# A SHORT PROOF OF THE DEUTSCH-SAGAN CONGRUENCE FOR CONNECTED NONCROSSING GRAPHS

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## 1. Introduction

Let $N_n$ be the number of connected noncrossing graphs on $n$ vertices. Flajolet and Noy [6] showed that for $n \geq 2$,

$$N_n = \frac{1}{n-1} \sum_{i=n-1}^{2n-3} \binom{3n-3}{n+i} \binom{i-1}{i-n+1}. \tag{1}$$

These numbers are sequence A007297 of the On-Line Encyclopedia of Integer Sequences [8]. Here is a table of small values of $N_n$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $N_n$ | 1 | 1 | 4 | 23 | 156 | 1162 | 9192 | 75819 | 644908 |

Deutsch and Sagan [4] conjectured that

$$N_n \equiv \begin{cases} 1 \pmod 3, & \text{if } n \text{ is a power of 3 or twice a power of 3,} \\ 2 \pmod 3, & \text{if } n \text{ is a sum of two distinct powers of 3,} \\ 0 \pmod 3, & \text{otherwise.} \end{cases} \tag{2}$$

They noted that the first two cases are not hard to prove using Lucas's theorem for the residue of a binomial coefficient modulo a prime. A complicated proof of the Deutsch-Sagan conjecture was given by Eu, Liu, and Yeh [5].

Here we give a simpler proof of Deutsch and Sagan's conjecture using Lagrange inversion. We then discuss some numbers related to the $N_n$ that arose in Eu, Liu, and Yeh's proof, given by sums similar to (1) and then we show how these sums can be evaluated explicitly.

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## 2. Proof of the congruence

We start by representing $N_n$ as a coefficient of a power series. We use the notation $[x^m]u(x)$ to denote the coefficient of $u(x)$ in the power series $u(x)$.

**Lemma 1.** *For $n \geq 1$,*

$$N_{n+1} = \frac{1}{n}[x^{n-1}]\frac{1}{(1-x)^{n+2}(1-2x)^n}. \tag{3}$$

*Proof.* We rewrite (1) as

$$N_{n+1} = \frac{1}{n}\sum_{k=0}^{n-1}\binom{3n}{n-1-k}\binom{n+k-1}{k}. \tag{4}$$

We have

$$1-2x = (1-x)\left(1-\frac{x}{1-x}\right),$$

so

$$\begin{aligned}\frac{1}{(1-x)^{n+2}(1-2x)^n} &= (1-x)^{-2n-2}\left(1-\frac{x}{1-x}\right)^{-n}\\ &= \sum_{j=0}^{\infty}\binom{n+j-1}{j}\frac{x^j}{(1-x)^{2n+2+j}}\\ &= \sum_{j=0}^{\infty}\binom{n+j-1}{j}\sum_{k=0}^{\infty}\binom{2n+1+j+k}{k}x^{j+k}.\end{aligned}$$

Then the coefficient of $x^{n-1}$ is

$$\sum_{j=0}^{n-1}\binom{n+j-1}{j}\binom{3n}{n-1-j},$$

which by (4) is $nN_{n+1}$. □

Next, recall the following form of Lagrange inversion [9, p. 42, equation (5.65)].

**Lagrange Inversion Theorem, First Form.** *Let $G(t)$ be a formal power series and let $f = f(x)$ be the unique formal power series satisfying $f = xG(f)$. Then for any formal power series $\Phi(x)$,*

$$[x^n]\Phi(f) = \frac{1}{n}[x^{n-1}]\Phi'(x)G(x)^n.$$

The Deutsch-Sagan conjecture is an immediate consequence of the following result.

**Theorem 2.** *Let* $F = \sum_{m=0}^{\infty} x^{3^m}$. *Then*

$$\sum_{n=1}^{\infty} N_n x^n \equiv F + F^2 \pmod 3.$$

*Proof.* We apply the Lagrange inversion theorem with $G(x) = 1/(1-x)(1-2x)$ and $\Phi(x) = 1/(1-x)$, so that $\Phi'(x) = 1/(1-x)^2$. Then by Lemma 1, together with the fact that $N_1 = 1$, we have

$$\sum_{n=0}^{\infty} N_{n+1} x^n = \frac{1}{1-\alpha} \tag{5}$$

where $\alpha$ is the unique formal power series satisfying

$$\alpha = \frac{x}{(1-\alpha)(1-2\alpha)}. \tag{6}$$

By (6), $\alpha - 3\alpha^2 + 2\alpha^3 = x$, so we have $\alpha(x) \equiv x - 2\alpha(x)^3 \equiv x + \alpha(x)^3 \equiv x + \alpha(x^3)$ (mod 3). Iterating this congruence gives

$$\alpha(x) \equiv x + x^3 + \alpha(x^{27}) \equiv \cdots \equiv \sum_{m=0}^{\infty} x^{3^m} \pmod 3. \tag{7}$$

By (6),

$$\frac{1}{1-\alpha} = x^{-1}(\alpha - 2\alpha^2) \equiv x^{-1}(\alpha + \alpha^2),$$

so by (5),

$$\sum_{n=1}^{\infty} N_n x^n \equiv \alpha + \alpha^2 \pmod 3. \tag{8}$$

Then Deutsch and Sagan's congruence (2) follows directly from (8) and (7). ∎

## 3. Eu, Liu, and Yeh's congruences

In their proof of the Deutsch-Sagan conjecture, Eu, Liu, and Yeh [5, Lemmas 1–4] found the residues modulo 3 for four auxiliary sequences which they define by

$$\begin{aligned} f_1(n) &= \sum_i \binom{3n+1}{n+i+1}\binom{i}{n} \\ f_2(n) &= \sum_i \binom{3n}{n+i+1}\binom{i}{n} \\ f_3(n) &= \sum_i \binom{3n}{n+i}\binom{i}{n} \\ f_4(n) &= \sum_i \binom{3n-1}{n+i+1}\binom{i}{n-1}, \end{aligned}$$

with $f_4(0) = 0$. We will also consider a fifth sum

$$f_5(n) = \sum_i \binom{3n}{n+i+1}\binom{i}{n-1},$$

with $f_5(0) = 1$.

The first few values of these sums are as follows:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $f_1(n)$ | 1 | 6 | 48 | 420 | 3840 | 36036 | 344064 | 3325608 | 32440320 |
| $f_2(n)$ | 0 | 1 | 9 | 82 | 765 | 7266 | 69930 | 679764 | 6659037 |
| $f_3(n)$ | 1 | 5 | 39 | 338 | 3075 | 28770 | 274134 | 2645844 | 25781283 |
| $f_4(n)$ | 0 | 1 | 7 | 58 | 515 | 4746 | 44758 | 428772 | 4154403 |
| $f_5(n)$ | 1 | 4 | 30 | 256 | 2310 | 21504 | 204204 | 1966080 | 19122246 |

Eu, Liu, and Yeh noted that $f_1(n) = f_2(n) + f_3(n)$ and that $f_2(n)$ is the number of edges in all noncrossing connected graphs on $n+1$ vertices for $n \geq 1$ (sequence A045741). The sequence $f_5(n)$ is sequence A091527 in the OEIS, and $f_1(n)$, $f_3(n)$, and $f_4(n)$ do not currently appear in the OEIS.

To derive Eu, Liu, and Yeh's congruences by our method, we consider the more general sequence $h_{j,k,l}(n)$, where $j$, $k$, $l$, and $n$ are arbitrary integers, defined by

$$h_{j,k,l}(n) = \sum_{i=0}^{n+j-k+l} \binom{3n+j}{n+j-k+l-i}\binom{n-l+i}{i}.$$

Then if $3n+j \geq 0$ and $n \geq l$, replacing the summation index $i$ with $i-n+l$ gives

$$h_{j,k,l}(n) = \sum_{i=n-l}^{2n+j-k} \binom{3n+j}{n+i+k}\binom{i}{n-l}.$$

Thus $f_1 = h_{1,1,0}$, $f_2 = h_{0,1,0}$, $f_3 = h_{0,0,0}$, $f_4 = h_{-1,1,1}$, and $f_5 = h_{0,1,1}$. A straightforward computation, as in the proof of Lemma 1, shows that

$$\begin{aligned} h_{j,k,l}(n) &= [x^n]\frac{x^{k-l-j}}{(1-x)^{n+k}(1-2x)^{n-l+1}} \\ &= [x^n]\frac{x^{k-l-j}}{(1-x)^k(1-2x)^{-l+1}} \cdot \frac{1}{(1-x)^n(1-2x)^n} \end{aligned}$$

Now we use the following form of Lagrange inversion (see, e.g., [7, equation (4.4)]; a closely related formula is [3, p. 150, Theorem D]):

**Lagrange Inversion Theorem, Second Form.** *Let $G(x)$ be a formal power series and let $f = f(x)$ be the unique formal power series satisfying $f = xG(f)$. Then for any formal Laurent series $\Psi(x)$ and any integer $n$,*

$$[x^n]\frac{\Psi(f)}{1-xG'(f)} = [x^n]\Psi(x)G(x)^n.$$

*or equivalently*

$$[x^n]\frac{\Psi(f)}{1-fG'(f)/G(f)} = [x^n]\Psi(x)G(x)^n.$$

As in the proof of Theorem 2, we apply this to the equation $\alpha = xG(\alpha)$, where $G(x) = 1/(1-x)(1-2x)$. Then

$$\frac{1}{1-\alpha G'(\alpha)/G(\alpha)} = \frac{(1-\alpha)(1-2\alpha)}{1-6\alpha+6\alpha^2}.$$

Now let

$$H_{j,k,l} = \sum_{n=k-j-l}^{\infty} h_{j,k,l}(n)x^n.$$

Then taking

$$\Psi(x) = \frac{x^{k-j-l}}{(1-x)^k(1-2x)^{-l+1}}$$

we obtain

$$H_{j,k,l} = \frac{(1-2\alpha)^l\alpha^{k-j-l}}{(1-6\alpha+6\alpha^2)(1-\alpha)^{k-1}}, \tag{9}$$

and thus

$$H_{j,k,l} \equiv (1+\alpha)^l(1-\alpha)^{1-k}\alpha^{k-j-l} \pmod 3. \tag{10}$$

Now let $F_m = \sum_{n=0}^{\infty} f_m(n)x^n$. Then by (10) we have

$$\begin{aligned} F_1 &= H_{1,1,0} \equiv 1 \pmod 3 \\ F_2 &= H_{0,1,0} \equiv \alpha \pmod 3 \\ F_3 &= H_{0,0,0} \equiv 1-\alpha \pmod 3 \\ F_4 &= H_{-1,1,1} \equiv \alpha+\alpha^2 \pmod 3 \\ F_5 &= H_{0,1,1} \equiv 1+\alpha \pmod 3). \end{aligned} \tag{11}$$

Then Eu, Liu, and Yeh's congruences follow immediately from these congruences and (7). We don't state Eu, Liu, and Yeh's congruences here since they are easy to read off from the congruences in (11) and (7).

## 4. Evaluation of the sums

There are simple explicit formulas for the sums $f_1$, $f_2$, $f_3$, $f_4$, and $f_5$ (and there is a similar formula for $N_n$ that we will give in section 5), though these formulas don't seem to yield simpler proofs for the congruences than the proofs we have already given.

**Theorem 3.** *The sequences $f_i(n)$ for $i$ from 1 to 5 are given by the following explicit formulas:*

$$\begin{aligned} f_1(n) &= 2^{2n}\binom{\frac32 n}{n} \\ f_2(n) &= 2^{2n-1}\binom{\frac32 n}{n} - 2^{2n-1}\binom{\frac32 n-\frac12}{n} \\ f_3(n) &= 2^{2n-1}\binom{\frac32 n}{n} + 2^{2n-1}\binom{\frac32 n-\frac12}{n} \\ f_4(n) &= -\frac{2^{2n-1}}{3}\binom{\frac32 n}{n} + 2^{2n-1}\binom{\frac32 n-\frac12}{n}, \text{ for } n>0 \\ f_5(n) &= 2^{2n}\binom{\frac32 n-\frac12}{n}. \end{aligned}$$

The evaluation of these sums is based on a binomial coefficient identity that is equivalent to a terminating case of a hypergeometric series evaluation called Kummer's theorem [2, p. 9, Theorem 2.3]. There are many ways to prove this identity; we give here a proof using Lagrange inversion. A short self-contained proof was given by Wildon [10].

**Lemma 4.**

$$\sum_{k=0}^{n} \binom{2n+a}{n-k}\binom{a+k-1}{k} = 2^{2n}\binom{\frac{1}{2}a+n-\frac{1}{2}}{n} \tag{12}$$

*Proof.* Let $S$ be the left side of (12). Then $S$ is equal to the coefficient of $x^n$ in $1/(1-x)^{n+1}(1-2x)^a$ since

$$\begin{aligned}\frac{1}{(1-x)^{n+1}(1-2x)^a} &= \frac{1}{(1-x)^{n+a+1}\left(1-\dfrac{x}{1-x}\right)^a} \\ &= \sum_k \frac{x^k}{(1-x)^{n+a+k+1}}\binom{a+k-1}{k} \\ &= \sum_{j,k} x^{j+k}\binom{n+a+j+k}{j}\binom{a+k-1}{k} \\ &= \sum_m x^m \sum_k \binom{n+a+m}{m-k}\binom{a+k-1}{k}.\end{aligned}$$

Let us take $G(x) = 1/(1-x)$ and $\Psi(x) = 1/(1-x)(1-2x)^a$ in the second form of Lagrange inversion. The solution of $f = xG(f)$ is

$$f = \frac{1-\sqrt{1-4x}}{2}$$

and $\Psi(x)/\big(1 - xG'(x)/G(x)\big) = (1-2x)^{-a-1}$, so

$$\begin{aligned}S &= [x^n]\frac{1}{(1-2f)^{a+1}} = [x^n]\frac{1}{\left(\sqrt{1-4x}\right)^{a+1}} \\ &= (-4)^n\binom{-(a+1)/2}{n} = 2^{2n}\binom{\frac{1}{2}a+n-\frac{1}{2}}{n}.\end{aligned}$$

□

We can now prove Theorem 3. Let

$$\begin{aligned}T_1(n,i) &= \binom{3n+1}{n+i+1}\binom{i}{n} \\ T_2(n,i) &= \binom{3n}{n+i+1}\binom{i}{n} \\ T_3(n,i) &= \binom{3n}{n+i}\binom{i}{n} \\ T_4(n,i) &= \binom{3n-1}{n+i+1}\binom{i}{n-1}\end{aligned}$$

$$T_5(n,i) = \binom{3n}{n+i+1}\binom{i}{n-1},$$

so that for $m = 1, \ldots, 5$ we have $f_m(n) = \sum_i T_m(n,i)$. Then by Lemma 4 we have

$$f_1(n) = \sum_i T_1(n,i) = 2^{2n}\binom{\frac{3}{2}n}{n}$$

$$f_5(n) = \sum_i T_5(n,i) = 2^{2n}\binom{\frac{3}{2}n - \frac{1}{2}}{n}.$$

Also, it is easy to check that $T_2(n,i)+T_3(n,i) = T_1(n,i)$, as noted in [5], that $T_3(n,i)-T_2(n,i-1) = T_5(n,i-1)$ for $n > 0$, and that $T_4(n,i) = \frac{2}{3}T_5(n,i) - \frac{1}{3}T_3(n,i+1)$. Thus $f_2(n) + f_3(n) = f_1(n)$, $f_3(n) - f_2(n) = f_5(n)$, and $f_4(n) = \frac{2}{3}f_5(n) - \frac{1}{3}f_3(n)$ for $n \geq 1$. We can then solve for $f_2$, $f_3$ and $f_4$ in terms of $f_1$ and $f_5$, and we can check that the formulas given in Theorem 3 also hold for $f_m(0)$ if $m \neq 4$. □

## 5. More Lagrange inversion

We can also prove Theorem 3, and derive a related formula for $N_n$, by Lagrange inversion.

Let us define the power series $\beta$ in $x$ by

$$\beta = \frac{x}{\sqrt{1-4\beta}}. \tag{13}$$

If we define the power series $\alpha = \alpha(x)$ by $\beta = \alpha - \alpha^2$ in (13) (together with the condition $\alpha(0) = 0$) we see that $\alpha$ satisfies

$$\alpha - \alpha^2 = \frac{x}{1-2\alpha},$$

so $\alpha = x/(1-\alpha)(1-2\alpha)$ and thus this $\alpha$ is the same power series as the $\alpha$ discussed in sections 2 and 3. Applying the second form of Lagrange inversion, we have for any power series $\Psi(x)$,

$$[x^n]\frac{1-4\beta}{1-6\beta}\Psi(\beta) = [x^n]\frac{\Psi(x)}{(1-4x)^{n/2}}.$$

Then taking $\Psi(x) = x^i(1-4x)^{r-1}$ gives

$$\frac{(1-4\beta)^r}{1-6\beta}\beta^i = \sum_{n=0}^{\infty} 2^{2n-2i}(-1)^{n-i}\binom{-\frac{1}{2}n + r - 1}{n-i}.$$

Since $\beta = \alpha - \alpha^2$ and $(-1)^{n-i}\binom{-n/2+r-1}{n-i} = \binom{3n/2-r-i}{n-i}$, we may write this as

$$\frac{(1-2\alpha)^{2r}(\alpha-\alpha^2)^i}{1-6\alpha+6\alpha^2} = \sum_{n=0}^{\infty} 2^{2n-2i}\binom{\frac{3}{2}n-r-i}{n-i}x^n. \tag{14}$$

Then in the notation of section 3, by (9) and the formulas for the $F_m$ given in (11), we have

$$\begin{aligned} F_1 &= \frac{1}{1-6\alpha+6\alpha^2} \\ F_2 &= \frac{\alpha}{1-6\alpha+6\alpha^2} \\ F_3 &= \frac{1-\alpha}{1-6\alpha+6\alpha^2} \\ F_4 &= \frac{\alpha-2\alpha^2}{1-6\alpha+6\alpha^2} \\ F_5 &= \frac{1-2\alpha}{1-6\alpha+6\alpha^2} \end{aligned}$$

from which the formulas of Theorem 3 can be obtained: $F_1$ and $F_5$ can be evaluated by (14), $F_2$ and $F_3$ are linear combinations of $F_1$ and $F_5$, and $F_4 = -\frac{1}{6}F_1 + \frac{1}{2}F_5 - \frac{1}{3}$.

Similarly, the first form of Lagrange inversion gives

$$[x^n]\Phi(\beta) = \frac{1}{n}[x^{n-1}]\frac{\Phi'(x)}{(1-4x)^{n/2}}.$$

Let us take $\Phi(x) = (1-4x)^r$. Then we have

$$(1-4\beta)^r = 1 + \sum_{n=1}^{\infty}(-4)^n\frac{r}{n}\binom{-n/2+r-1}{n-1}x^n = 1 - \sum_{n=1}^{\infty} 2^{2n}\frac{r}{n}\binom{\frac{3}{2}n-r-1}{n-1}x^n,$$

so

$$(1-2\alpha)^{2r} = 1 - \sum_{n=1}^{\infty} 2^{2n}\frac{r}{n}\binom{\frac{3}{2}n-r-1}{n-1}x^n. \tag{15}$$

From (5) it follows that $\sum_{n=1}^{\infty} N_n x^n = x/(1-\alpha)$, and by (6),

$$\frac{x}{1-\alpha} = \alpha - 2\alpha^2 = \tfrac{1}{2}(1-2\alpha) - \tfrac{1}{2}(1-2\alpha)^2,$$

so by (15), for $n \geq 1$ we have

$$N_n = \frac{1}{2}\left[\frac{2^{2n}}{n}\binom{\frac{3}{2}n-2}{n-1} - \frac{2^{2n-1}}{n}\binom{\frac{3}{2}n-\frac{3}{2}}{n-1}\right]$$
$$= \frac{2^{2n-1}}{n}\binom{\frac{3}{2}n-2}{n-1} - \frac{2^{2n-2}}{n}\binom{\frac{3}{2}n-\frac{3}{2}}{n-1}. \tag{16}$$

An equivalent formula was stated by Mark van Hoeij in the OEIS entry for sequence A007297. The first term on the right side of (16), $(2^{2n-1}/n)\binom{3n/2-2}{n-1}$ is twice sequence A078531, and the negative of the second term, $(2^{2n-2}/n)\binom{3n/2-3/2}{n-1}$ is sequence A085614. We note also that if $n = 2m+1$ then

$$\frac{2^{2n-1}}{n}\binom{\frac{3}{2}n-2}{n-1} = 2\frac{m!\,(6m)!}{(2m)!\,(2m+1)!\,(3m)!}$$

and if $n = 2m+2$ then

$$\frac{2^{2n-2}}{n}\binom{\frac{3}{2}n-\frac{3}{2}}{n-1} = 6\frac{m!\,(6m+1)!}{(2m)!\,(2m+2)!\,(3m)!}.$$

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