On an explicit representation of central $(2k+1)$-nomial coefficients

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We propose an explicit representation of central $(2k+1)$-nomial coefficients in terms of finite sums over trigonometric constructs. The approach utilizes the diagonalization of circulant boolean matrices and is generalizable to all $(2k+1)$-nomial coefficients, thus yielding a new family of combinatorical identities.

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I. INTRODUCTION

In the first volume of *Monthly* in 1894, De Volson Wood asked the question *"An equal number of white and black balls of equal size are thrown into a rectangular box, what is the probability that there will be contiguous contact of white balls from one end of the box to the opposite end?"* [1]. Though this was one of the first and most clearly visualizable examples in a set of combinatorical problems which later would give rise to the field of percolation theory [2], to this date it still eludes a definite solution.

Mathematically, this problem is linked to counting walks in random graphs, and recently an approach was proposed which translates this combinatorically hard problem into taking powers of a specific type of circulant boolean matrices [3]. Interestingly, here a specific type of sum over powers of fractions of trigonometric functions with fractional angle appears, specifically $\sum_{l=1}^{m-1}(\sin[kl\pi/m]/\sin[l\pi/m])^n$ for any given $k, m \in \mathbb{N} : m > 1, 1 \leq k \leq \lfloor m/2 \rfloor$. The latter shows a striking similarity to the famous Kasteleyn product formula for the number of tilings of a $2n \times 2n$ square with 1×2 dominos [4, 5].

Numerically one can conjecture that these constructs yield integer numbers which are multiples of central $(2k+1)$-nomial coefficients and, thus, provide an explicit representation of the integer sequences of multinomial coefficients in terms of finite sums over real-valued elementary functions. Although various recursive equations addressing these coefficients do exist, and the Almkvist-Zeilberger algorithm [6] allows for a systematic derivation of recursions for multinomial coefficients in the general case, no such explicit representation has yet been proposed.

Here, we provide a simple proof of the aforementioned conjecture and propose an explicit representation of central $(2k+1)$-nomial coefficients. To that end, let

$$P(x) = 1 + x + x^2 + \cdots + x^{2k} \tag{1}$$

be a finite polynomial of even degree $2k, k \in \mathbb{N}, k \geq 1$ in $x \in \mathbb{Q}$. Using the multinomial theorem and collecting terms with the same power in x, the nth power of $P(x)$ is then given by

$$P(x)^n = (1 + x + x^2 + \cdots + x^{2k})^n = \sum_{l=0}^{2kn} p_l^{(n)} x^l \tag{2}$$

with

$$p_l^{(n)} = \sum_{\substack{n_i \in [0,n] \forall i \in [0,2k] \\ n_0 + n_1 + \cdots + n_{2k} = n \\ n_1 + 2n_2 + \cdots + 2kn_{2k} = l}} \binom{n}{n_0, n_1, \cdots, n_{2k}}. \tag{3}$$

The central $(2k+1)$-nomial coefficients $M^{(2k,n)}$ are then given by $M^{(2k,n)} = p_{kn}^{(n)}$.

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II. A TRACE FORMULA FOR CENTRAL $(2k+1)$-NOMIAL COEFFICIENTS

Consider the $(2kn+1)\times(2kn+1)$ circulant matrix

$$\mathbf{A} = \operatorname{circ}\big\{(\overbrace{1,\dots,1}^{2k+1},0,\dots,0)\big\} = \operatorname{circ}\left\{\Big(\sum_{l=0}^{2k}\delta_{j,1+l\,\mathrm{mod}(2kn+1)}\Big)_j\right\}. \tag{4}$$

Multiplying $\mathbf{A}$ by a vector $\mathbf{x} = (1,x,x^2,\dots,x^{2k}) \in \mathbb{Q}^{2k+1}$ will yield the original polynomial as the first element in the resulting vector $\mathbf{Ax}$. Similarly, taking the n'th ($n' \leq n$) power of $\mathbf{A}$ and multiplying the result with $\mathbf{x}$ will yield $P(x)^{n'}$ as first element, thus $\mathbf{A}^{n'}$ will contain the sequence of multinomial coefficients $p_l^{(n')}$ in its first row. Moreover, as the power of a circulant matrix is again circulant, this continuous sequence of non-zero entries in a give row will shift by one column to the right on each subsequent row, and wraps around once the row-dimension $(2kn+1)$ is reached. This behavior will not change even if one introduces a shift by m columns of the sequence of 1 in $\mathbf{A}$, as this will correspond to simply multiplying the original polynomial by x^m. Such a shift, however, will allow, when correctly chosen, to bring the desired central multinomial coefficients on the diagonal of $\mathbf{A}^{n'}$.
We can formalize this approach in the following

Lemma 1. *Let*

$$A^{(m)} = \operatorname{circ}\left\{\Big(\sum_{l=0}^{2k}\delta_{j,1+(m+l)\mathrm{mod}(2kn+1)}\Big)_j\right\} \tag{5}$$

with $m \in \mathbb{N}_0$ be circulant boolean square matrices of dimension $2kn+1$ with $k,n \in \mathbb{N}$ and $k,n \geq 1$. The central $(2k+1)$-nomial coefficients are given by

$$M^{(2k,n)} = \frac{1}{2kn+1}\operatorname{Tr}\big[A^{(2kn-k)}\big]^n. \tag{6}$$

Proof. Let $B = \operatorname{circ}\big\{(0,1,0,\dots,0)\big\}$ be a $(2kn+1)\times(2kn+1)$ cyclic permutation matrix, such that

$$\begin{aligned} B^0 &\equiv I = B^{2kn+1} \\ B^m &= B^r \quad \text{with } r = m\,\mathrm{mod}(2kn+1) \\ B^n B^m &= B^{(nm)\,\mathrm{mod}(2kn+1)}, \end{aligned} \tag{7}$$

where I denotes the $(2kn+1)$-dimensional identity matrix. The set of powers of the cyclic permutation matrix, $\{B^m\}, m \in [0, 2kn+1]$, then acts as a basis for the circulant matrices $A^{(m)}$.
Let us first consider the case $m = 0$. It can easily be shown that

$$A^{(0)} = I + \sum_{l=1}^{2k} B^l.$$

Applying the multinomial theorem and ordering with respect to powers of B, we have for the nth power of $A^{(0)}$

$$\begin{aligned} \big(A^{(0)}\big)^n &= \sum_{n_0+n_1+\cdots+n_{2k}=n}\binom{n}{n_0,n_1,\cdots,n_{2k}} I^{n_0} B^{n_1+2n_2+\cdots+2kn_{2k}} \\ &= b_0^{(n)} I + \sum_{l=1}^{2kn} b_l^{(n)} B^l \\ &\equiv \operatorname{circ}\Big\{\big(b_0^{(n)}, b_1^{(n)},\dots,b_{2kn}^{(n)}\big)\Big\}, \end{aligned} \tag{8}$$

where $b_l^{(n)} = p_l^{(n)}$.
For $m > 0$, we have

$$A^{(m)} = B^m + \sum_{l=1}^{2k} B^{m+l} \equiv B^m\Big(I + \sum_{l=1}^{2k} B^l\Big),$$

and obtain, with (8), for the nth power

$$\big(A^{(m)}\big)^n = B^{mn}\Big(b_0^{(n)} I + \sum_{l=1}^{2kn} b_l^{(n)} B^l\Big). \tag{9}$$

Using (7), the factor B^{mn} shifts and wraps all rows of the matrix to the right by mn columns, so that

$$\begin{aligned}\big(A^{(m)}\big)^n &= b_0^{(n,m)} I + \sum_{l=1}^{2kn} b_l^{(n,m)} B^l \\ &\equiv \mathrm{circ}\Big\{\big(b_0^{(n,m)}, b_1^{(n,m)}, \ldots, b_{2kn}^{(n,m)}\big)\Big\},\end{aligned}$$

where $b_l^{(n,m)} = b_{(l-nm)\mathrm{mod}(2kn)}^{(n)}$ with $b_l^{(n)} = p_l^{(n)}$ given by (3).
For $m = 0$, the desired central multinomial coefficients can be found in column (kn), i.e. $M^{(2k,n)} = b_{kn}^{(n)}$. Observing that

$$(l - n(2kn-k))\mathrm{mod}(2kn) \equiv (l + kn)\mathrm{mod}(2kn)\,,$$

a shift by $m = 2kn - k$ yields

$$M^{(2k,n)} = b_{(l+kn)\mathrm{mod}(2kn)}^{(n)} = b_l^{(n,2kn-k)}. \tag{10}$$

That is, setting $l = 0$, the central $(2k+1)$-nomial coefficients $M^{(2k,n)}$ reside on the diagonal of $\big(A^{(2kn-k)}\big)^n$. Taking the trace of $\big(A^{(2kn-k)}\big)^n$ thus proves (6). □

With Lemma 1, the sequences of central $(2k+1)$-nomial coefficients $M^{(2k,n)}$ are given in terms of the trace of powers of the $(2kn+1)\times(2kn+1)$-dimensional circulant boolean matrix

$$\begin{aligned}A^{(2kn-k)} &= \mathrm{circ}\big\{(1, \overbrace{1,\ldots,1}^{k}, 0, \ldots, 0, \overbrace{1,\ldots,1}^{k})\big\} \\ &= \mathrm{circ}\left\{\Big(\sum_{l=0}^{k} \delta_{j,1+l} + \sum_{l=0}^{k-1} \delta_{j,1+2kn-l}\Big)_j\right\}.\end{aligned} \tag{11}$$

This translates the original problem not just into one of matrix algebra, but, in effect, significantly reduces its combinatorical complexity to finding powers of circulant matrices.

III. A SUM FORMULA FOR CENTRAL $(2k+1)$-NOMIAL COEFFICIENTS

Not only can circulant matrices be represented in terms of a simple base decomposition using powers of cyclic permutation matrices (see above), but circulant matrices also allow for an explicit diagonalization [7]. The latter will be utilized to prove the main result of this contribution, namely

Proposition 1. *The sequence of central $(2k+1)$-nomial coefficients $M^{(2k,n)}$ with $k, n \in \mathbb{N}, k, n > 0$ is given by*

$$M^{(2k,n)} = \frac{1}{2kn+1}\left\{(2k+1)^n + \sum_{l=1}^{2kn}\left(\frac{\sin\left[\frac{(2k+1)l}{2kn+1}\pi\right]}{\sin\left[\frac{l}{2kn+1}\pi\right]}\right)^n\right\}. \tag{12}$$

Proof. As $A^{(2kn-k)}$ is a circulant matrix, we can utilize the circulant diagonalization theorem to calculate its nth power. The latter states that all circulants $c_{ij} = \mathrm{circ}(c_j)$ constructed from an arbitrary N-dimensional vector c_j are diagonalized by the same unitary matrix $\mathbf{U}$ with components

$$u_{rs} = \frac{1}{\sqrt{N}} \exp\left[-\frac{2\pi i}{N_N}(r-1)(s-1)\right], \tag{13}$$

$r, s \in [1, N]$. Moreover, the N eigenvalues are explicitly given by

$$E_r(\mathbf{C}) = \sum_{j=1}^{N} c_j \exp\left[-\frac{2\pi i}{N}(r-1)(j-1)\right], \tag{14}$$

such that

$$c_{ij} = \sum_{r,s=1}^{N} u_{ir} e_{rs} u^*_{sj} \tag{15}$$

with $e_{rs} = \mathrm{diag}[E_r(\mathbf{C})] \equiv \delta_{rs} E_r(\mathbf{C})$ and u^*_{rs} denoting the complex conjugate of u_{rs}.
Using (11), the eigenvalues of $A^{(2kn-k)}$ are

$$\begin{aligned}
&E_r(A^{(2kn-k)}) \\
&= \sum_{j=1}^{2kn+1} \left\{ \sum_{l=0}^{k} \delta_{j,1+l} + \sum_{l=0}^{k-1} \delta_{j,1+2kn-l} \right\} e^{-2\pi i\,(r-1)(j-1)/(2kn+1)} \\
&= \sum_{l=1}^{k+1} e^{-2\pi i\,(l-1)(r-1)/(2kn+1)} + \sum_{l=2kn+2-k}^{2kn+1} e^{-2\pi i\,(l-1)(r-1)/(2kn+1)} \\
&= 1 + 2\sum_{l=1}^{k} \cos\left[2\frac{r-1}{2kn+1} l\pi\right] \\
&= \begin{cases} 1 + 2\sin\left[\frac{k(r-1)}{2kn+1}\pi\right]\cos\left[\frac{(1+k)(r-1)}{2kn+1}\pi\right] \big/ \sin\left[\frac{r-1}{2kn+1}\pi\right] & r > 1 \\ 2k+1 & r = 1. \end{cases}
\end{aligned}$$

Using the product-to-sum identity for trigonometric functions, the last equation can be simplified, yielding

$$E_r(A^{(2kn-k)}) = \begin{cases} \sin\left[\frac{2k+1}{2kn+1}(r-1)\pi\right] \big/ \sin\left[\frac{1}{2kn+1}(r-1)\pi\right] & r > 1 \\ 2k+1 & r = 1. \end{cases} \tag{16}$$

With this, Eqs. (13) and (15), one obtains for the elements of the nth power of $A^{(2kn-k)}$

$$\begin{aligned}
&\left(A^{(2kn-k)}\right)^n_{pq} \\
&= \sum_{r,s=1}^{2kn+1} u_{pr} E^n_r u^*_{rq} \\
&= \frac{1}{2kn+1} \left\{ E^n_0 + \sum_{r=2}^{2kn+1} E^n_r e^{-2\pi i\,(r-1)(p-q)/(2kn+1)} \right\} \\
&= \frac{1}{2kn+1} \left\{ (2k+1)^n + \sum_{r=1}^{2kn} \left(\frac{\sin\left[\frac{2k+1}{2kn+1} r\pi\right]}{\sin\left[\frac{1}{2kn+1} r\pi\right]} \right)^n e^{-2\pi i\, r(p-q)/(2kn+1)} \right\}.
\end{aligned}$$

Taking the trace, finally, proves (12). □

Equation (12) is remarkable in several respects. First, it provides a general, explicit representation of the sequences of central $(2k+1)$-nomial coefficients in terms of a linearly growing, but finite, sum, thus effectively translating a combinatorical problem into an analytical one. Note specifically $k = 1, n \in \mathbb{N}$ yields the sequence of central trinomial coefficients (OEIS A002426), $k = 2$ the sequence of central pentanomial coefficients (OEIS A005191), and $k = 3$ the sequence of central heptanomial coefficients (OEIS A025012). Secondly, utilizing trigonometric identities, this explicit representation may help to formulate general recurrences not just for coefficients of a given sequence, but between different central multinomial sequences. Moreover, by using different shift parameters m (see proof of Lemma 1), each $(2k+1)$-nomial coefficient could potentially be represented in a similarly explicit analytical form, thus allowing for a fast numerical calculation of arbitrary $(2k+1)$-nomial coefficients.
Finally, Proposition 1 establishes a direct link between central $(2k+1)$-nomial coefficients and the nth-degree Fourier series approximation of a function via the Dirichlet kernel $D_k[\theta]$. Using the trigonometric representation of Chebyshev polynomials of the second kind,

$$U_{2k}[\cos(\alpha)] = \frac{\sin[(2k+1)\alpha]}{\sin[\alpha]},$$

equation (12) takes the form

$$M^{(2k,n)} = \frac{1}{2kn+1} \left\{ (2k+1)^n + \sum_{l=1}^{2kn} \left(U_{2k} \left[\cos\left(\tfrac{1}{2kn+1} l\pi \right) \right] \right)^n \right\}. \tag{17}$$

Observing $U_{2k}[\cos(l\pi/(2kn+1))] \equiv D_k[2l\pi/(2kn+1)]$ makes explicit the link between central $(2k+1)$-nomial coefficients and the Dirichlet kernels of fractional angles.
Returning to the original problem by De Volson Wood, however, a definite solution is still at large. Here, relation (12) allows so far only for a representation of the combinatorical complexity in terms of an interesting finite analytical construct, thus, in principle, expressing a hard combinatorical problem in a trigonometric framework.

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