A meet-in-the-middle algorithm for finding extremal restricted additive 2-bases

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Abstract

An additive 2-basis with range n is *restricted* if its largest element is $n/2$. Among the restricted 2-bases of given length k, the ones that have the greatest range are *extremal restricted*. We describe an algorithm that finds the extremal restricted 2-bases of a given length, and we list them for lengths up to $k = 41$.

1 Introduction

Let n be a positive integer. An **additive 2-basis for** n, or more briefly a **basis for** n, is a set of integers $A_k = \{0 = a_0 < a_1 < \cdots < a_k\}$ such that every integer in $[0, n]$ is the sum of two of its elements, not necessarily distinct. The **length** of the basis is k. The largest possible n for a basis A_k is its **range** and denoted $n_2(A_k)$. The maximum range among all bases of length k is $n_2(k)$, and a basis that attains this maximum is **extremal**.

A basis A_k is **admissible** if $n_2(A_k) \geq a_k$, **restricted** if $n_2(A_k) \geq 2a_k$, and **symmetric** if $a_i + a_{k-i} = a_k$ for all $0 \leq i \leq k$. Since for any basis $n_2(A_k) \leq 2a_k$, a restricted basis has in fact $n_2(A_k) = 2a_k$ exactly.

The maximum range among restricted bases is called the **extremal restricted range** and denoted $n_2^*(k)$, and an **extremal restricted basis** is one that attains this maximum. For many values of k, at least some of the extremal bases are restricted, so that $n_2^*(k) = n_2(k)$. This is not always true: a counterexample is $k = 10$, where $n_2^*(10) = 44$, but $n_2(10) = 46$ (see Table 1).

Similarly, the maximum range among symmetric bases can be called the **extremal symmetric range**, and a basis that attains this maximum can be called **extremal symmetric basis**. Extremal symmetric bases are known up to $k = 30$ due to Mossige [5].

All extremal bases of lengths $k \leq 24$ are currently known [3]. Interestingly, all of them are either *both* symmetric and restricted, or *neither*. Three questions arise naturally:

1. If an extremal basis is symmetric, is it necessarily restricted?

range	notes	basis
44	R,A	0 1 2 3 7 11 15 17 20 21 22 †
44	R,S	0 1 2 3 7 11 15 19 20 21 22
44	R,S	0 1 2 5 7 11 15 17 20 21 22
44	R,A	0 1 2 5 7 11 15 19 20 21 22 †
44	R,S	0 1 2 5 8 11 14 17 20 21 22
44	R,A	0 1 3 4 6 11 13 18 19 21 22 ‡
44	R,S	0 1 3 4 9 11 13 18 19 21 22
44	R,A	0 1 3 4 9 11 16 18 19 21 22 ‡
46	NR,A	0 1 2 3 7 11 15 19 21 22 24
46	NR,A	0 1 2 5 7 11 15 19 21 22 24

Table 1: With length $k = 10$, there are eight extremal restricted bases and two extremal nonrestricted bases, listed by Wagstaff [10]. The two bases marked with † are mirror images of each other; similarly the two bases marked with ‡. Notes: R = restricted, NR = nonrestricted; S = symmetric, A = asymmetric.

2. If an extremal basis is restricted, is it necessarily symmetric?

3. Does every $k \geq 15$ have an extremal basis that is symmetric?

The first question is answered affirmatively by a theorem of Rohrbach [7]. The second question was posed by Riddell and Chan [6, p. 631], but to our knowledge has not been answered in general; with $k \leq 24$ the answer is yes.

The third question has appeared in a stronger form: it was suggested that all extremal bases with $k \geq 15$ might be symmetric [1]. This was later disproven by Challis and Robinson, since $k = 21$ has three extremal bases: one symmetric, two asymmetric [2]. The question remains whether every $k \geq 15$ has at least one extremal basis that is symmetric.

In this work we describe an efficient algorithm for finding all extremal restricted bases of a given length k. The algorithm is based on the idea that a restricted basis can be constructed by concatenating two shorter admissible bases, one of them as a mirror image. With this method we have computed all extremal restricted bases of lengths $k \leq 41$.

Note that we have included $a_0 = 0$ in a basis, similarly to Wagstaff [10]. If 0 is excluded, the equivalent condition is that every integer in $[0, n]$ is the sum of *at most* two elements of the basis [8, p. 3.1]. Excluding the zero is perhaps more usual in current literature, but including it is more convenient for our purposes. The zero element is not counted in the length of a basis.

2 Related work

Our search algorithm builds on a combination of existing ideas. Rohrbach discusses symmetric bases, and the proof of his Satz 1 is based on the observation that if a basis is mirrored from a_k, then its pairwise sums are mirrored from $2a_k$ [7]. We shall exploit a generalization of this for asymmetric restricted bases.

Riddell and Chan discuss the connection between symmetric and restricted bases [6]. Mossige notes that symmetric bases A_k can be efficiently searched by scanning through admissible bases of length $A_{\lceil k/2 \rceil}$ [5]. For symmetric bases this is sufficient; the second half of a symmetric basis is a mirror image of the first half, and then Rohrbach's theorem ensures that the constructed set A_k is a basis for $2a_k$. For asymmetric restricted bases, a similar search can be conducted separately for the two halves of the basis (prefix and suffix). However, since Rohrbach's theorem does not apply to asymmetric bases, the construction does not automatically yield a basis for $2a_k$. This must be checked separately.

The final ingredient is the "gaps test" by Challis [1]. Based on a simple combinatorial argument, it prunes the search tree of admissible bases, if they are required to have a range of at least a given target value T. In section 4 we shall prove lower bounds for the ranges of the prefix and the mirrored suffix. With these lower bounds the gaps test prunes the search tree very efficiently.

3 Definitions and initial results

If A and B are sets of integers, we define

$$A + B := \{a + b : a \in A, b \in B\},$$

and if b is an integer, we define the **mirror image** of A with respect to b as

$$b - A := \{b - a : a \in A\}.$$

The set of integers **generated by** A is

$$2A := A + A = \{a + a' : a, a' \in A\}.$$

It is straightforward to verify that

$$2(b - A) = 2b - 2A.$$

By $[c, d]$ we denote the consecutive integers $\{c, c + 1, \ldots, d\}$. Now the condition that A is a basis for n is succintly stated as follows:

$$2A \supseteq [0, n].$$

If $A_k = \{a_0 < \cdots < a_k\}$ is a basis and $i < k$, then $A_i = \{a_0, \ldots, a_i\}$ is a **partial basis**. We state without proof three easy observations (see [1] and [8]):

Lemma 1. *If a basis is restricted, then it is admissible.*

Lemma 2. *If a basis is extremal, then it is admissible.*

Lemma 3. *If a basis A_k is admissible, then for all $i < k$ the partial basis A_i is admissible, and $a_{i+1} \le n_2(A_i) + 1$.*

The first question posed in the introduction is now answered by the following theorem, essentially the same as Rohrbach's Satz 1 [7, p. 4].

Theorem 4. *If A_k is an extremal basis and it is symmetric, then it is restricted.*

Proof. Let $a_k = \max\{A_k\}$. By Lemma 2, A_k is admissible; thus $2A_k \supseteq [0, a_k]$. By symmetry $A_k = a_k - A_k$, thus

$$2A_k = 2(a_k - A_k) = 2a_k - 2A_k \supseteq [a_k, 2a_k].$$

Combining the above observations we have $2A_k \supseteq [0, 2a_k]$, thus A_k is restricted. □

Note that if A_k is a restricted basis with range n, then its largest element is exactly $a_k = n/2$. Exploiting the idea of mirroring from the largest element we obtain the following theorem.

Theorem 5. *If A_k is a restricted basis with range n, then $a_k - A_k$ is also a restricted basis for n.*

Proof. Since A_k is a basis for n, it follows that $2A_k \supseteq [0, n]$. Now

$$2(a_k - A_k) = 2a_k - 2A_k = n - 2A_k \supseteq [0, n],$$

thus $a_k - A_k$ is a basis for n. Its largest element is $a_k - 0 = a_k = n/2$, thus it is restricted. □

Theorem 5 implies that asymmetric restricted bases always form pairs that are mirror images of each other. Two such pairs are seen in Table 1.

4 Prefix and suffix of a restricted basis

Let A_k be a restricted basis with range n and length $k \ge 3$. Then by Theorem 5 the mirror image $B_k = a_k - A_k$ is also a restricted basis with range n. Choose now an arbitrary **pivot index** i such that $0 < i < k - 1$. Split A_k into a **prefix** $A_i = \{a_0 < \cdots < a_i\}$ and a **suffix** $R = \{a_{i+1} < \cdots < a_k\}$. The prefix is a partial basis of A_k. The suffix can be mirrored from a_k to obtain another basis

$$B_j = a_k - R = \{b_0 < \cdots < b_j\},$$

where $j = k - 1 - i$, and $b_h = a_k - a_{k-h}$ for all $0 \le h \le j$. Now B_j is a partial basis of B_k.

By Lemma 1 both A_k and B_k are admissible, and then by Lemma 3

$$n_2(A_i) \geq a_{i+1} - 1,$$
$$n_2(B_j) \geq b_{j+1} - 1.$$

We have now lower bounds for the ranges $n_2(A_i)$ and $n_2(B_j)$, but the bounds depend on a_{i+1} and b_{j+1}. However, these values can further be bounded from below:

$$a_{i+1} = a_k - b_j \geq a_k - (n_2(B_{j-1}) + 1) \geq a_k - n_2(j-1) - 1,$$
$$b_{j+1} = a_k - a_i \geq a_k - (n_2(A_{i-1}) + 1) \geq a_k - n_2(i-1) - 1,$$

where $n_2(i-1)$ and $n_2(j-1)$ are the maximum ranges of bases of lengths $i-1$ and $j-1$, respectively. These maximum ranges are currently known up to length 24.

Combining these bounds we can state a necessary condition for A_k being a restricted basis with range n.

Theorem 6. *If A_k is a restricted basis with range n, and i is an index such that $0 < i < k-1$, and $i + j = k - 1$, then:*

1. *The prefix A_i is an admissible basis such that $n_2(A_i) \geq a_k - n_2(j-1) - 2$.*
2. *The mirrored suffix $B_j = a_k - \{a_{i+1}, \ldots, a_k\}$ is an admissible basis such that $n_2(B_j) \geq a_k - n_2(i-1) - 2$.*

Example 7. Let $k = 10$ and $n = 44$, and choose $i = 5$ (thus $j = 4$). If A_{10} is a restricted basis for 44, then $a_{10} = 22$.

Since $n_2(3) = 8$, b_4 cannot be greater than $8 + 1 = 9$; in other words, $a_6 = 22 - b_4$ cannot be smaller than $22 - 9 = 13$; thus $n_2(A_5) \geq 12$.

Similarly, since $n_2(4) = 12$, a_5 cannot be greater than $12 + 1 = 13$; in other words, $b_5 = 22 - a_5$ cannot be smaller than $22 - 13 = 9$; thus $n_2(B_4) \geq 8$.

The lower bounds are the ones given by Theorem 6. Consider now a restricted basis A_{10} and its mirror image B_{10}, shown in right-to-left order:

$$\begin{array}{rcccccccccccl} & \multicolumn{6}{c}{\overbrace{\hphantom{0\ \ 1\ \ 3\ \ 4\ \ 6\ \ 11}}^{A_5}} & \multicolumn{5}{c}{\overbrace{\hphantom{13\ \ 18\ \ 19\ \ 21\ \ 22}}^{R}} & \\ A_{10} = & 0 & 1 & 3 & 4 & 6 & 11 & 13 & 18 & 19 & 21 & 22 & \\ & 22 & 21 & 19 & 18 & 16 & 11 & 9 & 6 & 3 & 1 & 0 & = B_{10} \\ & & & & & & & \multicolumn{5}{c}{\underbrace{\hphantom{13\ \ 18\ \ 19\ \ 21\ \ 22}}_{B_4}} & \end{array}$$

The prefix A_5 has range 12, and the mirrored suffix B_4 has range 10. Both ranges are within the bounds required by Theorem 6.

The second part of Theorem 6 also provides an upper bound for the range of a restricted basis:

$$\begin{aligned} n_2(A_k) = 2a_k &\le 2n_2(B_j) + 2n_2(i-1) + 4 \\ &\le 2n_2(j) + 2n_2(i-1) + 4. \end{aligned}$$

Choosing $i = \lfloor k/2 \rfloor$ this yields the following bounds, for even and odd values of k.

Corollary 8. *If $r > 1$ is an integer, then*

$$\begin{aligned} n_2^*(2r) &\le 4n_2(r-1) + 4, \\ n_2^*(2r+1) &\le 2n_2(r-1) + 2n_2(r) + 4. \end{aligned}$$

5 Search algorithm

Suppose that k and n are given, and the task is to enumerate every restricted basis of length k and range n (if any such bases exist). Choose a pivot index i, for example $i = \lfloor k/2 \rfloor$.

A straightforward method would be to enumerate all admissible prefix bases A_i, all admissible mirrored suffix bases B_j, and for each pair (A_i, B_j) check whether $A_i \cup (n/2 - B_j)$ happens to be a basis for n, that is, whether it generates all integers in $[0, n]$. For large k this is not feasible, as the number of admissible bases of length i increases rapidly (see A167809 in [9]).

However, Theorem 6 gives definite lower bounds for the ranges of the prefix A_i and the mirrored suffix B_j. Thus only a tiny fraction of all admissible prefixes and mirrored suffixes need to be considered, as seen in the following example.

Example 9. Let $k = 25$ and $n = 228$. We want to know whether there are any restricted bases with these values, and to list them if there are. Choose $i = 12$ (thus $j = 12$). The last element of a restricted basis must be $a_k = n/2 = 114$. There are 15 752 080 admissible bases of length 12, but we only need to consider the prefixes A_i such that $n_2(A_i) \ge 114 - n_2(11) - 2 = 58$; there are only 187 such prefixes.

Admissible bases with a given length and a given minimum range can be enumerated with the algorithm ("K-program") described by Challis [1]. Combining these ingredients we obtain Algorithm 1, which enumerates all restricted bases of given length k and range n.

If $n_2^*(k)$ is not known, Algorithm 1 can be run with different values of n, starting from the upper bound for $n_2^*(k)$ provided by Corollary 8. If no solutions are found, n is then decreased in steps of 2, until for some n there are solutions. Only even values of n need to be considered, since the range of a restricted basis is always even.

Algorithm 1 List restricted bases of length k and range n

Require: $k \geq 3$

$i \leftarrow \lfloor k/2 \rfloor$ {Choose pivot index}
$j \leftarrow k - i - 1$
$n_a \leftarrow n_2(i-1)$ {Lookup from A001212}
$n_b \leftarrow n_2(j-1)$ {Lookup from A001212}
$\mathcal{A} \leftarrow \{A_i : n_2(A_i) \geq n/2 - n_b - 2\}$ {List prefixes with Challis algorithm [1]}
$\mathcal{B} \leftarrow \{B_j : n_2(B_j) \geq n/2 - n_a - 2\}$ {List mirrored suffixes with Challis algorithm}
for all $A_i \in \mathcal{A}$ **do**
  for all $B_j \in \mathcal{B}$ **do**
    $R \leftarrow n/2 - B_j$ {Mirror from $n/2$}
    $A_k \leftarrow A_i \cup R$ {Concatenate}
    if $2A_k \supseteq [0, n]$ **then** {Generate pairwise sums and check range}
      print A_k
    end if
  end for
end for

Example 10. Let $k = 25$. By Corollary 8, $n_2^*(25) \leq 240$. For $n = 240$ the search algorithm finds no solutions. Then n is reduced in steps of 2, until for $n = 228$ the algorithm returns one solution:

$$A_{25} = \{0, 1, 3, 4, 6, 10, 13, 15, 21, 29, 37, 45, 53, \\ 61, 69, 77, 85, 93, 99, 101, 104, 108, 110, 111, 113, 114\}$$

By construction, this is an extremal restricted basis, so now we know that $n_2^*(25) = 228$.

6 Results

Using the search algorithm described in the previous section, we performed an exhaustive search for extremal restricted bases of lengths $k = 25, \ldots, 41$. The bases are listed in Table 3. For ease of reference, previously known extremal restricted bases of lengths $k = 1, \ldots, 24$ are listed in Table 2.

For lengths $25, \ldots, 29$ the extremal restricted bases are the extremal symmetric bases listed by Mossige [5]. For lengths $31, \ldots, 41$ they equal the bases given by Challis and Robinson's construction [2, p. 6]. Note that while the aforementioned construction gives a lower bound for the extremal restricted range, exhaustive search gives the exact range.

With $k = 30$, there are six extremal restricted bases with range 316. Four of them are symmetric and were listed by Mossige, but two are asymmetric. This is perhaps unexpected, and shows that at least one of the questions 2 and 3 stated in the introduction must be answered negatively. It is currently not known whether $n_2(30)$ is 316 or greater.

k	$n_2^*(k)$		basis																				
1	2	S	0	1																			
2	4	S	0	1	2																		
3	8	S	0	1	3	4																	
4	12	S	0	1	3	5	6																
5	16	S	0	1	3	5	7	8															
6	20	S	0	1	2	5	8	9	10														
6	20	S	0	1	3	5	7	9	10														
7	26	S	0	1	2	5	8	11	12	13													
7	26	S	0	1	3	4	9	10	12	13													
8	32	S	0	1	2	5	8	11	14	15	16												
9	40	S	0	1	3	4	9	11	16	17	19	20											
10	44		see Table 1																				
11	54	S	0	1	3	4	9	11	16	18	23	24	26	27									
11	54	S	0	1	3	5	6	13	14	21	22	24	26	27									
12	64	S	0	1	3	4	9	11	16	21	23	28	29	31	32								
13	72	S	0	1	3	4	9	11	16	20	25	27	32	33	35	36							
14	80	S	0	1	3	4	5	8	$\cdots$	+6	$\cdots$	32	35	36	37	39	40						
15	92	S	0	1	3	4	5	8	$\cdots$	+6	$\cdots$	38	41	42	43	45	46						
16	104	S	0	1	3	4	5	8	$\cdots$	+6	$\cdots$	44	47	48	49	51	52						
17	116	S	0	1	3	4	5	8	$\cdots$	+6	$\cdots$	50	53	54	55	57	58						
18	128	S	0	1	3	4	5	8	$\cdots$	+6	$\cdots$	56	59	60	61	63	64						
19	140	S	0	1	3	4	5	8	$\cdots$	+6	$\cdots$	62	65	66	67	69	70						
20	152	S	0	1	3	4	5	8	$\cdots$	+6	$\cdots$	68	71	72	73	75	76						
21	164	S	0	1	3	4	5	8	$\cdots$	+6	$\cdots$	74	77	78	79	81	82						
21	164	S	0	1	3	4	6	10	13	15	21	$\cdots$	+8	$\cdots$	61	67	69	72	76	78	79	81	82
22	180	S	0	1	3	4	6	10	13	15	21	$\cdots$	+8	$\cdots$	69	75	77	80	84	86	87	89	90
23	196	S	0	1	3	4	6	10	13	15	21	$\cdots$	+8	$\cdots$	77	83	85	88	92	94	95	97	98
24	212	S	0	1	3	4	6	10	13	15	21	$\cdots$	+8	$\cdots$	85	91	93	96	100	102	103	105	106

Table 2: Extremal restricted bases of lengths $k = 1, \ldots, 24$. S = symmetric, A = asymmetric. $+c$ indicates several elements with a repeated difference of c.

k	$n_2^*(k)$		basis
25	228	S	0 1 3 4 6 10 13 15 21 $\cdots$ +8 $\cdots$ 93 99 101 104 108 110 111 113 114
26	244	S	0 1 3 4 6 10 13 15 21 $\cdots$ +8 $\cdots$ 101 107 109 112 116 118 119 121 122
26	244	S	0 1 3 4 5 8 11 15 16 $\cdots$ +9 $\cdots$ 106 107 111 114 117 118 119 121 122
27	262	S	0 1 3 4 5 8 11 15 16 $\cdots$ +9 $\cdots$ 115 116 120 123 126 127 128 130 131
28	280	S	0 1 3 4 5 8 11 15 16 $\cdots$ +9 $\cdots$ 124 125 129 132 135 136 137 139 140
29	298	S	0 1 3 4 5 8 11 15 16 $\cdots$ +9 $\cdots$ 133 134 138 141 144 145 146 148 149
30	316	S	0 1 3 4 5 8 11 15 16 25 34 $\cdots$ +9 $\cdots$ 124 133 142 143 147 150 153 154 155 157 158
30	316	S	0 1 2 5 6 8 13 14 17 19 29 $\cdots$ +10 $\cdots$ 129 139 141 144 145 150 152 153 156 157 158
30	316	A	0 1 2 5 6 8 13 14 17 19 29 $\cdots$ +10 $\cdots$ 129 133 139 141 146 150 152 154 155 157 158
30	316	A	0 1 3 4 6 8 12 17 19 25 29 $\cdots$ +10 $\cdots$ 129 139 141 144 145 150 152 153 156 157 158
30	316	S	0 1 3 4 6 8 12 17 19 25 29 $\cdots$ +10 $\cdots$ 129 133 139 141 146 150 152 154 155 157 158
30	316	S	0 1 3 4 7 8 9 16 17 21 24 $\cdots$ +11 $\cdots$ 134 137 141 142 149 150 151 154 155 157 158
31	338	S	0 1 3 4 7 8 9 16 17 21 24 $\cdots$ +11 $\cdots$ 145 148 152 153 160 161 162 165 166 168 169
32	360	S	0 1 3 4 7 8 9 16 17 21 24 $\cdots$ +11 $\cdots$ 156 159 163 164 171 172 173 176 177 179 180
33	382	S	0 1 3 4 7 8 9 16 17 21 24 $\cdots$ +11 $\cdots$ 167 170 174 175 182 183 184 187 188 190 191
34	404	S	0 1 3 4 7 8 9 16 17 21 24 $\cdots$ +11 $\cdots$ 178 181 185 186 193 194 195 198 199 201 202
35	426	S	0 1 3 4 7 8 9 16 17 21 24 $\cdots$ +11 $\cdots$ 189 192 196 197 204 205 206 209 210 212 213
36	448	S	0 1 3 4 7 8 9 16 17 21 24 $\cdots$ +11 $\cdots$ 200 203 207 208 215 216 217 220 221 223 224
37	470	S	0 1 3 4 7 8 9 16 17 21 24 $\cdots$ +11 $\cdots$ 211 214 218 219 226 227 228 231 232 234 235
38	492	S	0 1 3 4 7 8 9 16 17 21 24 $\cdots$ +11 $\cdots$ 222 225 229 230 237 238 239 242 243 245 246
39	514	S	0 1 3 4 7 8 9 16 17 21 24 $\cdots$ +11 $\cdots$ 233 236 240 241 248 249 250 253 254 256 257
40	536	S	0 1 3 4 7 8 9 16 17 21 24 35 46 $\cdots$ +11 $\cdots$ 222 233 244 247 251 252 259 260 261 264 265 267 268
40	536	S	0 1 2 5 7 10 11 19 21 22 25 29 30 $\cdots$ +13 $\cdots$ 238 239 243 246 247 249 257 258 261 263 266 267 268
41	562	S	0 1 2 5 7 10 11 19 21 22 25 29 30 $\cdots$ +13 $\cdots$ 251 252 256 259 260 262 270 271 274 276 279 280 281

Table 3: Extremal restricted bases of lengths $k = 25, \ldots, 41$. S = symmetric, A = asymmetric. $+c$ indicates several elements with a repeated difference of c.

- If $n_2(30) = 316$, then we have here two extremal bases that are restricted, but asymmetric; this would answer question 2 negatively.

- If $n_2(30) > 316$, then there must be some (currently unknown) nonrestricted bases with range greater than 316, but they cannot be symmetric (for if they were, they would be restricted by Theorem 4). This would answer question 3 negatively.

As an example of the time requirement, with $k = 41$ and $n = 562$, the algorithm generates 5 514 prefixes of length 20 and range at least $n/2 - n_2(19) - 2 = 139$. These were enumerated in 120 CPU hours on parallel 2.6 GHz Intel Xeon processors, with a C++ implementation of the Challis algorithm. Since 41 is odd, we have $j = i$, and the mirrored suffixes are the same as the prefixes. The concatenation phase of the algorithm (lines 7 to 15) took 1.8 seconds with a Matlab implementation.

7 Discussion

Restricted bases are an interesting class of additive bases for two reasons. On one hand, searching for the extremal solutions among restricted bases is enormously faster than searching among all additive bases, as illustrated in the previous sections. This efficiency stems from Theorem 5, which places a very strong constraint on any extremal restricted basis: that its mirror image must also be a restricted basis (possibly different). Thus restricted additive bases can be seen as a generalization of symmetric additive bases.

On the other hand, among lengths $k = 1, \ldots, 24$, in almost every case at least one of the extremal bases is restricted (with the sole exception of $k = 10$). The reason for this is not known, and it is not known whether this regularity continues for $k > 24$. The case of $k = 30$, discussed in the previous section, suggests that there may be surprises waiting to be found.

For simplicity, we have always taken $i = \lfloor k/2 \rfloor$ in our search algorithm. Further research is needed to find the optimal pivot index i that minimizes the search work.

While Theorem 5 as such does not apply to nonrestricted bases, it would be interesting to know if it could be generalized in such a way that applies to them. Such a generalization might provide an improved search method for extremal additive bases in the nonrestricted case.

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2000 *Mathematics Subject Classification*: Primary 11B13.
Keywords: additive basis, restricted basis.

(Concerned with sequences A001212, A006638, and A167809.)
