of type $\mathbb{A}_{n-1}$ and actually an indecomposable projective representation (also as a Λ-module), thus $A^{(1)} = P(i)$ for some i with $1 \leq i < n$.

Denote by $\mathcal{V}_i$ the sincere antichains A such that $A^{(1)} = P(i)$. For $2 \leq j \leq r$, we have $\mathrm{Hom}(A^{(1)}, A^{(j)}) = 0$, thus $(A^{(1)})_j = 0$. It follows that $(A^{(2)}, \ldots, A^{(r)})$ is an antichain with support in $[1, i-1] \cup [i+1, n]$. Altogether, we see that any element of A has support either in $[1, i]$ or in $[i+1, i]$. The elements of A with support in $[1, i]$ but different from $A^{(1)}$ form an arbitrary antichain with support in $[2, i-1]$, thus the number of elements is $a(\mathbb{A}_{i-2})$, at least if $i \geq 2$. Note that $a(\mathbb{A}_{i-2}) = a_{i-1}(\mathbb{A}_{i-1})$.

The elements of A with support in $[i+1, n]$ form a sincere antichain for $\mathbb{B}_{n-i}$, thus the number of such antichains is $a_{n-i}(\mathbb{B}_{n-i})$. This shows that for $i \geq 2$, the set $\mathcal{V}_i$ has cardinality $a_{i-1}(\mathbb{A}_{i-1})a_{n-i}(\mathbb{B}_{n-i})$. This formula holds true also for $i = 1$, since the number of elements of $\mathcal{V}_1$ is $a_{n-1}(\mathbb{B}_{n-1})$ and $a_0(\mathbb{A}_0) = 1$. Thus we see that

$$\begin{aligned} v(\mathbb{B}_n) &= \sum\nolimits_{i=1}^{n-1} a_{i-1}(\mathbb{A}_{i-1})a_{n-i}(\mathbb{B}_{n-i}) \\ &= -a_{n-1}(\mathbb{A}_{n-1}) + \sum\nolimits_{i=1}^{n} a_{i-1}(\mathbb{A}_{i-1})a_{n-i}(\mathbb{B}_{n-i}) \\ &= -\frac{1}{n}\binom{2n-2}{n-1} + \binom{2n-2}{n-1} = \binom{2n-2}{n-2}. \end{aligned}$$

Thus

$$\boxed{v(\mathbb{B}_n) = \binom{2n-2}{n-2}.}$$

Altogether we see:

$$u(\mathbb{B}_n) + v(\mathbb{B}_n) = \binom{2n-2}{n-1} + \binom{2n-2}{n-2} = \binom{2n-1}{n-1}.$$

Remarks. (1) Note that we have

$$\boxed{u(\mathbb{B}_n) = \binom{2n-2}{n-1} = a_{n-1}(\mathbb{B}_n), \quad v(\mathbb{B}_n) = \binom{2n-2}{n-2} = a_{n-2}(\mathbb{B}_{n+1}),}$$

(2) The calculation of $v(\mathbb{B}_n)$ shows the following relationship between the cases $\mathbb{A}$ and $\mathbb{B}$:

$$\boxed{a_{n-1}(\mathbb{B}_n) = a_{n-2}(\mathbb{B}_{n+1}) + a_{n-1}(\mathbb{A}_{n-1})}$$

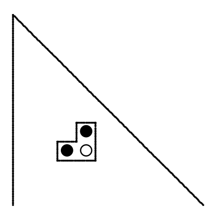
4. The hook formula for support-tilting modules.

Proposition (hook formula). *Let $\Delta = \mathbb{A}, \mathbb{B}, \mathbb{D}, \mathbb{E}$. Then*

$$a_s(\Delta_n) = a_s(\Delta_{n-1}) + a_{s-1}(\Delta_n)$$

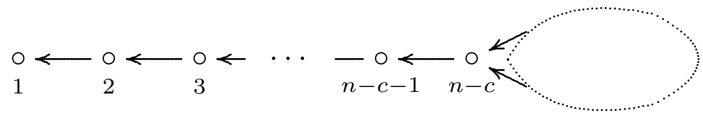
for all $n \geq m$ and $1 \leq s \leq n-c$, where $m = 1,2,3,4$ and $c = 0,1,2,3$ for $\Delta = \mathbb{A}, \mathbb{B}, \mathbb{D}, \mathbb{E}$, respectively.

Here we use the convention that $\mathbb{B}_1 = \mathbb{A}_1, \mathbb{D}_2 = \mathbb{A}_1 \sqcup \mathbb{A}_1, \mathbb{E}_3 = \mathbb{A}_2 \sqcup \mathbb{A}_1, \mathbb{E}_4 = \mathbb{A}_4, \mathbb{E}_5 = \mathbb{D}_5$. In the triangles depicted in section 6 and 7, this equality concerns the following kind of hooks:

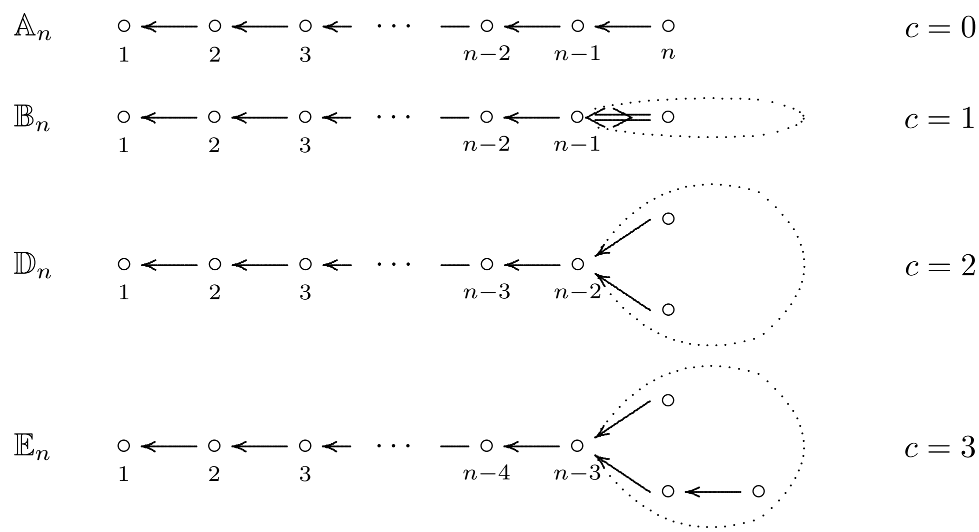


The hook formula asserts that the sum of the values at the positions marked by bullets is the value at the position marked by the circle.

The various assertions concern the following general situation: Up to the choice of an orientation, we deal with an artin algebra Λ with the following valued quiver with n vertices:



on the left, we have a quiver of type $\mathbb{A}_{n-c}$ with arrows $i \leftarrow i+1$, the remaining c vertices are in the dotted "cloud" to the right, all arrows between the cloud and the $\mathbb{A}_{n-c}$–quiver end in the vertex $n-c$. We denote by Q' the valued quiver obtained by deleting the vertex 1 and the arrow ending in 1; let Λ' be the corresponding factor algebra of Λ.



Lemma. *Let* $1 \leq s \leq n - c$. *Then*

$$a_s(\Lambda) = a_s(\Lambda') + a_{s-1}(\Lambda).$$

Proof: The support-tilting modules T for Λ with 1 not in the support are just the support-tilting modules for Λ'. Thus let us consider the set $\mathcal{S}_s(\Lambda; 1)$ of the basic support-tilting Λ-modules T with support-rank s and $T_1 \neq 0$ and construct a bijection

$$\alpha \colon \mathcal{S}_s(\Lambda; 1) \longrightarrow \mathcal{S}_{s-1}(\Lambda),$$

where $\mathcal{S}_{s-1}(\Lambda)$ is the set of basic support-tilting Λ-modules T with support-rank $s - 1$. This will establish the formula.

Let X be an indecomposable representation with support-rank $s \leq n - c$ and $X_1 \neq 0$, thus the support of X is contained in the $\mathbb{A}_{n-c}$-subquiver, therefore X is thin and its support is an interval of the form $[1, v]$ with $1 \leq v \leq n - c$ (a module is said to be *thin* provided the composition factors are pairwise non-isomorphic; in our setting thin indecomposable modules are uniquely determined by the support, thus we may write just $X = [1, v]$).

Let T be a module in $\mathcal{S}_s(\Lambda; 1)$. At least one of the indecomposable direct summand of T, say X, satisfies $X_1 \neq 0$ and we choose X of largest possible length. We claim that $T_w = 0$ for any arrow $v \leftarrow w$. Assume, for the contrary, that there is an indecomposable direct summand Y of T with $Y_w \neq 0$. The maximality of X shows that $Y_1 = 0$. But then $\operatorname{Ext}(Y, X) \neq 0$ contradicts the fact that T has no self-extensions (namely, if the support of X and Y is disjoint, then the arrow $v \leftarrow w$ yields directly a non-trivial extension of X by Y; if the support of X and Y is not disjoint, then there is a proper non-zero factor module of X which is a proper submodule of Y, thus there is a non-zero map $X \to Y$ which is neither injective nor surjective — again we obtain a non-trivial extension of X by Y). Thus the support of T is the disjoint union of the set $\{1, 2, \ldots, v\}$ and a set S'' which does not contain a vertex w with an arrow $v \leftarrow w$.

The indecomposable direct summands of T with support in $\{1, 2, \ldots, v\}$ yield a tilting module for this $\mathbb{A}_v$-quiver, and X is the indecomposable projective-injective representation of this $\mathbb{A}_v$-quiver. Deleting X from this tilting module, we obtain a support-tilting representation of $\mathbb{A}_v$ with support-rank $v - 1$.

Thus if we write $T = X \oplus T'$, then T' is a support-tilting Λ-module with support-rank $s - 1$ (namely, it is the direct sum of a support-tilting module with support properly contained in $\{1, 2, \ldots, v\}$ and a support-tilting module with support S''. We define $\alpha(T) = T'$, this yields the map

$$\alpha \colon \mathcal{S}_s(\Lambda; 1) \longrightarrow \mathcal{S}_{s-1}(\Lambda)$$

we are looking for. It remains to be shown that α is surjective and that we can recover T from $\alpha(T)$.

Thus, let T' be in $\mathcal{S}_{s-1}(\Lambda)$. Then there are at least $c+1$ vertices outside of the support of T'.

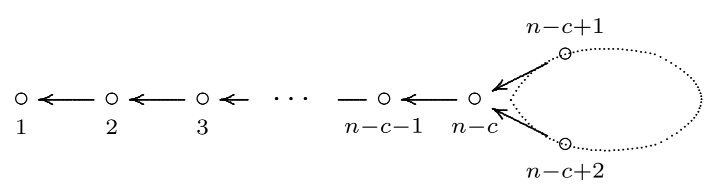
Case 1: These are the vertices in the cloud and precisely one additional vertex, say i (with $1 \le i \le n - c$). Note that in this case $s = n - c$. Let $T = T' \oplus [1, n - c]$. Since T' is a support-tilting module of $\mathbb{A}_{n-c}$ with support-rank $n - c - 1$ and $[1, n - c]$ is the indecomposable projective-injective representation of $\mathbb{A}_{n-c}$, we see that $T = T' \oplus [1, n - c]$ is a tilting module for $\mathbb{A}_{n-c}$.

Case 2: At least two vertices between 1 and $n - c$ do not belong to $\operatorname{Supp} T'$, say let $i < j$ be the smallest such numbers. Then let $T = T' \oplus [1, j - 1]$. $\square$

Proposition 2 (modified hook formula).

$$\begin{aligned} a_{n-1}(\mathbb{D}_n) &= a_{n-1}(\mathbb{D}_{n-1}) + a_{n-2}(\mathbb{D}_n) + a_{n-2}(\mathbb{A}_{n-2}), \\ a_{n-2}(\mathbb{E}_n) &= a_{n-2}(\mathbb{E}_{n-1}) + a_{n-3}(\mathbb{E}_n) + a_{n-3}(\mathbb{A}_{n-2}), \end{aligned}$$

Again, we consider a general setting, namely we consider an artin algebra Λ with the following valued quiver with n vertices and we assume that $c \ge 2$:



on the left, we have a quiver of type $\mathbb{A}_{n-c}$ with arrows $i \leftarrow i + 1$, the remaining c vertices are in the dotted "cloud" to the right, there are precisely two vertices in the cloud, namely $n - c + 1$ and $n - c + 2$ with arrows $n - c \leftarrow n - c + 1$ and $n - c \leftarrow n - c + 2$ and there is no other arrows between the cloud and the $\mathbb{A}_{n-c}$ quiver. Again, we denote by Q' the valued quiver obtained by deleting the vertex 1 and the arrow ending in 1 and by Λ' the corresponding factor algebra of Λ and we show:

Lemma.

$$a_{n-c+1}(\Lambda) = a_{n-c+1}(\Lambda') + a_{n-c}(\Lambda) + a_{n-c}(\mathbb{A}_{n-c}).$$

The proof follows closely the previous proof. The support-tilting modules T for Λ with 1 not in the support are just the support-tilting modules for Λ'. We construct a surjection α from the set $\mathcal{S}_{n-c+1}(\Lambda;1)$ of the support-tilting Λ-modules T with support-rank $n-c+1$ and $T_1 \neq 0$ onto the set $\mathcal{S}_{n-c}$ of support-tilting Λ-modules T with support-rank $n-c$. In the present setting, α will not be injective, but there will be pairs in $\mathcal{S}(\Lambda;1)$ which are identified by α, the number of such pairs will be just $a_{n-c}(\mathbb{A}_{n-c})$.

As above, one shows that any module T in $\mathcal{S}_{n-c+1}$ is of the form $T = X \oplus T'$ where X is indecomposable, $X_1 \neq 0$ and X is of maximal possible length. Note that the support of X is contained either in $\{1,2,\dots,n-c+1\}$ or in $\{1,2,\dots,n-c,n-c+2\}$. In particular, X is uniquely determined (since the support of T cannot contain all the vertices $1,2,\dots,n-c+2$). As above, the mapping α will be the deletion of the summand X.

Let Z be the indecomposable module with support $\{1,2,\dots,n-c+1\}$ and Z' the indecomposable module with support $\{1,2,\dots,n-c,n-c+2\}$. Starting with a tilting module T' for $\mathbb{A}_{n-c}$, we may form the direct sums $Z \oplus T'$ and $Z' \oplus T'$. Then these are elements of $\mathcal{S}_{n-c+1}(\Lambda;1)$ which both are mapped under α to the same module T'. These are the $a_{n-c}(\mathbb{A}_{n-c})$ pairs of elements of $\mathcal{S}(\Lambda;1)$ which are identified by α.

It follows that $\mathcal{S}(\Lambda;1)$ has cardinality $a_{n-c}(\Lambda) + a_{n-c}(\mathbb{A}_{n-c})$. $\square$

Corollary.

$$a_{n-1}(\mathbb{D}_n) = \begin{bmatrix} 2n-3 \\ n-1 \end{bmatrix}.$$

Proof: We start with the previous observation

$$\begin{aligned} a_{n-1}(\mathbb{D}_n) &= a_{n-1}(\mathbb{D}_{n-1}) + a_{n-2}(\mathbb{D}_n) + a_{n-2}(\mathbb{A}_{n-2}) \\ &= \frac{3n-7}{n-1}\binom{2n-5}{n-3} + \frac{3n-6}{2n-4}\binom{2n-4}{n-2} + \frac{1}{n-1}\binom{2n-4}{n-2} \end{aligned}$$

Write

$$\begin{aligned} \binom{2n-5}{n-3} &= \frac{n-1}{2n-3}\binom{2n-3}{n-1}, \\ \binom{2n-4}{n-2} &= \frac{(n-2)(n-1)}{(2n-4)(2n-3)}\binom{2n-3}{n-1}. \end{aligned}$$

One easily shows that

$$\frac{3n-7}{n-1} \cdot \frac{(n-2)(n-1)}{(2n-4)(2n-3)} + \frac{3n-6}{2n-4} \cdot \frac{n-1}{2n-3} + \frac{1}{n-1} \cdot \frac{n-1}{2n-3} \;=\; \frac{3n-4}{2n-3}.$$

As a consequence, we get

$$\frac{3n-7}{n-1}\binom{2n-5}{n-3} + \frac{3n-6}{2n-4}\binom{2n-4}{n-2} + \frac{1}{n-1}\binom{2n-4}{n-2} = \frac{3n-4}{2n-3}\binom{2n-3}{n-1} = \begin{bmatrix} 2n-3 \\ n-1 \end{bmatrix}.$$

$\square$

5. Summation formulas.

An immediate consequence of the previous section is the following assertion

Proposition. *Let* $\Delta = \mathbb{A}$, *or* $\mathbb{B}$ *and* $n \geq 0$, *or* $\Delta = \mathbb{D}$ *and* $n \geq 2$. *If* $1 \leq s \leq n-1$, *then*

$$\sum\nolimits_{i=0}^{s} a_i(\Delta_n) = a_s(\Delta_{n+1}).$$

Proof, by induction: For $s = 0$, both sides are equal to 1. For $s \geq 1$, we write

$$\begin{aligned}\sum\nolimits_{i=0}^{s} a_i(\Delta_n) &= a_s(\Delta_n) + \sum\nolimits_{i=0}^{s-1} a_i(\Delta_n)\\ &= a_s(\Delta_n) + a_{s-1}(\Delta_{n+1})\\ &= a_s(\Delta_{n+1})\end{aligned}$$

at the end, we have used the hook formula.

Corollary. *Let* $\Delta = \mathbb{A}, \mathbb{B}$ *or* $\mathbb{D}$. *Then*

$$\boxed{a(\Delta_n) = a_n(\Delta_n) + a_{n-1}(\Delta_{n+1})}$$

Case $\mathbb{A}_n$**.**

$$a(\mathbb{A}_n) = \frac{1}{n+1}\binom{2n}{n} + \frac{3}{n+2}\binom{2n}{n-1} = \frac{1}{n+2}\binom{2n+2}{n+1}.$$

Case $\mathbb{B}_n$**.**

$$a(\mathbb{B}_n) = \binom{2n-2}{n-1} + \binom{2n-1}{n} = \binom{2n}{n}.$$

Case $\mathbb{D}_n$**.**

$$a(\mathbb{D}_n) = \begin{bmatrix} 2n-2 \\ n-2 \end{bmatrix} + \begin{bmatrix} 2n-2 \\ n-1 \end{bmatrix} = \begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix}.$$

6. The three triangles $\mathbb{A}, \mathbb{B}, \mathbb{D}$ and the Catalan, Pascal, Lucas triangles.

6.1. The triangle of type $\mathbb{A}$, this is A009766.

$$a_s(\mathbb{A}_n) = \frac{n-s+1}{n+1}\binom{n+s}{s}$$

$n \backslash s$	0	1	2	3	4	5	6	7	8	9	sum
0	1										1
1	1	1									2
2	1	2	2								5
3	1	3	5	5							14
4	1	4	9	14	14						42
5	1	5	14	28	42	42					132
6	1	6	20	48	90	132	132				429
7	1	7	27	75	165	297	429	429			1430
8	1	8	35	110	275	572	1001	1430	1430		4862
9	1	9	44	154	429	1001	2002	3432	4862	4862	16796

6.2. The triangle of type $\mathbb{B}$, this is A059481.

$$a_s(\mathbb{B}_n) = \binom{n+s-1}{s}$$

$n \backslash s$	0	1	2	3	4	5	6	7	8	9	sum
0	1										1
1	1	1									2
2	1	2	3								6
3	1	3	6	10							20
4	1	4	10	20	35						70
5	1	5	15	35	70	126					252
6	1	6	21	56	126	252	462				924
7	1	7	28	84	210	462	924	1716			3432
8	1	8	36	120	330	792	1716	3432	6435		12870
9	1	9	45	165	495	1287	3003	6435	12870	24310	48620

6.3. The triangle of type $\mathbb{D}$

$$a_s(\mathbb{D}_n) = \begin{cases} \begin{bmatrix} n+s-2 \\ s \end{bmatrix} & \text{for } 0 \leq s < n, \\ \begin{bmatrix} 2n-2 \\ n-2 \end{bmatrix} & \text{for } s = n. \end{cases}$$

$n \backslash s$	0	1	2	3	4	5	6	7	8	9	sum
0	.										.
1	.	.									.
2	1	2	1								4
3	1	3	5	5							14
4	1	4	9	16	20						50
5	1	5	14	30	55	77					182
6	1	6	20	50	105	196	294				672
7	1	7	27	77	182	378	714	1122			2508
8	1	8	35	112	294	672	1386	2640	4290		9438
9	1	9	44	156	450	1122	2508	5148	9867	16445	35750

6.1′. The sheared Catalan triangle A008315

$t \backslash s$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2								
3	1	3	2							
4	1	4	5							
5	1	5	9	5						
6	1	6	14	14						
7	1	7	20	28	14					
8	1	8	27	48	42					
9	1	9	35	75	90	42				

$$\binom{t}{s} - \binom{t}{s-1} = \frac{t-2s+1}{t-s-1}\binom{t}{s}$$

6.2′. The Pascal triangle A007318, left of the staircase line is the increasing part.

$t \backslash s$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
9	1	9	36	84	126	126	84	36	9	1

$$\binom{t}{s}$$

6.3′. The Lucas triangle A029635, left of the staircase line is the increasing part.

$t \backslash s$	0	1	2	3	4	5	6	7	8	9
0	·									
1	1	2								
2	1	3	2							
3	1	4	5	2						
4	1	5	9	7	2					
5	1	6	14	16	9	2				
6	1	7	20	30	25	11	2			
7	1	8	27	50	55	36	13	2		
8	1	9	35	77	105	91	49	15	2	
9	1	10	44	112	182	196	140	64	17	2

$$\begin{bmatrix} t \\ s \end{bmatrix}$$

Remarks concerning the presentation of the triangles.

In the triangle of type $\mathbb{D}$ and in the corresponding Lucas triangle some values are left open (this is indicated by a dot).

In the Lucas triangle, this concerns the value at the position $(0,0)$. This value which would be denoted by $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ should be one of the numbers 1 or 2 (in OEIS A029635, the number is chosen to be 2). Note that here we deal with the product $\frac{0}{0}\binom{0}{0}$: whereas $\binom{0}{0} = 1$ is well-defined, there is the ambiguous fraction $\frac{0}{0}$.

In the triangle of type $\mathbb{D}$, the positions $(0,0), (0,1), (1,1)$ are left open, since the series of Dynkin diagram $\mathbb{D}_n$ starts with $n = 2$ (but see A129869); by definition $\mathbb{D}_2 = \mathbb{A}_1 \sqcup \mathbb{A}_1$ and $\mathbb{D}_3 = \mathbb{A}_3$. As a consequence, also the corresponding entries in the summation sequence are missing.

6.4. Some observations concerning the triangles $\mathbb{A}, \mathbb{B}, \mathbb{D}$.

The sum sequence occurs as a diagonal.

In the $\mathbb{A}$-triangle, the sum sequence is the same sequence as the main diagonal (and these are just the Catalan numbers):

$$a(\mathbb{A}_n) = a_{n+1}(\mathbb{A}_{n+1})$$

In the $\mathbb{B}$-triangle, the sum sequence is the same sequence as the second diagonal

$$a(\mathbb{B}_n) = a_n(\mathbb{B}_{n+1})$$

In the $\mathbb{D}$-triangle, the sum sequence is the same sequence as the fourth diagonal

$$a(\mathbb{D}_n) = a_{n-1}(\mathbb{D}_{n+2})$$

The main diagonal uses the same sequence as one of the other diagonals.

In the $\mathbb{A}$-triangle, this concerns the main diagonal and the second diagonal:

$$a_n(\mathbb{A}_n) = a_{n-1}(\mathbb{A}_n)$$

In the $\mathbb{B}$-triangle, this concerns the main diagonal and the second diagonal:

$$a_n(\mathbb{B}_n) = a_{n-1}(\mathbb{B}_{n+1})$$

In the $\mathbb{D}$-triangle, this concerns the main diagonal and the fifth diagonal:

$$a_n(\mathbb{D}_n) = a_{n-2}(\mathbb{D}_{n+2})$$

It may be of interest to exhibit explicit bijections between the corresponding sets of support-tilting modules. It seems that only in the case $\mathbb{A}$, this can be done easily!

6.5. Comparison between the Lucas triangle and the $\mathbb{D}$-triangle.

The difference between the number $\begin{bmatrix} 2n-2 \\ n \end{bmatrix}$ and $\begin{bmatrix} 2n-2 \\ n-2 \end{bmatrix}$ seems to be of interest:

$$\begin{bmatrix} 2n-2 \\ n \end{bmatrix} - \begin{bmatrix} 2n-2 \\ n-2 \end{bmatrix} = \frac{1}{n}\binom{2n-2}{n-1}$$

This means that

$$\boxed{\begin{bmatrix} 2n-2 \\ n \end{bmatrix} - a_n(\mathbb{D}_n) = a_{n-1}(\mathbb{A}_{n-1})}$$

Proof: We show that

$$\frac{3n-4}{n}\binom{2n-2}{n-2} + \frac{1}{n}\binom{2n-2}{n-1} \ = \ \frac{3n-2}{2n-2}\binom{2n-2}{n}$$

We rewrite

$$\binom{2n-2}{n-2} = \frac{n}{2n-2}\binom{2n-2}{n},$$
$$\binom{2n-2}{n-1} = \frac{n}{n-1}\binom{2n-2}{n}.$$

The assertion now follows from the equality

$$\frac{3n-4}{n}\cdot\frac{n}{2n-2} \ + \ \frac{1}{n}\cdot\frac{n}{n-1} \ = \ \frac{3n-2}{2n-2}.$$

Here is a table of these numbers

n	$\begin{bmatrix} 2n-2 \\ n \end{bmatrix}$	$\begin{bmatrix} 2n-2 \\ n-2 \end{bmatrix}$	$\frac{1}{n}\binom{2n-2}{n-1}$
2	2	1	1
3	7	5	2
4	25	20	5
5	91	77	14
6	336	294	42
7	1254	1122	132
8	4719	4290	429
9	17875	16445	1430

7. The exceptional cases.

Here are the numbers $a_s(\Delta_n)$ and $a(\Delta_n)$ in the exceptional cases $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2$ (we add some suitable additional rows in order to stress the induction scheme):

s	0	1	2	3	4	5	6	7	8	sum
$\mathbb{E}_3 = \mathbb{A}_2 \sqcup \mathbb{A}_1$	1	3	4	2						10
$\mathbb{E}_4 = \mathbb{A}_4$	1	4	9	14	14					42
$\mathbb{E}_5 = \mathbb{D}_5$	1	5	14	30	55	77				182
$\mathbb{E}_6$	1	6	20	50	110	228	418			833
$\mathbb{E}_7$	1	7	27	77	187	429	1001	2431		4160
$\mathbb{E}_8$	1	8	35	112	299	728	1771	4784	17342	25080
$\mathbb{B}_3$	1	3	6	10						20
$\mathbb{F}_4$	1	4	10	24	66					105
$\mathbb{G}_2$	1	2	5							8

Appendix. The Ingalls-Thomas bijections.

(A.1) Let Λ be a hereditary artin algebra. We need some further definitions: Given a class $\mathcal{X}$ of modules, we denote by $\operatorname{add}\mathcal{X}$ the class of modules which are direct summands of direct sums of modules in $\mathcal{X}$. If $\mathcal{X} = \{X\}$ for a single module X, we write $\operatorname{add} X$ instead of $\operatorname{add}\{X\}$ (the same convention will be used in similar situations). Given a class $\mathcal{X}$ of modules, let $\mathcal{G}(\mathcal{X})$ be the subcategory of all modules which are generated by modules in $\operatorname{add}\mathcal{X}$, and let $\mathcal{H}(\mathcal{X})$ be the subcategory of all modules which are cogenerated by modules in $\operatorname{add}\mathcal{X}$.

The modules M, M' are said to be *Morita equivalent* provided $\operatorname{add}(M) = \operatorname{add}(M')$. Note that basic modules which are Morita equivalent are actually isomorphic.

Starting with an antichain $A = \{A_1, \ldots, A_t\}$, its Ext-*quiver* has t vertices, and there is arrow $i \to j$ provided $\operatorname{Ext}(A_i, A_j) \neq 0$. The antichains $A = \{A_1, \ldots, A_t\}$ and $A' = \{A'_1, \ldots, A'_{t'}\}$ are said to be isomorphic, provided the modules $\bigoplus_i A_i$ and $\bigoplus_j A'_j$ are isomorphic.

A full subcategory $\mathcal{A}$ of $\operatorname{mod}\Lambda$ is called a *thick* subcategory provided it is closed under kernels, cokernels and extensions. Note that a thick subcategory is an abelian category, and the inclusion functor $\mathcal{A} \to \operatorname{mod}\Lambda$ is exact.

If $\mathcal{C}$ is a subcategory and $C \in \mathcal{C}$, then C is said to be a *cover* of $\mathcal{C}$ provided $\mathcal{C} \subseteq \mathcal{G}(C)$, and C is said to be a *cocover* of $\mathcal{C}$ provided $\mathcal{C} \subseteq \mathcal{H}(C)$.

A *torsion class* in $\operatorname{mod}\Lambda$ is a class of modules which is closed under factor modules and extensions. A *torsionfree class* in $\operatorname{mod}\Lambda$ is a class of modules which is closed under submodules and extensions.

There is the following well-known fact (see, for example [R3]): *A sincere module without self-extensions is faithful,* thus any module X without self-extensions is a faithful $\Lambda(X)$-module.

Here is the main result of the appendix:

(A.2) Theorem. *Let Λ be a hereditary artin algebra. There are bijections between the following data:*
(1) *Isomorphism classes of* Ext*-directed antichains.*
(2) *Thick subcategories with a cover.*
(3) *Isomorphism classes of normal modules without self-extensions.*
(4) *Morita equivalence classes of support-tilting modules.*
(5) *Torsion classes with a cover.*

If Λ is in addition representation-finite, then
(1′) *All antichains are* Ext*-directed.*
(2′) *All thick subcategories have a cover.*
(5′) *All torsion classes have a cover.*

In the case when Λ is the path algebra of a quiver, bijections between the sets (2), (4) and (5) were exhibited by Ingalls-Thomas in [IT]. Actually, a bijection between (4) and (5) has been known before, and the decisive idea of [IT] was to relate the support-tilting modules to thick subcategories. A bijection between (1) and (2) was exhibited already in 1976, see [R1], for a bijection between (1) and (3), we will refer below to [DR3].

Let us write down such bijections in detail, with an outline of the proofs.

(A.3) The bijections between (1), (2) and (3).

From (1) to (2): *If A is an antichain, take $\mathcal{F}(A)$,* this is the set of all Λ-modules with a filtration with factors in A. This is an abelian category with exact embedding functor and obviously closed under extensions, its simple objects are just the elements of A; the process of considering the elements of A as objects in $\mathcal{F}(A)$ is called *simplification* in [R1]. In case the antichain A is Ext-directed, the category $\mathcal{F}(A)$ is equivalent to the module category of a finite-dimensional algebra, thus it has projective generators. Such a projective generator is a cover for $\mathcal{F}(A)$.

For the step (1) to (2), we also may refer to [DR3]. Namely, an antichain A with directed Ext-quiver is a standardizable set as considered in [DR3] and the proof of Theorem 2 in [DR3] asserts that there is a quasi-hereditary algebra B such that the subcategory $\mathcal{F}(A)$ is equivalent to the category of Δ-filtered B-modules. Since the standardizable set A consists of pairwise orthogonal modules, the same is true for the Δ-modules of B, and consequently the Δ-modules of B are just the simple M-modules. This shows that the category of Δ-filtered B-modules is the whole category $\operatorname{mod} B$.

From (2) to (1): *If $\mathcal{A}$ is a thick subcategory with a cover, let $\mathcal{S}(\mathcal{A})$ be the set of simple objects in $\mathcal{A}$,* one from each isomorphism class. Then $\mathcal{S}(\mathcal{A})$ is an Ext-directed antichain.

From (2) to (3): *If $\mathcal{A}$ is a thick subcategory with a cover, let P be a minimal projective generator of $\mathcal{A}$.* Then P is a normal module without self-extensions.

If we start with (1), say with an Ext-directed antichain A, and use [DR3] in order to find an equivalence $\eta\colon \mathcal{F}(A) \to \operatorname{mod} B$, the proof of Theorem 2 in [DR3] first constructs indecomposable objects in $\mathcal{F}(A)$ which correspond under η to the indecomposable projective B-modules. In this way, one constructs a minimal projective generator for the abelian category $\mathcal{F}(A)$.

From (3) to (1). Let N be a normal module without self-extensions. Write $N = \bigoplus_i N_i$ with indecomposable modules N_i. For any i, let $u_i\colon U_i \to N_i$ be a minimal right $\mathcal{N}_i$-approximation of N_i, where $\mathcal{N}_i = \operatorname{add}(\{N_j \mid j \neq i\}$. Since N is normal, the map u_i cannot be surjective. Since Λ is hereditary, it follows that u_i is injective and we denote by $p_i\colon N_i \to \Delta(i)$ the cokernel of u_i. Since u_i is not surjective, we see that $\Delta(i) \neq 0$. We claim that the modules $\Delta(i)$ are pairwise orthogonal bricks. Let $h\colon N(j) \to \Delta(i)$ be a map, and form the induced exact sequence

$$\begin{tikzcd}
0 \arrow[r] & U_i \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & N_j \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & U_i \arrow[r, "u_i"] & N_i \arrow[r, "p_i"] & \Delta(i) \arrow[r] & 0
\end{tikzcd}$$

Since U_i belongs to $\mathcal{N}_i$ and N has no self-extensions, we have $\operatorname{Ext}(N_j, U_i) = 0$, thus the upper sequence splits. It follows that there is a map $h'\colon N_j \to N_i$ such that $h = p_i h'$. First of all, it follows that for any endomorphism g of $\Delta(i)$, there is an endomorphism $g'\colon N_i \to N_i$ with $gp_i = p_i g'$. Since all non-zero endomorphisms of N_i are invertible, the same is true for $\Delta(i)$. In this way, we see that $\Delta(i)$ is a brick. Second, let $g\colon \Delta(j) \to \Delta(i)$ be a homomorphism with $j \neq i$. Then there is $g'\colon N_j \to N_i$ such that $gp_j = p_i g'$. Since u_i is a left $\mathcal{N}_i$-approximation, it follows that $g' = u_i g''$ for some $g''\colon N_j \to U_i$. But then $gp_j = p_i g' = p_i u_i g'' = 0$ and therefore $g = 0$.

Thus, $\Delta = \{\Delta(i) \mid i\}$ is an antichain. Using induction on the length $|N_i|$ of N_i, we see that N_i belongs to $\mathcal{F}(\Delta)$. Namely, if N_i is of length 1, then $U_i = 0$ since $\Delta(i) \neq 0$. If $|N_i| \geq 2$, then U_i is a direct sum of modules of the form N_j with $|N_j| < |N_i|$, thus by induction U_i belongs to $\mathcal{F}(\Delta)$ and therefore also N_i belongs to $\mathcal{F}(\Delta)$.

The surjective map $p_i\colon N_i \to \Delta(i)$ yields a surjective map $\operatorname{Ext}(N, N_i) \to \operatorname{Ext}(N, \Delta(i))$, thus $\operatorname{Ext}(N, \Delta(i)) = 0$ for all i, and therefore $\operatorname{Ext}(N, M) = 0$ for all $M \in \mathcal{F}(\Delta)$. This shows that the objects N_i are indecomposable projective objects in $\mathcal{F}(\Delta)$; actually, N_i is the projective cover of $\Delta(i)$ in $\mathcal{F}(\Delta)$. As usual, one sees now that $\operatorname{Ext}(\Delta(i), \Delta(j) \neq 0$ if and only if N_j is a direct summand of U_i. If N_j is a direct summand of U_i, then, in particular, $|N_j| < |N_i|$. This shows that the Ext-quiver of Δ is directed.

Starting with an Ext-directed antichain A in (1), and going via (2) to (3), we obtain the minimal projective generator P of $\mathcal{F}(A)$. Going from (3) to (1), we attach to P the antichain Δ whose elements are just the simple objects in $\mathcal{F}(A)$, but these are just the elements of A. Conversely, starting in (3) say with a normal module N without self-extensions, then going to (1), we attach to it the antichain Δ. Going via (2) to (3), we form a minimal projective generator in $\mathcal{F}(\Delta)$. But N is up to isomorphism the only minimal projective generator in $\mathcal{F}(\Delta)$. $\square$

(A.4) The bijection between (3) and (4).

From (4) to (3): *If T is a support-tilting module, let $\nu(T)$ be its normalization.* This clearly is a normal module without self-extensions. Here we use that any module M can be written in the form $M = M' \oplus M''$ where M' is normal and generates M'' (this of course is trivial), and that such a decomposition is unique up to isomorphism (this is not so obvious); the module M' is called a *normalization* of the module M. The uniqueness was first shown by Roiter [Ro] and then also by Auslander-Smalø [AS], see also [R4]. The uniqueness shows that the map ν going from (4) to (3) is well-defined.

Let us show that ν is injective when we are dealing with support-tilting modules. We claim the following: if T, T' are support-tilting modules with $\nu(T) = \nu(T')$,, then T and T' are Morita equivalent. For the proof, we may replace Λ by the support algebra $\Lambda(T) = \Lambda(T')$, thus we may assume that T, T' are tilting modules. Now, if T' is generated by $\nu(T') = \nu(T)$, thus therefore by T. But it is well-known (and easy to see) that the only modules without self-extensions which are generated by a tilting module module T belong to $\operatorname{add}(T)$. Similarly, we see that T belongs to $\operatorname{add}(T')$.

In order to see that ν is also surjective, we need to find for any normal module N without self-extensions a support-tilting module T with $\nu(T) = N$. This we will show next.

From (3) to (4): If N is a module without self-extensions, there is a module N', with the following properties: first, N' is generated by N, and second, $N \oplus N'$ is a support-tilting module; we call N' a *factor complement* for N (this is the dual version of forming a Bongartz complement, see for example [R3]).

Here is the construction of a factor complement N' of a module without self-extensions (we follow [R3]). Let $\Lambda(N)$ be the support algebra for N and Z an injective cogenerator for $\operatorname{mod}\Lambda(N)$. Choose an epimorphism $Y \to Z$ with kernel in $\operatorname{add} N$ such that $\operatorname{Ext}(Y, N) = 0$. Such an epimorphism can be obtained as a universal foundation of Z by N (sometimes also called a universal extension of Z by N from below): take exact sequences $0 \to N \to Y_i \to Z \to 0$ such that the corresponding elements in $\operatorname{Ext}(Z, N)$ form a k-basis, and form the direct sum of these sequences. The induced sequence with respect to the diagonal inclusion $u\colon Z \to \bigoplus_i Z$

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_i N \arrow[r] & \bigoplus_i Y_i \arrow[r] & \bigoplus_i Z \arrow[r] & 0 \\
0 \arrow[r] & \bigoplus_i N \arrow[r] \arrow[u, equal] & Y \arrow[r, "g"] \arrow[u] & Z \arrow[r] \arrow[u, "u"'] & 0
\end{tikzcd}$$

yields a universal foundation $g\colon Y \to Z$. In general, given a universal foundation $g\colon Y \to Z$ of Z by N, say with kernel N', the module Y is generated by N. Namely, since N has no self-extensions, it is a faithful $\Lambda(N)$-module, thus Z is generated by N. An epimorphism $h\colon N^t \to Z$ yields a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & N' \arrow[r] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \\
0 \arrow[r] & N' \arrow[r] \arrow[u, equal] & N'' \arrow[r] \arrow[u, "h'"'] & N^t \arrow[r] \arrow[u, "h"'] & 0
\end{tikzcd}$$

Since $\operatorname{Ext}(N, N) = 0$, the lower sequence splits, thus N'' belongs to $\operatorname{add} N$. Since h is surjective, also h' is surjective, thus Y is generated by N.

Actually, if we choose a minimal direct summand $\phi(N)$ of N' such that $N \oplus \phi(N)$ is a support tilting module, then $\phi(N)$ is uniquely determined by N and may be called a *minimal factor complement* for N. Thus, going from (3) to (4), we may attach to a normal module N without self-extension the multiplicity-free support-tilting module $N \oplus \phi(N)$.

Of course, in case N is normal, then N is the normalization of $N \oplus Y$. Thus starting with a normal module N without self-extensions, then going from (3) to (4) and back to (3), we obtain N. On the other hand, let T be support-tilting. From (4) to (3) we take $\nu(T)$. From (3) to (4), we add to $\nu(T)$ a factor complement, say N'. But T and $T' = \nu(T) \oplus N'$ both are support-tilting modules with $\nu(T) = \nu(T')$ and generated by this module, thus they are Morita equivalent. $\square$

(A.5) The bijection between (4) and (5).

First, we show the following: If T is a suport-tilting module and $\mathcal{G} = \mathcal{G}(T)$, then $\operatorname{add} T$ is the class of the Ext-projective modules in $\mathcal{G}$. Tilting theory asserts that $\mathcal{G}$ is the class of $\Lambda(T)$-modules M such that $\operatorname{Ext}(T, M) = 0$. Let M be in $\mathcal{G}$ and $g\colon T' \to M$ be a right T-approximation of M. Then g is surjective and the kernel M' of g satisfies $\operatorname{Ext}(T, M') = 0$, thus belongs to $\mathcal{G}$. If M is Ext-projective, then the exact sequence $0 \to M' \to T' \to M \to 0$ splits, thus M is in $\operatorname{add} T$. This shows that the Ext-projective modules in $\mathcal{G}$ are just the modules in $\operatorname{add} T$.

From (4) to (5); If T is a module without self-extensions, let $\mathcal{G}(T)$ be the class of modules generated by T. Then it is well-known (and easy to see) that T is a torsion class. Of course, T is a cover for $\mathcal{G}(T)$.

From (5) to (4): If $\mathcal{C}$ is a torsion class with a cover C, then we attach to it a module T such that $\operatorname{add} T$ is the class of Ext-projective modules in $\mathcal{G}$. In order to do so, we need to know that the class $\mathcal{E}$ of Ext-projective modules in $\mathcal{C}$ is finite, say $\mathcal{G} = \operatorname{add} T$ for some module T. We also have to show that T is support-tilting.

With C also its normalization $\nu(C)$ is a cover. A normal cover of a torsion class has no self-extension (see Proposition 1 of [R4]). Let B be a factor complement for $\nu(C)$. As we have seen, $T = \nu(C) \oplus B$ is a support-tilting module. Since B is generated by $\nu(C)$, we have $\mathcal{G}(T) = \mathcal{G}(\nu(C)) = \mathcal{G}(C) = \mathcal{C}$. But we have shown already that $\operatorname{add} T$ is the class of Ext-projective modules in $\mathcal{G}(T)$.

From (4) to (5) to (4): Let us start with a support-tilting module T and attach to it $\mathcal{G} = \mathcal{G}(T)$. As we have seen, the class of Ext-projectives in $\mathcal{G}$ is $\operatorname{add} T$. We choose T' with $\operatorname{add} T' = \operatorname{add} T$. But this just means that T, T' are Morita equivalent.

From (5) to (4) to (5). We start with a torsion class $\mathcal{C}$ with a cover, we choose a support-tilting module T with $\mathcal{C} = \mathcal{G}(T)$, thus we are back at $\mathcal{C}$. $\square$

(A.6) Duality.

Using duality, the sets (1), (2) and (4) are preserved. Of course, the dual concept of a thick subcategory with a cover is a thick subcategory with a cocover (note that an abelian k-category with finitely many simple objects and finite-dimensional Ext-groups has a cover if and only if it has a cocover).

Dualizing (3) we get:

(6) *The isomorphism classes of conormal modules without self-extensions.*

Dualizing (5) we get:

(7) *The torsionfree classes with a cocover.*

Both assertions are equivalent to (1), ... ,(5). □

Remark: The bijections between the set (2) of thick subcategories $\mathcal{A}$ and the sets (1), (3) and (6) of isomorphism classes of modules can be reformulated as follows: In an abelian category we may look at the semi-simple, the projective and the injective objects: the set of simple objects in $\mathcal{A}$ is an antichain in $\operatorname{mod}\Lambda$, a minimal projective generator in $\mathcal{A}$ is a normal module without self-extensions, a minimal injective cogenerator is a conormal module without self-extensions. These are the procedures to obtain from a thick subcategory the corresponding antichain, as well as a normal or conormal module without self-extensions.

Conversely, let us start with (1), (3) or (6). It has been mentioned already that starting with an antichain A, we take the full subcategory $\mathcal{F}(A)$ of all modules with a filtration with factors in A. Starting with a normal module P without self-extensions, the corresponding thick subcategory $\mathcal{A}$ consists of all modules which arise as the cokernel of a map in $\operatorname{add} P$ (in this way, we specify projective presentations of the objects in $\mathcal{A}$). Dually, starting with a conormal module I without self-extensions, the corresponding thick subcategory $\mathcal{A}$ consists of all modules which arise as the kernel of a map in $\operatorname{add} I$ (in this way, we specify injective presentations of the objects in $\mathcal{A}$). □

(A.7) The support.

Proposition. *The bijections which we have constructed preserve the support.*

□

(A.8) Sincere modules and subcategories.

Specializing the Ingalls-Thomas bijections to sincere modules, it follows from (A.7) that we get *bijections between:*

(1) *Isomorphism classes of* Ext*-directed sincere antichains.*
(2) *Thick subcategories with a sincere generator.*
(3) *Isomorphism classes of normal sincere partial tilting modules.*
(4) *Morita equivalence classes of tilting modules.*
(5) *Torsion classes with a sincere generator.*
(6) *Isomorphism classes of conormal sincere partial tilting modules.*
(7) *Torsionfree classes with a sincere cogenerator.* □

Of course, conversely this special case implies the general case. □

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