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HOPF ALGEBRAS OF m-PERMUTATIONS, $(m+1)$-ARY TREES, AND m-PARKING FUNCTIONS

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Abstract. The m-Tamari lattice of F. Bergeron is an analogue of the clasical Tamari order defined on objects counted by Fuss-Catalan numbers, such as m-Dyck paths or $(m+1)$-ary trees. On another hand, the Tamari order is related to the product in the Loday-Ronco Hopf algebra of planar binary trees. We introduce new combinatorial Hopf algebras based on $(m+1)$-ary trees, whose structure is described by the m-Tamari lattices.

In the same way as planar binary trees can be interpreted as sylvester classes of permutations, we obtain $(m+1)$-ary trees as sylvester classes of what we call m-permutations. These objects are no longer in bijection with decreasing $(m+1)$-ary trees, and a finer congruence, called metasylvester, allows us to build Hopf algebras based on these decreasing trees. At the opposite, a coarser congruence, called hyposylvester, leads to Hopf algebras of graded dimensions $(m+1)^{n-1}$, generalizing noncommutative symmetric functions and quasi-symmetric functions in a natural way. Finally, the algebras of packed words and parking functions also admit such m-analogues, and we present their subalgebras and quotients induced by the various congruences.

Contents

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1. Introduction

Among the so-called combinatorial Hopf algebras, the Loday-Ronco algebra of planar binary trees [23] plays a prominent role. Aside from its operadic interpretation (it is the free dendriform algebra on one generator), it admits the Connes-Kreimer algebra as a quotient, and shares also many features with the Hopf algebra of standard Young tableaux, which itself explains the famous Littlewood-Richardon rule for multiplying Schur functions.

Indeed, both algebras can be defined in similar ways, from two congruences on words over a totally ordered alphabet, respectively the sylvester[1] congruence [15] and the plactic congruence [24, 35, 5]. In both cases, the product of two basis elements spans an interval of an order on the relevant combinatorial objects, respectively the Tamari order on binary trees, and the weak (Melnikov [27]) order on standard Young tableaux.

[1]Litteral translation of the French adjective *sylvestre*, and only an indirect tribute to the great algebraic combinatorist James Joseph Sylvester.

Recently, the Tamari order has been generalized to an infinite series of lattices, the m-Tamari orders [1], defined on combinatorial objects counted by Fuss-Catalan numbers, for example $(m+1)$-ary trees. This raises the question of the existence of a Hopf algebra on $(m+1)$-ary trees, for which the products of basis elements correspond to m-Tamari intervals.

We shall give a positive answer to this question. Actually, there are "m-analogues" of most classical combinatorial Hopf algebras (permutations, packed words, parking functions ...), the $(m+1)$-ary trees being obtained as sylvester classes of m-permutations, as for the Loday-Ronco algebra obtained for $m=1$. Regarding combinatorial Hopf algebras as generalizations of the algebra of symmetric functions, we can see that these m-analogues are related to its Adams operations ψ^m, raising the variables to the power m.

These m-analogues lead naturally to new congruences, called hyposylvester and metasylvester, which in turn lead to a wealth of new combinatorial Hopf algebras.

This article is structured as follows. We first recall the basic constructions of combinatorial Hopf algebras from congruences and polynomial realizations, and then establish a criterion for a congruence to be compatible with the Hopf algebra structure (Section 2). This criterion, which generalizes the one of [34], encompasses all previously known examples.

The new congruences are presented in Section 3. The metasylvester congruence accounts for the fact that $(m+1)$-ary decreasing trees are no longer in bijection with m-permutations for $m>1$. Thus, one may expect intermediate algebras of decreasing trees lying between naked $(m+1)$-ary trees and m-permutations. We shall see that metasylvester classes of m-permutations are parametrized by $(m+1)$-ary decreasing trees, and that sylvester classes are unions of metasylvester classes.

The hyposylvester congruence lies between the sylvester congruence and the hypoplactic congruence. Its introduction is motivated by the fact that there are always 2^{n-1} hypoplactic classes of m-permutations for any m, though one may have expected to see new algebras of graded dimensions $(m+1)^{n-1}$ in the diagram of homomorphisms. The hyposylvester congruence does exactly this. For ordinary permutations $(m=1)$, its classes coincide with the hypoplactic classes, which may explain why it has been overlooked.

The graded dimensions of the various Hopf algebras are computed by means of explicit formulas for the number of congruence classes of a given evaluation. These results are of course new for the new congruences, but also for the sylvester congruence, where it arises as an application of our previous results on noncommutative Lagrange inversion [33].

In Section 4, we introduce the Hopf algebras of m-permutations, ${}^m\mathbf{FQSym}$ and its dual, and describe its quotients and subalgebras induced by the various congruences. The algebras of trees, obtained from the sylvester and metasylvester congruences are discussed in full detail, the remaining cases being merely sketched. It should be noted, however, that the isomorphism of the hypoplactic quotient (or commutative image) with quasi-symmetric functions is given by the power-sum plethysm operator (or Adams operator ψ^m), and that all this m-combinatorics appears to be related to a noncommutative lift of these plethysm operations. We shall also see that the sylvester subalgebra of ${}^m\mathbf{FQSym}$ is isomorphic to a Hopf subalgebra of the Loday-Ronco algebra, which proves that the product of two basis elements spans an interval of an ideal of the Tamari order, which is precisely the m-Tamari order.

In Section 5, we introduce Hopf algebras of m-packed words, generalizing **WQSym** and its dual. The sylvester quotients and subalgebras generalizes the free tridendriform algebra on one generator (based on Schröder trees, counted by the little Schröder numbers). These algebras are based on m-Schröder paths, or on plane trees with no vertex of arity m or less, counted by generalized Runyon numbers. For the other congruences, we limit our description to the calculation of the graded dimensions.

Finally, in Section 6.3, we describe the natural Hopf algebra structures on m-parking functions (in the sense of Bergeron [1]), and investigate subalgebras and quotients. The hyposylvester quotient is interesting, it is based on $(2m+1)$-ary trees. Nondecreasing m-parking functions, which are in bijection with ordinary nondecreasing parking functions with repeated letters, are also particularly interesting. They are involved in the description of the images of the noncommutative Lagrange series by the adjoints ϕ_m of the plethysm operators, and the dual Hopf algebra ${}^m\mathbf{CQSym}^*$ can be identified as the plethysm $\psi^m(\mathbf{CQSym}^*)$. We conclude with the observation that the number of hypoplactic classes of m-parking functions can be obtained by means of m-Narayana polynomials, reciprocals of those giving the number of sylvester classes of m-packed words, and provide some tables of these polynomials and of various numbers of classes.

2. Congruences and polynomial realizations

2.1. **Polynomial realizations.** It is often the case that a combinatorial Hopf algebra admits what we call a *polynomial realization*. This means that some basis can be faithfully represented by some (in general, noncommutative) polynomials[2], so that the product rule of the algebra is induced by the ordinary multiplication of polynomials (or concatenation of words), and the coproduct is obtained by taking the disjoint union of two copies of the alphabet, endowed with some extra structure (in general, a total order, but other binary relations may be involved [10]).

The basis elements of a polynomial realization are obtained by summing words sharing some characteristic. This characteristic may be the result of a *normalization algorithm*, such as standardization [5], packing [30], parkization [31], or an equivalence class for some congruence on the free monoid. Examples include the plactic congruence [5, 35, 24], the hypoplactic congruence [18, 28], the sylvester congruence [15], and the stalactic congruence [31]. Sometimes, there is an analogue of the Robinson-Schensted correspondence associated with a congruence, and the Q-symbol can play the role of a normalization algorithm.

[2]More precisely, as elements of an inverse limit of polynomial algebras in the category of graded rings, as in the case of, *e.g.*, ordinary symmetric functions.

2.2. **Basic normalization algorithms.** In the sequel, we shall only need a countable totally ordered alphabet A, usually labeled by the positive integers. We denote by A^* the free monoid generated by A.

All algebras will be taken over a field $\mathbb{K}$ of characteristic 0. The notation $\mathbb{K}\langle A\rangle$ means the free associative algebra over A when A is finite, and the inverse limit $\varprojlim_B \mathbb{K}\langle B\rangle$, where B runs over finite subsets of A, when A is infinite.

The *evaluation* $\mathrm{ev}(w)$ of a word w is the sequence whose i-th term is the number of occurrences of the letter a_i in w.

The *packed evaluation* $\mathrm{pev}(w)$ of w if the composition obtained by removing the zeros in $\mathrm{ev}(w)$.

The *standardized word* $\mathrm{std}(w)$ of a word $w \in A^*$ is the permutation obtained by iteratively scanning w from left to right, and labelling $1, 2, \ldots$ the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. Alternatively, $\sigma = \mathrm{std}(w)^{-1}$ can be characterized as the unique permutation of minimal length such that $w\sigma$ is a nondecreasing word. For example, $\mathrm{Std}(bbacab) = 341625$.

The *packed word* $u = \mathrm{pack}(w)$ of a word $w \in A^*$ where $A = \{a_1 < a_2 < \cdots\}$ is obtained by the following process. If $b_1 < b_2 < \cdots < b_r$ are the letters occuring in w, u is the image of w by the homomorphism $b_i \mapsto a_i$. A word u is *packed* if $\mathrm{pack}(u) = u$. We denote by PW the set of packed words.

Packed words of a given evaluation $I \vDash n$ form a permutation representation of the symmetric group $\mathfrak{S}_n$ (acting on the right), and also of the 0-Hecke algebra $H_n(0)$. The 0-Hecke algebra, acting by non-invertible sorting operators, is more convenient, as it allows to distinguish compositions with the same underlying partition. The noncommutative characteristic of this action is the product of noncommutative complete functions S^I [18]. Its dimension is therefore the multinomial coefficient $n!S^I(\mathbb{E})$, where $S_i(\mathbb{E})$ is by definition $1/i!$. For example, the characteristic of PW is

$$\mathrm{ch}(\mathrm{PW}) = \sum_I S^I = (1 - S_1 - S_2 - S_3 - \cdots)^{-1} \tag{1}$$

which implies immediately that the exponential generating series of $|\mathrm{PW}_n|$ is

$$\sum_{n\geq 0} |\mathrm{PW}_n| \frac{t^n}{n!} = (1 - (e^t - 1))^{-1} = (2 - e^t)^{-1}. \tag{2}$$

This simple calculation will be generalized several times in the sequel.

For a word w over a totally ordered alphabet in which each element has a successor, we defined in [31] a notion of *parkized word* $\mathrm{park}(w)$, a parking function which reduces to $\mathrm{std}(w)$ when w is a word without repetition.

For $w = w_1 w_2 \cdots w_n$ on $\{1, 2, \ldots\}$, we set

$$d(w) := \min\{i | \#\{w_j \leq i\} < i\}\,. \tag{3}$$

If $d(w) = n + 1$, then w is a parking function and the algorithm terminates, returning w. Otherwise, let w' be the word obtained by decrementing all the values of w greater than $d(w)$. Then $\mathrm{park}(w) := \mathrm{park}(w')$. Since w' is smaller than w in the lexicographic order, the algorithm terminates and always returns a parking function.

For example, let $w = (3,5,1,1,11,8,8,2)$. Then $d(w) = 6$ and the word $w' = (3,5,1,1,10,7,7,2)$. Then $d(w') = 6$ and $w'' = (3,5,1,1,9,6,6,2)$. Finally, $d(w'') = 8$ and $w''' = (3,5,1,1,8,6,6,2)$, which is a parking function. Thus $\mathrm{park}(w) = (3,5,1,1,8,6,6,2)$.

For a word w over the alphabet $\{1,2,\ldots\}$, we denote by $w[k]$ the word obtained by replacing each letter i by the integer $i+k$. If u and v are two words, with u of length k, one defines the *shifted concatenation* $u \bullet v = u \cdot (v[k])$ and the *shifted shuffle* $u \Cup v = u ⧢ (v[k])$, where $w_1 ⧢ w_2$ is the usual shuffle product defined recursively by

- $w_1 ⧢ \epsilon = w_1, \quad \epsilon ⧢ w_2 = w_2$,
- $au ⧢ bv = a(u ⧢ bv) + b(au ⧢ v)$,

where $w_1 = a{\cdot}u$ and $w_2 = b{\cdot}v$, and both a and b are letters and $\cdot$ means concatenation.

For example,

$$12 \Cup 21 = 12 ⧢ 43 = 1243 + 1423 + 1432 + 4123 + 4132 + 4312\,. \tag{4}$$

2.3. Combinatorial Hopf algebras from elementary characteristics.

2.3.1. *Permutations.* Taking the standardization as characteristic, we obtain the algebra $\mathbf{FQSym}(A)$ of Free Quasi-Symmetric functions, an algebra spanned by the noncommutative polynomials [5]

$$\mathbf{G}_\sigma(A) := \sum_{\substack{w \in A^n \\ \mathrm{std}(w)=\sigma}} w, \tag{5}$$

where σ is a permutation in the symmetric group $\mathfrak{S}_n$. The multiplication rule is

$$\mathbf{G}_\alpha \mathbf{G}_\beta = \sum_{\gamma \in \alpha * \beta} \mathbf{G}_\gamma, \tag{6}$$

where the convolution $\alpha * \beta$ of $\alpha \in \mathfrak{S}_k$ and $\beta \in \mathfrak{S}_l$ is the sum in the group algebra of $\mathfrak{S}_{k+l}$ [25]

$$\alpha * \beta = \sum_{\substack{\gamma = uv \\ \mathrm{std}(u)=\alpha;\, \mathrm{std}(v)=\beta}} \gamma\,. \tag{7}$$

FQSym is therefore a polynomial realization of the Malvenuto-Reutenauer Hopf algebra [25].

Let $\mathbf{F}_\sigma = \mathbf{G}_{\sigma^{-1}}$. If one denotes by A' and A'' two mutually commuting alphabets isomorphic to A as ordered sets, and by $A' + A''$ the ordinal sum, the coproduct is defined by the ordinal sum of alphabets

$$\Delta \mathbf{F}_\sigma = \mathbf{F}_\sigma(A' + A'') = \sum_{u \cdot v = \sigma} \mathbf{F}_{\mathrm{std}(u)} \otimes \mathbf{F}_{\mathrm{std}(v)}\,, \tag{8}$$

(where $\cdot$ means concatenation) under the identification $U(A) \otimes V(A) = U(A')V(A'')$.

A scalar product is defined by

$$\langle \mathbf{F}_\sigma\,,\, \mathbf{G}_\tau \rangle = \delta_{\sigma,\tau}\,, \tag{9}$$

and one then has, for all $F, G, H \in \mathbf{FQSym}$

$$\langle FG, H \rangle = \langle F \otimes G, \Delta H \rangle \tag{10}$$

that is, **FQSym** is self-dual. The product formula in the $\mathbf{F}_\sigma$ basis is

$$\mathbf{F}_\alpha \mathbf{F}_\beta = \sum_{\sigma \in \alpha \Cup \beta} \mathbf{F}_\sigma. \tag{11}$$

2.3.2. *Packed words.* Taking the packing algorithm as characteristic, we define polynomials

$$\mathbf{M}_u := \sum_{\mathrm{pack}(w)=u} w\,, \tag{12}$$

for all packed word u. Under the abelianization map $\chi:\ \mathbb{K}\langle A\rangle \to \mathbb{K}[X]$, sending the noncommuting variables a_i to mutually commuting variables x_i, the $\mathbf{M}_u$ are mapped to the monomial quasi-symmetric functions M_I ($I = (|u|_a)_{a\in A}$ being the evaluation vector of u).

The $\mathbf{M}_u$ span a subalgebra of $\mathbb{K}\langle A\rangle$, called **WQSym** for Word Quasi-Symmetric functions [13], consisting in the invariants of the noncommutative version [6] of Hivert's quasi-symmetrizing action [14], which is defined by $\sigma \cdot w = w'$ where w' is such that $\mathrm{std}(w') = \mathrm{std}(w)$ and $\chi(w') = \sigma \cdot \chi(w)$. Hence, two words are in the same $\mathfrak{S}(A)$-orbit iff they have the same packed word.

The product of **WQSym** is given by

$$\mathbf{M}_{u'} \mathbf{M}_{u''} = \sum_{u \in u' *_W u''} \mathbf{M}_u\,, \tag{13}$$

where the *convolution* $u' *_W u''$ of two packed words is defined as

$$u' *_W u'' = \sum_{\substack{u = v\cdot w \\ \mathrm{pack}(v)=u',\, \mathrm{pack}(w)=u''}} u\,. \tag{14}$$

Evaluating $\mathbf{M}_u$ on an ordinal sum, we get

$$\mathbf{M}_u(A' + A'') = \sum_{0\le k\le \max(u)} \mathbf{M}_{(u|_{[1,k]})}(A')\mathbf{M}_{\mathrm{pack}(u|_{[k+1,\max(u)})}(A''), \tag{15}$$

where $u|_B$ denotes the subword obtained by restricting u to the subset B of the alphabet, so that the coproduct Δ is given by

$$\Delta \mathbf{M}_u(A) = \sum_{0\le k\le \max(u)} \mathbf{M}_{(u|_{[1,k]})} \otimes \mathbf{M}_{\mathrm{pack}(u|_{[k+1,\max(u)})}. \tag{16}$$

2.3.3. *Parking functions.* Let $A = \{a_1 < a_2 < \dots\}$. Taking the parkization algorithm as characteristic, we can define polynomials

$$\mathbf{G}_{\mathbf{a}}(A) = \sum_{\mathrm{park}(w)=\mathbf{a}} w, \tag{17}$$

$\mathbf{a}$ being a parking function.

The $\mathbf{G}_{\mathbf{a}}(A)$ span a subalgebra of $\mathbb{K}\langle A\rangle$. This algebra is denoted by $\mathbf{PQSym}^*$, and its product rule is given by

$$\mathbf{G}_{\mathbf{a}'} \mathbf{G}_{\mathbf{a}''} = \sum_{\mathbf{a} \in \mathbf{a}' *_P \mathbf{a}''} \mathbf{G}_{\mathbf{a}}\,, \tag{18}$$

where the *convolution* $\mathbf{a}' *_P \mathbf{a}''$ of two parking functions is defined as

$$\mathbf{a}' *_P \mathbf{a}'' = \sum_{\substack{\mathbf{a}=u\cdot v \\ \mathrm{park}(u)=\mathbf{a}',\, \mathrm{park}(v)=\mathbf{a}''}} \mathbf{a}\,. \tag{19}$$

The coproduct is again given by the ordinal sum

$$\Delta(\mathbf{G})(A) = \mathbf{G}(A' + A''). \tag{20}$$

The dual basis of $\mathbf{G_a}$ is denoted by $\mathbf{F_a}$. The product of **PQSym** is a shifted shuffle

$$\mathbf{F}_{\mathbf{a}'}\mathbf{F}_{\mathbf{a}''} := \sum_{\mathbf{a}\in\mathbf{a}'\Cup\mathbf{a}''} \mathbf{F}_{\mathbf{a}}\,. \tag{21}$$

2.4. **Hopf algebras from congruences.** Further examples of combinatorial Hopf algebras can be constructed by quotienting polynomial realizations by various congruences on A^*, or dually, by summing permutations, packed words or parking functions over congruence classes. All known examples are special cases of the following construction.

Let us define a *shuffle-like Hopf algebra* H as a graded connected Hopf algebra, with a basis b_w indexed by some class of words over the alphabet A of positive integers such that the product in this basis is given by the shifted shuffle and the coproduct is of the form

$$\Delta b_w = \sum_{w=w_1\cdot w_2} b_{a(w_1)} \otimes b_{a(w_2)}, \tag{22}$$

where a is an algorithm such that $a(u\cdot\sigma) = a(u)\cdot\sigma$ for any permutation σ and such that $\mathrm{pack}(a(v)) = \mathrm{pack}(v)$.

For example, **FQSym**, **WQSym**, and **PQSym** are shuffle-like Hopf algebras since their product in the $\mathbf{F}$ bases is given by the shifted shuffle and the algorithm used in their coproduct is pack for the first two algebras and park for the last one. We shall see that all m-generalizations of these algebras also are shuffle-like Hopf algebras .

The following result generalizes and refines [34][Chap. 2.1].

Theorem 2.1. *Let H be a shuffle-like Hopf algebra . Let $\equiv$ be a congruence on A^* such that*

- *two words are equivalent iff they have same evaluation and their packed words are equivalent,*
- *they are such that*

$$u \equiv v \implies u|_I \equiv v|_I, \tag{23}$$

for any interval I of A, the notation $w|_I$ meaning the restriction of w to I, obtained by erasing the letters not in I.

Then, the elements

$$P_u = \sum_{w\equiv u} b_w \tag{24}$$

span a Hopf subalgebra A of H.

Moreover, its dual Hopf algebra A^* can be identified with the quotient of H^* by the relations $b^*_w = b^*_{w'}$ iff $w \equiv w'$.

Proof – Let us first consider a product $P_{u_1}P_{u_2}$ and an element b_w obtained by expanding this product on the b basis. Note that the P_u are sums over disjoint sets of b_w, so that any b_w occurs in exactly one P_u. Let us now consider $w' \equiv w$. Let k be the size of u_1, ℓ the size of u_2. Then the restrictions of both w and w' to $I_1 = [1, k]$ are congruent, and the same holds for the restrictions to $I_2 = [k+1, k+\ell]$. So $w|_{I_1}$ and $w'|_{I_1}$ are both congruent to u_1 and $w|_{I_2}$ and $w'|_{I_2}$ are both congruent to $u_2[k]$ since $w|_{I_2} = w_2[k]$. So $b_{w'}$ is also obtained in the expansion of $P_{u_1}P_{u_2}$, so that the product is indeed a sum of terms P_u.

Let us now consider the coproduct of a P_u and an element $b_{w'} \otimes b_{w''}$ appearing in it. Consider the set S of words w with $w \equiv u$ and such that the same $b_{w'} \otimes b_{w''}$ appears in the coproduct of b_w. Write for each such word $w = w_1 \cdot w_2$, so that $a(w_1) = w'$ and $a(w_2) = w''$.

Consider now another element $b_{v'} \otimes b_{v''}$ with $v' \equiv w'$ and $v'' \equiv w''$. We want to prove that the coefficient of this element in the coproduct of P_u is the same as the coefficient of $b_{w'} \otimes b_{w''}$ is the same coproduct.

Since $\mathrm{ev}(v') = \mathrm{ev}(w')$ and $\mathrm{ev}(v'') = \mathrm{ev}(w'')$, one defines two permutations as the permutations with smallest number of inversions so that $v' = w' \cdot \sigma_1$ and $v'' = w'' \cdot \sigma_2$.

Then let $S' = S \cdot \sigma_1 \bullet \sigma_2$. The prefixes v_1 of the elements of S' satisfy $v_1 = w_1 \cdot \sigma_1$ and also satisfy $a(v_1) = v'$ by the properties of the algorithm a. So $\mathrm{pack}(v_1) = \mathrm{pack}(v') \equiv \mathrm{pack}(w') = \mathrm{pack}(w_1)$, and since $\mathrm{ev}(v_1) = \mathrm{ev}(w_1)$, we conclude that $v_1 \equiv w_1$. The same holds for the suffixes, so that all the v also belong to the same equivalence class.

As for the dual Hopf algebra, if one has an injective Hopf algebra map from A to S, then its transpose is a surjective Hopf algebra map from S^* to A^*, hence realizing the dual A^* as claimed. Indeed, the P_u are disjoint sums of b_w, so that the first matrix has one 1 in each column, and its transpose has one 1 in each row. Thus, each b^*_w is sent to exactly one element, always the same one for a given equivalence class. ∎

For example, the plactic, hypoplactic, and sylvester congruences all satisfy the statement of Theorem 2.1, so that, we directly conclude that quotienting, *e.g.*, **FQSym*** by any of these equivalences gives rise to (already known) Hopf algebras. Let us first recall some details of these cases before presenting two new congruences that arose when studying generalizations of permutations.

2.5. **The plactic congruence.** The plactic congruence [20] is generated by the Knuth relations

$$\left\{ \begin{array}{lllll} aba \equiv baa & , & bba \equiv bab & & \text{for } a < b, \\ acb \equiv cab & , & bca \equiv bac & & \text{for } a < b < c\ . \end{array} \right. \tag{25}$$

The fact that it satisfies the conditions of Theorem 2.1 is the starting point of Schützenberger's proof of the Littlewood-Richardson rule (see [24] for a simplified presentation), which has been reinterpreted in [5] in terms of the algebra **FSym** (Free Symmetric functions), based on standard Young tableaux.

The plactic monoid remained a mysterious and singular object until its true nature was revealed with the advent of quantum groups, and in particular of crystal bases [19, 22].

The number of plactic classes of words of packed evaluation I can be computed as a sum of Kostka numbers [26]

$$\sum_{\lambda \vdash n} K_{\lambda\mu} \tag{26}$$

if I is a composition of n whose underlying partition is μ.

2.6. **The hypoplactic congruence.** The hypoplactic congruence does for the fundamental basis of quasi-symmetric functions what the plactic congruence does for Schur functions. It was originally derived from a degenerate quantum group [18].

It is generated by the plactic relations, and the following quartic relations (derived from the presentation of the quantum group)

$$\left\{ \begin{array}{rclcrclc} baba & \equiv & abab & , & baca & = & abac & \text{for } a < b < c, \\ cacb & \equiv & acbc & , & cbab & \equiv & bacb & \text{for } a < b < c, \\ badc & \equiv & dbca & , & acbd & \equiv & cdab & \text{for } a < b < c < d\ . \end{array} \right. \tag{27}$$

This may look complicated, but there is a much simpler description: two words w and w' are congruent if and only if they have the same evaluation, and their standardizations $\mathrm{std}(w)$ and $\mathrm{std}(w')$ have same recoils [28].

It is interesting to observe that the hypoplactic congruence is also generated by the following nonlocal relations

$$ac\cdots b \quad \equiv \quad ca\cdots b \quad (a \le b < c), \tag{28}$$

$$b\cdots ac \quad \equiv \quad b\cdots ca \quad (a < b \le c), \tag{29}$$

where the dots may be any letters [28].

Hypoplactic classes of words are parametrized by *quasi-ribbon tableaux*. A *quasi-ribbon tableau* of shape I is a ribbon diagram r of shape I filled by letters in such a way that each row of r is nondecreasing from left to right, and each column of r is strictly increasing from *top to bottom*. A word is said to be a *quasi-ribbon word* of shape I if it can be obtained by reading from *bottom to top* and from left to right the columns of a quasi-ribbon diagram of shape I. Thus, hypoplactic classes of permutations are just recoil classes. Denoting by $\mathrm{RC}(\sigma)$ the recoil composition of a permutation σ and by $\mathrm{DC}(\sigma) = \mathrm{RC}(\sigma^{-1})$ its descent composition, we see that the sums

$$R_I = \sum_{\mathrm{RC}(\sigma)=I} \mathbf{F}_\sigma = \sum_{\mathrm{DC}(\sigma)=I} \mathbf{G}_\sigma \tag{30}$$

are just the noncommutative ribbon Schur functions [11, 5], and span a Hopf subalgebra isomorphic to **Sym**. Dually, the hypoplactic quotient of **FQSym** is isomorphic to $QSym$, the class of a $\mathbf{F}_\sigma$ being identified with F_I, $I = \mathrm{RC}(\sigma)$.

The number of hypoplactic classes of words of packed evaluation I is clearly $2^{\ell(I)-1}$. So hypoplactic classes of packed words are parametrized by *segmented compostions*, that is, compositions with two kinds of separators between the parts. There are 3^{n-1}

segmented compositions of $n > 0$, and the resulting pair of Hopf algebras are the free tricubical algebra on one generator and its dual [30].

Hypoplactic classes of parking functions correspond to *parking quasi-ribbons*, that is, to quasi-ribbon words which are parking functions. There is however a simpler parametrization [31].

Define a *segmented word* as a finite sequence of non-empty words, separated by vertical bars, *e.g.*, 232|14|5|746.

The parking quasi-ribbons can be represented as segmented nondecreasing parking functions such that the bars can only occur as $\cdots a|b\cdots$ with $a < b$.

The hypoplactic Hopf algebras obtained from **PQSym** have graded dimensions given by the little Schröder numbers, these are the free triduplicial algebra on one generator and its dual [33].

2.7. **The sylvester congruence.** The sylvester congruence [15] is generated by the relations

$$ac\cdots b \equiv ca\cdots b \quad (a \le b < c) \tag{31}$$

that is, the first hypoplactic relation in the nonlocal presentation (28).

It is proved in [15] that the equivalence classes of words under this congruence are parametrized by the words whose standardized words avoid the pattern 132.

The sylvester Hopf algebras obtained from **FQSym** are the Loday-Ronco algebra of planar binary trees, and its (isomorphic) dual.

From **WQSym**, one obtains the free tridendriform algebra on one generator (self-dual), whose graded dimensions are given by the little Schröder numbers (see [31, 33]).

2.8. **Counting sylvester classes.** As for the plactic and hypoplactic congruences, the number of sylvester classes of words of packed evaluation I can be computed. This relies upon the noncommutative Lagrange inversion formula [32, 33].

Let **PBT** denote the Loday-Ronco algebra of planar binary trees [23], and $\mathbf{P}_T$ its natural basis (notations are as in [15]). Let $\mathbf{Q}_T \in \mathbf{PBT}^*$ be the dual basis of $\mathbf{P}_T$. Let $Q_T \in QSym$ be the image of $\mathbf{Q}_T$ by the adjoint of the Hopf algebra morphism $\mathbf{Sym} \to \mathbf{PBT}$ sending S_n to the left comb with n (internal) vertices. Expanding Q_T in the monomial basis M_I of $QSym$, we get that the coefficient of M_I in Q_T is then 1 or 0, according to whether there is a word of evaluation I which has sylvester shape T or not. By duality, the number of terms in the expansion

$$S^I = \sum_T \langle S^I, \mathbf{Q}_T \rangle \mathbf{P}_T \tag{32}$$

is equal to the number of sylvester classes of evaluation I.

Now,

$$S^I = \sum_{J \le I} R_J, \tag{33}$$

and $\langle R_J, \mathbf{Q}_T \rangle$ can be computed from [33, Prop. 3.2]. Under the bijection between nondecreasing parking functions and binary trees described there, each R_J is an interval of the Tamari order, consisting of all nondecreasing parking functions of packed evaluation $\overline{J}$. The number of these is the coefficient of $S^{\overline{J}}$ in the Lagrange series g, which is the unique noncommutative symmetric series satisfying the functional equation

$$g = \sum_{n\geq 0} S_n g^n. \tag{34}$$

To summarize,

Theorem 2.2. *The number of sylvester classes of evaluation I is*

$$\sum_{J\leq \overline{I}} \langle M_J, g\rangle, \tag{35}$$

where $\langle\cdot,\cdot\rangle$ is here the duality bracket between $QSym$ and **Sym**.

∎

Defining a basis E_I of $QSym$ by

$$E_I = \sum_{J\leq I} M_J \tag{36}$$

and denoting by L_I its dual basis in **Sym**, we can rewrite this number as

$$\langle E_{\overline{I}}, g\rangle \ = \ \text{coefficient of } L_{\overline{I}} \text{ in } g. \tag{37}$$

Example 2.3. Take $I = (112)$. Then, $\bar{I} = (211)$, and the term of degree 4 in g is

$$g_4 = S^4 + 3S^{31} + 2S^{22} + 3S^{211} + S^{13} + 2S^{121} + S^{112} + S^{1111}. \tag{38}$$

The compositions $J \leq (211)$ are $(211), (31), (22), (4)$. Summing the coefficients of the correponding S^J, we obtain $3+3+2+1 = 9$ sylvester classes of words of evaluation (112). And indeed, representatives of these classes (avoiding the pattern 132) are

$$1233,\ 2133,\ 2313,\ 2331,\ 3123,\ 3213,\ 3231,\ 3312,\ 3321. \tag{39}$$

Example 2.4. The number of sylvester classes of permutations is

$$\langle E_{1^n}, g\rangle = \sum_{J\vDash n} \langle M_I, g\rangle \tag{40}$$

which is the sum of the coefficients of g_n in the basis S^I, the nth Catalan number. Indeed, under the specialization $S_n = t^n$ ($A = \{t\}$), the functional equation for g becomes

$$g(t) = \sum_{n\geq 0} t^n g(t)^n = (1 - tg(t))^{-1} \tag{41}$$

so that $g(t)$ reduces to the generating series of Catalan numbers. For $n = 4$, we can check that $1+3+2+3+1+2+1+1 = 14$.

Example 2.5. To find the number of sylvester classes of packed words, we have to compute

$$\left\langle \sum_{I \vDash n} E_{\overline{I}}, g \right\rangle = \left\langle \sum_{\lambda \vdash n} 2^{n-\ell(\lambda)} m_\lambda, g \right\rangle \tag{42}$$

where the m_λ are the monomial symmetric functions. This can be evaluated by noting that

$$\begin{aligned} \sum_{\lambda \vdash n} 2^{n-\ell(\lambda)} m_\lambda &= \frac{1}{(1-t)^n} \sum_{\lambda \vdash n} (1-t)^{\ell(\lambda)} m_\lambda |_{t=1/2} \\ &= \frac{1}{(1-t)^n} h_n((1-t)X)|_{t=1/2} \end{aligned} \tag{43}$$

Thus, the number of sylvester classes of packed words is obtained by putting $x = 2$ in the polynomial

$$\begin{aligned} N_n(x) &= \frac{1}{(1-t)^n} \frac{h_n((1-t)(n+1))}{n+1} \Big|_{t=1-1/x} \\ &= \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} \binom{2n-k}{n-k} (1-x)^k x^{n-k} \end{aligned} \tag{44}$$

a (reciprocal) Narayana polynomial, which gives back the little Schröder numbers, as expected.

3. New congruences

The m-generalizations of the classical combinatorial Hopf algebras introduced in the sequel lead naturally to two new congruences.

3.1. The hyposylvester congruence.

Let A be a totally ordered alphabet. Let $\sim$ be the equivalence relation on A^* defined by

$$w = \cdots ac \cdots \sim w' = \cdots ca \cdots \ (a < c \in A) \tag{45}$$

if a letter b such that $a < b < c$ occurs in w, and

$$w = \cdots ab \cdots a \cdots \sim w' = \cdots ba \cdots a \cdots \ (\ a < b \in A), \tag{46}$$

where the dots represent arbitrary letters. We have

Proposition 3.1. *The equivalence relation $\sim$ is a congruence on A^*, that is, $w \sim w'$, then $uwv \sim uw'v$ for all $u, v \in A^*$.* ■

Clearly, $\sim$ is a quotient of the sylvester congruence. We will call it the *hyposylvester congruence*, and denote it by $\equiv_{\mathrm{hs}}$. Therefore, any hyposylvester class is a disjoint union of sylvester classes.

The hyposylvester relations are homogeneous, and since they depend only on *comparisons* between letters, we have:

Lemma 3.2. *Two words w, w' are in the same hyposylvester class if and only if*

$$\mathrm{ev}(w) = \mathrm{ev}(w') \quad \textit{and} \quad \mathrm{pack}(w) \equiv_{hs} \mathrm{pack}(w'). \tag{47}$$

Thus, the number of classes of any evaluation α depends only on the composition $I = \mathrm{pack}(\alpha)$. ∎

Let us now give a formula for the number of hyposylvester classes of (packed) evaluation I.

Let us interpret (45) and (46) as rewriting rules, orienting them from left to right so that $w' > w$ lexicographically. The number of classes of packed evaluation I is at most the number of words of evaluation I admitting no such elementary rewriting, since there is at least one such word in each class. So these maximal words are precisely those in which each occurence of a letter j, except perhaps for the last one, is followed by smaller letters. This last occurence is either followed by a smaller letter, or by $j+1$, or by nothing. Building the word by inserting its letters sequentially in decreasing order of magnitude, we see that at each step, we have $i_k + 1$ choices to place the letters k, since they must necessarily be consecutive, immediately to the left of an occurence of $k+1$ or at the end of the word.

Example 3.3. To find all maximal words of evaluation $I = (222)$, we first insert 33, then a block 22 in 3 possible ways: 2233, 3223, 3322, and finally, a block 11, resulting into the 3^2 possibilities

$$112233,\ 211233,\ 223311,\ 311223,\ 321123,\ 322311,\ 331122,\ 332112,\ 332211. \tag{48}$$

Thus, the number of hyposylvester classes of evaluation I is at most

$$(i_2+1)(i_3+1)\cdots(i_r+1). \tag{49}$$

Let us now prove that there are as many classes as maximal words by providing a hyposylvester class invariant: the class of any packed word u of evaluation I is the set of all words v in which the last occurence of each letter k has exactly as many letters $k+1$ to its left as in u. Write $u \equiv' v$ when this is the case. Indeed, by a direct check on the elementary rewritings, all words v in the hyposylvester class of u must have this property, so that $u \equiv_{\mathrm{hs}} v$ implies $u \equiv' v$. But clearly, the number of $\equiv'$-classes of evaluation I is given by (49). Thus, $\equiv'$ coincides with $\equiv_{\mathrm{hs}}$.

Summarizing, we have:

Theorem 3.4. *The number of hyposylvester classes of packed evaluation $(i_1, \ldots, i_r)$ is equal to*

$$(i_2+1)(i_2+1)\ldots(i_r+1). \tag{50}$$

The hyposylvester class of a packed word u is the set of all words v in which the last occurence of each letter k has exactly as many letters $k+1$ to its left as in u. Moreover, each class has a unique element that do not have any elementary rewriting increasing its number of inversions. ∎

Example 3.5. For permutations, we obtain 2^{n-1} classes coinciding with the hypoplactic classes, since (45) reduces to the nonlocal hypoplactic relations, and (46) is never applied.

Example 3.6. To compute the generating series $c(t)$ of the number c_n of hyposylvester classes of packed words of length n, we can proceed as follows. Let χ_0 and χ_1 be the characters of **Sym** defined by

$$\chi_0(S_n) = t^n, \quad \chi_1(S_n) = (n+1)t^n \quad (n \geq 0)\,. \tag{51}$$

and define a linear map α by

$$\alpha(S^{i_1 i_2 \cdots i_r}) = \chi_0(S_{i_1})\chi_1(S^{i_2\cdots i_r}), \tag{52}$$

Let

$$H = \sum_I S^I, \text{ so that } c(t) = \alpha(H). \tag{53}$$

Clearly,

$$H = 1 + \left(\sum_{n\geq 1} S_n\right) H, \text{ and } \alpha(H) = 1 + \chi_0\left(\sum_{n\geq 1} S_n\right)\chi_1(H) \tag{54}$$

so that

$$c(t) = 1 + t\cdot\frac{1}{1-t}\cdot\frac{1}{1-\sum_{n\geq 1}(n+1)t^n} = \frac{1-3t+2t^2}{1-4t+3t^2}. \tag{55}$$

The first terms are

$$c(t) = 1 + t + 3\,t^2 + 10\,t^3 + 34\,t^4 + 116\,t^5 + 396\,t^6 + 1352\,t^7 + 4616\,t^8 + \ldots \tag{56}$$

(sequence A007052).

Example 3.7. It will be proved in Section 6.4 that the number of hyposylvester classes of parking functions of length n is equal to the number of ternary trees with n (internal) nodes.

As the hyposylvester congruence satisfies the conditions of Theorem 2.1, there are hyposylvester quotients and subalgebras of **FQSym**, **WQSym**, and **PQSym**, which will be investigated in the sequel together with their m-analogues.

3.2. **The metasylvester congruence.** The metasylvester congruence is defined by the nonlocal relations

$$ab\cdots a \;\equiv\; ba\cdots a \ (a<b), \tag{57}$$

$$b\cdots ac\cdots b \;\equiv\; b\cdots ca\cdots b \ (a<b<c)\,. \tag{58}$$

These relations are compatible with the concatenation of words, so that they indeed define a congruence $\equiv_{\mathrm{ms}}$ of the free monoid A^*. The quotient will be called the metasylvester monoid.

This congruence is very close to both the sylvester and hyposylvester congruences. First, it is also compatible with restriction to intervals (exchanging a and c requires an extra letter in the interval $[a,c]$ to be performed), and with any increasing morphism of ordered alphabets (the relations involve only comparisons between values, not the actual values), so that it satisfies the statement of Theorem 2.1 as well, and we can use it to build new Hopf algebras.

We shall use the same technique as for the hyposylvester monoid (see Section 3.1) to provide first a characterization of some canonical elements of the classes, and then an invariant of the classes, thus proving all statements of Theorem 3.9 below in one stroke.

First, consider the packed words w having no elementary rewriting raising their number of inversions. In other words, consider the words w avoiding both patterns $ac\dots a$ and $b\dots ac\dots b$. Then, all the letters 1 are necessarily consecutive, because of the first pattern. Now, if a word w avoids both patterns, its restriction to the letters greater than 1 avoids the patterns as well. From this, we can conclude that the number of metasylvester classes of packed evaluation I is at most

$$(1+i_r)(1+i_{r-1}+i_r)\cdots(1+i_2+\cdots+i_r). \tag{59}$$

As in the hyposylvester case, we have an invariant of metasylvester classes, provided by the following algorithm:

Algorithm 3.8. *Let w be a word. The algorithm is defined recursively.*

- *Step 1: label the root with the maximal letter n of w. Put $i = n-1$.*
- *Step 2: consider the smallest $j > i$ such that the last occurrence of i lies* between *two occurrences of j.*
  - *Step 2.1: if such a j exists, the $(k+1)$th subtree of j, where k is the number of occurences of j to the left of the last occurence of i becomes the tree obtained by inserting the restriction of w to i and all letters that are already in the $(k+1)$th subtree of j. End the algorithm if $i = 1$. Otherwise, start again Step 2 with $i = i-1$.*
  - *Step 2.2: if no such j exists, if this occurrence is to the left of all maximal letters of w, then the leftmost node of the root becomes the tree obtained by inserting the restriction of w to i and all letters that are already in the leftmost subtree of the root. The same happens on the rightmost node if the last occurrence of i is to the right of all maximal letters of w. End the algorithm if $i = 1$. Otherwise, start again Step 2 with $i = i-1$.*

For example, with $w = 1413324343$, we obtain

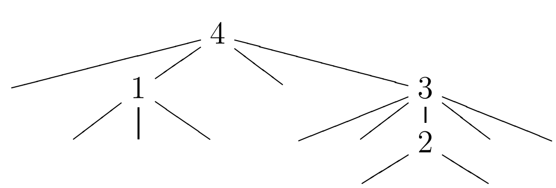


FIGURE 1. The decreasing tree of 1413324343.

Let now be two words w and w' congruent by one elementary rewriting exchanging a small value a with a greater one. Then all steps of the algorithm up to $a+1$ are the same, and the insertion of a follows the same path for both words. Indeed, j is the same, or does not exist for both words, and then the induction applies. So this algorithm provides an invariant, showing that there are at least as many metasylvester classes of evaluation I as decreasing trees where node k has $1 + i_k$ children, since all such trees can be obtained from at least one word.

Finally, the number of decreasing trees where node k has $1+i_k$ children is the same as in Equation (59) since letter $k-1$ has indeed $(1 + i_k + \cdots + i_r)$ places to go in the decreasing tree consisting of nodes greater than or equal to k.

Summarizing, we get:

Theorem 3.9. *The number of metasylvester classes of packed evaluation* $(i_1, \ldots, i_r)$ *is equal to*

$$(1 + i_r)(1 + i_{r-1} + i_r) \cdots (1 + i_2 + \cdots + i_r). \tag{60}$$

The metasylvester class of a packed word u is the set of all words v giving the same result by the previous algorithm. Moreover, each class has a unique element admitting no elementary rewriting increasing its number of inversions.

For permutations of size n, we get $n!$ and the congruence is trivial.

For packed words of length n, the formula yields $\frac{1}{2}(n + 1)!$. Indeed, one has the induction $P_{n+1} = (n + 1)P_n$: write $MS(I)$ for the number of classes of evaluation $I = (i_1, \ldots, i_r)$. Then if $I' = (1+i_1, i_2, \ldots, i_r)$ and $I'' = (1, I)$ then $MS(I') = MS(I)$ and $MS(I'') = n.MS(I)$. Since all compositions of size $n + 1$ are obtained by one of these two processes, we conclude to the induction and the formula.

For parking functions of length n, the number of classes appears to be given by Sequence A132624 of [37]. It would be interesting to have a direct proof of this result.

3.3. **Notes on both congruences.**

3.3.1. *The canonical elements of the metasylvester congruence.* Let us first describe two alternative ways to compute the maximal word of a given metasylvester class.

Consider a decreasing tree where the node labelled k has $1 + i_k$ children, and complete it by adding to each node the missing leaves. Then read the nodes following the infix reading: read the leftmost subtree of the root, then the root, then the next subtree of the root, then the root, and so on, ending by reading the rightmost subtree of the root.

The word obtained by this algorithm is the same as the following one: starting from a decreasing $(m + 1)$-ary tree, complete it as above, then label each sector between two edges of a given node by the value of the node and read now each sector in left-right order.

For example, both readings of the tree represented in Figure 1 yield the word 4114433233. It is indeed the maximal word of its class since it cannot have any of the patterns $\cdots ac \cdots$ and $b \cdots ac \cdots b$. We shall call this word the *canonical word* of its class.

Proposition 3.10. *The metasylvester canonical words are the words avoiding the pattern $a \cdots b \cdots a$ with $a < b$ (*i.e.*, a letter cannot be between two occurences of a smaller one). Such words will be identified with their decreasing trees.*

Proof – Assume first that a word w contains a pattern $a\cdots b\cdots a$. Choose an occurrence of this pattern such that $a\cdots b$ is of minimal length r. If $r=2$, this is a pattern $ab\cdots a$, and w cannot be canonical. If $r>2$, let x be the letter to the left of b. Then, $x>a$ or $x<a$. The first possibility is excluded by the minimality assumption, and if $x<a$, then $a\cdots xa\cdots a$ is a pattern $b\cdots ac\cdots a$, so that again, w in not canonical. Conversely, if w does not contain $a\cdots b\cdots a$, then in particular, it cannot contain the patterns $ab\cdots a$ or $b\cdots ac\cdots b$. ∎

3.3.2. *The special case where all i_k are equal.* In the case where all parts i_k of the evaluation are equal to an integer m, which shall be called later the set of m-permutations, the resulting decreasing trees are in fact all decreasing $(m+1)$-ary trees. In the special cases $m=1$ or $m=2$, one can simplify the output of the algorithm computing the decreasing tree of a word: one can forget about the empty subtrees by making the convention that the first subtree is to the left, the second one to the middle and the third one to the right of their root. For example, with $m=2$ and $n=9$, one gets the following tree by inserting 556119367322474898, whose canonical word is 559611673322744988.

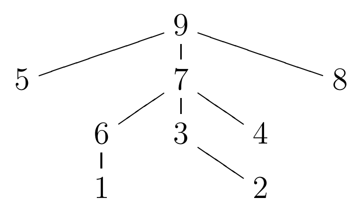


Figure 2. The decreasing ternary tree of 556119367322474898.

Note also that in the case of permutations ($m=1$), this algorithm reduces to the well-known algorithm computing the decreasing (binary) tree of a permutation.

3.3.3. *Relating the (*)-sylvester congruences.* The hyposylvester classes are unions of sylvester classes, which are themselves unions of metasylvester classes. In particular, the canonical elements of the hyposylvester classes are canonical elements of both the sylvester and metasylvester classes and the canonical elements of the sylvester classes are canonical elements of the metasylvester classes.

Hence the hyposylvester classes are in bijection with a set of decreasing trees which is easily described: these are the decreasing trees such that, for all i smaller than n, node i is a child of $i+1$ but not a rightmost child, or is the rightmost node of the decreasing tree restricted to nodes greater than or equal to i. This implies clearly that the number of hyposylvester classes of evaluation I is as in Equation (49).

One can do the same with sylvester classes: they are also in bijection with a set of decreasing trees: the decreasing trees where, for all $i\neq n$, node i cannot be to the left of node $i+1$ and not a child of $i+1$. One can also understand this result as follows: there is only one way to label a given tree so as to obtain a representative of a sylvester class.

4. Hopf algebras from m-Permutations

4.1. **Definitions.** We have seen that the Loday-Ronco algebra can be constructed from the sylvester congruence by summing over sylvester classes of permutations, which are counted by the Catalan numbers [15]. Similarly, $(m+1)$-ary trees, which are counted by the m-Fuss-Catalan numbers, are in natural correspondence with sylvester classes of m-permutations.

Definition 4.1. *An m-permutation of n is a word of evaluation (m^n), that is, a shuffle of m ordinary permutations of n. We denote by $\mathfrak{S}_n^{(m)}$ the set of m-permutations of n.*

Their number is therefore $(mn)!/m!^n$.

Lemma 4.2. *The map $w \mapsto \mathrm{std}(w)^{-1}$ induces a bijection between $\mathfrak{S}_n^{(m)}$ and the set of permutations $\tau \in \mathfrak{S}_{mn}$ whose descents are multiples of m.*

Proof – Clearly, $\mathrm{std}(w)$ belongs to the shifted shuffle of n factors $12\dots m$. ■

For example, $w = 223411341342$ is a 3-permutation in $\mathfrak{S}_4^{(3)}$. Its standardization is $\sigma = 4\,5\,7\,10\,1\,2\,8\,11\,3\,9\,12\,6$, whose inverse is

$$\sigma^{-1} = \begin{array}{|c|c|c|ccccccccc} \hline 5 & 6 & 9 & \multicolumn{1}{c}{} \\ \cline{1-6} \multicolumn{2}{c}{} & \multicolumn{1}{|c|}{1} & 2 & 12 & \multicolumn{1}{|c}{} \\ \cline{3-9} \multicolumn{5}{c}{} & \multicolumn{1}{|c|}{3} & 7 & 10 & \multicolumn{1}{|c}{} \\ \cline{6-12} \multicolumn{8}{c}{} & \multicolumn{1}{|c|}{4} & 8 & 11 & \multicolumn{1}{|c}{} \\ \cline{9-11} \end{array} \tag{61}$$

The restriction of the left weak order to permutations whose descents are multiple of m induce an order on m permutations which will be called the (left) m-weak order.

4.2. **m-Standardization.** Let A be an infinite, totally ordered alphabet. We denote by ${}^mA^n$ the set of words w of length mn whose number of occurences $|w|_a$ of each letter is a multiple of m. These words will be called m-words.

With an m-word w, we can associate an m-permutation $\mathrm{std}_m(w)$ called its m-standardization. The process is similar to the usual standardization: scan iteratively the word from left to right, label 1 the first m occurences of the smallest letter, then 2 the next m occurences of this letter or the first m occurences of the next one, and so on. For example, $\mathrm{std}_2(aabacbcbab) = 1132535424$.

Define now, for $\alpha \in \mathfrak{S}_n^{(m)}$,

$$\mathbf{G}_\alpha(A) = \sum_{w \in {}^mA^n;\ \mathrm{std}_m(w) = \alpha} w. \tag{62}$$

Note that $\mathbf{G}_\alpha$ belongs to **WQSym**:

$$\mathbf{G}_\alpha = \sum_{u \in \mathrm{PW}_n^{(m)},\, \mathrm{std}_m(u) = \alpha} \mathbf{M}_u\,. \tag{63}$$

where $\mathrm{PW}_n^{(m)}$ denotes the set of packed m-words of length nm.

4.3. **The Hopf algebra of Free m-quasi-symmetric functions.** Let ${}^m\mathbf{FQSym}_n^*$ be the linear span of the $\mathbf{G}_\alpha$ for $\alpha \in \mathfrak{S}_n^{(m)}$.

Proposition 4.3. *The $\mathbf{G}_\alpha$ span a subalgebra of $\mathbb{Z}\langle A\rangle$, and their product is given by*

$$\mathbf{G}_\alpha \mathbf{G}_\beta = \sum_{\gamma \in \alpha \star_m \beta} \mathbf{G}_\gamma \tag{64}$$

where the m-convolution $\alpha \star_m \beta$ is the set of m-permutations $\gamma = uv$ such that $\mathrm{std}_m(u) = \alpha$ and $\mathrm{std}_m(v) = \beta$.

Proof – Clearly, for any word w,

$$\mathrm{std}_m(\mathrm{pack}(w)) = \mathrm{std}_m(w), \tag{65}$$

so that

$$\mathbf{G}_\alpha \mathbf{G}_\beta = \sum_{\substack{\mathrm{std}_m(u)=\alpha \\ \mathrm{std}_m(v)=\beta}} \mathbf{M}_u \mathbf{M}_v = \sum_{\substack{w=uv \\ \mathrm{std}_m(\mathbf{a}(u))=\alpha \\ \mathrm{std}_m(\mathbf{a}(v))=\beta}} \mathbf{M}_w = \sum_{\gamma \in \alpha \star_m \beta} \mathbf{G}_\gamma \tag{66}$$

∎

For example,

$$\mathbf{G}_{2121} \mathbf{G}_{11} = \mathbf{G}_{212133} + \mathbf{G}_{313122} + \mathbf{G}_{323211}. \tag{67}$$

Note 4.4. The product $\mathbf{G}_\sigma \mathbf{G}_\tau$ of two m-permutations of respective sizes i and j consists in partitionning the set $\{1, \ldots, i+j\}$ into two sets A and B of sizes i and j and change σ and τ by increasing morphisms of alphabets from respectively $\{1, \ldots, i\}$ to A and $\{1, \ldots, j\}$ to B.

Similarly, applying the coproduct of **WQSym** yields

$$\begin{aligned} \Delta \mathbf{G}_\alpha &= \sum_{\mathrm{std}_m(u)=\alpha} \sum_{k=0}^{n} \mathbf{M}_{\mathrm{pack}(u[1..k])} \otimes \mathbf{M}_{\mathrm{pack}(u[k+1..n])} \\ &= \sum_{k=0}^{n} \mathbf{G}_{\mathrm{std}_m(\alpha[1..k])} \otimes \mathbf{G}_{\mathrm{std}_m(\alpha[k+1..n])} \end{aligned} \tag{68}$$

where for a word w, $w[i..j]$ denotes the subword of w whose letters belong to the interval $[i, j]$.

Summarizing, we have:

Theorem 4.5. *${}^m\mathbf{FQSym}^*$ is a Hopf subalgebra of* **WQSym**, *whose product and coproduct are given by* (64) *and* (68). ∎

4.4. **Commutative image.** The commutative image operation consists in mapping our noncommuting variables a_i to commuting indeterminates x_i.

For $m=1$, the commutative image of $\mathbf{G}_\sigma$ is the quasi-symmetric function $F_{\mathrm{RC}(\sigma)}$. There is an analogue of this property for general m. As already observed, for $\alpha \in \mathfrak{S}_n^{(m)}$, the recoils of $\mathrm{std}(\alpha)$ are all divisible by m. Hence, its recoils composition I is of the form $I = mJ$, where J is a composition of n, and we have

$$\mathbf{G}_\alpha(X) = \psi^m(F_J(X)) \tag{69}$$

where ψ^m is the algebra morphism $x_i \mapsto x_i^m$ (power-sum plethysm operator). Indeed, this follows from (63), since for a packed m-word u, $\mathbf{M}_u(X) = \psi^m(M_J)$.

4.5. **Duality.** Let $\mathbf{F}_\alpha$ be the dual basis of $\mathbf{G}_\alpha$ in the dual Hopf algebra ${}^m\mathbf{FQSym}$. From (64) and (68), we obtain by duality

$$\mathbf{F}_\alpha \mathbf{F}_\beta = \sum_{\gamma \in \alpha \Cup \beta} \mathbf{F}_\gamma, \tag{70}$$

and

$$\Delta \mathbf{F}_\alpha = \sum_{\substack{\alpha = u\cdot v \\ u,v \in A^{(m)}}} \mathbf{F}_{\mathrm{std}_m(u)} \otimes \mathbf{F}_{\mathrm{std}_m(v)}. \tag{71}$$

where u and v run over m-word factorizations of α.

For example,

$$\begin{aligned} \mathbf{F}_{11}\mathbf{F}_{1212} = \mathbf{F}_{112323} + \mathbf{F}_{121323} + \mathbf{F}_{123123} + \mathbf{F}_{123213} + \mathbf{F}_{123231} \\ +\mathbf{F}_{211323} + \mathbf{F}_{213123} + \mathbf{F}_{213213} + \mathbf{F}_{213231} + \mathbf{F}_{231123} \\ +\mathbf{F}_{231213} + \mathbf{F}_{231231} + \mathbf{F}_{232113} + \mathbf{F}_{232131} + \mathbf{F}_{232311}. \end{aligned} \tag{72}$$

The calculation of coproducts is easy, since, as $\alpha = u\cdot v$ is an m-permutation, if its prefix u is an m-word, then v also is an m-word. For example, with $m=2$, all $\mathbf{F}_\alpha$ with α a permutation of 1122 are primitive, except

$$\Delta \mathbf{F}_{1122} = \mathbf{F}_{1122} \otimes 1 + \mathbf{F}_{11} \otimes \mathbf{F}_{11} + 1 \otimes \mathbf{F}_{1122}, \tag{73}$$

and

$$\Delta \mathbf{F}_{2211} = \mathbf{F}_{2211} \otimes 1 + \mathbf{F}_{11} \otimes \mathbf{F}_{11} + 1 \otimes \mathbf{F}_{2211}. \tag{74}$$

Note 4.6. *The Hopf algebra is a shuffle-like Hopf algebra in the sense of Section 2.4 since its product is the shifted shuffle and the coproduct amount to an algorithm that commutes with the action of a group and is compatible with the packing algorithm.*

The product formulas for $\mathbf{F}$ and $\mathbf{G}$ imply that ${}^m\mathbf{FQSym}$ is free and cofree, so that by [8], it is self-dual.

Note that the number of terms in a product $\mathbf{F}_\alpha \mathbf{F}_\beta$ with (α,β) in $\mathfrak{S}_n^{(m)} \times \mathfrak{S}_p^{(m)}$ is equal to

$$\binom{m(n+p)}{mn} \tag{75}$$

whereas the number of terms in a coproduct of an element in $\mathfrak{S}_n^{(m)}$ does not only depend on m and n. On the dual side, for the $\mathbf{G}$ basis, the number of terms in a product or a coproduct depends only on the sizes of the m-permutations. Thus, an explicit isomorphism between ${}^m\mathbf{FQSym}$ and ${}^m\mathbf{FQSym}^*$ cannot be expected to be as simple as in the case $m = 1$.

4.6. **Noncommutative symmetric functions.** The dual of the Hopf quotient $\psi^m(QSym)$ of ${}^m\mathbf{FQSym}^*$ is the Hopf subalgebra of ${}^m\mathbf{FQSym}$ freely generated by the

$$S_n^{(m)} \mapsto \mathbf{F}_{1^m 2^m \dots n^m} \tag{76}$$

whose coproduct is given by

$$\Delta S_n^{(m)} = \sum_{i+j=n} S_i^{(m)} \otimes S_j^{(m)}. \tag{77}$$

This algebra is clearly isomorphic to **Sym**, and the ribbon basis is embedded as

$$R_I^{(m)} = \sum_{C(\mathrm{std}(\alpha)^{-1})=mI} \mathbf{F}_\alpha. \tag{78}$$

The adjoint ϕ_m of ψ^m is the algebra morphism of **Sym** such that $\phi_m(S_{nm}) = S_n$ and $\phi_m(S_k) = 0$ if m does not divide k. On the ribbon basis, for a composition I of mn,

$$\phi_m(R_I) = \epsilon_m(I) R_K \tag{79}$$

where K is the greatest composition of n such that $mK \leq I$, and $\epsilon_m(I) = (-)^{\ell(I)-\ell(K)}$.

For example,

$$\phi_2(R_{23342}) = -R_{1321}. \tag{80}$$

4.7. **Plactic quotient.** The number of plactic classes of m-permutations is given by triangle A188403 of [37]: Number of $(nm) \times m$ binary arrays with rows in nonincreasing order, n ones in every column and no more than 2 ones in any row. By definition, it is the number of semi-standard tableaux of weight (m^n). It is also the number of Yamanouchi words of length mn with descents multiple of m, or the number of standard tableaux of size mn with recoils multiple of m.

4.8. **Hypoplactic quotient.** The number of hypoplactic classes of m-permutations of n is always 2^{n-1}, and the quotients/subalgebras remain isomorphic to $QSym$ and **Sym**.

However, one would have expected to see new algebras in the diagram of morphisms for $m > 1$. These will be provided by the hyposylvester congruence.

4.9. **Hyposylvester quotient.** Applying Theorem 3.4 to evaluation $I = (m^n)$, we obtain:

Corollary 4.7. *The number of hyposylvester classes of m-permutations of n is*

$$(m+1)^{n-1}. \tag{81}$$

■

For ordinary permutations ($m = 1$), we obtain 2^{n-1} classes, which in this case coincide with the hypoplactic classes, since (45) and (46) imply that two permutations are equivalent iff they have the same recoils.

Theorem 4.8. *The hyposylvester classes of m-permutations are intervals of the m-weak order.*

Proof – From the above discussion, we know that two m-permutations α, β in the same hyposylvester class have their last occurence of each letter i between the same occurences of $i+1$. The same is true for any element of the interval $[\alpha, \beta]$. Moreover, each class has a unique minimal element and a unique maximal element. Thus, hyposylvester classes are intervals. ■

Since the hyposylvester congruence satisfies the hypotheses of Theorem 2.1, we obtain Hopf algebras of graded dimensions $(m+1)^{n-1}$.

The sums of $\mathbf{F}_\alpha$ over hyposylvester classes define a ribbon-like basis, for which the product is the sum of $m+1$ totally associative operations. The resulting algebra can therefore be identified with the free $As^{(m+1)}$-algebra on one generator.

Indeed, encode a hyposylvester class as a word $w = w_1 \dots w_{n-1}$ on the alphabet $A_m := \{0, \dots, m\}$ such that the rightmost letter i has w_i letters $i+1$ to its left. If an m-permutation satisfies this property, we write $w = \mathrm{GD}(\sigma)$.

Let w be a word on A_m and define

$$\mathbf{R}_w := \sum_{\mathrm{GD}(\sigma)=\mathrm{w}} \mathbf{F}_\sigma. \tag{82}$$

We then directly have

Theorem 4.9. *The $\mathbf{R}_w$ generate a Hopf subalgebra ${}^m\mathbf{NCSF}$ of ${}^m\mathbf{FQSym}$. Their product is given by the formula*

$$\mathbf{R}_w \mathbf{R}_{w'} := \sum_{k \in A_m} \mathbf{R}_{w \cdot k \cdot w'}. \tag{83}$$

Moreover, all operations sending $(\mathbf{R}_w, \mathbf{R}_{w'})$ to $\mathbf{R}_{w \cdot k \cdot w'}$ are associative and mutually associative, so that ${}^m\mathbf{NCSF}$ can be identified with the free $As^{(m+1)}$-algebra on one generator. ■

Moreover, to mimic the case $m=1$, we shall define F_w as the dual basis of the $\mathbf{R}_w$ and define the dual of ${}^m\mathbf{NCSF}$ as ${}^m\mathbf{QSym}$. Then consider the set of all F_w indexed by words on the subalphabet $\{0,m\}$. This set spans a commutative subalgebra of ${}^m\mathbf{QSym}$: there is a direct isomorphism of Hopf algebras sending these elements to the usual F_I of $QSym$. In particular, this proves that ${}^m\mathbf{NCSF}$ cannot be self-dual since its dual contains a non-trivial commutative subalgebra. Moreover, a simple computation in the case $m=2$ also shows that ${}^m\mathbf{QSym}$ is not commutative:

$$F_1F_\emptyset = F_{10}+F_{02}+F_{21} \qquad F_\emptyset F_1 = F_{01}+F_{20}+F_{12}. \tag{84}$$

Finally, since the hypoplactic classes are unions of hyposylvester classes, one obtains an algebra isomorphic to $QSym$ by quotienting the F_w by the congruence identifying 0 with itself and all the other letters with 1. One then obtains a binary word encoding the descents of a composition.

4.10. **Sylvester quotient: the Hopf algebra of plane $(m+1)$-ary trees.**

4.10.1. *Number of sylvester classes of m-permutations.* Applying (37), we shall see that the number of sylvester classes of m-permutations in $\mathfrak{S}_n^{(m)}$ is equal to the Fuss-Catalan number counting the number of $(m+1)$-ary trees with n internal nodes.

Indeed, let us apply this formula to $I=(m^n)$ (m-permutations). Here, E_{m^n} is symmetric, it is

$$E_{m^n} = \sum_{J\le m^n} M_J = \sum_{I\vDash n} M_{mI} = \psi^m(h_n) \tag{85}$$

so that we are reduced to a scalar product in Sym:

$$\begin{aligned}\langle \psi^m(h_n), g_{mn}\rangle &= \left\langle \psi^m(h_n), \frac{1}{mn+1}h_{mn}((mn+1)X)\right\rangle \\ &= \left\langle h_n, \frac{1}{mn+1}h_n((mn+1)X)\right\rangle\end{aligned} \tag{86}$$

which is

$$\frac{1}{mn+1}h_n(mn+1) = \frac{1}{mn+1}\binom{n(m+1)}{n}. \tag{87}$$

4.10.2. *Functional equations, parking functions, and m-Dyck words.* Denoting by ϕ_m the adjoint of ψ^m, the number of sylvester classes of m-permutations can be written as well

$$\langle \psi^m(h_n), g_{mn}\rangle = \langle h_n, \phi_m(g_{mn}\rangle. \tag{88}$$

Recall that ψ^m is well-defined on $QSym$ ($\psi^m(M_I)=M_{mI}$), and that ϕ_m is defined on $\mathbf{Sym}$, it is the algebra morphism sending S_{mn} to S_n and the other S_i to 0.

Let $g^{(m)}=\phi_m(g)\in\mathbf{Sym}$. Then, the coefficient of S^I in $g^{(m)}$ is the number of nondecreasing parking functions of evaluation mI. The functional equation satisfied by $g^{(m)}$

$$g^{(m)} = \sum_{n\ge 0} S_n \left(g^{(m)}\right)^{mn} \tag{89}$$

becomes multiplicity-free if one considers S_0 as a letter, and the solution X of the new functional equation

$$X = S_0 + S_1X^m + S_2X^{2m} + S_3X^{3m} + \cdots \tag{90}$$

is, as in [32],

$$X = \sum_{\pi\in \mathrm{NDPF}_n^{(m)}} S^{\mathrm{ev}(\pi)\cdot 0^m} \tag{91}$$

where $\mathrm{NDPF}_n^{(m)}$ denotes the set of nondecreasing m-parking functions (see Section 6.3 for their definition) of length n, which are in bijection with ordinary nondecreasing parking functions of length mn whose evaluation is multiple of m (by repeating m times each letter).

For example, with $m=2$

$$X = S^0 + S^{100} + S^{20000} + S^{11000} + S^{10100} + \cdots \tag{92}$$

The term of degree 2 yields the evaluations (2000), (1100), (10010) corresponding to the 2-parking functions 11, 12, 13 or to the ordinary parking functions 1111 ,1122, 1133.

Introducing the operator

$$\Omega:\ S^{(i_1,i_2,\ldots,i_r)} \longmapsto S^{(i_1+1,i_2,\ldots,i_r)}, \tag{93}$$

Equation (91) can be rewritten in the form

$$X = S_0 + (\Omega X)X^m =: S_0 + T_{m+1}(X,X,\ldots,X) \tag{94}$$

with an $(m+1)$-linear operator T_{m+1}. Thus, the solution by iterated substitution can be expressed as a sum over $(m+1)$-ary trees, each tree corresponding to exactly one parking function.

Note that setting $S_n = a^nb$ as in [32], and $D = Xb$, the functional equation for X becomes that of the series of m-Dyck words

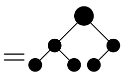
$$Db = b + a(Db)^{m+1}\,. \tag{95}$$

4.10.3. *Bijection with $(m+1)$-ary trees.* Now, by definition, sylvester classes of m-permutations are parametrized by the shapes of the decreasing trees of permutations of mn whose descents are multiple of m, whose sum in the basis $\mathbf{G}_\sigma$ is

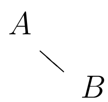
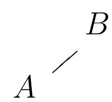
$$S^{mm\cdots m} = \sum_{I\vDash n} R_{mI}. \tag{96}$$

According to [33], the expansion of a ribbon R_{mI} on the basis $\mathbf{P}_T$ of $\mathbf{PBT}$ is the sum of the binary trees corresponding to nondecreasing parking functions of evaluation $m\bar{I}$ under the duplicial bijection of [33].

Encoding the shape of the binary search tree of an m-permutation α by a nondecreasing parking function and decoding it into an $(m+1)$-ary tree, we obtain a parametrization of sylvester classes of m-permutations by $(m+1)$-ary trees. For example, the sylvester class of the 2-permutation 213132 is encoded by the binary tree

(97) $T =$ 

Recall from [33] that the duplicial correspondence between binary trees and non-decreasing parking functions is given by

(98) $A\backslash B =$  $\to \mathbf{P}^{\alpha} \succ \mathbf{P}^{\beta}, \quad A/B =$  $\to \mathbf{P}^{\alpha} \prec \mathbf{P}^{\beta}$

where $\prec$ and $\succ$ are the duplicial operations defined in [33]:

$$\mathbf{P}^{\alpha} \succ P^{\beta} = \mathbf{P}^{\alpha\cdot\beta[k]}, \quad \mathbf{P}^{\alpha} \prec \mathbf{P}^{\beta} = \mathbf{P}^{\alpha\cdot\beta[\max(\alpha)-1]} \tag{99}$$

if α is of length k. For example, $\mathbf{P}^{12}\mathbf{P}^{113} = \mathbf{P}^{12335}$, and $\mathbf{P}^{12} \prec \mathbf{P}^{113} = \mathbf{P}^{12224}$.

Decomposing T with the over-under operations, we find

$$T = ((\bullet/\bullet)\backslash\bullet)/(\bullet\backslash(\bullet/\bullet) \tag{100}$$

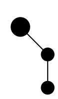
so that the corresponding parking function is

$$\tau = ((1 \prec 1) \succ 1) \prec (1 \succ (1 \prec 1) = 113344. \tag{101}$$

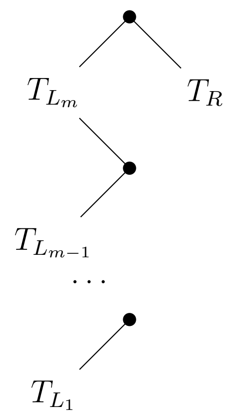
The 2-parking function 134 has evaluation (10110), and the corresponding ternary tree is the one evaluating to $S^{1011000}$ in the iterative solution of (91):

$$\begin{aligned} S^{1011000} &= T_3(S^0, S^0, S^{1100}) = T_3(S^0, S^0, T_3(S^0, S^{100}, S^0)) \\ &= T_3(S^0, S^0, T_3(S^0, T_3(S^0, S^0, S^0), S^0)) \end{aligned} \tag{102}$$

which is

(103) 

4.10.4. *m-binary trees.* The shapes of the binary trees obtained as the binary search trees of m-permutations are known: these are the left-right flips of the m-binary trees defined in [36, 3]. We shall slightly modify their definition (it amounts to a symmetry through the vertical axis), so that our m-binary trees are binary trees with the following structure, where all T_L and T_Rs are m-binary trees.



This definition reveals the recursive structure of these objects, and provides a direct bijection with $(m+1)$-ary trees : T_{L_i} becomes the i-th subtree of the root and T_R becomes the rightmost (last) subtree of the root. If one then restricts the Tamari order to our m-binary trees (which form an upper ideal), one gets the m-Tamari lattice defined by Bergeron [1] up to reversal.

More precisely, denote by ${}^m\mathfrak{R}$ the set of permutations whose recoils are multiple of m, so that the number of sylvester classes of m-permutations of size n is equal to the number of sylvester classes of elements of ${}^m\mathfrak{R}$ of size mn. Note that the elements of ${}^m\mathfrak{R}$ form an initial interval of the weak order, so their sylvester classes also form an initial interval of the Tamari order. Its greatest element is a very special tree: it is the binary search tree of the permutation

$$m(n-1)+1,\ \ldots,\ mn,\ m(n-2)+1,\ \ldots,\ m(n-1),\ \ldots,\ 1,\ \ldots,\ m. \tag{104}$$

It is the (m,n) comb-tree, a tree consisting of a root with $n-1$ right nodes that all have as left subtrees a sequence of $m-1$ left nodes, hence with mn nodes (see the example for $m=2$ and $n=4$ below).

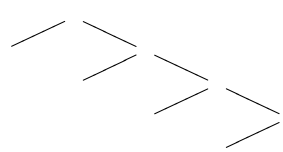


FIGURE 3. The binary search tree of 78563412.

4.10.5. *Hopf algebras and the m-Tamari lattice.* Now, since ${}^m\mathbf{FQSym}$ is a shuffle-like Hopf algebra and since the sylvester congruence satisfies the hypothesis of Theorem 2.1, we have

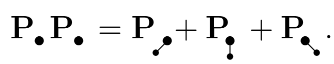
Theorem 4.10. *The sums over sylvester classes*

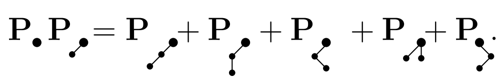
$$\mathbf{P}_T = \sum_{\mathrm{sylv}(\alpha)=T} \mathbf{F}_\alpha \tag{105}$$

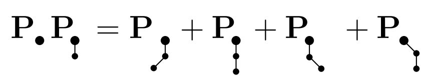
in ${}^m\mathbf{FQSym}$ *span a Hopf algebra* ${}^m\mathbf{PBT}$*, of graded dimension given by the Fuss-Catalan sequence. The dual basis* $\mathbf{Q}_T$ *of* $\mathbf{P}_T$ *in* ${}^m\mathbf{PBT}^*$ *is formed of the sylvester classes* $\bar{\mathbf{G}}_\alpha$ *for* $\mathrm{sylv}(\alpha) = T$.

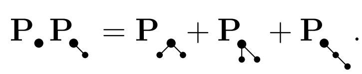
Moreover, the product of two basis elements $\mathbf{P}_{T'}\mathbf{P}_{T''}$ *in* ${}^m\mathbf{PBT}$ *spans an interval of the* m*-Tamari order. The bounds of this interval are obtained through the bijection with* m*-binary trees: order recursively the nodes of the* $(m+1)$*-ary trees from left to right in the following way. The first* m *subtrees in that order, then the root, then the right subtree. The lower (resp. upper) bound of the product* $\mathbf{P}_{T'}\mathbf{P}_{T''}$ *consists in glueing* T *to the left-most node of* T' *(resp.* T' *to the right-most node of* T*), hence copying the equivalent property in* $\mathbf{PBT}$. ■

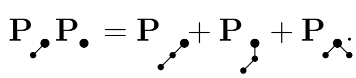
Example 4.11. Using the above encoding of sylvester classes by trees, we can now give a few examples of products and coproduct of ternary trees. With $m = 2$, we have the products

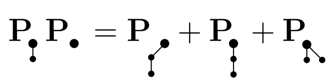
(106) 

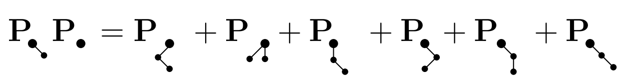
(107) 

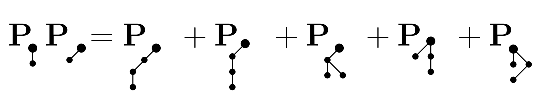
(108) 

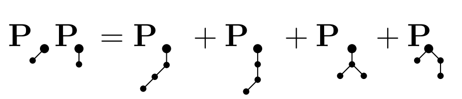
(109) 

(110) 

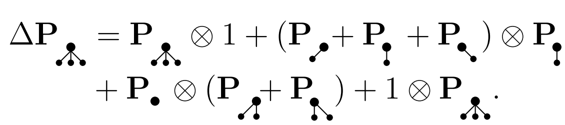
(111) 

(112) 

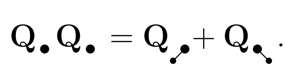
(113) 

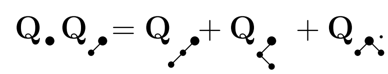
(114) 

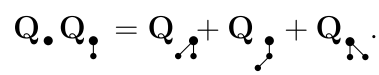
and the coproduct

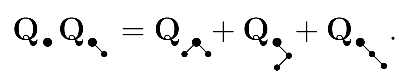
(115) 

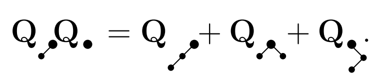
On the $\mathbf{Q}$ basis, we get

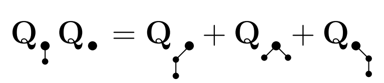
(116) 

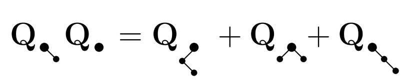
(117) 

(118) 

(119) 

(120) 

(121) 

(122) 

Note 4.12. Finally, as for **PBT**, the algebra ${}^m\mathbf{PBT}$ is generated by the m-binary trees whose root has no right child, hence through the bijection, ${}^m\mathbf{PBT}$ is generated by the $(m+1)$-ary trees with no right (last) subtree.

One can see this directly, since in a product $\mathbf{P}_T\mathbf{P}_{T'}$, the tree corresponding to the permutation having the maximal number of inversions is the concatenation of the shifted maximal word of T' and the maximal word of T, hence belongs to the tree where T' is glued to the rightmost node of T (which is the root when T has no right subtree attached to it).

The same holds for the same reason for the basis Q_T, except that this time, the maximal element corresponds to glueing T to the rightmost node of T'.

4.10.6. *Isomorphism between* ${}^m\mathbf{FQSym}$ *and a subalgebra of* **FQSym**. If one standardizes an m-permutation on letters $1, \ldots, n$, one obtains a permutation of size mn. The map

$$\iota_m : \ \mathbf{F}_\alpha \to \mathbf{F}_{\mathrm{std}(\alpha)} \tag{123}$$

is then an algebra morphism, since the products in the bases $\mathbf{F}$ of ${}^m\mathbf{FQSym}$ and **FQSym** are given by the shifted shuffle which is compatible with standardization. If one restricts to permutations whose recoil composition has parts multiple of m, we then get an algebra isomorphism.

Note that the coproducts are not the same, so it is only an *algebra* morphism. It is even worse than that: consider the mapping sending $\mathbf{F}_\sigma$ to itself if its recoils are multiple of m and to 0 otherwise. Its kernel is an ideal of **FQSym** but not a coideal as one can see by computing the smallest counter-example

$$\Delta\mathbf{F}_{1423}. \tag{124}$$

4.11. **Metasylvester subalgebra and quotient: Hopf algebras of decreasing $(m+1)$-ary trees.** As observed in Section 3.3.2, metasylvester classes of m-permutations are in bijection with decreasing $(m+1)$-ary trees. Thus, for $m>1$, new Hopf algebras, based on decreasing $(m+1)$-ary trees can be constructed. For $m=1$, decreasing binary trees are in bijection with permutations, and the construction would be only a recoding of **FQSym**.

4.11.1. *The metasylvester subalgebra.* As already mentioned, we shall identify the canonical metasylvester m-permutations (avoiding the pattern $1\cdots 2\cdots 1$) with their decreasing trees, which can also be characterized as follows:

Definition 4.13. *An m-permutation τ of n is a decreasing $(m+1)$-ary tree if it is either the empty word, or of the form $u_1nu_2\cdots nu_{m+1}$ where the packed words of $u_1,\ldots,u_{m+1}$ are decreasing $(m+1)$-ary trees. We shall denote by $\mathrm{DT}^{(m)}$ the set of these m-permutations.*

Up to reversal of the order, this is the same notion as the generalized Stirling permutations of [17], that correspond to increasing trees.

Regarding τ as a packed word, one can associate with it a plane tree with sectors labeled by the letters of τ, as in [16]. Then, τ satisfies these conditions iff this tree is a $(m+1)$-ary tree. Labeling the vertices with the labels of their sectors, one obtains indeed a decreasing tree.

We can now set, for a decreasing $(m+1)$-ary tree τ

$$\mathbf{P}_\tau = \sum_{\alpha\equiv_{\mathrm{ms}}\tau} \mathbf{F}_\alpha \tag{125}$$

For example, with $m=2$, we have

$$\begin{aligned}
\mathbf{P}_{113322} &= \mathbf{F}_{112332} + \mathbf{F}_{113232} + \mathbf{F}_{113322}.\\
\mathbf{P}_{311322} &= \mathbf{F}_{131232} + \mathbf{F}_{131322} + \mathbf{F}_{311232} + \mathbf{F}_{311322}.\\
\mathbf{P}_{331122} &= \mathbf{F}_{133122} + \mathbf{F}_{131322} + \mathbf{F}_{331122}.
\end{aligned} \tag{126}$$

Note that those three $\mathbf{P}$ elements represent one sylvester class.

As $\equiv_{\mathrm{ms}}$ satisfies the hypotheses of Theorem 2.1, we can state:

Theorem 4.14. *The $\mathbf{P}_\tau$ span a Hopf subalgebra ${}^m\mathcal{DT}$ of ^{m}**FQSym**. The dual Hopf algebra ${}^m\mathcal{DT}$ is the metasylvester quotient ${}^m\mathbf{FQSym}^*/\equiv_{\mathrm{ms}}$, based on the equivalence classes $\mathbf{Q}_\tau = \overline{\mathbf{G}_\alpha}$ $(\alpha\equiv_{\mathrm{ms}}\tau)$.*

For example, with $m=2$, we have

$$\mathbf{P}_{11}\mathbf{P}_{1122} = \mathbf{P}_{112233} + \mathbf{P}_{211233} + \mathbf{P}_{221133} + \mathbf{P}_{223113} + \mathbf{P}_{223311}. \tag{127}$$

$$\mathbf{P}_{11}\mathbf{P}_{2112} = \mathbf{P}_{113223} + \mathbf{P}_{311223} + \mathbf{P}_{321123} + \mathbf{P}_{322113} + \mathbf{P}_{322311}. \tag{128}$$

$$\mathbf{P}_{11}\mathbf{P}_{2211} = \mathbf{P}_{113322} + \mathbf{P}_{311322} + \mathbf{P}_{331122} + \mathbf{P}_{332112} + \mathbf{P}_{332211}. \tag{129}$$

$$\Delta\mathbf{P}_{311223} = \mathbf{P}_{311223}\otimes 1 + 1\otimes\mathbf{P}_{311223}. \tag{130}$$

$$\Delta \mathbf{P}_{322311} = \mathbf{P}_{322311} \otimes 1 + \mathbf{P}_{2112} \otimes \mathbf{P}_{11} + 1 \otimes \mathbf{P}_{322311}. \tag{131}$$

$$\Delta \mathbf{P}_{113322} = \mathbf{P}_{113322} \otimes 1 + \mathbf{P}_{1122} \otimes \mathbf{P}_{11} + \mathbf{P}_{11} \otimes \mathbf{P}_{2211} + 1 \otimes \mathbf{P}_{113322}. \tag{132}$$

$$\mathbf{Q}_{11}\mathbf{Q}_{1122} = \mathbf{Q}_{112233} + \mathbf{Q}_{221133} + \mathbf{Q}_{331122}. \tag{133}$$

$$\mathbf{Q}_{11}\mathbf{Q}_{2112} = \mathbf{Q}_{113223} + \mathbf{Q}_{223113} + \mathbf{Q}_{332112}. \tag{134}$$

$$\mathbf{Q}_{11}\mathbf{Q}_{2211} = \mathbf{Q}_{113322} + \mathbf{Q}_{223311} + \mathbf{Q}_{332211}. \tag{135}$$

$$\Delta \mathbf{Q}_{321123} = \mathbf{Q}_{332112} \otimes 1 + \mathbf{Q}_{2112} \otimes \mathbf{Q}_{11} + \mathbf{Q}_{11} \otimes \mathbf{Q}_{2211} + 1 \otimes \mathbf{Q}_{321123}. \tag{136}$$

As we shall now see, ${}^m\mathcal{DT}$ is also a subalgebra of ${}^m\mathbf{FQSym}^*/\equiv_{\rm ms}$. This property, which does not hold for the sylvester congruence, implies immediately the product and coproduct rules for decreasing trees.

Note 4.15. As for ${}^m\mathbf{PBT}$ (see Note 4.12), both the algebra ${}^m\mathcal{DT}$ and its dual ${}^m\mathcal{DT}^*$ are generated by decreasing trees whose roots have no right (last) subtree. This time, the maximal element of a product $\mathbf{P}_\tau\mathbf{P}_{\tau'}$ corresponds to glueing the decreasing tree τ to the right of the shifted decreasing tree τ'.

On the $\mathbf{Q}$ basis, the same property holds and the maximal decreasing tree corresponds to glueing the decreasing tree τ' to the right of the shifted decreasing tree τ.

4.11.2. *${}^m\mathcal{DT}$ as subalgebra of ${}^m\mathbf{FQSym}^*$.*

Proposition 4.16. *If α and β are decreasing $(m+1)$-ary trees, the product $\mathbf{G}_\alpha\mathbf{G}_\beta$ contains only decreasing $(m+1)$-ary trees. Thus these trees form a subalgebra of ${}^m\mathbf{FQSym}$.*

Moreover, this subalgebra is also a sub-coalgebra.

Proof – The m-convolution of two m-permutations avoiding the pattern $1\dots2\dots1$ cannot contain this pattern, and the restriction of such an m-permutation to an interval cannot contain it as well. ∎

For example, we have (with, as always, $m=2$)

$$\mathbf{G}_{11}\mathbf{G}_{2112} = \mathbf{G}_{113223} + \mathbf{G}_{223113} + \mathbf{G}_{332112}. \tag{137}$$

Corollary 4.17. *(i) The product in ${}^m\mathcal{DT}$ is obtained by selecting the decreasing trees in the shifted shuffle*

$$\mathbf{P}_{\tau'}\mathbf{P}_{\tau''} = \sum_{\tau\in\tau'\Cup\tau'',\ \tau\ \mathit{tree}} \mathbf{P}_\tau\,. \tag{138}$$

(ii) The coproduct in ${}^m\mathcal{DT}$ is given by

$$\Delta \mathbf{P}_\tau = \sum_{\substack{\tau = u\cdot v \\ \mathrm{std}_m(u),\mathrm{std}_m(v)\in \mathrm{DT}^{(m)}}} \mathbf{P}_{\mathrm{std}_m(u)} \otimes \mathbf{P}_{\mathrm{std}_m(v)}. \tag{139}$$

where u and v run over m-word factorizations of τ.
(iii) The product in the basis $\mathbf{Q}_\tau$ of ${}^m\mathcal{DT}$ are those of the basis $\mathbf{G}_\tau$ with $\tau \in \mathrm{DT}^{(m)}$.

4.11.3. *Naked trees.*

Proposition 4.18. *The shapes of the trees occuring in a product $\mathbf{P}_\tau\mathbf{P}_{\tau'}$ depend only on the shapes of the factors. The same holds for any coproduct $\Delta\mathbf{P}_\tau$.*

Then, the map sending the decreasing trees to their shapes is a Hopf (quotient) morphism, so that, if one defines $\mathbf{Q}'_T$ as the class of the decreasing trees $\mathbf{P}_\tau$ of its shape, these $\mathbf{Q}'_T$ span a Hopf quotient of ${}^m\mathcal{DT}$.

Proof – Let us consider an $(m+1)$-ary tree and consider two decreasing trees of this shape. Those two trees are obtained from each other by renumbering their values so that their maximal words are obtained from each other by multiplication to the left by a permutation. More precisely, all decreasing posets of the same underlying poset can be obtained by a chain of elementary transpositions exchanging two consecutive values (see, *e.g.*, [5] for a refined structure on posets: a 0-Hecke module structure).

Thus, we just have to prove that if τ and τ' are two m-permutations, both maximal elements of their metasylvester classes, with the same shape as decreasing trees, and are obtained from one another by an elementary transposition σ_i (acting on their values hence to their left), then, their left and right products by a given element and their coproduct when restricted to maximal elements of metasylvester classes happen to have the same shapes. Note that it means in particular that the rightmost value i is not between two values of $i+1$. Consider the products $\mathbf{F}_\sigma\mathbf{F}_\tau$ and $\mathbf{F}_\sigma\mathbf{F}_{\tau'}$ for any σ maximal in its metasylvester class. Then, match the elements of both products, according to the positions of the letters coming from σ in the shuffles $\sigma \Cup \tau$ and $\sigma \Cup \tau'$. First, either these elements are both maximal elements of metasylvester classes, or none is. In the first case, they must have the same shape, since when applying the algorithm computing the decreasing tree of an m-permutation, all steps are identical up to the exchange of i and $i+1$, because the rightmost i is not between two values of $i+1$. The same holds for the product to the right by a given $\mathbf{F}_\sigma$.

The discussion of the coproduct is easy. First, start again with two m-permutations, maximal, with the same shape as decreasing trees, and differing by a single transposition σ_i. Then, thanks to Section 3.3.1 that gives a way to compute the maximal element of a class given its decreasing tree, we directly conclude that those two m-permutations can be de-contatenated at the exact same places. Moreover, if the last occurrence of i is not between two occurrences of $i+1$, then it is also the case for any prefix and suffix of these m-permutations, hence proving both coproducts always have same shape.

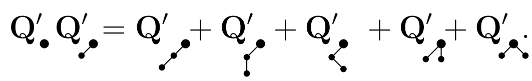
The Hopf structure is now implied by the previous discussion. ∎

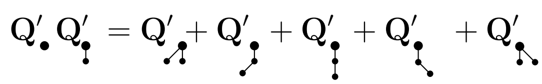
As an example of the quotient property, we have

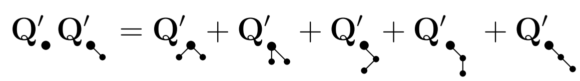
$$\mathbf{P}_{223113}\mathbf{P}_{11} = \mathbf{P}_{22311344} + \mathbf{P}_{22431134} + \mathbf{P}_{42231134} + \mathbf{P}_{42243113}. \tag{140}$$

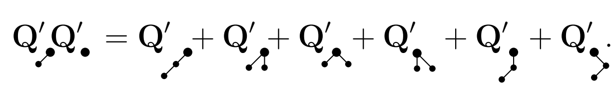
$$\mathbf{P}_{113223}\mathbf{P}_{11} = \mathbf{P}_{11322344} + \mathbf{P}_{11432234} + \mathbf{P}_{41132234} + \mathbf{P}_{41143223}. \tag{141}$$

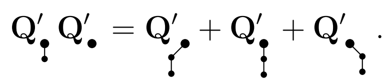
Note that the quotient basis is defined as $\mathbf{Q}'$ and not as $\mathbf{P}'$. We shall understand right after the examples why it is so. Let us now compute a few products of the $\mathbf{Q}'$ with $m = 2$.

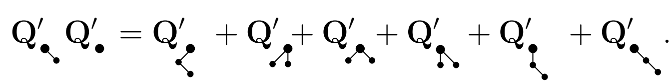
(142)


(143)


(144)


(145)


(146)


(147)


Note 4.19. Thanks to Note 4.15, it is immediate that the maximal term in a product $\mathbf{Q}'_T\mathbf{Q}'_{T'}$ is obtained by glueing T as the rightmost child of the rightmost node of T'. So the algebra of the $\mathbf{Q}'$ is free on the trees whose root do not have a right child.

Note in particular that the maximal term in a product of $\mathbf{Q}'$ is equal to the maximal term in a product of $\mathbf{Q}$ of ${}^m\mathbf{PBT}$, hence the choice of notations. Note also that this implies that the algebra of the $\mathbf{Q}'$ is isomorphic *as an algebra* to ${}^m\mathbf{PBT}^*$.

Dually, we have

Theorem 4.20. *Let $\mathbf{P}'_T$ be the sum of all decreasing trees $\mathbf{G}_\tau$ of shape T. Then, the $\mathbf{P}'_T$ span a Hopf subalgebra of ${}^m\mathcal{DT}^*$ and ${}^m\mathbf{FQSym}^*$.*

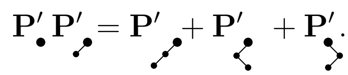
Moreover, this algebra is isomorphic to ${}^m\mathbf{PBT}$.

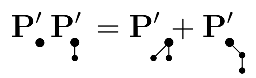
Proof – As we have shown before, ${}^m\mathcal{DT}$ is isomorphic to the quotient algebra of ${}^m\mathbf{FQSym}$ where one sends to zero all the $\mathbf{F}_w$ where w is not the maximal word of a metasylvester class. So ${}^m\mathcal{DT}$ is isomorphic to the algebra generated by the $\mathbf{F}_w$ where w are the maximal words of a metasylvester class, understanding that the products and coproducts are computed by killing in their expansions the non-maximal words. Moreover, the isomorphism is direct: send each $\mathbf{P}_\tau$ to $\mathbf{F}_\tau$.

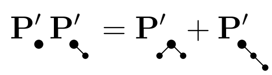
Given this, since there is a well-defined quotient on the $\mathbf{P}_\tau$ sending classes of maximal words to their shape, this means dually that the $\mathbf{Q}'_T$ define a Hopf subalgebra of ${}^m\mathcal{DT}^*$.

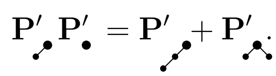
Consider now the algebra of the $\mathbf{P}'$, and a product $\mathbf{P}'_T\mathbf{P}'_{T'}$. Thanks to Note 4.15, we see that the maximal term in this product is obtained by glueing T' as the rightmost child of the rightmost node of T'. So the algebra of the $\mathbf{P}'$ is free on the trees whose root do not have a right child. Since it is the same for the dual basis $\mathbf{Q}'$, we conclude that the algebra generated by the $\mathbf{P}'$ is free and cofree of dimension given by the Fuss-Catalan numbers, and is therefore isomorphic to the algebra ${}^m\mathbf{PBT}$ since there exists at most one free and cofree algebra given its series of dimensions [8]. ■

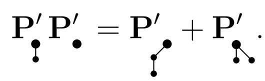
For example, with $m = 2$,

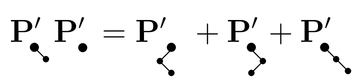
(148) 

(149) 

(150) 

(151) 

(152) 

(153) 

4.11.4. *Functional equations.* In [16], it has been shown that the basis of the Loday-Ronco algebra, that is the sums over sylvester classes of permutations, could be obtained by iterated substitution in a simple functional equation in **FQSym**:

$$X = 1 + B(X, X)\,, \tag{154}$$

where for $\alpha \in \mathfrak{S}_k$, $\beta \in \mathfrak{S}_l$, and $n = k + l$,

$$B(\mathbf{G}_\alpha, \mathbf{G}_\beta) = \sum_{\substack{\gamma=u(n+1)v \\ \mathrm{std}(u)=\alpha,\mathrm{std}(v)=\beta}} \mathbf{G}_\gamma\,. \tag{155}$$

Clearly,

$$\partial B(\mathbf{G}_\alpha, \mathbf{G}_\beta) = \mathbf{G}_\alpha \mathbf{G}_\beta\,, \tag{156}$$

where the derivation ∂ is defined by

$$\partial \mathbf{G}_\sigma = \mathbf{G}_{\sigma'}, \tag{157}$$

σ' being the word obtained by erasing the letter n in $\sigma \in \mathfrak{S}_n$.

The formal sum X of all decreasing $(m+1)$-ary trees in ${}^m\mathbf{FQSym}^*$ satisfies a functional equation generalizing that of [16]

$$X = 1 + T_{m+1}(X, X, \ldots, X), \tag{158}$$

where the $(m+1)$-linear map T_{m+1} is defined, for $\alpha_1,\ldots,\alpha_{m+1}$ of respective degrees k_i, with $n=k_1+\cdots+k_{m+1}$, by

$$T_{m+1}(\mathbf{G}_{\alpha_1},\mathbf{G}_{\alpha_2},\ldots,\mathbf{G}_{\alpha_{m+1}}) = \sum_{\mathrm{pack}(u_i)=\alpha_i} \mathbf{G}_{u_1(n+1)u_2(n+1)\cdots(n+1)u_{m+1}}. \tag{159}$$

Since ∂ is a derivation, the functional equation lifts the differential equation

$$\frac{dx}{dt} = x^{m+1},\ x(0)=1 \tag{160}$$

for the exponential generating series

$$x(t) = (1-mt)^{-1/m} \tag{161}$$

of the number of decreasing $(m+1)$-ary trees.

The map $\phi:\ \mathbf{G}_\alpha \mapsto t^n/n!$ for $\alpha\in\mathfrak{S}_n^{(m)}$ is a character, and $\phi(X)=x(t)$. Using its q-analogue as in [16], one may easily derive q-enumerations of decreasing trees.

The basis $\mathbf{Q}_\tau$ is generated by the $(m+1)$-ary tree solution of the functional equation.

The sylvester classes of m-permutations can also be generated by a functional equation, as in [16]. Let us illustrate this for $m=2$ (ternary trees).

Let X and Y be the sums in **FQSym** of all $\mathbf{G}_\sigma$ with even descents, of respectively even and odd lengths.

These series are in **Sym**:

$$\begin{aligned} X &= 1+S^2+S^{22}+S^{222}\cdots = (1-S_2)^{-1}, && (162)\\ Y &= S^1+S^{21}+S^{221}+\cdots = XS_1. && (163)\end{aligned}$$

Note that the number of sylvester classes of evaluation 2^n1 is $\frac{1}{n+1}\binom{3n+1}{n}$ (1, 2, 7 30, 143, ..., which is A006013, the Hilbert series of the free triplicial algebra on one generator).

With the usual derivation ∂ of **FQSym** or **Sym**, we have the system

$$\begin{aligned} \partial X &= YX, && (164)\\ \partial Y &= X+Y^2. && (165)\end{aligned}$$

The fixed point equation is

$$\begin{aligned} X &= 1+B(Y,X), && (166)\\ Y &= B(Y,Y)+B(X,1). && (167)\end{aligned}$$

Setting $A=t\mathsf{E}$ in the differential system yields

$$\begin{aligned} x' &= yx, && (168)\\ y' &= x+y^2 && (169)\end{aligned}$$

with $x(0)=1$ and $y(0)=0$. This implies $(y/x)'=1$ so that $y=tx$ and $x=(1-\frac{1}{2}t^2)^{-1}$ as expected.

Setting $A = t$ in the fixed point system gives back the OGFs for the numbers of sylvester classes. The system becomes

$$x = 1 + tyx, \tag{170}$$
$$y = tx + ty^2, \tag{171}$$

which implies $x = 1 + t^2x^3$, the OGF for ternary trees.

In terms of dendriform operations,

$$X = 1 + Y \succ \bullet \prec X, \tag{172}$$
$$Y = Y \succ \bullet \prec Y + X \succ \bullet. \tag{173}$$

5. Hopf algebras from m-packed words

5.1. **m-Packed words.** An m-packed word of degree n is a packed m-word, *i.e.*, a packed word of length mn whose evaluation is multiple of m. Their set will be denoted by $\mathrm{PW}_n^{(m)}$.

The noncommutative characteristic of the set of m-packed words of evaluation mI is S^{mI}, so that

$$\mathrm{ch}(\mathrm{PW}^{(m)}) = \sum_I S^{mI} = (1 - S_m - S_{2m} - S_{3m} - \cdots)^{-1}, \tag{174}$$

and the exponential generating function of $a_n(m) = |\mathrm{PW}_n^{(m)}|$ is

$$\left(1 - \sum_{n\geq 1} \frac{t^n}{(mn)!}\right)^{-1} = \sum_{n\geq 0} a_n(m) \frac{t^n}{(mn)!}. \tag{175}$$

Example 5.1. For $m = 2$, the sequence is [37, A094088]:

$$1, 1, 7, 121, 3907, 202741, 15430207, 1619195761, 224061282907 \tag{176}$$

whose exponential generating series is

$$\frac{1}{2 - \cosh(x)} \quad \text{(even coefficients)}. \tag{177}$$

We can observe that these numbers are given by evaluating at $x = 2$ the reciprocal 2-Eulerian polynomials, counting 2-permutations according to their number of descents (plus one):

$$7 = 5 + 1 \times 2^1,\ 121 = 61 + 28 \times 2^1 + 2^2, \ldots \tag{178}$$

Indeed, the natural action of the 0-Hecke algebra on $\mathrm{PW}_n^{(m)}$ has noncommutative characteristic $\sum_{I \vDash n} S^{2I}$, and

$$\sum_I S^{2I} \left(\frac{t}{1-t}\right)^{\ell(I)} = \sum_J t^{\ell(J)} R_{2J}. \tag{179}$$

5.2. **The Hopf algebras ^{m}WQSym.** Going back to the definition of **WQSym** (see Formulas (13) and (16)), it is immediate that the $\mathbf{M}_u$ for words u whose evaluation is multiple of m span a Hopf subalgebra of **WQSym**, which will be denoted by ^{m}**WQSym**.

Its dual is the quotient of **WQSym*** by the Hopf ideal generated by the $\mathbf{N}_u$ such that $u \notin \mathrm{PW}^{(m)}$.

The sylvester quotient of **WQSym** is known to be the free tridendriform algebra on one generator [30, 29]. We shall now investigate the case of ^{m}**WQSym**. Note that this algebra is a shuffle-like Hopf algebra since it is a subalgebra of **WQSym** which is itself a shuffle-like Hopf algebra .

5.3. **Sylvester classes.** To find the number of sylvester classes of m-packed words, we have to compute

$$\left\langle \sum_{I \vDash n} E_{m\overline{I}}, g \right\rangle = \left\langle \sum_{\lambda\vdash n} 2^{n-\ell(\lambda)} m_\lambda, \phi_m(g) \right\rangle \tag{180}$$

One can give directly a q-enumeration by remarking that

$$\sum_{\lambda\vdash n} 2^{n-\ell(\lambda)} m_\lambda = \frac{1}{(1-t)^n} \sum_{\lambda\vdash n} (1-t)^{\ell(\lambda)} m_\lambda|_{t=1/2} = \frac{1}{(1-t)^n} h_n((1-t)X)|_{t=1/2} \tag{181}$$

Thus, the number of sylvester classes of m-packed words is obtained by putting $x = 2$ in the polynomial

$$\begin{aligned} N_n^{(m)}(x) &= \frac{1}{(1-t)^n} \frac{h_n((1-t)(mn+1))}{mn+1}\Big|_{t=1-1/x} \\ &= \frac{1}{mn+1} \sum_{k=0}^{n} \binom{mn+1}{k} \binom{(m+1)n-k}{n-k} (1-x)^k x^{n-k} \end{aligned} \tag{182}$$

For example, with $m = 2$, the first polynomials are

$$\begin{aligned} &1, \quad x+2, \quad x^2+6x+5, \quad x^3+12x^2+28x+14, \\ &x^4+20x^3+90x^2+120x+42, \ldots \end{aligned} \tag{183}$$

See Figure 5 for a triangular array-like representation of these polynomials. Putting $x = 2$, we obtain the required numbers: for $m = 2$, these are

$$1, 4, 21, 126, 818, 5594, 39693, 289510, \ldots \tag{184}$$

(sequence A003168, number of ways to dissect a convex $(2n+2)$-gon with non-crossing diagonals so that no $(2m+1)$-gons ($m > 0$) appear).

For $m = 3$ we get a new sequence:

$$1, 5, 34, 267, 2279, 20540, 192350, 1853255 \tag{185}$$

The coefficients of $N_n^{(m)}$ are the generalized Runyon numbers [4, 38]. Their generating function is the functional inverse of

$$f_m(z) = \frac{z}{(1+z)(1+xz)^m} \tag{186}$$

Their combinatorial interpretation is known [4]: $R^m_{n,k}$ is m-Dyck paths (*i.e.*, paths from $(0,0)$ to $(mn,0)$ by steps $u=(1,1)$ and $d=(1,-m)$) with k peaks.

Note that the reciprocal polynomials $x^nN_n^{(m)}(1/x)$ are the m-Narayana polynomials in the sense of [38]. As we shall see below, their evaluation at $x=2$ yields the number of hypoplactic classes of m-parking functions.

Finally, setting $x=1+q$ instead of just $x=2$ provides the following q-analogues (for $m=2$)

$$\begin{aligned}&1,\quad q+3,\quad q^2+8q+12,\quad q^3+15q^2+55q+55,\\&q^4+24q^3+156q^2+364q+273,\dots\end{aligned}\tag{187}$$

See Figure 10 for a triangular array-like representation of these polynomials. Their coefficients form the triangle A102537 (number of dissections of a convex $(2n+2)$-gon by $k-1$ noncrossing diagonals into $(2j+2)$-gons, $1\le j\le n-1$).

For $m=3$, we obtain new polynomials, encoding a new triangle:

$$\begin{aligned}&1,\quad q+4,\quad q^2+11q+22,\quad q^3+21q^2+105q+140,\\&q^4+34q^3+306q^2+969q+969,\dots\end{aligned}\tag{188}$$

See Figure 11 for a triangular array-like representation of these polynomials. These are m-generalizations of the triangle A033282, obtained for $m=1$ (number of diagonal dissections of a convex n-gon into $k+1$ regions).

5.4. **Sylvester quotient of ^{m}WQSym.** Since the algebra ^{m}**WQSym** is a subalgebra of **WQSym**, its sylvester quotient is a subalgebra of the sylvester quotient of **WQSym**, which is known to be the free tridendriform algebra on one generator [30].

Its bases may be labeled by m-Schröder paths, which for $m=1$ are in bijection with reduced plane trees [38].

Alternatively, m-Schröder paths are in bijection with the natural generalization of Schröder trees: plane trees in which each internal vertex has at least $m+1$ children.

5.5. **Hypoplactic quotient of ^{m}WQSym.** Since $\mathrm{ch}(\mathrm{PW}_n^{(m)})=\sum_{I\vDash n}S^{mI}$, the number of hypoplactic classes of length mn is

$$\sum_{I\vDash n}2^{\ell(I)-1}=3^{n-1}.\tag{189}$$

The hypoplactic subalgebras of ^{m}**WQSym*** and quotients of ^{m}**WQSym** are thus always isomorphic to the free tricubical algebra on one generator and its dual.

5.6. **Hyposylvester quotient of ^{m}WQSym.** For m-packed words, the generating series of the number of hyposylvester classes is

$$\begin{aligned}c_m(t)&=1+\sum_{r\ge1}\sum_{i_1,\dots,i_r\ge1}(mi_2+1)\cdots(mi_r+1)t^{i_1+\cdots i_r}\\&=1+\frac{t(1-t)}{1-(m+3)t+2t^2}.\end{aligned}\tag{190}$$

Indeed, with α and χ_1 defined as in (52) and (51),

$$c_m(t^m) = \alpha(H_m), \text{ where } H_m = \sum_I S^{mI} = 1 + \left(\sum_{n\geq 1} S_{mn}\right) H_m. \tag{191}$$

so that

$$c_m(t^m) = 1 + \frac{t^m}{1-t^m}\frac{1}{1-\sum_{n\geq 1}(mn+1)t^{mn}}\,. \tag{192}$$

These series are not in [37], but their inverses are: A001835 ($m = 2$), A004253 ($m = 3$), A001653 ($m = 4$), A049685 ($m = 5$), A070997 ($m = 6$), A070998 ($m = 7$), A072256 ($m = 8$), A078922 ($m = 9$), A077417 ($m = 10$), A085260 ($m = 11$), A001570 ($m = 12$), A160682 ($m = 13$), A157456 ($m = 14$), A161595 ($m = 15$), ...

Actually, it can be easily proved that the hyposylvester subalgebras are free, and these sequences give their numbers of generators by degree (coefficients of the series $1 - c_m(t)^{-1}$).

Thanks to their formula, it is easy to generalize their definition: the coefficient of t^n in $c_m(t)$ is the number of compositions of $n-1$ with $m+2$ type of ones and m types of the other values.

5.7. **Metasylvester quotient of ^{m}WQSym.** From (60), we get that the number of metasylvester classes of m-packed words in $\mathrm{PW}_n^{(m)}$ is

$$\prod_{k=1}^{n-1}(mk+2). \tag{193}$$

The resulting algebras will be investigated in another publication.

6. m-PARKING FUNCTIONS

6.1. **Definition.** There are several definitions of generalized parking functions in the literature. We shall use the one of Bergeron [1]: these are the words over the positive integers such that the sorted word satisfies $a_i \leq k(i-1)+1$. We shall denote their set by $\mathrm{PF}_n^{(m)}$. Their number is $(mn+1)^{n-1}$.

6.2. **Frobenius characteristic.** The Frobenius characteristic of the permutation representation of $\mathfrak{S}_n$ on $\mathrm{PF}_n^{(m)}$ is related to that of ordinary parking functions of length mn whose evaluation is multiple of m:

$$\mathrm{ch}(\mathrm{PF}_n^{(m)}) = \frac{1}{mn+1}h_n\left((mn+1)X\right) = \phi_m(g_{mn}) = \phi_m(g_{mn})\,, \tag{194}$$

where $g_{mn} = \mathrm{ch}(PF_{mn})$ (ordinary parking functions) and ϕ_m is the adjoint of the plethysm operator ψ^m. This can be deduced as usual from the noncommutative 0-Hecke characteristic $g^{(m)}$, whose functional equation is

$$g^{(m)} = \sum_{n\geq 0} S_n(g^{(m)})^{mn}. \tag{195}$$

The solution is given in [32] in the form

$$g^{(m)} = F(r,m)F(r-1,m)^{-1} \tag{196}$$

where r is an arbitrary integer, and

$$F(r,m) = \sum_{n\geq 0} S_n((mn+r)A)\,. \tag{197}$$

For example,

$$g_3^{(2)} = S_3 + 2S^{21} + 4S^{12} + 5S^{111} \tag{198}$$

the sum of the dimensions of the representations is

$$1 + 2\times 3 + 4\times 3 + 5\times 6 = 49 = (2\times 3+1)^{3-1} \tag{199}$$

and the sum of the coefficients is

$$1+2+4+5 = 12 \tag{200}$$

the number of nondecreasing 2-parking functions, see (206) below.

As already mentioned, the relation $g^{(m)} = \phi_m(g)$ remains true for the noncommutative characteristic, where ϕ_m is now interpreted as the adjoint of the Adams operation $\psi^m(M_I) = M_{mI}$ of $QSym$.

6.3. **The Hopf algebra ^{m}PQSym.** The m-parking functions admit Hopf algebra structures. Indeed, there is a notion of m-parkization: to m-parkize a word w, let $u = w^{\uparrow}$ be the sorted word. If u_i is the first letter violating the constraint $u_i \leq k(i-1)+1$, shift it and all the letter on its right by the minimal amount $k(i-1)+1-u_i$ so that the constraint gets satisfied, and iterate the process so as to obtain a nondecreasing m-parking function. Finally, put back each letter at its original place.

Now, let $\mathbf{a}$ be an m-parking function and define

$$\mathbf{G_a} := \sum_{m-\mathrm{park}(w)=\mathbf{a}} w. \tag{201}$$

Then, as it is the case for $m=1$ (see Section 2.3.3), these $\mathbf{G_a}$ span a subalgebra of $\mathbb{K}\langle A\rangle$. This algebra is denoted by ${}^m\mathbf{PQSym}^*$.

It is then straightforward that the dual basis $\mathbf{F_a}$ has its product defined by shifted shuffle and its coproduct is given by deconcatenation and m-parkization.

In particular, ${}^m\mathbf{PQSym}$ is a shuffle-like Hopf algebra on the $\mathbf{F}$ basis.

6.4. **Hyposylvester classes of m-parking functions.** The number of hyposylvester classes of m-parking functions of size n can be computed as follows.

Let χ_0 and χ_1 be the characters of **Sym** defined in (51), α as in (52), so that the generating series of the number $a_n(m)$ of hyposylvester classes of m-parking functions is

$$X(t) := \sum_{n\geq 0} a_n(m)t^m = \alpha(g_m)\,. \tag{202}$$

From (195), we obtain, setting $Y = \chi_1(g_m)$,

$$X(t) = \sum_{n\geq 0} t^n \chi_1(g_m)^{mn} = \frac{1}{1-tY^m}, \tag{203}$$

since χ_1 is a character. Applying again χ_1 to (195), we obtain a second equation

$$Y = \sum_{n\geq 0} (n+1)t^n Y^{mn} = \frac{1}{(1-tY^m)^2} \tag{204}$$

and combining both, we have $Y^m = X^{2m}$, so that $X(t)$ satisfies the functional equation

$$X = 1 + tX^{2m+1} \quad (X(0) = 1). \tag{205}$$

Thus, we have proved:

Proposition 6.1. *The number of hyposylvester classes of m-parking functions is equal to the number of $(2m+1)$-ary trees with n nodes.* ∎

6.5. **Nondecreasing m-parking functions.** Ordinary nondecreasing parking functions are counted by Catalan numbers, and admit simple bijections with noncrossing partitions and binary trees. The Hopf algebra **CQSym** obtained by summing parking functions having the same sorted word can be identified with the free duplicial algebra on one generator [33].

Now, nondecreasing m-parking functions of length n are in bijection with ordinary nondecreasing parking functions of length nm whose evaluation is multiple of m: just repeat each letter m times, *e.g.*, with $m = 2$, $115 \to 111155$.

As already seen in Section 4.10.1, they are enumerated by the Fuss-Catalan numbers, and are therefore in bijection with $(m+1)$-ary trees or m-Dyck paths.

Arguing as in [32], we can see that the reciprocal coefficients of $\phi_m(G)$, where G is the noncommutative q-Lagrange series, enumerate nondecreasing m-parking functions by the shifted sum of their entries. For example, with $m = 2$ and $n = 3$, the enumeration of nondecreasing 2-parking functions of size 3

$$111, 112, 113, 114, 115, 122, 123, 124, 125, 133, 134, 135 \tag{206}$$

by sum of their *entries minus one* is

$$1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + q^6. \tag{207}$$

These polynomials do not seem to be a known family of q-Fuss-Catalan numbers.

Nondecreasing m-parking functions are also in bijection with m-divisible noncrossing partitions. Maximal chains in this lattice correspond to m-parking functions of length $n-1$ [7].

6.6. **Hopf algebras.** The sums $\mathbf{P}^\pi$ of all m-parking functions $\mathbf{F_a}$ with a given reordering π span a Hopf subalgebra ${}^m\mathbf{CQSym}$. Its dual is a commutative algebra realized as usual (basis $\mathcal{M}_\pi$). The Adams operation ψ^m map $\mathcal{M}_\pi \in {}^m\mathbf{CQSym}^*$ to $\mathcal{M}_{\pi^m\uparrow} \in \mathbf{CQSym}^*$ (an ordinary NDPF). The dual map ϕ_m is an algebra morphism which maps $\mathbf{P}^{\pi^m\uparrow}$ to $\mathbf{P}^\pi$ and $\mathbf{P}^\tau$ to 0 if τ is not m-divisible.

6.7. **Hypoplactic and hyposylvester quotients.**

6.7.1. *m-Parking functions.* The hypoplactic quotient of **PQSym** is a Hopf algebra of graded dimension given by the little Schröder numbers, not isomorphic to the free tridendriform algebra on one generator (as it is not self-dual). This algebra, denoted by **SQSym**, is the free triduplicial algebra on one generator [33].

The number of hypoplactic classes of m-parking functions is given by the Fuss-Schröder numbers obtained as follows. Let

$$F(y) = 1 + \frac{ty^m}{1 - ty^m}. \tag{208}$$

Then,

$$P_{n,m}(x) = \frac{1}{mn+1}[t^{mn}]F^{mn+1} \tag{209}$$

enumerates nondecreasing m-parking functions according to the number of different letters, and the number of hypoplactic classes is

$$\frac{1}{2}P_{n,m}(2). \tag{210}$$

For $m = 2$, this is sequence A034015:

$$1, 1, 5, 33, 249, 2033, 17485, 156033, 1431281, 13412193, 127840085, \ldots \tag{211}$$

The polynomials $P_{n,m}/x$ are

$$\begin{aligned} &1, 2x+1, 5x^2+6x+1, 14x^3+28x^2+12x+1,\\ &42x^4+120x^3+90x^2+20x+1, \ldots \end{aligned} \tag{212}$$

These are the m-Narayana polynomials whose reciprocals $N_n^{(m)}(x)$ evaluated at $x = 2$ yield the enumeration of sylvester classes of m-packed words.

They are given by

$$\sum_{i=0}^{n} \frac{1}{n+1}\binom{2n+2}{i}\binom{n+1}{i+1}x^i. \tag{213}$$

Indeed, nondecreasing m-parking functions correspond to m-divisible non-crossing partitions under the standard bijection, and the number of those is known [7].

Thus:

Proposition 6.2. *The number of hyposylvester classes of m-parking functions of length n is*

$$\sum_{i=0}^{n} \frac{1}{n+1}\binom{2n+2}{i}\binom{n+1}{i+1}2^i. \tag{214}$$

6.8. **Narayana triangles.**

6.8.1. *Standard m-Narayana polynomials.* These triangles show the number of nondecreasing m-parking functions with a given number of different letters (parameter p): Figures 4 to 6.

(215)

$n\backslash p$	1	2	3	4	5	6
1	1					
2	1	1				
3	1	3	1			
4	1	6	6	1		
5	1	10	20	6	1	
6	1	15	50	50	15	1

FIGURE 4. The usual Narayana triangle.

(216)

$n\backslash p$	1	2	3	4	5	6
1	1					
2	1	2				
3	1	6	5			
4	1	12	28	14		
5	1	20	90	120	42	
6	1	30	220	550	495	132

FIGURE 5. The Narayana triangle at $m = 2$.

(217)

$n\backslash p$	1	2	3	4	5	6
1	1					
2	1	3				
3	1	9	12			
4	1	18	66	55		
5	1	30	210	455	273	
6	1	45	510	2040	3060	1428

FIGURE 6. The Narayana triangle at $m = 3$.

6.8.2. *Modified Narayana triangles.* These triangles show the number of hypoplactic classes of m-parking functions according to their number of recoils: Figures 7 to 9.

(218)

$n\backslash p$	1	2	3	4	5	6
1	1					
2	2	1				
3	5	5	1			
4	14	21	9	1		
5	42	84	56	14	1	
6	132	330	300	120	20	1

FIGURE 7. The $x = 1 + q$ Narayana triangle at $m = 1$.

(219)

$n\backslash p$	1	2	3	4	5	6
1	1					
2	3	2				
3	12	16	5			
4	55	110	70	14		
5	273	728	702	288	42	
6	1428	4760	6160	3850	1155	132

FIGURE 8. The $x = 1 + q$ Narayana triangle at $m = 2$.

(220)

$n\backslash p$	1	2	3	4	5	6
1	1					
2	4	3				
3	22	33	12			
4	140	315	231	55		
5	969	2907	3213	1547	273	
6	7084	26565	39270	28560	10200	1428

FIGURE 9. The $x = 1 + q$ Narayana triangle at $m = 3$.

6.8.3. *Modified Narayana triangles (second version).* The triangle at $m = 1$ is the same as in Figure 7 since the Narayana triangle at $m = 1$ is symmetric. These triangles enumerate the number of sylvester classes of m-packed words by their size minus their number of different letters: Figures 10 and 11.

(221)

$n\backslash p$	1	2	3	4	5	6
1	1					
2	3	2				
3	12	8	1			
4	55	55	15	1		
5	273	364	145	24	1	
6	1428	2380	1400	350	35	1

FIGURE 10. The reversed $x = 1 + q$ Narayana triangle at $m = 2$.

7. Conclusions and perspectives

7.1. **Other algebras.** We have focused our attention on Hopf algebras of trees, and mainly investigated the sylvester and metasylvester congruences. A detailed study of the other ones (plactic, hypoplactic, hyposylvester and stalactic) will be done in another publication. The colored versions, as well as generalizations of the cambrian congruences need also to be explored. In the plactic case, one may expect to obtain at some point information on power-sum plethysms of Schur functions.

(222)

$n\backslash p$	1	2	3	4	5	6
1	1					
2	4	1				
3	22	11	1			
4	140	105	21	1		
5	969	969	306	34	1	
6	7084	8855	3850	700	50	1

FIGURE 11. The reversed $x = 1 + q$ Narayana triangle at $m = 3$.

7.2. **Bidendriform structures.** All the algebras presented here, as ${}^m\mathbf{FQSym}$, as ${}^m\mathbf{WQSym}$, or as ${}^m\mathbf{PQSym}$ and their sylvester, metasylvester, and hyposylvester subalgebras are bidendriform bialgebras. This can be proved by means of a generalization of Theorem 2.1 stating that a subalgebra of a bidendriform bialgebra is itself bidendriform if the monoid satisfies the conditions of Theorem 2.1 and one extra condition: the congruence never modifies the last letter of a word. This result and a summary of techniques studying bidendriform structures will be the subject of a forthcoming paper.

7.3. **Operads.** Finally, it is highly probable that the new algebras constructed in this paper are related to some operads, old or new. Their identification would certainly shed new light on the relationship between operads and combinatorial Hopf algebras.

8. Appendix: Tables

We present here for small m tables of the number of m-permutations along with their numbers of classes for the sylvester, hyposylvester, and meta-sylvester congruences (Figures 12 to 15), and the same data for m-packed words (Figures 16 to 19), m-parking functions (Figures 20 to 24). And, for comparison, multiple parking functions (shuffles of ordinary parking functions, Figures 25 to 28).

When a sequence is in OEIS, its number is written to the right of it (except for the hyposylvester classes of m-permutations enumerated by powers of integers).

(223)

$m\backslash n$	1	2	3	4	5	
1	1	2	6	24	120	$A000142$
2	1	6	90	2520	113400	$A000680$
3	1	20	1680	369600	168168000	$A014606$
4	1	70	34650	63063000	305540235000	$A014608$
5	1	252	756756	11732745024	623360743125120	$A014609$

FIGURE 12. m-permutations.

(224)

$m\backslash n$	1	2	3	4	5	
1	1	2	5	14	42	$A000108$
2	1	3	12	55	273	$A001764$
3	1	4	22	140	969	$A002293$
4	1	5	35	285	2530	$A002294$
5	1	6	51	506	5481	$A002295$

FIGURE 13. Sylvester classes of m-permutations (Fuss-Catalan numbers).

(225)

$m\backslash n$	1	2	3	4	5
1	1	2	4	8	16
2	1	3	9	27	81
3	1	4	16	64	256
4	1	5	25	125	625
5	1	6	36	216	1296

FIGURE 14. Hyposylvester classes of m-permutations.

(226)

$m\backslash n$	1	2	3	4	5	
1	1	2	6	24	120	$A000142$
2	1	3	15	105	945	$A001147$
3	1	4	28	280	3640	$A007559$
4	1	5	45	585	9945	$A007696$
5	1	6	66	1056	22176	$A008548$

FIGURE 15. Metasylvester classes of m-permutations.

(227)

$m\backslash n$	1	2	3	4	5	
1	1	3	13	75	541	$A000670$
2	1	7	121	3907	202741	$A094088$
3	1	21	1849	426405	203374081	
4	1	71	35641	65782211	323213457781	
5	1	253	762763	11872636325	633287284180541	

FIGURE 16. m-packed words.

(228)

$m\backslash n$	1	2	3	4	5	
1	1	3	11	45	197	A001003
2	1	4	21	126	818	A003168
3	1	5	34	267	2279	
4	1	6	50	484	5105	
5	1	7	69	793	9946	

FIGURE 17. Sylvester classes of m-packed words.

(229)

$m\backslash n$	1	2	3	4	5	
1	1	3	10	34	116	A007052
2	1	4	18	82	374	A052913
3	1	5	28	158	892	
4	1	6	40	268	1796	
5	1	7	54	418	3236	

FIGURE 18. Hyposylvester classes of m-packed words

(230)

$m\backslash n$	1	2	3	4	5	
1	1	3	12	60	360	A001710
2	1	4	24	192	1920	A002866
3	1	5	40	440	6160	A034000
4	1	6	60	840	15120	A000407
5	1	7	84	1428	31416	A034323

FIGURE 19. Metasylvester classes of m-packed words

(231)

$m\backslash n$	1	2	3	4	5	
1	1	3	16	125	1296	A000272
2	1	5	49	729	14641	A052750
3	1	7	100	2197	65536	A052752
4	1	9	169	4913	194481	A052774
5	1	11	256	9261	456976	A052782

FIGURE 20. m-parking functions

(232)

$m\backslash n$	1	2	3	4	5
1	1	3	13	69	417
2	1	5	40	407	4797
3	1	7	82	1239	21810
4	1	9	139	2789	65375
5	1	11	211	5281	154661

FIGURE 21. Sylvester classes of m-parking functions (none in [37]).

(233)

$m\backslash n$	1	2	3	4	5	
1	1	3	12	55	273	$A001764$
2	1	5	35	285	2530	$A002294$
3	1	7	70	819	10472	$A002296$
4	1	9	117	1785	29799	$A059967$
5	1	11	176	3311	68211	$A230388$

FIGURE 22. Hyposylvester classes of m-parking functions.

(234)

$m\backslash n$	1	2	3	4	5	
1	1	3	11	45	197	$A001003$
2	1	5	33	249	2033	$A034015$
3	1	7	67	741	8909	
4	1	9	113	1649	26225	
5	1	11	171	3101	61381	

FIGURE 23. Hypoplactic classes of m-parking functions.

(235)

$m\backslash n$	1	2	3	4	5	
1	1	3	14	87	669	$A132624$
2	1	5	45	585	9944	
3	1	7	94	1879	50006	
4	1	9	161	4353	158035	
5	1	11	246	8391	386211	

FIGURE 24. Metasylvester classes of m-parking functions (the first row in conjecturally A132624).

(236)

$m\backslash n$	1	2	3	4	5
1	1	3	16	125	1296
2	1	7	136	5293	347776
3	1	21	1933	483209	257484501
4	1	71	36136	68501421	349901224576
5	1	253	765766	12012527625	648203695298171

*A*000272

FIGURE 25. Multiparking functions (shuffles of m ordinary PFs).

(237)

$m\backslash n$	1	2	3	4	5
1	1	3	13	69	417
2	1	4	24	179	1532
3	1	5	38	360	3919
4	1	6	55	628	8235
5	1	7	75	999	15262

FIGURE 26. Sylvester classes of multiparking functions (none in [37], not even the first one since $n = 6$ gives 2759).

(238)

$m\backslash n$	1	2	3	4	5
1	1	3	12	75	273
2	1	4	21	126	818
3	1	5	32	233	1833
4	1	6	45	382	3498
5	1	7	60	579	6017

*A*001764
*A*003168

FIGURE 27. Hyposylvester classes of multiparking functions.

(239)

$m\backslash n$	1	2	3	4	5
1	1	3	14	87	669
2	1	4	27	254	3048
3	1	5	44	551	8919
4	1	6	65	1014	20598
5	1	7	90	1679	40977

*A*132624

FIGURE 28. Metasylvester classes of multiparking functions.

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