The Scattering Variety

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Abstract

The so-called Scattering Equations which govern the kinematics of the scattering of massless particles in arbitrary dimensions has recently been cast into a system of homogeneous polynomials. We study these as affine and projective geometries which we call Scattering Varieties by analyzing such quantities as Hilbert series, Euler characteristic and singularities. Interestingly, we find such structures as affine Calabi-Yau threefolds as well as singular K3 and del Pezzo surfaces.

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1 Introduction

Recently, a programme was launched [1–7] to study the kinematics of massless particle scattering in arbitrary dimensions. The initial motivation of this programme was the reduction of the tree-level S-matrix of a wide range of theories as an integral over the moduli space of maps from the N-punctured sphere into the null light-cone in momentum space [8–13].

Suppose we are given N null-vectors in D-dimensions corresponding to the momenta of N massless particles:

$$\{k_1^\mu,\ k_2^\mu,\ \ldots,\ k_N^\mu\}\ , \qquad k^2 = 0\ . \tag{1.1}$$

The map that retrieves the scattering data from the N-punctured sphere with complex coordinate z and punctures σ_a is obtained from [1]:

$$\sigma_a \mapsto k_a^\mu = \frac{1}{2\pi i}\oint_{|z-\sigma_a|=\epsilon} dz \frac{P^\mu(z)}{\prod\limits_{b=1}^{N}(z-\sigma_b)}\ , \tag{1.2}$$

where $P^\mu(z)$ is a collection of D (indexed by μ) degree $N-2$ polynomials.

The massless condition on k^μ clearly translates to that $P(\sigma_a)^2 = 0$ for all $a = 1, 2, \ldots, N$, so that $P(z)\cdot\partial_z P(z) = 0$. This subsequently translates to a set of equations

$$\sum_{b\neq a} \frac{k_a \cdot k_b}{z_a - z_b} = 0\ , \quad a, b = 1, 2, \ldots, N\ . \tag{1.3}$$

This set of equations, governing the kinematics of our problem, was dubbed the **scattering equations** in [2].

Our main focus will be the study of (1.3). In fact, the nice paper [6] has just reduced these scattering equations to a system of homogeneous polynomials; it is with the geometry of this system, to which *cit. ibid.* already alluded, that we shall be chiefly concerned.

Nomenclature: Throughout this work, we will adhere to the nomenclature of [6]. We label the N particles with the set of indices

$$A = \{1, 2, 3, \ldots, N\}\ , \tag{1.4}$$

for the momenta $k_{a\in A}$ and introduce complex variables $z_{a\in A}$. Then, consider all subsets S of A with m elements and define

$$k_S := \sum_{b\in S} k_b \ ; \text{ and } z_S := \prod_{a\in S} z_a \ , \text{ such that } S \subset A \ , \ |S| = m \ . \tag{1.5}$$

The insight of [6] is that the scattering equations (1.3) is *equivalent* to the polynomial system:

$$V := \{\tilde{h}_m = 0\} \ , \quad \text{with} \quad \tilde{h}_m := \sum_{\substack{S\subset A\\ |S|=m}} k_S^2 z_S \ , \quad 2 \le m \le N-2 \ , \tag{1.6}$$

for null conserved momenta. That is, the number of independent parameters k satisfy the constraints

$$k_a \cdot k_a = 0 \ , \quad \text{and} \quad \sum_a k_a = 0 \ . \tag{1.7}$$

Thus defined, V can be seen as a polynomial ideal consisting of a set of $N-2-2+1 = N-3$ polynomials, each homogenous of degree m in the N complex variables z_a, and such that k_a are parameters obeying $k_a \cdot k_a = 0$. In other words, V defines an algebraic variety $\mathcal{V}$ in $\mathbb{CP}^{N-1}$ with parametres k_a^μ constrained to satisfy (1.7).

Properties: It should also be pointed out that each polynomial in V, despite being of degree exceeding 2, is linear in each variable z_a considered separately. Hence, we have an ideal which is square-free and multilinear in all the coordinates.

The homogeneous polynomials (1.6) have the peculiar characteristic that when all coefficients $k_S^2 \neq 0$ then the values of z_a are distinct for all $a \in A$. This is reminiscent of the way the scattering factorize when one of the Mandelstam variables vanishes and has been demonstrated in [6].

Moreover, the scattering equation has the nice property to be Möbius invariant and [6] demonstrated that the equivalent set of polynomials (1.6) form an irreducible representation of the Möbius algebra, $\mathfrak{sl}_2(\mathbb{C})$.

Summary: In this work, we draw on the insight of [6] and explore the geometry of algebraic varieties $\mathcal{V}_\varphi$ defined from polynomial systems defining irreducible representations of the Möbius algebra. We found that they are all affine Calabi-Yau manifolds and, for the case of the variety $\mathcal{V}$ corresponding to the scattering equation, the Hilbert series coefficients in the Second Kind has the pattern of the *Mahonian triangles*. The corresponding varieties are the Conifold, a singular K3 surface and Fano varieties for the number of scattering particles $4, 5, 6$ and 7 respectively. For the varieties $\mathcal{V}_\varphi$ defined from highest weight polynomial of degree 3, we found that the Hilbert series coefficients correspond to the *cyclotomic* polynomials.

The plan of the paper is as follows. In the following section, we review the irreducible representations of the Möbius algebra and its relation to the scattering equation. We then present our results in the subsequent section, describing the geometries obtained. The last section presents our conclusions.

2 The Polynomial Systems

This section reviews the relation of the scattering equations to the irreducible representations of the Möbius algebra that has been presented in [6]. We will not present the derivations in full length but will summarise the most important conclusions. We do so to set our notations and motivate the geometrical considerations that are presented in the subsequent section. We will mainly follow the notations from [6] with small variations for the sake clarity.

2.1 Möbius Invariance

The scattering equation (1.3) is invariant under Möbius transformations and, consequently, so is (1.6). This can be seen as follows. For complex numbers α, β, γ and δ satisfying $\alpha\delta - \beta\gamma \neq 0$, we have

$$z_a \to \zeta_a = \frac{\alpha z_a + \beta}{\gamma z_a + \delta} \ , \quad a \in A \ ; \tag{2.1}$$

then ζ_a are also solutions of (1.3) when z_a are solutions themselves. In fact, [6] showed that the polynomials (1.6) form a basis of an $(N-3)$-dimensional representation of the Möbius algebra, $\mathfrak{sl}_2(\mathbb{C})$, in a way that we now present.

Let us consider the following operators acting on the ring of polynomials in $z_a, a \in A$ generating the Möbius group $PSL(2,\mathbb{C})$

$$\begin{aligned} L_0 &= -\sum_{a\in A} z_a \frac{\partial}{\partial z_a} + \frac{N}{2} \ , \\ L_1 &= \sum_{a\in A} z_a - z_a^2 \frac{\partial}{\partial z_a} \ , \\ L_{-1} &= -\sum_{a\in A} \frac{\partial}{\partial z_a} \ . \end{aligned} \tag{2.2}$$

We can easily see that they satisfy the following $\mathfrak{sl}_2(\mathbb{C})$ commutation relations (Möbius algebra)

$$[L_1, L_{-1}] = 2L_0 \ , \quad [L_0, L_{\pm 1}] = \mp L_{\pm 1} \ . \tag{2.3}$$

How does this relate to the scattering equation? A straightforward calculation shows that the above operators act on the $\tilde{h}_m$ polynomials (1.6) in the following way:

$$\begin{aligned} L_0 \tilde{h}_m &= (\tfrac{1}{2}N - m)\, \tilde{h}_m \ , \\ L_1 \tilde{h}_m &= (m-1)\, \tilde{h}_{m+1} \ , \\ L_{-1} \tilde{h}_m &= -\,(N - m - 1)\, \tilde{h}_{m-1} \ , \end{aligned} \tag{2.4}$$

with $L_{-1}\tilde{h}_2 = 0$ and $L_1 \tilde{h}_{N-2} = 0$. Therefore, m takes the following values $2 \leq m \leq N-2$ and we have $N-3$ polynomials. The corresponding quadratic Casimir for this representation is given by

$$L^2 := L_0^2 - \tfrac{1}{2} L_1 L_{-1} - \tfrac{1}{2} L_{-1} L_1 \ , \tag{2.5}$$

and takes value $(\frac{1}{2} - 2)(\frac{1}{2}N - 1)$. Therefore, the representation spanned by the $\tilde{h}_m$ polynomials form an irreducible representation of Möbius spin $\frac{1}{2}N - 2$.

Furthermore, the above operators have the nice property that they generate the set of all polynomials $\tilde{h}_m$ starting from the lowest degree polynomial $\tilde{h}_2$. More precisely, repeated action of L_1 will generate $\tilde{h}_m$ for $m > 2$,

$$(L_{-1})^r \tilde{h}_2 = r! \tilde{h}_{2+r} \ . \tag{2.6}$$

The polynomials $\tilde{h}_2$ have the property that $L_{-1}\tilde{h}_2 = 0$ and the series terminates at $L_1 \tilde{h}_{N-2} = 0$, making it closed under the Möbius representation.

2.2 Irreducible Representations

In fact, the ring of polynomials in $z_a, a \in A$ defines an infinite-dimensional representation of the Möbius algebra (2.3) and we shall see that the Möbius invariant subspace of polynomials provides a graded finite-dimensional representation that decomposes into irreducible subspaces [6].

Drawing on the above observations for $\tilde{h}_m$, we can define more general irreducible representations from starting with different lowest degree polynomials. The eigenvalues of the L_0 operator are the largest for these lowest degree polynomials and they are therefore referred to as the highest weight polynomials.

Let us define the Möbius invariant space of polynomials $\mathcal{F}^N$ (i.e., the space is invariant under (2.2)), which is spanned by $\{z_S : S \subset A\}$.

Let us also write $\mathcal{F}^N_m$ the subset of $\mathcal{F}^N$ consisting of polynomials of degree m, hence $\mathcal{F}^N_m$ is spanned by $\{z_S : S \subset A, |S| = m\}$. A generic polynomial $\varphi_m \in \mathcal{F}^N_m$ can therefore be written as,

$$\varphi_m := \sum_{\substack{S \subset A \\ |S|=m}} \lambda_S z_S \ , \tag{2.7}$$

where λ_S are tensors with indices in S. Explicitly, this implies that the tensor indices $\lambda_{i_1 \dots i_m}$ for $S = \{i_1 \dots i_m\}$ are totally symmetric in $i_1 \dots i_m$ and vanishes if any two indices are equal. In addition, we impose the constraint that $\lambda_S \neq 0$ for all subsets S as we will not consider cases when some of the Mandelstam variables vanish.

In order to define an irreducible representation of the Möbius algebra, we select a highest weight polynomial $\varphi_n \in \mathcal{F}_n^N$ with the property that,

$$L_{-1}\varphi_n = 0 \ . \tag{2.8}$$

This implies the following constraints on the tensor coefficients

$$\sum_{\substack{S\subset A \ | \ a\in S, \\ |S|=n}} \lambda_S = 0 \ , \quad \text{for each} \quad a \in A \ . \tag{2.9}$$

We will reserve the notation n for indices of highest weight polynomials, whereas the full series of polynomials will be noted with an index m, hence $n = \min(m)$.

Let us write the set of highest weight polynomials as

$$\mathcal{H}_n^N := \{\varphi_n \in \mathcal{F}_n^N \ | \ L_{-1}\varphi_n = 0\} \ . \tag{2.10}$$

Then, acting on any $\varphi_n \in \mathcal{H}_n^N$ with L_1 repeatedly, we generate an irreducible representation of the Möbius algebra

$$(L_1)^r\varphi_n = r!\varphi_{n+r} \ . \tag{2.11}$$

Generating the polynomials this way will imply some structure on the tensor coefficient λ_S. We have, for the full set of polynomials,

$$\varphi_m = \sum_{\substack{S\subset A \\ |S|=m}} \lambda_S^{(n)} z_S \ , \quad \text{with} \quad \lambda_S^{(n)} = \sum_{\substack{U\subset S \\ |U|=n}} \lambda_U \ . \tag{2.12}$$

In general, it has been shown [6] that the operators (2.3) acts on the polynomials generated this way in the following manner

$$\begin{aligned} L_0\varphi_m &= \left(\tfrac{1}{2}N - m\right)\varphi_m \ , \\ L_1\varphi_m &= (m - n + 1)\,\varphi_{m+1} \ , \\ L_{-1}\varphi_m &= -\,(N - m - n - 1)\,\varphi_{m-1} \ . \end{aligned} \tag{2.13}$$

Moreover, the series terminates at $L_1\varphi_{N-n} = 0$ and thus the index m range is given by $n \leq m \leq N - n$ giving a set of $N - 2n + 1$ polynomials. The Casimir operator takes value

$$L^2\varphi_m = (\tfrac{1}{2}N - n)(\tfrac{1}{2}N - n + 1)\varphi_m \ . \tag{2.14}$$

Thus we have an irreducible representation of the Möbius algebra of spin $\frac{1}{2}N - n$. Geometrically, we have the important result that

The set of polynomials $\{\varphi_m\}$ define an algebraic variety $\mathcal{V}_\varphi$ in $\mathbb{CP}^{N-1}$ of dimension $2n-1$.

2.3 Fixing Möbius Invariance

Möbius invariance allows us to fix some of the z variables from the homogeneous system of equations. Following, [6] we choose to set $z_1 \to \infty$ and $z_N \to 0$. Fixing these points amounts to acting on every polynomials with the following operators,

$$L_{z_1} = \frac{\partial}{\partial z_1} \ , \quad \text{and} \quad L_{z_N} = 1 - z_N \frac{\partial}{\partial z_N} \ , \tag{2.15}$$

which corresponds to $z_1 \to \infty$ and $z_N \to 0$ to respectively. For implementing both conditions, we act with the product of both operators $L_{z_1} \cdot L_{z_N}$. This operator decreases the degree by 1 and it is straightforward to realise that the new set of polynomials thus defined is still a subset of $\mathcal{F}^N$. However, they do not lead to another irreducible representation as the highest weight condition (2.9) is not satisfied for the lowest degree polynomial (hence it is also not a Möbius invariant subspace of $\mathcal{F}^N$).

The resulting set of polynomials can be written in the following manner using similar notations as previously,

$$\varphi^*_m := \lim_{z_1 \to \infty} \frac{1}{z_1} \varphi_{m+1} = \sum_{\substack{S \subset A' \\ |S| = m}} \varphi_{S^1} z_S \ , \quad n-1 \leq m \leq N-n-1 \ , \tag{2.16}$$

where $S^1 = S \cup 1$ and $A' = \{a \in A : a \neq 1, N\}$. The $*$ notation is adopted to designate the system of fixed Möbius invariance. Each set contains $N - 2n + 1$ polynomials and these sets of polynomials $\{\varphi^*_m\}$ thus define algebraic varieties $\mathcal{V}^*_\varphi$ in $\mathbb{CP}^{N-3}$ of dimension $2n-3$. We can see that fixing Möbius invariance this way decreases the dimension of the variety by two.

For the case $n = 2$ the (affine) dimension is 1 and therefore we have a 0-dimensional (projective) variety, that is, a set of discrete points. The degree of each polynomials is $\{1, \ldots, N-3\}$ and from Bézout's theorem, we therefore expect to have $(N-3)!$ points, counting multiplicity.

3 The Scattering Varieties

Thus far we have encountered the following varieties of note which we summarize for convenience:

Variety	Defining Polynomials	Projective Dim
$\mathcal{V}_\varphi$	$\{\varphi_m := \sum_{\substack{S\subset A\\ \lvert S\rvert=m}} \lambda_S^{(n)} z_S \mid \lambda_S^{(n)} = \sum_{\substack{U\subset S\\ \lvert U\rvert=n}} \lambda_U\}_{m=n,\ldots,N-n} \in \mathbb{CP}^{N-1}$	$2n-2$
$\mathcal{V}_\varphi^*$	$\{\varphi_m^* := \lim_{z_1\to\infty} \frac{1}{z_1}\varphi_{m+1}\}_{m=n-1,\ldots,N-n-1} \in \mathbb{CP}^{N-3}$	$2n-4$

Table 1: Summary of defining polynomials for the varieties under consideration.

We will now study these algebraic varieties in detail. Noting that for the case of $n = 2$ $\mathcal{V}_\varphi^*$ is a discrete set of points corresponding to the solutions of the scattering equations, and noting that this is only one member of the family, we will refer to the general cases of $\mathcal{V}_\varphi^*$ and $\mathcal{V}_\varphi$ as **scattering varieties**.

Indeed, the polynomials (1.6) from the scattering equation corresponds to the case $n = 2$ for the above irreducible representation of the Möbius algebra with spin $\frac{1}{2}N - 2$. To see this, let us look at the coefficient $k_S := \sum_{b\in S} k_b$. For null vectors, all of the coefficients k_S^2 can be expressed in terms of the $\frac{1}{2}N(N-1)$ quadric ones $2k_a \cdot k_b$. The highest weight polynomial $\tilde{h}_2$ thus becomes

$$\tilde{h}_2 = \sum_{\substack{S\subset A\\ |S|=2}} \lambda_S z_S \ , \tag{3.1}$$

with $\lambda_{ab} = 2k_a \cdot k_b$. The operators L_1 then generates the full set $\{\tilde{h}_m\}$ following (2.6) and we obtain,

$$\tilde{h}_m = \sum_{\substack{S\subset A \\ |S|=m}} \lambda_S^{(2)} z_S \ , \quad \text{with} \quad \lambda_S^{(2)} = \sum_{\substack{U\subset S \\ |U|=2}} \lambda_U \ . \tag{3.2}$$

In fact, [6] showed that this kind of representations is uniquely defined and, therefore, $\{\varphi_m\}$ has to have the form of the scattering equation (1.6) when starting from highest weight φ_2. The constraints on the k parameters (1.7) are precisely those leading to the condition of highest weight (2.9). This is seen as follows. Combining the null condition with the conservation of momenta, we have

$$\sum_{\substack{b\in A \\ b\neq a}} k_a \cdot k_b = k_a \cdot \sum_{b\in A} k_b = 0 \ , \tag{3.3}$$

and thus

$$\sum_{\substack{S\subset A \ | \ a\in S, \\ |S|=2}} \lambda_S = 0 \ , \quad \text{for each} \quad a \in A \ , \tag{3.4}$$

when $\lambda_{ab} = 2k_a \cdot k_b$. This is precisely the highest weight condition (2.9) for $n = 2$.

This deep relation of the scattering equation with irreducible representations of the Möbius algebra motivates the study of the full class of varieties $\mathcal{V}_\varphi$. The question of physical significance of the varieties for $n \neq 2$ still remains open. Nevertheless, we hope that an understanding of the geometrical structures will shed light into possible interpretations. We therefore consider every $\mathcal{V}_\varphi$ without any discrimination.

3.1 Hilbert Series

Let us first find the dimension, degree and Hilbert series of these varieties. Start by listing all possibles $N = 2, 3, 4, 5, 6, \ldots$ and tabulate the results. We can extract such geometrical quantities using standard computational geometric package, such as [16] as well as software interfacing with Mathematica [15]. The tensor coefficients from (2.9) are kept as generic non-vanishing parameters $\lambda_S \neq 0$ satisfying the highest weight condition (2.9).

For convenience of presentation, let us adopt the standard nomenclature that the

Hilbert series be in Second Kind, that is, for an affine variety V,

$$H(t) = (1-t)^{-\dim(V)} \sum_{i=0}^{k} a_i t^i \ , \tag{3.5}$$

with the power of the denominator being the dimension of the variety (the First Kind would have the dimension of the ambient space as the power instead). This is in turn abbreviated to simply the sequence of coefficients $\{a_0, \ a_1, \ \ldots, \ a_k\}$. In this second kind, a useful fact is that the sum over a_i, i.e., the numerator evaluated at $t = 1$, is simply the degree of the variety.

3.1.1 The variety $\mathcal{V}_\varphi$

Using this notation, it is expedient to tabulate our findings of the various Hilbert series for $\mathcal{V}_\varphi$, obtained through explicit computation [15, 16]. First, before imposing the Möbius transformation, we obtain the results from Table 2.

N	$n=2 \quad \dim(\mathcal{V}_\varphi)=3$	$n=3 \quad \dim(\mathcal{V}_\varphi)=5$	$n=4 \quad \dim(\mathcal{V}_\varphi)=7$
4	$1,1$	$-$	$-$
5	$1,2,2,1$	$-$	$-$
6	$1,3,5,6,5,3,1$	$1,1,1$	$-$
7	$1,4,9,15,20,22,20,15,9,4,1$	$1,2,3,3,2,1$	$-$
8	$\ldots$	$1,3,6,9,11,11,9,6,3,1$	$1,1,1,1$
9	$\ldots$	$\ldots$	$1,2,3,4,4,3,2,1$
10	$\ldots$	$\ldots$	$1,3,6,10,14,17,18,17,14,10,6,3,1$

Table 2: Dimension and Hilbert series for $\mathcal{V}_\varphi$. Note that we have recorded the *affine* dimension here, whereby embedding $\mathcal{V}_\varphi$ into $\mathbb{C}^N$.

A few observations are immediate. First, from combinatorics, all the numerators are **palindromic** in that $a_i = a_{k-i}$ for all i. This means, by a theorem of Stanley [17, 20], that all our varieties V are, in fact, *affine Calabi-Yau*.

Next, these sequences are well-known. The $n = 2$ case corresponds to the so-called *Mahonian triangles* [21]; they have a nice generating function which allows us to analytically write the Hilbert series as a function of $N \geq 4$

$$H(t;N)_{n=2} = (1-t)^{-3} \prod_{j=1}^{N-3} \sum_{i=0}^{j} t^i = (1-t)^{-3} \sum_k T_{N-3,k} t^k \ , \tag{3.6}$$

where $T_{N-3,k}$ are our Mahonian numbers. One combinatorial definition [21] of these numbers is that it is the number of permutations $\pi = (\pi(1), \ldots, \pi(k))$ of $\{1, \ldots, k\}$ such that the so-called major index $\sum\limits_{i:\pi(i)>\pi(i+1)}$ is equal to k. This nice analytical formula deduced from our examples allows us to conjecture its validity for any number of scattering particles N.

Similarly, the $n = 3$ case corresponds to a k-generalization of lattice permutations. As a function of $N \geq 7$ (the initial case of $N = 6$ has the numerator $1 + t + t^2$ which does not obey the following generating function):

$$H(t;N)_{n=3} = (1-t)^{-5} \prod_{j=1}^{N-4} C_{j+1}(t) \ ; \qquad C_j(t) := \prod_{0<k<j, \ \gcd(k,j)=1} (t - e^{\frac{2\pi i j}{k}}) \ , \tag{3.7}$$

where in the above $C_j(t)$ are the *cyclotomic* polynomials. Again, a nice analytical formula supports the conjecture for its validity up to any N value.

3.1.2 The variety $\mathcal{V}_\varphi^*$

After fixing by two Möbius transformations, i.e., $z_1 \to \infty$ and $z_N \to 0$, the complex (affine) dimension of the variety uniformly drops by 2. Moreover, we can apply the third and last of the Möbius transformations and set one more z to 1, whereby dropping the dimension further by 1, which in the $n = 2$ case of the scattering equations, gives the requisite zero, viz., discrete set of points. The resulting Hilbert series for the varieties $\mathcal{V}_\varphi^*$ are summarized in Table 3.

We observe that these Hilbert series are simply a shift of the above table. For the case $n = 2$, we recover the same Hilbert series as for the $\mathcal{V}_\varphi$ case shifted from N to $N + 1$. For the other cases, the Hilbert series for n correspond to that of $\mathcal{V}_\varphi$ for $n - 1$

N	$n=2 \quad \dim(V)=1$	$n=3 \quad \dim(V)=3$	$n=4 \quad \dim(V)=5$
4	1	$-$	$-$
5	$1,1$	$-$	$-$
6	$1,2,2,1$	$1,1$	$-$
7	$1,3,5,6,5,3,1$	$1,2,2,1$	$-$
8	$1,4,9,15,20,22,20,15,9,4,1$	$1,3,5,6,5,3,1$	$1,1,1$
9	$\ldots$	$1,4,9,15,20,22,20,15,9,4,1$	$1,2,3,3,2,1$
10	$\ldots$	$\ldots$	$1,3,6,9,11,11,9,6,3,1$

Table 3: Affine dimension and Hilbert series for $\mathcal{V}_{\varphi}^{*}$.

shifted from N to $N+2$. This is reminiscent from the fact that the number of variables decreases by 2 when Möbius invariance is fixed. Thus, we found again the Mahonian numbers however now for $n=2$ and $n=3$ and the cyclotomic polynomials for $n=4$.

3.2 Geometric Considerations

Are the varieties of geometrical significance, or, at least, of some familiarity? Let us focus on the cases of affine dimension 3: these are the cases of $n=2$ and $n=3$, respectively before and after Möbius fixing, with a shift of two in N. The first case has $H(t)=(1-t)^{-3}(1+t)=(1-t)^{-4}(1-t^2)$, which is the Hilbert series of the famous *conifold* as as an affine (non-compact) CY3 [22], given by a quadric in $\mathbb{C}^4$.

The next case has $H(t)=(1-t)^{-3}(1+2t+2t+t^2)$; guided by the plethystic logarithm [22], we can recast this into Euler form, viz., $H(t)=(1-t^2)(1-t^3)/(1-t)^5$, the intersection, as per definition, of a quadric and a cubic in $\mathbb{C}^5$. Upon projectivizing to $\mathbb{CP}^4$, this is a complete intersection which gives a K3 surface of degree 6 and (geometric) genus 4 [18].

Continuing in similar fashion, we can identify all the dimension 3 (projective dimension 2) cases as complete intersection complex surfaces. Using the standard notation [19]

$$[n|k_1, k_2, \ldots, k_m] := \{\text{intersection of } m \text{ degree } k_i \text{ hypersurfaces in } \mathbb{CP}^n\} \ , \qquad (3.8)$$

and noting that the K3 condition (vanishing of the first Chern class) requires that $\sum_i k_i = n + 1$, while the Fano condition, that the left-hand side be greater, we can summarize our (projective) geometries as in Table 4.

Geometry	Type	Hilbert Series	Degree
$[3\|2]$	Base of Conifold	$1, 1$	$2!$
$[4\|2,3]$	K3	$1, 2, 2, 1$	$3!$
$[5\|2,3,4]$	Fano	$1, 3, 5, 6, 5, 3, 1$	$4!$
$[6\|2,3,4,5]$	Fano	$1, 4, 9, 15, 20, 22, 20, 15, 9, 4, 1$	$5!$

Table 4: Geometry of the affine 3-dimensional (and hence, projective dimension 2) varieties.

We emphasize that the above are projective varieties of complex dimension 2, the affine (unprojectivize) varieties are all of dimension 3 and by the palindromic nature of the Hilbert series of Second Kind, are all affine Calabi-Yau threefolds. In other words, we can consider our 3-dimensional Calabi-Yau varieties as complex cones over some compact base surface, precisely in the same way as in the Calabi-Yau singularities of AdS_5/CFT_4.

3.2.1 Singularity

Calculating the Euler number χ of the above *projective* varieties unveils more information about the nature of the geometry considered. Indeed, we found the following,

$$\chi\left[\mathcal{V}_\varphi(n=2, N=4)\right] = 4 \qquad (3.9)$$

$$\chi\left[\mathcal{V}_\varphi(n=2, N=5)\right] = 29 \qquad (3.10)$$

$$\chi\left[\mathcal{V}_\varphi(n=2, N=6)\right] = 502 \qquad (3.11)$$

The astute reader would see the 29 for above K3 surface ($N = 5$) and be puzzled. Here we have a K3 surface as a complete intersection of a quadric and a cubic in $\mathbb{CP}^4$

and should the Euler number not be the standard 24? Upon further inspection of the defining polynomials from (3.2), we see that the coefficients λ to these polynomials in z are *not generic* and obey physical constraints. In other words, we are *not* in a generic point in the complex structure moduli space of K3 surfaces but quite special ones.

In fact, we will now see that the scattering variety is *not smooth* and contains singularities which will indeed deviate the values of the Euler number. Taking the Jacobian matrix for the variety (3.2), we have

$$J = \begin{pmatrix} \frac{\partial \tilde{h}_n}{\partial z_1} & \cdots & \frac{\partial \tilde{h}_n}{\partial z_N} \\ \vdots & \cdots & \vdots \\ \frac{\partial \tilde{h}_{N-n}}{\partial z_1} & \cdots & \frac{\partial \tilde{h}_{N-n}}{\partial z_N} \end{pmatrix} \tag{3.12}$$

Using indices $i = 1, \ldots, N$ for z and $m = n, \ldots, N-n$ for $\tilde{h}$, we have

$$\frac{\partial \tilde{h}_m}{\partial z_i} = \sum_{\substack{S \subset A \ | \ i \in S, \\ |S| = m}} \lambda_S z_{S \setminus \{i\}} \ , \tag{3.13}$$

and realise that each polynomial from the Jacobian corresponds to the limit where the variable $z_i \to \infty$ as this is the effect of the action from the operators $L_{\mathrm{z_i}} = \frac{\partial}{\partial z_i}$ similarly as for z_1. From the ideal generated by these partial derivatives, viz., the Jacobian ideal, we therefore have, for each $i = 1, \ldots, N$, $N - 2n + 1$ polynomials of degree ranging from $n-1$ to $N-n-1$. This defines an algebraic variety in $\mathbb{CP}^{N-1}$ which we will now study.

Again, let us adhere to the important case of $n = 2$, continuing along the strand of thought from Table 4. Explicitly, for $N = 4$, we have only one homogenous degree 2 polynomial $\langle \tilde{h}_2 \rangle$, so

$$
\begin{aligned}
V_4 = \mathcal{V}_\varphi(n=2, N=4) = \\
\{\lambda_{12}z_1z_2 + \lambda_{13}z_1z_3 + \lambda_{14}z_1z_4 + \lambda_{23}z_2z_3 + \lambda_{24}z_2z_4 + \lambda_{34}z_3z_4\} \\
\text{with} \quad \begin{cases} \lambda_{12} + \lambda_{13} + \lambda_{14} = 0 \\ \lambda_{12} + \lambda_{23} + \lambda_{24} = 0 \\ \lambda_{13} + \lambda_{23} + \lambda_{34} = 0 \\ \lambda_{14} + \lambda_{24} + \lambda_{34} = 0 \end{cases} ,
\end{aligned}
\tag{3.14}
$$

and the Jacobian variety is

$$
\begin{aligned}
J_4 = \{&\lambda_{12}z_2 + \lambda_{13}z_3 + \lambda_{14}z_4, \lambda_{12}z_1 + \lambda_{23}z_3 + \lambda_{24}z_4, \\
&\lambda_{13}z_1 + \lambda_{23}z_2 + \lambda_{34}z_4, \lambda_{14}z_1 + \lambda_{24}z_2 + \lambda_{34}z_3\} \ .
\end{aligned}
\tag{3.15}
$$

We readily see that for *arbitrary generic* choices of λ, the only point in J_4 is when all $z_i = 0$, which in $\mathbb{C}^4$ is the origin and in $\mathbb{CP}^3$ is excluded. Hence, the quadric projective variety $[3|2]$ is smooth and as an affine variety, realized as the conifold, there is our familiar singularity at the origin which is the tip of the cone.

However, for λ satisfying the constraints in (3.14), we can find a one-parametre family of solutions to $V_4 = J_4 = 0$, this is the **singular locus** on our non-generic variety:

$$
(z_1, z_2, z_3, z_4) = (x, x, x, x) \ , \quad x \in \mathbb{C}
\tag{3.16}
$$

with a convenient set of solutions for λ as

$$
\{\lambda_{14} \to -\lambda_{12} - \lambda_{13} \ , \lambda_{23} \to -\lambda_{12} - \lambda_{13} \ , \lambda_{24} \to \lambda_{13} \ , \lambda_{34} \to \lambda_{12}\} \ .
$$

Affine geometrically, the singular locus is a ray from the origin while projective, it is a single point $[1:1:1]$ on the quadric in $\mathbb{CP}^3$.

Next, for $N = 5$, we have a quadric intersecting a cubic

$$V_5 = \left\{ \begin{array}{l} \lambda_{12} z_1 z_2 + \lambda_{13} z_1 z_3 + \lambda_{14} z_1 z_4 + \lambda_{15} z_1 z_5 + \lambda_{23} z_2 z_3 \\ +\lambda_{24} z_2 z_4 + \lambda_{25} z_2 z_5 + \lambda_{34} z_3 z_4 + \lambda_{35} z_3 z_5 + \lambda_{45} z_4 z_5 \ , \\ (\lambda_{12} + \lambda_{13} + \lambda_{23}) z_1 z_2 z_3 + (\lambda_{12} + \lambda_{14} + \lambda_{24}) z_1 z_2 z_4 \\ +(\lambda_{12} + \lambda_{15} + \lambda_{25}) z_1 z_2 z_5 + (\lambda_{13} + \lambda_{14} + \lambda_{34}) z_1 z_3 z_4 \\ +(\lambda_{13} + \lambda_{15} + \lambda_{35}) z_1 z_3 z_5 + (\lambda_{14} + \lambda_{15} + \lambda_{45}) z_1 z_4 z_5 \\ +(\lambda_{23} + \lambda_{24} + \lambda_{34}) z_2 z_3 z_4 + (\lambda_{23} + \lambda_{25} + \lambda_{35}) z_2 z_3 z_5 \\ +(\lambda_{24} + \lambda_{25} + \lambda_{45}) z_2 z_4 z_5 + (\lambda_{34} + \lambda_{35} + \lambda_{45}) z_3 z_4 z_5 \end{array} \right\} , \tag{3.17}$$

with similar constraints on the coefficients:

$$\left\{ \begin{array}{l} \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{15} = 0 \\ \lambda_{12} + \lambda_{23} + \lambda_{24} + \lambda_{25} = 0 \\ \lambda_{13} + \lambda_{23} + \lambda_{34} + \lambda_{35} = 0 \\ \lambda_{14} + \lambda_{24} + \lambda_{34} + \lambda_{45} = 0 \\ \lambda_{15} + \lambda_{25} + \lambda_{35} + \lambda_{45} = 0 \end{array} \right. . \tag{3.18}$$

This is our non-generic K3 surface upon projectivization. The Jacobian becomes

$$
\begin{aligned}
J_5 = \{&\lambda_{12}z_2 + \lambda_{13}z_3 + \lambda_{14}z_4 + \lambda_{15}z_5, \lambda_{12}z_1 + \lambda_{23}z_3 + \lambda_{24}z_4 + \lambda_{25}z_5, \\
&\lambda_{13}z_1 + \lambda_{23}z_2 + \lambda_{34}z_4 + \lambda_{35}z_5, \lambda_{14}z_1 + \lambda_{24}z_2 + \lambda_{34}z_3 + \lambda_{45}z_5, \\
&\lambda_{15}z_1 + \lambda_{25}z_2 + \lambda_{35}z_3 + \lambda_{45}z_4, \\
&(\lambda_{12} + \lambda_{13} + \lambda_{23})z_2z_3 + (\lambda_{12} + \lambda_{14} + \lambda_{24})z_2z_4 + (\lambda_{12} + \lambda_{15} + \lambda_{25})z_2z_5 \\
&+ (\lambda_{13} + \lambda_{14} + \lambda_{34})z_3z_4 + (\lambda_{13} + \lambda_{15} + \lambda_{35})z_3z_5 + (\lambda_{14} + \lambda_{15} + \lambda_{45})z_4z_5, \\
&(\lambda_{12} + \lambda_{13} + \lambda_{23})z_1z_3 + (\lambda_{12} + \lambda_{14} + \lambda_{24})z_1z_4 + (\lambda_{12} + \lambda_{15} + \lambda_{25})z_1z_5 \\
&+ (\lambda_{23} + \lambda_{24} + \lambda_{34})z_3z_4 + (\lambda_{23} + \lambda_{25} + \lambda_{35})z_3z_5 + (\lambda_{24} + \lambda_{25} + \lambda_{45})z_4z_5, \\
&(\lambda_{12} + \lambda_{13} + \lambda_{23})z_1z_2 + (\lambda_{13} + \lambda_{14} + \lambda_{34})z_1z_4 + (\lambda_{13} + \lambda_{15} + \lambda_{35})z_1z_5 \\
&+ (\lambda_{23} + \lambda_{24} + \lambda_{34})z_2z_4 + (\lambda_{23} + \lambda_{25} + \lambda_{35})z_2z_5 + (\lambda_{34} + \lambda_{35} + \lambda_{45})z_4z_5, \\
&(\lambda_{12} + \lambda_{14} + \lambda_{24})z_1z_2 + (\lambda_{13} + \lambda_{14} + \lambda_{34})z_1z_3 + (\lambda_{14} + \lambda_{15} + \lambda_{45})z_1z_5 \\
&+ (\lambda_{23} + \lambda_{24} + \lambda_{34})z_2z_3 + (\lambda_{24} + \lambda_{25} + \lambda_{45})z_2z_5 + (\lambda_{34} + \lambda_{35} + \lambda_{45})z_3z_5, \\
&(\lambda_{12} + \lambda_{15} + \lambda_{25})z_1z_2 + (\lambda_{13} + \lambda_{15} + \lambda_{35})z_1z_3 + (\lambda_{14} + \lambda_{15} + \lambda_{45})z_1z_4 \\
&+ (\lambda_{23} + \lambda_{25} + \lambda_{35})z_2z_3 + (\lambda_{24} + \lambda_{25} + \lambda_{45})z_2z_4 + (\lambda_{34} + \lambda_{35} + \lambda_{45})z_3z_4\}\ .
\end{aligned}
\tag{3.19}
$$

Again, considering the linear terms, we see that for $(z_1, z_2, z_3, z_4, z_5) = (x, x, x, x, x)$ for $x \in \mathbb{C}$, we simply recover the Highest Weight condition (3.18). In fact, one can see that for all N, the set of linear constraints on the coefficients λ will exactly make the Jacobian vanish for the point $(x, x, \ldots, x)$ whereby making the projective point $[1 : 1 : \ldots : 1]$ always a singular point on the scattering variety $\mathcal{V}_\varphi(n = 2)$.

It is remarkable that the existence of this singularity is deeply rooted into the constraint from Highest Weight (2.9), and hence the fact that the polynomial system form an irreducible representation of the Möbius algebra.

3.3 Zero-Dimensional Varieties

Let us now focus on the most important case of projective dimension 0, i.e., the variety $\mathcal{V}^*_{n=2}$ which is a discrete set of points corresponding to the solutions of the scattering equations. As noted in Table 3, the numerators of the Hilbert series in Second Kind encode the Mahonian numbers $T_{N-3,k}$. Let us write I the ideal of the polynomial generators $\varphi_m^* := \lim_{z_1 \to \infty} \frac{1}{z_1}\varphi_{m+1} = \sum_{\substack{S\subset A' \\ |S|=m}} \varphi_{S^1} z_S$ (with $1 \leq m \leq N-3$, $S^1 = S \cup 1$ and $A' = \{a \in A : a \neq 1, N\}$) for $\mathcal{V}^*_{n=2}$. In other words, $\mathcal{V}^*_{n=2} \simeq \mathrm{Proj}(\mathbb{C}[z_2, ..., z_{N-1}]/I)$ as a projective variety. We then have the following structure due to the Mahonian numbers:

The global sections M of the variety $\mathcal{V}^*_{n=2}$ is a combination of graded vector spaces S with dimension $T_{N-3,i}$, that is

$$M = \mathbb{C}[z_2, ..., z_{N-1}]/I = \bigoplus_{i \geqslant 0} S_i \ .$$

Moreover, the Hilbert Functions $HF_S(i) := \dim_{\mathbb{C}} S_i = T_{N-3,i}$ for each graded pieces S_i.

This is implying that the dimension of each graded vector space is equals the number of permutations on $N-3$ elements with k inversions, which means that the k-th graded piece of the $\mathbb{C}[z_2, ..., z_{N-1}]$-module $\mathbb{C}[z_2, ..., z_{N-1}]/I$ is a $T_{N-3,k}$ dimensional vector space.

Explicit computation of the graded structure involves the resolution of the module M and the counting of Betti numbers. It turns out that, if assuming non-zero parameters λ_{ab} in the defining ideal I, namely we are choosing *generic points* in our parameter space corresponding to the zero-dimensional variety, then the resolution of the module M follows a predictable pattern as we now see.

As a concrete example, take the generic ideal for $N = 5$ as

$$I_5 = \langle \lambda_{12}z_2+\lambda_{13}z_3+\lambda_{14}z_4, (\lambda_{12}+\lambda_{13}+\lambda_{23})z_2z_3+(\lambda_{12}+\lambda_{14}+\lambda_{24})z_2z_4+(\lambda_{13}+\lambda_{14}+\lambda_{34})z_3z_4\rangle \ , \tag{3.20}$$

where we still have the highest weight condition (2.9) which translates as

$$\sum_{\substack{S \in A \setminus \{N\} \\ |S|=2}} \lambda_S = 0 \tag{3.21}$$

for the coefficients λ_{ab} in I_5. This can easily be seen by considering equation (2.9) for $a = N$ and realising that each terms $\lambda_S \mid N \in S$ are all individually included in the other constraints $a \neq N$. Thus, the sum of all equations (2.9) for $a = 1, \ldots, N-1$ will contain the vanishing sum of all terms $\lambda_S \mid N \in S$ and, thus, lead to (3.21).

The minimal resolution of the module $M = \mathbb{C}[z_2, ..., z_4]/I_5$ is

$$0 \leftarrow M \leftarrow R^1 \leftarrow R^2 \leftarrow R^1 \tag{3.22}$$

with Betti numbers

$i\backslash j$	0	1	2
0	1	1	0
1	0	1	1

. This Betti tally means that the j-th column of the i-th row gives the number of basis elements of degree $i+j$ in the free module $M[j]$ of shifted degree j. Therefore the Hilbert function $HF_S(k)$ can be represented by counting these Betti numbers: the column sum, here $(1,2,1)$, gives the total number of basis elements for $M[j]$.

Next, take $N = 6$. Now, we have defining ideal

$$\begin{aligned} I_6 = \langle &\lambda_{12}z_2 + \lambda_{13}z_3 + \lambda_{14}z_4 + \lambda_{15}z_5 \ , \\ &(\lambda_{12} + \lambda_{13} + \lambda_{23})z_2z_3 + (\lambda_{12} + \lambda_{14} + \lambda_{24})z_2z_4 + (\lambda_{13} + \lambda_{14} + \lambda_{34})z_3z_4 + \\ &(\lambda_{12} + \lambda_{15} + \lambda_{25})z_2z_5 + (\lambda_{13} + \lambda_{15} + \lambda_{35})z_3z_5 + (\lambda_{14} + \lambda_{15} + \lambda_{45})z_4z_5 \ , \\ &2(\lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{23} + \lambda_{24} + \lambda_{34})z_2z_3z_4 + 2(\lambda_{12} + \lambda_{13} + \lambda_{15} + \lambda_{23} + \lambda_{25} + \lambda_{35})z_2z_3z_5 + \\ &2(\lambda_{12} + \lambda_{14} + \lambda_{15} + \lambda_{24} + \lambda_{25} + \lambda_{45})z_2z_4z_5 + 2(\lambda_{13} + \lambda_{14} + \lambda_{15} + \lambda_{34} + \lambda_{35} + \lambda_{45})z_3z_4z_5\rangle \ , \end{aligned} \tag{3.23}$$

and again we have the condition $\sum_{\substack{S\in A\backslash\{N\} \\ |S|=2}} \lambda_S = 0$ on the λ coefficients. The minimal resolution:

$$0 \leftarrow M \leftarrow R^1 \leftarrow R^3 \leftarrow R^3 \leftarrow R^1 \tag{3.24}$$

with Betti numbers

$i\backslash j$	0	1	2	3
0	1	1	0	0
1	0	1	1	0
2	0	1	1	0
3	0	0	1	1

.

We clearly see and pattern allowing us to conjecture and tabulate the Hilbert functions as a vector of the dimension of the basis elements of the module $M[j]$:

N	$n = 2, \quad \text{Affine } \dim(\mathcal{V}^*_{n=2}) = 1$
5	$1, 2, 1$
6	$1, 3, 3, 1$
7	$1, 4, 6, 4, 1$
8	$1, 5, 10, 10, 5, 1$
9	$1, 6, 15, 20, 15, 6, 1$
10	$1, 7, 21, 35, 35, 21, 7, 1$
11	$1, 8, 28, 56, 70, 56, 28, 8, 1$
...	...

Table 5: Dimension of the basis elements of the module $M[j]$ for the varieties $\mathcal{V}^*_{n=2}$.

We conclude that the minimal resolution of the module M will be a exact sequence with the numbers of generators forming a Pascal's triangle. This is in congruence with known results on the Betti numbers of minimal resolutions of zero-dimensional ideals [23].

4 Discussion and Outlook

In this work, we presented geometrical properties of algebraic varieties built up from irreducible representations of the Möbius algebra in the hope that it will help to shed light on their physical meaning and, as a consequence, on the understanding of the scattering equation.

We found that these varieties are all *affine Calabi-Yau* manifolds with Hilbert series following very regular pattern, such as the Mahonian triangles or the cyclotomic polynomials. For the physical Möbius invariant three-dimensional varieties, we furthermore found that they consist of familiar geometries such as the *conifold* for the scattering of four particles, a cone over a *K3* surface for five particles and a cone over *Fano* surfaces (i.e., del Pezzo surfaces) for six and seven particles.

In addition, computation of the Euler number showed that not only are the affine Calabi-Yau spaces singular, but so too are their projectivizations to compact surfaces. A short computation unveiled that the singular points are related to the conditions for the polynomial system to be an irreducible representation of the Möbius algebra. The singularity consists of one point for the projective varieties and it would be expedient to understand its physical meaning. Indeed, the physical constraints on the momenta and hence coefficients in the scattering variety show that we are in special, singular points in the complex structure moduli space.

It would be interesting to develop further the geometrical properties, such as Hodge numbers, for each of the varieties. Furthermore, investigating the way the varieties degenerate when one (or more) of the polynomial coefficient λ_S vanishes and understanding the dependence of the geometrical properties of the varieties on the structure of these coefficients. Our study focused on generic parameters and very special momenta configurations might lead to extra geometrical structures. Understanding the physical meaning of the above varieties is of crucial importance and we hope that our analyses for the geometrical structure of the scattering variety offer a good starting point to unveil deeper connections.

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